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Serban N. BUZETEANU

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Serban N. BUZETEANU *)

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Institute of Mathematics, University of Bucharest

Str. Academiei 14, R-70109, Bucuresti, ROMANIA

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ŞERBAN N.BUZEŢEANU

Institute of Mathematics, University of Bucharest
Str.Academiei, Nr.14, R-70109, Bucureşti, ROMANIA

1. INTRODUCTION

Let $(F_n)_{n\geq 0}$ be the Fibonacci numbers defined recursively by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.

It is well known that the generating function of the sequence $(F_n)_{n\geq 0}$ is $f_1(z) = z/(1-z-z^2)$. In [2], S.W.Golomb showed that the sequence $(F_n^2)_{n\geq 0}$ verifies the recurrence

$$F_n^2 - 2F_{n+1}^2 - 2F_{n+2}^2 + F_{n+3}^2 = 0, \text{ for every } n \geq 0.$$

This implies that the generating function of the sequence $(F_n^2)_{n\geq 0}$ is $f_2(z) = (z-z^2)/(1-2z-2z^2+z^3)$.

The above result was generalized by J.Riordan [4], which proved that for every integer $r \geq 2$ the sequence $(F_n^r)_{n\geq 0}$ satisfies an appropriate $(r+1)$-th order linear recurrence with integer coefficients. This in turn was also generalized by L.Carlitz [1] which showed that for every recursive sequence $(u_n)_{n\geq 0}$ given by $u_0 = 0$, $u_1 = 1$, $u_{n+1} = pu_n - qu_{n-1}$ $(p^2-4q \neq 0)$, the sequence $(u_n^r)_{n\geq 0}$ satisfies a linear recurrence (of $(r+1)$-th order), for every integer $r \geq 2$. Moreover, if p and q are integers, then the sequence $(u_n^r)_{n\geq 0}$ also verifies a linear recurrence with integral coefficients.

All the above results were obtained by considering the associated generating functions. In this short note we shall extend the simple inductive argument of D. Jarden (according to D.E. Knuth [3]) for solving a more general form of Riordan's problem, i.e. to the secuence considered by Carlitz for $q = -1$.

2. SOME PRELIMINARY LEMMAS

Let $p \neq 0$ be a fixed real number. We consider the recurrent sequence $(x_n)_{n\geq 0}$ defined by :

$$(1)\qquad \begin{cases} x_0 = 0,\ x_1 = 1\ , \\ x_{n+1} = px_n + x_{n-1},\ n \geq 1\ . \end{cases}$$

In the following it will be convenient to extend the above recurrence also for negative indices, by the relation

$$(2)\qquad x_{-n} = -px_{-n+1} + x_{-n+2},\ n \geq 1.$$

With this convention we have the following lemma.

Lemma 1: Let $(x_n)_{n\in\mathbb{Z}}$ be a recurrent sequence defined by (1) and (2). Then

1^0 For every $n \geq 1$, we have $x_{-n} = (-1)^{n+1} x_n$.

2^0 For all integers n, k and s, we have:

$$x_{n+k} = x_{k-1}x_n + x_k x_{n+1} = x_k x_{n-1} + x_{k+1} x_n =$$
$$= x_{k+s-1}x_{n-s} + x_{k+s}x_{n-s+1}\ .$$

3^0 For all integers n, k and s, we have:

$$(-1)^k x_{n-k} = x_{k+1}x_n - x_k x_{n+1} = -x_k x_{n-1} + x_{k-1}x_n =$$
$$= (-1)^s (x_{k-s+1}x_{n-s} - x_{k-s}x_{n-s+1}).$$

Proof: 1^0 It is easily seen that by (2) and (1), $x_{-1} = x_1$ and $x_{-2} = -x_2$. Thus the equality $x_{-n} = (-1)^{n+1} x_n$ follows by induction on $n \geq 1$, using (2) in the inductive step.

2^0 For every fixed integral k, we can easily prove, by a two-fold induction on n (for $n \geq 0$ and $-n \leq -1$), that $x_{n+k} = x_{k-1}x_n + x_k x_{n-1}$. However, we shall prefer to prove the required relation by a matricial method similar to that of J.R. Silvester [5], which explains also <u>why</u> this formula holds.

Consider the matrix

$$A=\begin{pmatrix}0 & 1\\ 1 & p\end{pmatrix}=\begin{pmatrix}x_0 & x_1\\ x_1 & x_2\end{pmatrix}=A_1$$

and for every integral n put

$$A_n=\begin{pmatrix}x_{n-1} & x_n\\ x_n & x_{n+1}\end{pmatrix}.$$

We observe that A is invertible and

$$A^{-1}=\begin{pmatrix}-p & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}x_{-2} & x_{-1}\\ x_{-1} & x_0\end{pmatrix}=A_{-1}.$$

From this and (2) it follows, by induction on $n\geq 1$, that

$$A^{-n}=\begin{pmatrix}x_{-n-1} & x_{-n}\\ x_{-n} & x_{-n+1}\end{pmatrix},\quad A^n=\begin{pmatrix}x_{n-1} & x_n\\ x_n & x_{n+1}\end{pmatrix}.$$

Now the first two relations follow by identifying the (1,2) and (2,1) cells in the matrix $A^{n+k}=A^nA^k$. The third relation can be obtained from the first simply by noticing that $n+k=(n-s)+(k+s)=$ $=n'+k'$.

3^o This follows from 2^o, replacing k by -k and applying 1^o. This ends the proof of the lemma. □

In all that follow we shall suppose that $x_n\neq 0$, for every $n>0$.

By analogy with the binomial coefficients, we introduce for every integral $n\geq 0$ and k the numbers C(n,k) given by

$$C(n,k)=\begin{cases}1 & \text{, if } k=0\,,\\ (x_nx_{n-1}\ldots x_{n-k+1})/(x_kx_{k-1}\ldots x_1) & \text{, if } 1\leq k\leq n\,,\\ 0 & \text{, if } k<0 \text{ or } k>n\,.\end{cases}$$

The following lemma gives some elementary properties of the numbers C(n,k), similar to those of the binomial coefficients $\binom{n}{k}$.

<u>Lemma 2</u>: Let n, k be arbitrary integers with $n\geq 0$. Then:

1^o $C(n,k) = C(n,n-k)$.

2^o $x_kC(n,k) = x_nC(n-1,k-1)$ and

$x_{n-k}C(n,k) = x_nC(n-1,k)$.

3° $C(n,k) = x_{k-1}C(n-1,k)+x_{n-k+1}C(n-1,k-1) =$

$$= x_{k+1}C(n-1,k)+x_{n-k-1}C(n-1,k-1) .$$

Proof: 1° and 2° follows immediately from the definition of the numbers $C(n,k)$.

3° By lemma 1 -2° for $n-k$ instead of n we obtain:

$$x_n = x_{k-1}x_{n-k}+x_kx_{n-k+1} = x_{k+1}x_{n-k}+x_kx_{n-k-1}$$

Now the result follows by substituting the above expressions for x_n in the definition of $C(n,k)$. □

Corollary: Let p in (1) be a positive integer. Then the numbers $C(n,k)$ are non-negative integers for all integral n, k with $n \geq 0$. Moreover, $C(n,k)$ is a positive integer for $0 \leq k \leq n$.

Proof: This easily follows by induction on $n \geq 0$, using lemma $2-2^\circ$ in the inductive step. □

For every real x we denote by $\lceil x \rceil$ the smalest integer which is greater or equal to x, and by $[x]$ the integral part of x. Observe that for all positive integers n, m we have:

$$\lceil n/2 \rceil = [(n+1)/2] \text{ and } \lceil n/2 \rceil + \lceil m/2 \rceil = (n+m+1)/2 \text{ for } n+m \text{ odd.}$$

The following technical lemma will be essential in the inductive proof of the main result.

Lemma 3: Let $(x_n)_{n\geq 0}$ be a recurrent sequence defined by (1). Suppose that for some integer $r \geq 2$, the following relations holds:

$$(3) \qquad \sum_k (-1)^{\lceil (r-k)/2 \rceil} C(r,k)x_{n+k}^{r-1} = 0, \quad \text{for every } n \geq 0 .$$

Then the following relations are also true, for every $n \geq 0$:

1° $\sum_k (-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k)x_{n+k}^{r-1}\, x_k = 0.$

2° $\sum_k (-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k)x_{n+k}^{r-1} \cdot (-1)^k x_{r+1-k} = 0.$

$$3^\circ \quad \sum_k (-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k) x_{n+k}^{r-1} x_{k-1} = 0.$$

$$4^\circ \quad \sum_k (-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k) x_{n+k}^{r} = 0 .$$

Proof: 1° Since by lemma 2, $x_k C(r+1,k) = x_{r+1} C(r,k-1)$, it follows by that and (3);

$$\sum_k (-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k) x_{n+k}^{r-1} x_k =$$

$$= x_{r+1} \sum_k (-1)^{\lceil (r+1-k)/2 \rceil} C(r,k-1) x_{n+k}^{r-1} =$$

$$= x_{r+1} \sum_k (-1)^{\lceil (r-k)/2 \rceil} C(r,k) x_{(n+1)+k}^{r-1} = 0.$$

2° By lemma 2 we have $x_{r+1-k} C(r+1,k) = x_{r+1} C(r,k)$. Obviously, $\lceil (r-k+1)/2 \rceil + \lceil (r-k)/2 \rceil = ((r-k+1)+(r-k)+1)/2 = r-k+1$. Hence, we obtain succesively

$$\sum_k (-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k) x_{n+k}^{r-1} (-1)^k x_{r+1-k} =$$

$$= \sum_k (-1)^{\lceil (r+1+k)/2 \rceil} x_{r+1}\, C(r,k) x_{n+k}^{r-1} =$$

$$= (-1)^{r+1} x_{r+1} \sum_k (-1)^{\lceil (r-k)/2 \rceil} C(r,k) x_{n+k}^{r-1} = 0, \text{ by (3).}$$

3° By lemma 1 we have:

$$(-1)^k x_{r+1-k} = x_{k-1} x_{r+1} - x_k x_r .$$

Multiplying this relation by $(-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k) x_{n+k}^{r-1}$ and summing over integral k, we obtain by 1° and 2° :

$$x_{r+1} \sum_k (-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k) x_{n+k}^{r-1} x_{k-1} = 0 .$$

The result follows because, by hypothesis, $x_n \neq 0$ for every $n \geq 1$.

4° By lemma 1 we have:

$$x_{n+k} = x_{k-1} x_n + x_k x_{n+1} .$$

Multiplying this relation by $(-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k)x_{n+k}^{r-1}$ and summing over integral k, we obtain by 1° and 3°:

$$\sum_k (-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k)x_{n+k}^{r} = 0.$$

This concludes the proof of the lemma. □

3. THE MAIN RESULT

We can now state and prove the main result of this note, that is the following:

Theorem: Let $(x_n)_{n\geq 0}$ be a recurrent sequence defined by (1), with p a positive integer. Then for every integer $r \geq 2$, the sequence $(x_n^r)_{n\geq 0}$ satisfies a linear recurrence of (r+1)-th order with integral coefficients (of the type (3)). Moreover, for every $r \geq 2$, these coefficients can be effectively computed by some recursive relations.

Proof: We shall prove that for every integer $r \geq 1$, the sequence $(x_n^r)_{n\geq 0}$ satisfies the following linear recurrence of order r+1 :

(4) $\sum_k (-1)^{\lceil (r+1-k)/2 \rceil} C(r+1,k)x_{n+k}^{r} = 0$, for every $n \geq 0$

This relation will be proved by induction on $r \geq 1$.

For r = 1, it is clearly true by (1), (2) and the definition of the numbers C(n,k). Let now $r \geq 2$ and suppose (4) is true for r - 1, that is (3) is true. By lemma 3-4° it follows that (4) is also true for r, completing the inductive step.

Moreover, the coefficients C(r+1,k) are integral by the corollary following lemma 2, and can be recursively computed by lemma 2. □

4. REMARKS

1° We remark that for $n \geq 0$ and $r \geq 2$, the sums

$$S(r,n) = \sum_k (-1)^{\lceil (r-k)/2 \rceil} C(r,k)x_{n+k}^{r-1}$$

verifie the recurrence relation

$$S(r+1,n)=(x_{n+1}x_{r+1}+x_nx_r)S(r,n+1)+(-1)^{r+1}x_nS(r,n)$$

(this may be proved along the same lines as lemma 3: 1^o-4^o).

Since $S(2,n) = 0$ for every $n \geq 0$, it follows by induction on $r \geq 2$ that $S(r,n) = 0$ for all $r \geq 2$ and $n \geq 0$, giving thus an alternative proof of the theorem.

2^o Reversing the order of summation in (4), we notice that the generating function for the sequence $(x_n^r)_{n\geq o}$ is given by:

$$f_r(z) = \Big(\sum_{k=1}^{r} \alpha_k z^k\Big)/\Big(\sum_{\ell=o}^{r+1} (-1)^{\lceil \ell/2 \rceil} C(r+1,\ell)z^\ell\Big) ,$$

where $\alpha_k = \sum_{j=1}^{k} (-1)^{\lceil (k-j)/2 \rceil} C(r+1,k-j)x_j^r$ for $1 \leq k \leq r$.

In Riordan's and Carlitz's papers, the generating function $f_r(z)$ is not explicitly displayed, but merely it is given by a recurrence relation in terms of $f_j(z)$ for $1 \leq j < r$.

3^o For every reals $p,q \neq 0$, consider also the recurrent sequence $(y_n)_{n\geq o}$ given by $y_o= 0$, $y_1= 1$, $y_{n+1}= py_n+ pqy_{n-1}+pq^2y_{n-2}$. It can be proved that $(y_n^2)_{n\geq o}$ verifies a linear recurrence of order ≤ 6. Moreover, if p,q are integers, then the recurrence relation for $(y_n^2)_{n\geq o}$ has also integral coefficients. For $p=q=1$, we obtain thus the exact analog of Golomb's result for the Fibonacci sequence of second order. In terms of generating functions, this merely says that the generating function for the sequence $\big(F_n^{(2)^2}\big)_{n\geq o}$ given by $F_o^{(2)}= 0$, $F_1^{(2)}= 0$, $F_2^{(2)}= 1$, $F_{n+1}^{(2)}= F_n^{(2)}+ F_{n-1}^{(2)}+F_{n-2}^{(2)}$ $(n \geq 2)$, is $g_2(z) = (z^2-z^3-z^4-z^5)/(1-2z-3z^2-6z^3+z^4+z^6)$.

It can also be proved that for every integers $k \geq 2$ and $r \geq 2$, the sequence of r-th powers of k-th order Fibonacci numbers verifies a linear recurrence with integral coefficients. But this and some other related results will be proved in a future paper (an earlier version of some of these results can be found in [6]).

4° The results contained in Riordan's and Carlitz's papers were further generalized by A.F.Horadam [7].

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The present author considers in [4] the diophantine equation $|y^2-zyx-x^2| = 1$. For every fixed $z = p \geq 0$, all the solutions in positive integers x,y of the above equation are given by $(y,x)=(x_{k+1}, x_k)$ for some $k \geq 1$, where $(x_n)_{n\geq 0}$ is recursively defined by $x_0 = 0$, $x_1=1$, $x_{n+1}= px_n+x_{n-1}$ for $n \geq 1$. This immediately yields a generalization of the integer polynomial representation problem of Jones, quoted above, for more general sequences than that of Fibonacci.

In this note, we shall generalize all the above results concerning solutions of diophantine equations. This will be done in Section 2, where lemmas concerning the "reduction" of the solutions of the following diophantine equations

(1) $y^2-ayx-x^2 = \delta$

(2) $y^2-ayx-x^2 = -\delta$

(3) $y^2-ayx+x^2 = \delta$

(4) $y^2-ayx+x^2 = -\delta$

will be proved, for positive integers a and δ.

In Section 3, we apply the above results for obtaining polynomials with integer coefficients whose positive values coincides with some recursively defined sequences.Finally, in Section 4 we prove that for some linear recurrent sequences $(x_n)_{n\geq 1}$, the sequence $(x_{nk+r})_{n\geq 0}$ verifies also a linear recurrence for every $1 \leq r \leq k$. This result has a stronger "qualitative" form (see R.Vaidyanathaswamy [12] or Klarner [8]), but here we give explicit recurrences for the involved sequences. These recurrences are then used to give other integer polynomial representations of the sequences in Section 3.

2. SOLVING THE DIOPHANTINE EQUATIONS (1)-(4)

As stated in the Introduction, we prove now lemmas for "reducing" an arbitrary solution in positive integers of (1)-(4) to a "primitive" solution. This will be done mainly by finding a linear invertible transformation $T : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ such that if (y,x) is a solution in positive integers, then $(y',x') = T(y,x)$ is also a solution in integers (eventually negative!). If moreover $x > f(a,\delta)$ (some explicit limit depending on a and δ only), then $(y',x') = T(y,x)$ is a solution in positive integers with $x' < x$. It follows that starting with an arbitrary solution (y,x) in positive integers, we reach after a finite number of applications of the transformation T a "primitive" solution (that is, one with $0 < x \leq f(a,\delta)$). We obtain by the above the following finite procedure for solving any of the equations (1)-(4):

1) Find all "primitive" solutions. That is, for every $x = 1,2,\dots,f(a,\delta)$, solve the resulting quadratic equation for y and retain only the positive integers solutions. This produce a finite number of "primitive" solutions.

2) To find all solutions in positive integers, apply T^{-1} a finite number of times to any "primitive" solution. Thus, every solution in positive integers is of the form $(y,x) = T^{-n}(y_o,x_o)$, for some integer $n \geq 0$ and (y_o,x_o) a "primitive" solution.

Remarks. 1) Suppose that all the "primitive" solutions are of the form $(y_o,x_o) = (x_{k+1},x_k)$, where $(x_n)_{n\geq o}$ is some recursively defined sequence. If moreover (x_2,x_1) is a "primitive" solution and $T(x_{k+1},x_k) = (x_k,x_{k-1})$ for every $k \geq 1$, then the solutions of the equation coincides with the pairs of the form $(y,x)=(x_{k+1},x_k)$ for $k \geq 1$. This will be the case in almost all of ours examples in Sections 3 and 4.

2) By the above procedure, it follows that for every fixed a there exists only a finite number of values of δ such that one of the equations (1)-(4) has a solution in positive integers. Moreover, these δ can be effectively determined as $\pm(y_o^2 - ay_o x_o \pm x_o^2)$, where (y_o, x_o) is some "primitive" solution of the corresponding equation.

Lemma 1. Let (y,x) be a solution in positive integers of the equation (1). Then:

1^o $y > ax$, and consequently $y = (ax + \sqrt{(a^2+4)x^2+4\delta})/2$.

2^o $(y',x') = (x, y-ax)$ is a solution in positive integers of the equation (2).

3^o Let $(y',x') = (x,y-ax)$. If $x > a\sqrt{\delta}$, then $x' = y-ax > \sqrt{\delta}$. Moreover, in this case $(y'',x'') = (x', y'-ax')$ is a solution in positive integers of the equation (1) with $x'' < x$.

Proof. 1^o Follows immediately from $y(y-ax) = x^2+\delta > 0$ and the formula for solving a quadratic equation.

2^o Immediately by direct calculation and by 1^o.

3^o By 1^o we obtain:

$$x' = y-ax = (ax+\sqrt{(a^2+4)x^2+4\delta})/2 - ax =$$
$$= (\sqrt{(a^2+4)x^2+4\delta} - ax)/2 > \sqrt{\delta} \quad \text{for } x > a\sqrt{\delta} ,$$

as is easily seen by isolating $\sqrt{(a^2+4)x^2+4\delta}$ in one member of the inequality and squaring next both members of the resulting inequality.

Similarly:

$$x'' = y'-ax' = x-a(y-ax) = (a^2+1)x-ay =$$
$$= (a^2+1)x-(a/2)(ax+\sqrt{(a^2+4)x^2+4\delta}) =$$
$$= (1/2)((a^2+2)x-a\sqrt{(a^2+4)x^2+4\delta}) > 0, \text{ since } x > a\sqrt{\delta} .$$

Evidently $y'' = x' = y-ax > 0$, and by direct calculation we see that (y'',x'') is a solution in positive integers of the equation (1).

Finally,

$$x'' = y'-ax' = x-a(y-ax) =$$
$$=(1/2)((a^2+2)x-a\sqrt{(a^2+4)x^2+4\delta}) < x ,$$

since evidently $ax < \sqrt{(a^2+4)x^2+4\delta}$.

This concludes the proof of the lemma. □

Remark. The linear invertible transformation used in this case is $T(y,x) = (y'',x'') = (y-ax,(a^2+1)x-ay)$. Its inverse is $T^{-1}(y,x) = ((a^2+1)y+ax,ay+x)$ and $f(a,\delta) = a\sqrt{\delta}$ (see the remarks at the beginning of this section). Similar remarks hold for the following lemmas too (these will not more be explicitely stated).

Lemma 2. Let (y,x) be a solution in positive integers of the equation (2). Then:

1° If $x > \sqrt{\delta}$ then $y > ax$, and consequently $y=(ax+\sqrt{(a^2+4)x^2+4\delta})/2$.

2° If $x > \sqrt{\delta}$ then $(y',x') = (x,y-ax)$ is a solution in positive integers of the equation (1).

3° Let $(y',x') = (x,y-ax)$. If $x > a\sqrt{\delta}$, then $x'= y-ax > \sqrt{\delta}$. Moreover, in this case $(y'',x'') = (x',y'-ax')$ is a solution in positive integers of the equation (2) with $x'' < x$.

Proof. 1° Follows immediately from $y(y-ax) = x^2-\delta > 0$ for $x > \sqrt{\delta}$, and the formula for solving a quadratic equation.

2° Immediately by direct calculation and by 1°.

3° Absolutely analogous with the proof of 3° in Lemma 1. □

The equations (3) and (4) are obviously symmetric in x and y, and thus il will be sufficient to solve them with the additional hypothesis $y > x > 0$.

Lemma 3. Let (y,x) be a solution in positive integers of the equation (3), with $y > x$. Then:

1° $y > (a-1)x$

2° If $x > \sqrt{\delta}$, then $y-ax < 0$.

3° If $x > \sqrt{\delta}$, then $(y',x') = (x,ax-y)$ is a solution in positive integers of the equation (3), with $y' > x'$. Moreover, $x'=ax-y < x$.

Proof. 1° Using equation (3) we infer:

$$y(y-(a-1)x) = yx-x^2+\delta = (y-x)x+\delta > \delta > 0,$$

and consequently $y > (a-1)x$.

2° Immediately by $y(y-ax) = \delta - x^2 < 0$ for $x > \sqrt{\delta}$.

3° By direct calculation one verifies that $(y',x')=(x,ax-y)$ is a solution of equation (3). Now $x = y' > x' > 0$ by 1° and 2°, thus concluding the proof of the lemma. □

Lemma 4. Let (y,x) be a solution in positive integers of the equation (4), with $y > x$. Then:

1° $y - ax < 0$.

2° If $x > \delta$, then $y > (a-1)x$.

3° If $x > \delta$, then $(y',x') = (x,ax-y)$ is a solution in positive integers of the equation (4), with $y' > x'$. Moreover, $x'=ax-y < x$.

Proof. 1° By equation (4) we obtain:

$$y(y-ax) = -x^2-\delta < 0,$$

and consequently $y-ax < 0$.

2° By equation (4) we have:

$$y(y-(a-1)x)=yx-x^2-\delta=(y-x)x-\delta > 0,$$ since $y > x$ and $x > \delta$.

Thus $y > (a-1)x$.

3° Absolutely analogous with the proof of 3° in Lemma 3. □

Remarks. 1) Lemma 3 and Lemma 4 remains true for a an arbitrary integer (not necessarily positive!).

2) For every fixed $\delta > 0$, there exists a finite number of values of $a > 0$ such that the equation (4) has a solution (and consequently an infinity of solutions!). Indeed, from (4) we obtain $y = (ax+\sqrt{a^2x^2-4(x^2+\delta)})/2$. As y is integer, it follows that $a^2x^2-4(x^2+\delta) = b^2$ for some integer b. Solving for a and b we obtain:

$$a = (u+v)/x, \quad b = v-u$$

where u, v are integers with $1 \leq u \leq v$ and $uv = x^2+\delta$ (here (y,x) is a primitive solution of (4), with $y > x > 0$ and $x \leq \delta$).

Applying now lemmas 1 and 2 in Section 2 we obtain the following theorem, which includes some previous results of J.P.Jones ([6] and [7]).

Theorem 1. 1^oa) The solutions in positive integers of the equation $y^2-yx-x^2 = 1$ are given by : $(y,x) = (F_{2k+1},F_{2k})$ for some $k \geq 1$.

b) The solutions in positive integers of the equation $y^2-yx-x^2 = 1$ are given by : $(y,x) = (F_{2k},F_{2k-1})$ for some $k \geq 1$.

2^oa) The solutions on positive integers of the equation $y^2-yx-x^2 = 5$ are given by : $(y,x) = (L_{2k},L_{2k-1})$ for some $k \geq 1$.

b) The solutions in positive integers of the equation $y^2-yx-x^2 = -5$ are given by : $(y,x) = (L_{2k+1},L_{2k})$ for some $k \geq 0$.

3^oa) The solutions in positive integers of the equation $y^2-2yx-x^2 = 1$ are given by : $(y,x) = (P_{2k+1},P_{2k})$ for some $k \geq 1$.

b) The solutions in positive integers of the equation $y^2-2yx-x^2 = -1$ are given by : $(y,x) = (P_{2k},P_{2k-1})$ for some $k \geq 1$.

Proof. Immediate by the remarks at the beginning of Section 2 and lemmas 1 and 2. □

Using theorem 1, we obtain the following theorem concerning the representation of the above mentioned recurrent sequences by some integral polynomials. This theorem extends the results of J.P. Jones and will be generalized in Section 4.

Theorem 2. 1^o The set of all positive Fibonacci numbers is identical with the positive values of the polynomial

(11) $P_1(x,y) = x(2-(y^2-yx-x^2)^2)$,

as the variables x and y range over the positive integers.

2^o The set of all Lucas numbers is identical with the positive values of the polynomial

(12) $P_2(x,y) = x(1-((y^2-yx-x^2)^2-25))$,

as the variables x and y range over the positive integers.

3^o The set of all Pell numbers is identical with the positive values of the polynomial

(13) $P_3(x,y) = x(2-(y^2-2yx-x^2)^2)$,

as the variables x and y range over the positive integers.

Proof. We have only to observe that $y^2-yx-x^2 = 0$ is not solvable in positive integers and the right factor of (12)-(14) cannot be positive unless the corresponding equations in theorem 1 hold. (Here we are using an idea of Putnam [9].) Hence the theorem follows by theorem 1. □

We close this section by remarking that theorems like the preceding two-ones can now be easily proved for any recurrence of the form $x_{n+1} = ax_n \pm x_{n-1}$. This will be done in the next section for the sequences $(x_{nk+r})_{n\geq o}$, where $1 \leq r \leq k$ are arbitrarily fixed integers.

4. THE RECURRENCE $x_{n+1} = ax_n \pm x_{n-1}$

Let a be some fixed positive integer. We consider the following recurrent sequence $(x_n)_{n\geq 1}$ given by:

(14) $x_1 = \alpha_1$, $x_2 = \alpha_2$, $x_{n+2} = ax_{n+1} + x_n$ for $n \geq 1$.

The canonical sequence of the type (14) is:

(15) $y_1 = 1$, $y_2 = a$, $y_{n+2} = ay_{n+1} + y_n$ for $n \geq 1$.

Using induction on $n \geq 1$, it is easily verified that

(16) $x_{n+2} = y_{n+1}\alpha_2 + y_n\alpha_1$ for every $n \geq 1$.

We wish to find explicit linear recurrences for the sequences $(x_{nk+r})_{n\geq o}$, for arbitrarily fixed integers $1 \leq r \leq k$. To this end, the following recurrent sequence will be very useful:

(17) $z_1 = a$, $z_2 = a^2+2$, $z_{n+2} = az_{n+1} + z_n$ for $n \geq 1$.

Before stating the main result, we need the following technical lemma.

Lemma 5. Let $(x_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ be the sequences recursively defined by (14) and (17). Then for every $k\geq 1$ we have:

(18) $z_k x_{k+3} - z_{k+1}x_{k+2} = (-1)^k(a\alpha_2+2\alpha_1)$ and

(19) $z_{k+1}x_{k+3} - z_{k+2}x_{k+2} = (-1)^k(a\alpha_1-2\alpha_2)$

Proof. We observe that:

$$z_k x_{k+3} - z_{k+1}x_{k+2} = \det\begin{pmatrix} z_k & z_{k+1} \\ x_{k+2} & x_{k+3}\end{pmatrix}$$

and $\begin{pmatrix} z_i & z_{i+1} \\ x_{i+2} & x_{i+3}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a\end{pmatrix} = \begin{pmatrix} z_{i+1} & z_{i+2} \\ x_{i+3} & x_{i+4}\end{pmatrix}$ (by (14) and (17)).

From the last relation we obtain immediately by induction on $k\geq 1$:

(20) $$\begin{pmatrix} z_k & z_{k+1} \\ x_{k+2} & x_{k+3}\end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ x_3 & x_4\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a\end{pmatrix}^{k-1}.$$

Taking the determinants in both sides of (20) we obtain after some easy calculations the relation (18).

The proof of (19) is entirely analogous, based on the relations

$$z_{k+1}x_{k+3} - z_{k+2}x_{k+2} = \det\begin{pmatrix} z_{k+1} & z_{k+2} \\ x_{k+2} & x_{k+3}\end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} z_{i+1} & z_{i+2} \\ x_{i+2} & x_{i+3}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a\end{pmatrix} = \begin{pmatrix} z_{i+2} & z_{i+3} \\ x_{i+3} & x_{i+4}\end{pmatrix}. \quad \square$$

Remark. A technique somewhat similar to our proof of lemma 5 is used by P.Bruckman [3].

Theorem 3. For every integers $1\leq r\leq k$, the sequence $(x_{nk+r})_{n\geq 0}$ verifies the following linear recurrence:

(21) $x_{n+k} = z_k x_n + (-1)^{k+1}x_{n-k}$, for $n>k$, $n\equiv r \pmod k$.

Proof. We prove first by induction on $k\geq 1$ the following two relations:

(22) $x_{2k+1} = z_k\, x_{k+1} + (-1)^{k+1}\alpha_1$

(23) $x_{2k+2} = z_k\, x_{k+2} + (-1)^{k+1}\alpha_2$

They may be readily verified by direct calculation for $k = 1$. Suppose now they both are true for some $k \geq 1$. Then:

(24) $$x_{2k+3} = ax_{2k+2}+x_{2k+1}=(ax_{k+2}+x_{k+1})z_k+(-1)^{k+1}(a\alpha_2+\alpha_1) =$$
$$= z_k x_{k+3}+(-1)^{k+1}(a\alpha_2+\alpha_1)= z_{k+1}x_{k+2}+(-1)^{k+2}\alpha_1,$$

where the last equality holds by lemma 5 (18).

Hence (22) is true for k+1 instead of k.

Using now (23) and (24), we obtain using again lemma 5 (19) in the last step:

$$x_{2k+4} = ax_{2k+3}+x_{2k+2}=(az_{k+1}+z_k)x_{k+2}+(-1)^k(a\alpha_1-\alpha_2) =$$
$$= z_{k+2}x_{k+2}+(-1)^k(a\alpha_1-\alpha_2) = z_{k+1}x_{k+3}+(-1)^{k+2}\alpha_2$$

Thus (23) is also true for k+1 instead of k, and by induction (22) and (23) are true for any $k \geq 1$. Using now induction on $n \geq k+1$, we immediately obtain by (22) and (23) the relation (21). This completes the proof of the theorem. □

The above theorem has interesting consequences concerning the representation of Fibonacci, Lucas and Pell sequences by integral polynomials. But first, as usual, a lemma.

Lemma 6. Let $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ be the sequences recursively defined by (15) and (17). Then for every positive integers n and k we have:

(25) $y^2_{n+k} - z_k y_{n+k} y_n + (-1)^k y^2_n = (-1)^n\, y^2_k$

(26) $z^2_{n+k} - z_k z_{n+k} z_n + (-1)^k\, z^2_n = (-1)^{n+1}(a^2+4)y^2_k$

(27) $P^2_{n+k} - z_k P_{n+k} P_n + (-1)^k P^2_n = (-1)^n P^2_k$, where $(P_n)_{n\geq 1}$ is defined by (15) for $a = 2$.

Proof. We easily observe that $y_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and

$z_n = \alpha^n + \beta^n$, where α and β are the roots of the "characteristic" equation $z^2 - az - 1 = 0$. Substituting the above expressions for y_n and z_n, we easily obtain (25) and (26) using also $\alpha\beta = -1$.

The relation (27) follows by (25), observing that $P_n = y_n$ for $a = 2$.

The following theorem contains some few of the many analogs of theorem 1 which may now be stated and proved.

Theorem 4. 1°a) The solutions in positive integers of the equation $y^2 - 3yx + x^2 = 1$ are given by: $(y,x) = (F_{2k+2}, F_{2k})$ for some $k \geq 1$.

b) The solutions in positive integers $y > x$ of the equation $y^2 - 3yx + x^2 = -1$ are given by: $(y,x) = (F_{2k+1}, F_{2k-1})$ for some $k \geq 1$.

2°a) The solutions in positive integers of the equation $y^2 - 4yx - x^2 = 4$ are given by: $(y,x) = (F_{2k+3}, F_{2k})$ for some $k \geq 1$.

b) The solutions in positive integers of the equation $y^2 - 4yx - x^2 = -4$ are given by: $(y,x) = (F_{2k+2}, F_{2k-1})$ for some $k \geq 1$ and $(y,x) = (1,1)$.

3°a) The solutions in positive integers of the equation $y^2 - 3yx + x^2 = 5$ are given by: $(y,x) = (L_{2k+1}, L_{2k-1})$ for some $k \geq 1$.

b) The solutions in positive integers of the equation $y^2 - 3yx + x^2 = -5$ are given by: $(y,x) = (L_{2k+2}, L_{2k})$ for some $k \geq 0$.

4°a) The solutions in positive integers of the equation $y^2 - 4yx - x^2 = 20$ are given by: $(y,x) = (L_{2k+2}, L_{2k-1})$ for some $k \geq 1$.

b) The solutions in positive integers of the equation $y^2 - 4yx - x^2 = -20$ are given by: $(y,x) = (L_{2k+3}, L_{2k})$ for some $k \geq 0$.

5°a) The solutions in positive integers of the equation $y^2 - 6yx + x^2 = 4$ are given by: $(y,x) = (P_{2k+2}, P_{2k})$ for some $k \geq 1$.

b) The solutions in positive integers $y > x$ of the equation $y^2 - 6yx + x^2 = -4$ are given by: $(y,x) = (P_{2k+1}, P_{2k-1})$ for some $k \geq 1$.

6^oa) The solutions in positive integers of the equation $y^2-14yx-x^2 = 25$ are given by: $(y,x) = (P_{2k+3},P_{2k})$ for some $k \geq 1$.

b) The solutions in positive integers of the equation $y^2-14yx-x^2 = -25$ are given by: $(y,x) = (P_{2k+2},P_{2k-1})$ for some $k \geq 1$.

7^o Let a be a positive integer and $(y_n)_{n\geq 1}$ the recurrent sequence given by (15). Then, the solutions in positive integers of the equation $y^2-(a^2+2)yx+x^2= a^2$ are given by: $(y,x)=(y_{2k+2},y_{2k})$ for some $k \geq 1$.

Using theorem 3, all the above solutions can be recursively defined.

<u>Proof</u>. Immediate by the remarks at the beginning of Section 2 and lemmas 1,2,3,4 and 6.

<u>Remark</u>. Unfortunately, the present author was unable to find the "primitive" solutions of the above equations for <u>general</u> a. For every <u>fixed</u> a, results like theorem 1 and 4 may however be proven. Very few general results were proved (see 7^o of theorem 4 and Ş.Buzeţeanu [4]). I believe that for much more general equations as 7^o (theorem 4) for example, the converse of the implication in lemma 6 may be proven by means of lemmas 1-4.

Based on the results in lemma 6 and theorems 1 and 4, I make the following conjecture.

<u>CONJECTURE</u>

Let a be an arbitrary fixed positive integer and define sequences $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ by the recurrences (15) and (17). Then:

1) For every integer $k \geq 1$, $\Delta = z_k^2x^2+4((-1)^{k+1}x^2+y_k^2)$ is a perfect square <u>iff</u> $x = y_{2p}$ for some $p \geq 1$. In this case it follows by lemma 6 that $y = y_{2p+k}$ verifies $y^2-z_kyx+(-1)^k x^2= y_k^2$.

2) For every integer $k \geq 1$, $\Delta = z_k^2x^2-4((-1)^kx^2+y_k^2)$ is a perfect square <u>iff</u> $x = y_{2p-1}$ for some $p \geq 1$. In this case it follows by lemma

6 that $y = y_{2p-1+k}$ verifies $y^2 - z_k yx + (-1)^k x^2 = -y_k^2$.

3) For every integer $k \geq 1$, $\Delta = z_k^2 x^2 + 4((-1)^{k+1} x^2 + (a^2+4) y_k^2)$ is a perfect square iff $x = y_{2p-1}$ for some $p \geq 1$. In this case it follows by lemma 6 that $y = y_{2p-1+k}$ verifies $y^2 - z_k yx + (-1)^k x^2 = (a^2+4) y_k^2$.

4) For every integer $k \geq 1$, $\Delta = z_k^2 x^2 - 4((-1)^k x^2 + (a^2+4) y_k^2)$ is a perfect square iff $x = y_{2p}$ for some $p \geq 1$. In this case it follows by lemma 6 that $y = y_{2p+k}$ verifies $y^2 - z_k yx + (-1)^k x^2 = -(a^2+4) y_k^2$.

We remark that the "if" part of the CONJECTURE follows by lemma 6. The "only if" part must be checked only for $x \leq f(a, \delta)$ given by lemmas 1-4 (see also the remarks at the beginning of Section 2).

A similar conjecture can be stated for the Pell sequence (see lemma 6 for $a = 2$, relation (27)).

From the above theorem 4 we obtain immediately the following more "integral representations" of the Fibonacci, Lucas and Pell'sequences.

Theorem 5. 1° The set of all positive Fibonacci numbers having even, respectively odd index is identical with the positive values of the polynomial

$$Q_1(x,y) = x(1-(y^2-3yx+x^2-1)^2), \quad \text{respectively}$$
$$Q_2(x,y) = x(1-(y^2-3yx+x^2+1)^2),$$

as the variables x and y range over the positive integers.

2° The set of all positive Fibonacci numbers having even, respectively odd index is identical with the positive values of the polynomial

$$Q_3(x,y) = x(1-(y^2-4yx-x^2-4)^2), \quad \text{respectively}$$
$$Q_4(x,y) = x(1-(y^2-4yx-x^2+4)^2),$$

as the variables x and y range over the positive integers.

3° The set of all Lucas numbers having odd, respectively even index is identical with the positive values of the polynomial

$$R_1(x,y) = x(1-(y^2-3yx+x^2-5)^2) \text{ , respectively}$$
$$R_2(x,y) = x(1-(y^2-3yx+x^2+5)^2) \text{ ,}$$

as the variables x and y range over the positive integers.

4^0 The set of all Lucas numbers having odd, respectively even index is identical with the positive values of the polynomial

$$R_3(x,y) = x(1-(y^2-4yx-x^2-20)^2) \text{ , respectively}$$
$$R_4(x,y) = x(1-(y^2-4yx-x^2+20)^2) \text{ ,}$$

as the variables x and y range over the positive integers.

5^0 The set of all Pell numbers having even, respectively odd index is identical with the positive values of the polynomial

$$S_1(x,y) = x(1-(y^2-6yx+x^2-4)^2) \text{ , respectively}$$
$$S_2(x,y) = x(1-(y^2-6yx+x^2+4)^2) \text{ ,}$$

as the variables x and y range over the positive integer.

6^0 The set of all Pell numbers having odd, respectively even index is identical with the positive values of the polynomial

$$S_3(x,y) = x(1-(y^2-14yx-x^2-25)^2) \text{ , respectively}$$
$$S_4(x,y) = x(1-(y^2-14yx-x^2+25)^2) \text{ ,}$$

as the variables x and y range over the positive integers.

7^0 Let a be a positive integer and $(y_n)_{n\geq 1}$ the recurrent sequence given by (15). Then, the sequence $(y_{2n})_{n\geq 1}$ is identical with the positive values of the polynomial

$$T(x,y) = x(1-(y^2-(a^2+2)yx+x^2-a^2)^2) \text{ ,}$$

as the variables x and y range over the positive integer.

Proof. See the proof of theorem 2.

Remark. By the above theorem, we can represent separatelly the sequences $(F_{2n})_{n\geq 1}$ and $(F_{2n-1})_{n\geq 1}$, $(L_{2n})_{n\geq 1}$ and $(L_{2n-1})_{n\geq 1}$, $(P_{2n})_{n\geq 1}$ and $(P_{2n-1})_{n\geq 1}$. It will be of interest to give such "integral polynomial representations" for the sequence $(F_{nk+r})_{n\geq o}$, for arbitrary fixed integers $1 \leq r \leq k$ $(k \geq 3)$. Such representations exists by results in M.Davis [5].

Finally, we remark that results analogous to those in lemma 5, theorem 3 and lemma 6 hold for the recurrence $x'_{n+1}=ax'_n-x'_{n-1}$ instead of $x_{n+1} = ax_n+x_{n-1}$. We state these results without proof, since then are entirely similar to the above ones.

Let a be some fixed integer. We consider the following recurrent sequences:

(28) $x'_1 = \beta_1,\ x'_2 = \beta_2,\ x'_{n+2} = ax'_{n+1}-x'_n$ for $n \geq 1$.

(29) $y'_1 = 1,\ y'_2 = a,\ y'_{n+2} = ay'_{n+1}-y'_n$ for $n \geq 1$.

Using induction on $n \geq 1$, it is easily to verify that:

(30) $x'_{n+2} = y'_{n+1}\beta_2 - y'_n \alpha_1$ for every $n \geq 1$. Define also:

(31) $z'_1 = a,\ z'_2 = a^2-2,\ z'_{n+2}= az'_{n+1}-z'_n$ for $n \geq 1$.

<u>Lemma 7</u>. Let $(x'_n)_{n\geq 1}$ and $(z'_n)_{n\geq 1}$ be the sequences recursively defined by (28) and (31). Then for every $k \geq 1$ we have:

(32) $z'_k x'_{k+3} - z'_{k+1} x'_{k+2} = a\beta_2 - 2\beta_1$

(33) $z'_{k+1} x'_{k+3} - z'_{k+2} x'_{k+2} = 2\beta_2 - a\beta_1$.

<u>Theorem 6</u>. For every integers $1 \leq r \leq k$, the sequence $(x'_{nk+r})_{n\geq o}$ verifies the following linear recurrence:

(34) $x'_{n+k} = z'_k x'_n - x'_{n-k}$, for $n > k$, $n \equiv r \pmod k$.

<u>Lemma 8</u>. Let $(y'_n)_{n\geq 1}$ and $(z'_n)_{n\geq 1}$ be the sequences recursively defined by (29) and (31). Then for every positive integers n and k we have:

(35) $x'^2_{n+k} - z'_k x'_{n+k} x'_n + x'^2_n = x'^2_k$

(36) $z'^2_{n+k} - z'_k z'_{n+k} z'_n + z'^2_n = (a^2+4) z'^2_k$.

Now, the attentive reader can easily write down and prove analogous of theorems 4 and 5.

Final remarks. 1) By using succesively theorems 3 and 6, we can obtain some relations involving both sequences $(z_n)_{n\geq 1}$ and $(z'_n)_{n\geq 1}$.

2) Using the equivalence:

$$y^2-ayx\pm x^2 = \pm\delta \Longleftrightarrow (2y-ax)^2-(a^2\pm 4)x^2 = \pm 4\delta,$$

every result concerning the solutions of the equation $y^2-ayx\pm x^2=\pm\delta$ can be "translated" into a result concerning the solutions of the corresponding equation $z^2-(a^2\mp 4)x^2=\pm 4\delta$.

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