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In this paper we consider the viscous fluid flow in the (small) canals of a rigid porous solid with periodic structure. We prove that if ε, the characteristic length of the period, is sufficiently small, then the Navier-Stokes problem has unique solution, in both evolution and stationary cases.

1. PRELIMINARIES

Let Ω be an open conected bounded set in $\mathbb{R}^n$, n=2 or 3, locally located on one side of the boundary $\partial\Omega$, a(n-1)-dimensional manifold of class C^2, composed of a finite number of connected components. Let Γ be also a (n-1)-dimensional manifold of class C^2, composed of a finite number of connected components, included in $Y=[0,1[^n$ and which separates Y into two sets , Y_s (the solid part) and Y_f (the fluid part), with the property that repeating Y by periodicity, the union of all fluid parts is connected in $\mathbb{R}^n$ and of class C^2. We also assume that if Γ crosses the boundary of Y, then these intersections are reproduced identically on opposite faces of Y. Thus, if n=3, it is possible for the union of all the solid parts to be also connected.

Defining $\varphi : \mathbb{R}^n \longrightarrow Y$ by

$$\varphi(x_1, x_2, \ldots, x_n) = (\{x_1\}, \{x_2\}, \ldots, \{x_n\})$$

where $\{\cdot\}$ denotes the function which associates to any real number its fractional part, we say that a function f defined on $\mathbb{R}^n$ is Y-periodic iff $f = f \circ \varphi$.

Further, for any $\varepsilon \in]0,1[$ we denote

$$\varphi^{\varepsilon}(x) = \varphi(x/\varepsilon), \quad x \in \mathbb{R}^n$$

$$\Omega_{\varepsilon} = \{x \in \Omega \mid \varphi^{\varepsilon}(x) \in Y_f\} := \text{the fluid part of } \Omega$$

$$\Omega \setminus \overline{\Omega}_{\varepsilon} = \{x \in \Omega \mid \varphi^{\varepsilon}(x) \in Y_s\} := \text{the solid part of } \Omega$$

$$\Gamma_{\varepsilon} = \{x \in \Omega \mid \varphi^{\varepsilon}(x) \in \Gamma\}$$

$$(\partial\Omega)_{\varepsilon} = \overline{\Omega}_{\varepsilon} \cap \partial\Omega$$

Let us remark here that $\partial\Omega_{\varepsilon} = (\partial\Omega)_{\varepsilon} \cup \Gamma_{\varepsilon}$.

Let $\mathcal{V}_{\varepsilon}$ be the space (without topology)

$$\mathcal{V}_{\varepsilon} = \left\{ v \in \mathcal{D}(\Omega_{\varepsilon}) \mid \operatorname{div} v = 0 \text{ in } \Omega_{\varepsilon} \right\}$$

We denote by H_{ε} and V_{ε} the closures of $\mathcal{V}_{\varepsilon}$ in $L^2(\Omega_{\varepsilon})$ and $H^1_o(\Omega_{\varepsilon})$, respectively.

As usual the scalar products and norms in $L^2(\Omega)$ and $H^1_o(\Omega)$ are denoted respectively by $(\cdot,\cdot)$, $|\cdot|$ and $((\cdot,\cdot))$, $\|\cdot\|$. The norms in $L^p(\Omega)$ $(p \neq 2)$ and $H^m(\Omega)$ will be denoted by $|\cdot|_p$ and $\|\cdot\|_m$. To the corresponding notations associated to Ω_{ε} , we attach the index ε (for instance the norm in $L^4(\Omega_{\varepsilon})$ will be denoted by $|\cdot|_{4,\varepsilon}$).

In the sequel we shall prove that if ε is sufficiently small then the Navier-Stokes problem in the domain Ω_{ε} has unique solution. Although this result seems to be related to the classical "large ν , small f (and u_o)" case of uniqueness, neither it can be reduced to that, nor viceversa.

2. THE EVOLUTION CASE

For any $T > 0$, we consider the Navier-Stokes model of incompressible viscous fluids flows. That is, if the external force f, the initial velocity distribution u_o and the kinematic viscosity ν are given, we have to find the velocity field u and the pressure p, satisfying in some senses the system:

$$\text{(2.1)} \qquad \operatorname{div} u = 0 \quad \text{in } \Omega_\varepsilon \times]0,T[$$

$$\text{(2.2)} \qquad \frac{\partial u}{\partial t} + (u\nabla)u - \nu\Delta u = f - \nabla p \quad \text{in } \Omega_\varepsilon \times]0,T[$$

and the boundary and initial conditions

$$\text{(2.3)} \qquad u = 0 \quad \text{on } \partial\Omega_\varepsilon \times]0,T[$$

$$\text{(2.4)} \qquad u(0) = u^o \quad \text{in } \Omega_\varepsilon$$

The problem (2.1) - (2.4) has a well-known variational formulation:

Problem (E). For f, u_o and ν given with

$$\text{(2.5)} \qquad f \in L^2(0,T;V'), \; u_o \in H_\varepsilon, \; \nu > 0$$

to find u $L^2(0,T;V_\varepsilon)$ satisfying (2.4) and

$$\text{(2.6)} \qquad (u',v)_\varepsilon + \nu((u,v))_\varepsilon + b_\varepsilon(u,u,v) = \langle f,v\rangle_\varepsilon \quad (\forall) v \in V_\varepsilon$$

where b_ε is the trilinear continuous form on V defined by

$$(2.7) \qquad b_\varepsilon(u,v,w) = \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} (u_i \frac{\partial v_j}{\partial x_i} w_j)\,dx$$

Remark 2.1. If u is a solution of Problem (E) then $u' \in L^1(0,T;V'_\varepsilon)$ and hence u is almost everywhere equal to a function continuous from $[0,T]$ into V'_ε. These are the senses of (2.4) and of u' in (2.6). For any $v \in V_\varepsilon$, we naturally define $v_\varepsilon \in V$ by

$$(2.8) \qquad v_\varepsilon = \begin{cases} v & \text{in } \Omega_\varepsilon \\ 0 & \text{in } \Omega \setminus \Omega_\varepsilon \end{cases}$$

Thus the meaning of $\langle \cdot, \cdot \rangle_\varepsilon$ in (2.6) is

$$\langle f,v \rangle_\varepsilon = \langle f,v_\varepsilon \rangle_{(V',V)} : = \langle f,v_\varepsilon \rangle .$$

One can prove (see for instance [2] Ch.III) that there exists a solution of Problem (E) which satisfies

$$(2.9) \qquad u \in L^\infty(0,T;H_\varepsilon)$$

and which is continuous between the weak topologies of $[0,T]$ and H_ε. Moreover, if n=2 then Problem (E) has a unique solution satisfying (2.9).

In this section, from now on we consider only the case n=3.

Exactly, like in [1] we can prove that

$$(2.10) \qquad |u|_\varepsilon \leqslant C_1 \varepsilon \, \|u\|_\varepsilon \qquad (\forall) \; u \in H_0^1(\Omega_\varepsilon)$$

where C_1 is independent of ε and u.

Now let us consider the Sobolev inequality

$$(2.11) \qquad |v|_6 \leqslant C_2 \|v\| \qquad (\forall)\ v \in H_o^1(\Omega)$$

In order to obtain the corresponding Sobolev inequality in Ω_ε, for any $u \in H_o^1(\Omega_\varepsilon)$ we define $u_\varepsilon \in H_o^1(\Omega)$ like in (2.8). Then, taking $v = u_\varepsilon$ in (2.11) it follows

$$(2.12) \qquad |u|_{6,\varepsilon} \leqslant C_2 \|u\|_\varepsilon \qquad (\forall)\ u \in H_o^1(\Omega_\varepsilon)$$

Because $1/4 + 3/4 = 1$ and $1/4 = (1/4)/2 + (3/4)/6$, by the Holder inequalities we have

$$|u|_{4,\varepsilon} \leqslant |u|_\varepsilon^{1/4} |u|_{6,\varepsilon}^{3/4}$$

Getting (2.10) and (2.12) in it, we finally receive:

$$(2.13) \qquad |u|_{4,\varepsilon} \leqslant C_o^{1/2} \varepsilon^{1/4} \|u\|_\varepsilon \qquad (\forall)\ u \in H_o^1(\Omega_\varepsilon)$$

where C_o is independent of ε and u. Now we can prove:

Theorem 2.1. If $f \in W_1^{(1)}(0,T;H)$ and $u_o \in V_\varepsilon \cap H_2(\Omega_\varepsilon)$ such that $(\|u_o\|_\varepsilon + \|u_o\|_{2,\varepsilon})$ is essentially bounded with respect to ε, then for any $\varepsilon \in]0,1[$ sufficiently small (this phrase will be specified during the proof by the estimates (2.18) and (2.27)) there exists a solution u of Problem (E) which satisfies

$$(2.14) \qquad u \in L^\infty(0,T;V_\varepsilon) \text{ and } u' \in L^\infty(0,T;H_\varepsilon)$$

Proof. We apply the Galerkin method. As we already know (see [2] Ch.III) that the Galerkin approximation is converging strongly in $L^2(0,T;H_\varepsilon)$ to a solution of Problem (E), it remains only to obtain the a priori estimates corresponding to (2.14)

After differentiating (2.6) with respect to t, we take $v=u'$; it yields

$$(2.15)\quad \frac{d}{dt}|u'|_{\varepsilon}^{2} + 2b_{\varepsilon}(u',u,u')+2\nu\|u'\|_{\varepsilon}^{2} = 2\langle f',u'\rangle_{\varepsilon} \leqslant 2|f'|\cdot|u'|_{\varepsilon}$$

According to (2.13) we can estimate the non-linear term as follows:

$$(2.16)\quad |b_{\varepsilon}(u',u,u')| \leqslant |u'|_{4,\varepsilon}^{2}\|u\|_{\varepsilon} \leqslant C_{o}\varepsilon^{1/2}\|u'\|_{\varepsilon}^{2}\|u\|_{\varepsilon}$$

In this way (2.15) becomes

$$(2.17)\quad \frac{d}{dt}(|u'|_{\varepsilon}^{2} + 1)+(2\nu-C_{o}\varepsilon^{1/2}\|u\|_{\varepsilon})\|u'\|_{\varepsilon}^{2} \leqslant |f'|(|u'|_{\varepsilon}^{2} + 1)$$

Assuming that ε satisfies

$$(2.18)\quad 2\nu-C_{o}\varepsilon^{1/2}\operatorname*{ess\,sup}_{\varepsilon\in]0,1[}\|u_{o}\|_{\varepsilon} > 0$$

we deduce that there exists $T_{*}\in]0,T]$ such that

$$(2.19)\quad 2\nu-C_{o}\varepsilon^{1/2}\|u(t)\|_{\varepsilon} > 0 \quad \text{for a.a.}\ t\in[0,T_{*}],$$

and T_{*} is maximal with this property. Obviously, we want to prove that $T_{*} = T$.

Taking (2.19) into account, (2.17) becomes:

$$(2.20)\quad \frac{d}{dt}(|u'|_{\varepsilon}^{2} + 1) \leqslant |f'|(|u'|_{\varepsilon}^{2} + 1)$$

Because of Gronwall's inequality, (2.20) implies

(2.21) $(|u'(t)|_{\varepsilon}^{2}+1)\leqslant(|u'(0)|_{\varepsilon}^{2}+1)\exp(|f'|_{L^1(0,T;H)})$ a.e. on $[0,T_*]$

Therefore we need an estimation of $|u'(0)|_{\varepsilon}$. For this let t tends to zero in (2.6) and choose $v=u'(0)$; it follows

(2.22) $|u'(0)|_{\varepsilon}^{2}=\nu(\Delta u_o,u'(0))_{\varepsilon}-|u_o|_{\infty,\varepsilon}\|u_o\|_{\varepsilon}|u'(0)|_{\varepsilon}+\langle f(0),u'(0)\rangle_{\varepsilon}$

from which we derive

(2.23) $|u'(0)|_{\varepsilon}\leqslant C_3(1+\operatorname{ess\,sup}_{\varepsilon\in]0,1[}\|u_o\|_{\varepsilon})\operatorname{ess\,sup}_{\varepsilon\in]0,1[}\|u_o\|_{2,\varepsilon}+|f(0)|$

where C_3 is some constant. According to (2.23), from (2.21) it results that there exists a positive constant C_4, independent of t and ε, such that

(2.24) $|u'(t)|_{\varepsilon}\leqslant C_4$ a.e. on $[0,T_*]$

Since $b_{\varepsilon}(u,v,v)=0$ $(\forall)$ $u,v\in V_{\varepsilon}$, if we set $v=u$ in (2.6) it yields

(2.25) $\nu\|u\|_{\varepsilon}^{2}=\langle f,u\rangle_{\varepsilon}-(u',u)_{\varepsilon}\leqslant(|f|+|u'|_{\varepsilon})|u|_{\varepsilon}$

Using (2.10) and (2.24), we find from (2.25) that there exists a positive constant C_5, independent of t and ε, such that

(2.26) $\|u(t)\|_{\varepsilon}\leqslant C_5\varepsilon$ a.e. $[0,T_*]$

Choosing ε to satisfy also

(2.27) $2\nu-C_o\varepsilon^{1/2}C_5>0$

we finally obtain

(2.28) $2\nu - C_0\varepsilon^{1/2}\|u(t)\|_\varepsilon > 2\nu - C_0\varepsilon^{1/2}C_5 > 0$ a.e. on $[0,T_*]$

If we assume that $T_* < T$, then as t tends to T_* , from (2.28) we obtain that T_* is not the maximal element with the property (2.19). Hence $T_* = T$ and the estimations (2.24) and (2.26) imply (2.14).

Remark 2.2. If we continue u by zero in $\Omega \setminus \Omega_\varepsilon$, defining u_ε by (2.8), then from (2.26), using again (2.10), it follows that

(2.29) $\left\{ \frac{1}{\varepsilon^2} u_\varepsilon \right\}_\varepsilon$ is bounded in $L^\infty(0,T;H)$

It seems that with (2.29) we can start the study of the homogenization process (as $\varepsilon \to 0$) of Problem (E) like in [1] and [3].

Corollary 2.1. Under the conditions of Theorem 2.1, Problem (E) has a unique solution in $L^\infty(0,T;H_\varepsilon) \cap L^8(0,T;L^4(\Omega_\varepsilon))$.

Proof. Because $V_\varepsilon \subseteq L^4(\Omega_\varepsilon)$, then according to (2.9) and (2.14) it follows that Problem (E) has a solution in $L^\infty(0,T;H_\varepsilon) \cap L^8(0,T;L^4(\Omega_\varepsilon))$. But a solution of Problem (E) is surely unique with this property (see [2] Ch.III).

3. THE STATIONARY CASE

Naturally, for the stationary Navier-Stokes model of incompressible viscous fluid flow it is sufficient to ignore the time dependence. Then, for the given external force f and kinematic viscosity ν, we have to find the velocity field u and the pressure p, satisfying in some senses the system

(3.1) $\quad \operatorname{div} u = 0 \quad \text{in } \Omega_\varepsilon$

(3.2) $\quad (u\nabla)u - \nu\Delta u = f - \nabla p \quad \text{in } \Omega_\varepsilon$

and the boundary condition

(3.3) $\quad u = 0 \quad \text{on } \partial\Omega_\varepsilon$

The problem (3.1) - (3.3) has the following variational formulation:

Problem (S). For f and ν given with

(3.4) $\quad f \in V', \quad \nu > 0$

to find $u \in V_\varepsilon$, satisfying

(3.5) $\quad \nu((u,v))_\varepsilon + b_\varepsilon(u,u,v) = \langle f,v\rangle_\varepsilon \quad (\forall)\ v \in V_\varepsilon$

where b_ε is defined by (2.7), and the meaning of $\langle f,v\rangle_\varepsilon$ is the same as in Remark 2.1.

One can prove (see for instance [2] Ch.II) that Problem (S) has at least one solution. In this section $n \in \{2,3\}$ because there is no general uniqueness result.

In the stationary case we can prove straightly a uniqueness result similar to that of Corollary 2.1.

Theorem 3.1. If ε is sufficiently small so that

(3.6) $\quad c_o \varepsilon^{1/2} |f|_{V'} < \nu^2$

then there exists a unique solution of Problem (S).

Proof. Let u_1 be a solution of Problem (S). Then taking $v = u_1$ in (3.5) we obtain

(3.7) $$\|u_1\|_\varepsilon \leq |f|_{V'} / \nu$$

Let u_2 be a solution of Problem (S), possible different from u_1. If we subtract the equations (3.5) corresponding to u_1 and u_2, and if we denote by $w = u_1 - u_2$, then we have

(3.8) $$\nu((w,v))_\varepsilon + b_\varepsilon(u_1,w,v) + b_\varepsilon(w,u_1,v) = 0 \quad (\forall)\ v \in V_\varepsilon .$$

Since $b_\varepsilon(u,v,v)=0$ $(\forall)$ $u,v \in V_\varepsilon$, for $v=w$ the relation (3.8) reduces to

(3.9) $$\nu \|w\|_\varepsilon^2 = -b_\varepsilon(w,u_1,w) \leq |w|_{4,\varepsilon}^2 \|u_1\|_\varepsilon$$

Using (2.13) and (3.7), from (3.9) it follows

(3.10) $$(\nu - c_0 \varepsilon^{1/2} |f|_{V'}/\nu) \|w\|_\varepsilon^2 \leq 0$$

According to (3.6) it implies $\|w\|_\varepsilon = 0$, that is $u_1 = u_2$ in V .

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