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On the Duals of the Stable Rank 3 Reflexive Sheaves on $\mathbb{P}^3$ with c_3 Maximal

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Introduction

The aim of this note is to complete the results of [1]. Let $P = \mathbb{P}^3_k$ be the projective 3-space over an algebraically closed field k of characteristic 0. In [1] one proves some finer properties of the spectrum of the stable rank 3 reflexive sheaves on P which have stable restriction to a general plane. Here we consider the missing case (cf. [3], Theorem 0.1): the duals of the stable rank 3 reflexive sheaves on P with $c_1 = 0$ and $c_3 = c_2^2 - c_2$. We show that the spectrum of these duals has properties similar to those proved in [1]. We also determine their Chern classes.

The Chern classes of the stable rank 3 reflexive sheaves on P have been determined by MIRÓ-ROIG in her thesis [4]. The result is the following:

If $c_1 = 0$ then $c_2 \geqq 2$ and:

$$(0)\quad c_3 \in [-c_2^2+c_2, c_2^2-c_2] \setminus \bigcup_{q=1}^{B(0,c_2)} (c_2^2-(2q+1)c_2 + 2(q^2+q+1), c_2^2-(2q-1)c_2)$$

if $c_1 = -1$ then $c_2 \geqq 1$ and

$$(1)\quad c_3 \in [-c_2^2, c_2^2-2\ c_2+2] \setminus \bigcup_{q=1}^{B(-1,c_2)} (c_2^2-2(q+1)c_2 + (q+1)^2, c_2^2 - 2q\ c_2)$$

and if $c_1 = -2$ then $c_2 \geqq 2$ and:

(2) $$c_3 \in [-c_2^2+3c_2-4,\, c_2^2-3c_2+2] \setminus \bigcup_{q=1}^{B(-2,c_2)} (c_2^2-(2q+3)c_2+2(q+1)^2,\ c_2^2-(2q+1)c_2+2q)$$

where $B(c_1,c_2)$ are the largest integers for which the intervals are nonempty. Moreover, one has to take into account that $c_1c_2 \equiv c_3$ (mod 2).

We show that if $c_2 \geq 3$ and c_3 verifies (0) (but $c_3 \neq c_2^2-c_2$) then there exists a stable rank 3 reflexive sheaf E on P with $c_1(E)=0$, $c_2(E)=c_2$, $c_3(E)=c_2^2-c_2$ such that $c_3(E^*)=c_3$ (Example 0).

We also notice that if $c_2 \geq 2$ and c_3 verifies (1) then there exists a stable rank 3 reflexive sheaf E on P with $c_1(E)=-2$, $c_2(E)=c_2+1$, $c_3(E)=(c_2+1)^2-3(c_2+1)+2$ such that $c_3(E^*(-1))=c_3$ (Example 1).

We do not know if a similar result holds for the stable rank 3 reflexive sheaves on P with $c_1=-1$ and c_3 maximal.

Results

Let E be a stable rank 3 reflexive sheaf on P with Chern classes $c_1=0$, $c_2 \geq 2$ and $c_3=c_2^2-c_2$. If $c_2 \geq 4$ then by [3], Proposition 5.1, there is an exact sequence:

(3) $$0 \longrightarrow \Omega_P(1) \longrightarrow E \longrightarrow \mathcal{O}_{H_0}(-c_2+1) \longrightarrow 0$$

for some plane $H_0 \subset P$.

PROPOSITION. Let E be a stable rank 3 reflexive sheaf on P with $c_1=0$ and $c_2 \geq 3$ for which there exists an exact sequence (3). Let $k_{E^*}=(k_1,\ldots,k_m)$ be the spectrum of E^*. Here $m=c_2$.

(a) If 0 does not appear in the spectrum then $k_{m-2}=k_{m-1}=k_m=-$

(b) If $k_1 \leq -2$ then E^* has an unstable plane of order $-k_1$.

(c) If there is an i with $2 \leq i \leq m-1$ such that $k_{i-1}<k_i<k_{i+1} \leq$ then $k_1<k_2<\ldots<k_i$.

PROOF. (a) Dualizing (3) one gets an exact sequence:

$$(4) \qquad 0 \longrightarrow E^* \longrightarrow T_P(-1) \xrightarrow{\alpha} \mathcal{O}_{H_o}(c_2) \longrightarrow \mathcal{E}xt^1(E,\mathcal{O}_P) \longrightarrow 0$$

It follows that Coker (α) has finite length and $H^o(\alpha)$ is injective. We have $T_P(-1)|H_o \cong T_{H_o}(-1) \oplus \mathcal{O}_{H_o}$ and the morphism $H^o(T_P(-1)) \longrightarrow H^o(T_P(-1)|H_o)$ is an isomorphism. It follows that the cokernel of the morphism $\alpha|H_o : T_{H_o}(-1)\oplus \mathcal{O}_{H_o} \to \mathcal{O}_{H_o}(c_2)$ has finite length and that $H^o(\alpha|H_o)$ is injective. Let $F = \mathrm{Ker}(\alpha|H_o)$. F is a locally free $\mathcal{O}_{H_o}$-module of rank 2 with $c_1(F) = -c_2+1$ and $H^o(F) = 0$. By the Theorem of Grauert-Mülich we have $H^o(F|L) = 0$ for the generic lines $L\subset H_o$ of F.

Let $H\subset P$ be a plane which does not contain any singular point of E and such that the line $L = H\cap H_o$ is a generic line of F. We have an exact sequence:

$$0 \longrightarrow E^*|H \longrightarrow T_H(-1)\oplus \mathcal{O}_H \xrightarrow{\alpha|H} \mathcal{O}_L(c_2) \longrightarrow 0$$

Using the exact sequence:

$$0 \longrightarrow F|L \longrightarrow (T_{H_o}(-1)\oplus \mathcal{O}_{H_o})|L \xrightarrow{\alpha|L} \mathcal{O}_L(c_2) \longrightarrow 0$$

one deduces that $H^o(\alpha|L)$ is injective. The morphism $H^o(T_H(-1)\oplus \mathcal{O}_H) \longrightarrow H^o((T_H(-1)\oplus \mathcal{O}_H)|L)$ being an isomorphism one finds that $H^o(\alpha|H)$ is injective, hence $H^o(E^*|H) = 0$. The Theorem of Riemann-Roch on H implies that $h^1(E^*|H) = c_2-3$.

Put now:

$N_\ell = \mathrm{Im}\,(H^1(E^*(\ell)) \longrightarrow H^1(E^*(\ell)|H))$, $n_\ell = \dim_k N_\ell$

$R_\ell = \mathrm{Coker}\,(H^1(E^*(\ell)) \longrightarrow H^1(E^*(\ell)|H))$, $r_\ell = \dim_k R_\ell$.

For $\ell \leq -1$ we have an exact sequence:

$$0 \longrightarrow H^o(\mathcal{O}_L(c_2+\ell)) \longrightarrow H^1(E^*(\ell)|H) \longrightarrow H^1(T_H(\ell-1)\oplus \mathcal{O}_H(\ell)) .$$

If $\ell \neq -2$ then $H^1(T_H(\ell-1)\oplus \mathcal{O}_H(\ell)) = 0$. Using the commutative diagram:

$$\begin{array}{ccc} H^1(E^*(-2)) & \longrightarrow & H^1(T_P(-3)) = 0 \\ \downarrow & & \downarrow \\ H^1(E^*(-2)\,|\,H) & \longrightarrow & H^1(T_H(-3)\oplus \mathcal{O}_H(-2)) \end{array}$$

one finds that the composed map:

$$H^1(E^*(-2)) \longrightarrow H^1(E^*(-2)\,|\,H) \longrightarrow H^1(T_H(-3)\oplus \mathcal{O}_H(-2))$$

is 0. It follows that $N_\ell \subseteqq H^0(\mathcal{O}_L(c_2+\ell))$ for all $\ell \leqq -1$.

Now, if 0 does not appear in the spectrum then $n_{-2} = n_{-1}$ (see [1], Sect.2). It follows that $n_{-1} = 0$, hence $n_\ell = 0$ for all $\ell \leqq -1$, hence $k_m \leqq -1$. Furthermore:

$$\begin{aligned} \operatorname{card}\{j\,|\,k_j = -1\} &= r_{-1}-r_0 \geqq h^1(E^*(-1)\,|\,H) - h^1(E^*\,|\,H) = \\ &= c_2 - (c_2-3) = 3. \end{aligned}$$

(b) The definition of the spectrum implies that $H^2(E^*(-k_1-2)) = 0$ and $H^2(E^*(-k_1-3)) \neq 0$. Let $b \in H^2(E^*(-k_1-3))'$ be a nonzero element. For every plane $H \subset P$ we denote by b_H the image of b by the morphism $\delta' : H^2(E^*(-k_1-3))' \longrightarrow H^1(E^*(-k_1-2)\,|\,H)'$.

Using the exact sequence:

$$0 = H^2(E^*(-k_1-2))' \longrightarrow H^2(E^*(-k_1-3))' \longrightarrow H^1(E^*(-k_1-2)\,|\,H)'$$

one finds that $b_H \neq 0$.

Now, suppose that H does not contain any singular point of E. Let $L = H \cap H_0$ and let $\lambda_H \in H^0(\mathcal{O}_H(1))$ be an equation of the line L. As we already observed, for $\ell \neq -2$ there is an exact sequence:

$$H^0(\mathcal{O}_L(c_2+\ell)) \longrightarrow H^1(E^*(\ell)\,|\,H) \longrightarrow 0\ .$$

It follows that λ_H annihilates $H^1(E^*(\ell)\,|\,H)$ for $\ell \geq -1$, hence λ_H annihilates $H^1(E^*(\ell)\,|\,H)'$ for $\ell \geqq 0$. By [1], Lemma 2.4 and Remark 2.3, E^* has an unstable plane of order $-k_1$.

(c) By (b), there is an unstable plane $H_1 \subset P$ for E^* of order $-k_1$. We apply the reduction step:

$$0 \longrightarrow E' \longrightarrow E^* \longrightarrow \mathcal{O}_{H_1}(k_1)$$

E' is a stable rank 3 reflexive sheaf with $c_1(E') = -1$. Let $H \subset P$ be a plane which does not contain any of the singular points of E or E' or any of the points where the morphism $E^* \longrightarrow \mathcal{O}_{H_1}(k_1)$ is not surjective. Let $L = H \cap H_1$. We have an exact sequence:

$$0 \longrightarrow E'|H \longrightarrow E^*|H \longrightarrow \mathcal{O}_L(k_1) \longrightarrow 0$$

We cannot have $E' = \Omega_P(1)$ because $H^0(E^*|H) = 0$ for the general plane $H \subset P$. By the restriction theorem of Schneider ([3], Theorem 3.4) we may suppose that $E'|H$ is stable.

Now, using the results of [1], Sect.1, one proves in a standard manner that $k_1 < k_2 < \ldots < k_i$.

EXAMPLE 0. We shall construct sufficiently many examples of stable rank 3 reflexive sheaves on P with $c_1 = 0$, $c_2 \geqq 3$ and $c_3 = c_2^2 - c_2$ in order to show that their duals get all the possible values of c_3 except $c_2^2 - c_2$.

Let $H_o \subset P$ be a plane, $\alpha_o : T_{H_o}(-1) \longrightarrow \mathcal{O}_{H_o}(c_2)$ a morphism of $\mathcal{O}_{H_o}$-modules such that $H^0(\alpha_o)$ is injective and let $f \in H^0(\mathcal{O}_{H_o}(c_2))$ be such that $f \notin H^0(\mathrm{Im}\,\alpha_o)$ and $\mathcal{O}_{H_o}(c_2)/(\mathrm{Im}\,\alpha_o + \mathcal{O}_{H_o} f)$ has finite length. α_o and f define a morphism $T_{H_o}(-1) \oplus \mathcal{O}_{H_o} \longrightarrow \mathcal{O}_{H_o}(c_2)$, which composed with the restriction morphism $T_P(-1) \longrightarrow T_{H_o}(-1) \oplus \mathcal{O}_{H_o}$ gives us a morphism $\alpha : T_P(-1) \longrightarrow \mathcal{O}_{H_o}(c_2)$. Coker($\alpha$) has finite length and $H^0(\alpha)$ is injective. It follows that $E = (\mathrm{Ker}\,\alpha)^*$ is a stable rank 3 reflexive sheaf with $c_1(E) = 0$, $c_2(E) = c_2$, $c_3(E) = c_2^2 - c_2$.

We have : $\mathcal{E}xt^1(E, \mathcal{O}_P) = \mathrm{Coker}(\alpha) = \mathcal{O}_{H_o}(c_2)/(\mathrm{Im}\,\alpha_o + \mathcal{O}_{H_o} f)$. If

s = length $(\mathcal{E}xt^1(E,\mathcal{O}_p))$ then $c_3(E^*) = -c_2^2+c_2+2s$ ([3], Lemma 4.1).

Firstly, let Z be a closed subscheme of H_o consisting of $c_2^2-c_2+1$ simple points geometrically linked to a subscheme S of H_o consisting of c_2-1 colinear simple points by 2 nonsingular curves Y_1, Y_2 of degree c_2. This means that $Y_1 \cap Y_2 = Z \cup S$ as schemes.

We assert that there is a morphism $\alpha_o : T_{H_o}(-1) \longrightarrow \mathcal{O}_{H_o}(c_2)$ such that $\operatorname{Im}\alpha_o = I_Z(c_2)$. Indeed, by [6], Proposition 2.5, $I_Z(c_2)$ has a resolution:

$$0 \longrightarrow \mathcal{O}_{H_o}(-c_2+1) \oplus \mathcal{O}_{H_o}(-1) \longrightarrow \mathcal{O}_{H_o}^3 \longrightarrow I_Z(c_2) \longrightarrow 0$$

We may suppose that the morphism $\mathcal{O}_{H_o}(-1) \longrightarrow \mathcal{O}_{H_o}^3$ is a monomorphism of locally free $\mathcal{O}_{H_o}$-modules (this is equivalent to the fact that none of the points of Z lies on the line which contains S). It follows that $\operatorname{Coker}(\mathcal{O}_{H_o}(-1) \longrightarrow \mathcal{O}_{H_o}^3) \cong T_{H_o}(-1)$, hence we obtain an epimorphism $T_{H_o}(-1) \longrightarrow I_Z(c_2)$ which composed with the inclusion $I_Z(c_2) \longrightarrow \mathcal{O}_{H_o}(c_2)$ gives us the morphism α_o.

Now, an argument similar to one used in [2], Example 1.4, shows us that there is a set $T \subset Z$ consisting of $\frac{1}{2}(c_2-1)(c_2-2)$ points such that if $T \subset D \subseteq Z$ then there is an $f \in H^o(\mathcal{O}_{H_o}(c_2))$ vanishing at every point of $(Z \setminus D) \cup S$ but at no point of D. α_o and f satisfy the conditions stated at the beginning of our example. If E is the reflexive sheaf constructed using α_o and f then $s = \operatorname{card}(Z \setminus D)$. It follows that $c_3(E^*)$ gets all the even values from the interval $[-c_2^2+c_2, 2c_2-2]$.

Next, let q be such that $2 \leq q \leq \frac{1}{2}c_2+1$. Let $\alpha_1 : T_{H_o}(-1) \to \mathcal{O}_{H_o}(q)$ be a morphism such that $\operatorname{Im}\alpha_1 = I_{Z_1}(q)$, where Z_1 is a closed subscheme of H_o consisting of q^2-q+1 simple points lying on a nonsingular curve $C \subset H_o$ of degree q. Let $g \in H^o(\mathcal{O}_{H_o}(c_2-q))$ be such that g vanishes at no point of Z_1.

Let $\alpha_o : T_{H_o}(-1) \longrightarrow \mathcal{O}_{H_o}(c_2)$ be the composed map:

$$T_{H_o}(-1) \xrightarrow{\alpha_1} \mathcal{O}_{H_o}(q) \xrightarrow{g} \mathcal{O}_{H_o}(c_2)$$

If D_1 is a subset of Z_1 then there is an $f \in H^0(\mathcal{O}_{H_o}(c_2))$ vanishing at every point of D_1 but at no point of $Z_1 \setminus D_1$. Indeed:

$$\deg(\mathcal{O}_C(c_2) \otimes \mathcal{O}_C(-D_1)) \geqq c_2 q-(q^2-q+1) \geqq (q-1)(q-2)+1 = 2g(C) + 1$$

hence $\mathcal{O}_C(c_2) \otimes \mathcal{O}_C(-D_1)$ is very ample on C.

If E is the reflexive sheaf constructed using α_o and f then $s = c_2(c_2-q) + \text{card } D_1$. In this case, $c_3(E^*)$ gets all the even values from the interval:

$$\left[c_2^2-(2q-1)c_2,\ c_2^2-(2q-1)c_2+2(q^2-q+1)\right] .$$

EXAMPLE 1. We shall construct sufficiently many examples of stable rank 3 reflexive sheaves E on P with $c_1(E) = -2$, $c_2(E) = c_2+1$ $(c_2 \geqq 2)$ and $c_3(E) = (c_2+1)^2-3(c_2+1)+2$ in order to show that $c_3(E^*(-1))$ gets all the possible values.

If E is as above, then by [5], Theorem 5.2, there is an exact sequence:

(5) $$0 \longrightarrow \mathcal{O}_P(-1)^3 \longrightarrow E \longrightarrow \mathcal{O}_{H_o}(-c_2) \longrightarrow 0$$

for some plane $H_o \subset P$. Dualizing (5) one gets an exact sequence:

$$0 \longrightarrow E^*(-1) \longrightarrow \mathcal{O}_P^3 \xrightarrow{\alpha} \mathcal{O}_{H_o}(c_2) \longrightarrow \mathcal{E}xt^1(E,\mathcal{O}_P)(-1) \longrightarrow 0$$

One deduces that such an E can be constructed as it follows: let $f_o, f_1, f_2 \in H^0(\mathcal{O}_{H_o}(c_2))$ be linearly independent elements such that $\mathcal{O}_{H_o}(c_2)/(\mathcal{O}_{H_o}\cdot f_o+\mathcal{O}_{H_o}\cdot f_1+\mathcal{O}_{H_o}\cdot f_2)$ has finite length.

Let $\alpha : \mathcal{O}_P^3 \longrightarrow \mathcal{O}_{H_o}(c_2)$ be the morphism defined by f_o, f_1, f_2. Coker(α) is of finite length and $H^0(\alpha)$ is injective. $E = (\text{Ker}\,\alpha)^*$ is a stable rank 3 reflexive sheaf with the announced Chern classes. We have:

$$\mathcal{E}xt^1(E,\mathcal{O}_P) = \mathrm{Coker}(\alpha) = \mathcal{O}_{H_o}(c_2)/(\mathcal{O}_{H_o}\cdot f_o + \mathcal{O}_{H_o}\cdot f_1 + \mathcal{O}_{H_o}\cdot f_2).$$

If s = length $(\mathcal{E}xt^1(E,\mathcal{O}_P))$ then $c_1(E^*(-1)) = -1$, $c_2(E^*(-1)) = c_2$ and $c_3(E^*(-1)) = -c_2^2+2s$.

Firstly, let $f_1, f_2 \in H^o(\mathcal{O}_{H_o}(c_2))$ be such that they define nonsingular curves $Y_1, Y_2 \subset H_o$ meeting transversely at c_2^2 points. An argument similar to one used in [2], Example 1.4, shows us that if $\frac{1}{2}(c_2-1)(c_2-2)+1 \leqq r \leqq c_2^2$ then there is an $f_o \in H^o(\mathcal{O}_{H_o}(c_2))$ vanishing at exactly c_2^2-r points of $Y_1 \cap Y_2$. If E is the reflexive sheaf constructed using f_o, f_1, f_2 then $s = c_2^2-r$. It follows that $c_3(E^*(-1))$ gets all the odd values from the interval $[-c_2^2, 3c_2-4]$.

Next, let q be such that $1 \leqq q \leqq \frac{1}{2}(c_2+1)$. We choose $g_1, g_2 \in H^o(\mathcal{O}_{H_o}(q))$ such that they define nonsingular curves $Z_1, Z_2 \subset H_o$ meeting transversely at q^2 points. Let $h \in H^o(\mathcal{O}_{H_o}(c_2-q))$ be such that h vanishes at no point of $Z_1 \cap Z_2$.

If $D \subseteqq Z_1 \cap Z_2$ then there is an $f_o \in H^o(\mathcal{O}_{H_o}(c_2))$ vanishing at every point of D but at no point of $(Z_1 \cap Z_2) \setminus D$. Indeed :

$$\deg(\mathcal{O}_{Z_1}(c_2) \otimes \mathcal{O}_{Z_1}(-D)) \geqq c_2q-q^2 \geqq (q-1)(q-2) = 2g(Z_1)$$

hence $\mathcal{O}_{Z_1}(c_2) \otimes \mathcal{O}_{Z_1}(-D)$ is generated by its global sections.

If E is the reflexive sheaf constructed using $f_o, f_1 = g_1h$ and $f_2 = g_2h$ then $s = c_2(c_2-q) +$ card D. In this case, $c_3(E^*(-1))$ gets all the odd values from the interval $[c_2^2-2qc_2,\ c_2^2-2qc_2+2q^2]$.

REFERENCES

[1] I.COANDĂ, On the spectrum of a stable rank 3 reflexive sheaf on $\mathbb{P}^3$, Crelles J. (to appear).

[2] I.COANDĂ, The Chern classes of the stable rank 3 vector bundles on $\mathbb{P}^3$, Math.Ann., 273 (1985), pp.65-79.

[3] L.EIN - R.HARTSHORNE - H.VOGELAAR, Restriction theorems for stable rank 3 vector bundles on $\mathbb{P}^n$, Math.Ann., 258 (1982), pp. 541-569.

[4] R.MIRÓ-ROIG, Haces reflexivos sobre espacios proyectivos, Thesis, Barcelona (1985).

[5] C.OKONEK - H.SPINDLER, Reflexive Garben vom Rang $r>2$ auf $\mathbb{P}^n$, Crelles J., 344 (1983), pp.38-64.

[6] C.PESKINE - L.SZPIRO, Liaison des variétés algébriques I, Invent. Math., 26 (1974), pp.271-302.

[7] H.SPINDLER, Die Modulräume stabiler 3-Bündel auf $\mathbb{P}_3$ mit den Chernklassen $c_1=0$, $c_3=c_2^2-c_2$, Math.Ann., 256 (1981), pp.133-143.