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by

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INTRODUCTION

Let $\{X_i; p_{ij} : X_j \to X_i\}_{i \in I,\ i \leq j}$ be a directed projective system of discrete finite sets, where the connecting maps p_{ij} are onto. According to B. Mazur [14, 15], a distribution on $X = \varprojlim X_i$ with values in an abelian group A is a family of maps $\varphi_i : X_i \to A$, $i \in I$, subject to $\sum_{p_{ij}(x)=y} \varphi_j(x) = \varphi_i(y)$ for $i,j \in I$, $i \leq j$, $y \in X_i$.

Distributions are basic tools in the investigation of some arithmetic questions concerning cyclotomic fields (Iwasawa [1], Sinnott [19], Lang [11,12]), units in the modular function field (Kubert-Lang [6-10]) and Galois module structure of elementary abelian extensions (Mc Culloh [16, 17]).

The distributions defined above turn out to be special cases of the following more general measure theoretic concept.

DEFINITION. Given a distributive lattice L, with binary operations $\vee, \wedge$, and a smallest element 0, a map $\varphi : L \to A$ with values in the abelian group A is called a distribution on L if $\varphi(a \vee b) = \varphi(a) + \varphi(b)$ for arbitrary $a,b \in L$, $a \wedge b = 0$.

The present paper is a self contained approach of certain purely algebraic questions concerning the distributions on distributive lattices. A general algebraic framework is developed and some results of Sinnott, Kubert and Lang are suitably extended.

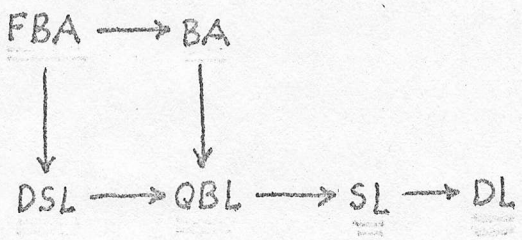
1. STONE DUALITY

Consider the category DL whose objects are the distributive lattices $\underline{L} = (L, \vee, \wedge, 0)$ with the customary binary operations $\vee, \wedge$, and a smallest element 0 w.r.t. the partial order $a \leq b \Leftrightarrow a = a \wedge b$; if L, L' are objects of DL then the set DL(L,L') of morphisms from L to L' consists of the maps $f : L \to L'$ subjects to the following conditions:

i) $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$ for $a,b \in L$, and $f(0) = 0$;

ii) f(L) is cofinal in L', i.e., for each $b \in L'$ there is $a \in L$ such that $b \leq f(a)$.

The next picture represents certain distinguished full subcategories of DL; the arrows stand for canonic embeddings



The objects of SL, called Stone lattices, are the distributive lattices L satisfying the following equivalent conditions:

(SL') for each $a \in L$, the set $a \vee a^{\perp} = \{a \vee c : a \wedge c = 0\}$ is cofinal in L, i.e., for each $b \in L$ there is $c \in L$ such that $b \leq a \vee c$ and $a \wedge c = 0$;

(SL") for each $a \in L$, the principal ideal $\langle a \rangle = \{b \in L : b \leq a\}$ is a boolean algebra with maximal element a.

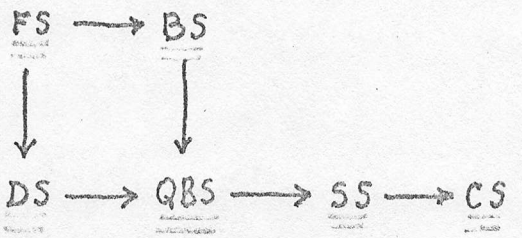
The object of QBL, called quasi-boolean lattices, are the Stone lattices L satisfying the suplementary condition

(QBL) there exists a subset A of L such that $a \neq 0$ for $a \in A$, $a \wedge b = 0$ for $a, b \in A$, $a \neq b$, and the ideal $\langle A \rangle = \{c \in L : \exists\, a_1, \ldots, a_n \in A,\ n \geq 1,\ c \leq \bigvee_{i=1}^{n} a_i\}$ generated by A is the whole L.

BA is the category of boolean algebras, while FBA is the full subcategory of finite boolean algebras. The objects of DSL, called discrete Stone lattices, are the Stone lattices L subject to the next condition

(DSL) for each $a \in L$, $\langle a \rangle$ is finite.

On the other hand, consider certain categories of topological spaces related each to other as in the picture



The objects of CS, called coherent spaces, are the toological spaces X satisfying the following conditions:

(S) X is sober, i.e., every irreducible closed subset of X is the closure of a unique point of X;

(C) the family K(X) of quasi-compact open subsets of X is closed under intersection, i.e., $A \cap B \in K(X)$ for $A, B \in K(X)$, and forms a base for the topology on X.

A morphism $f : X \to Y$ in CS, called a coherent map, is a continuous map subject to $f^{-1}(A) \in K(X)$ for $A \in K(Y)$.

The objects of SS, called Stone spaces, are the Hausdorff spaces X for which the family K(X) of compact open subsets of X forms a base for the topology on X. Obviously SS (with coherent maps as morphisms) is a full subcategory of CS.

The objects of QBS, called quasi-boolean spaces, are the Stone spaces X for which there is a partition $X = \dot{\bigcup}_{i \in I} X_i$ with $X_i \in K(X)$. The objects of BS, called boolean spaces, are the compact Stone spaces, i.e. compact totally disconnected spaces. Note that every continuous map between boolean spaces is coherent, since K(X) is the boolean algebra of all clopen subsets of X whether X is a boolean space. Note also that the objects of QBS are arbitrary direct sums in the category Top of topological spaces (not in CS!) of boolean spaces.

Finally, the objects of DS are the discrete topological spaces, while the objects of FS are the discrete finite spaces. Note that a morphism $f : X \to Y$ in DS is a map with the property that the fiber $f^{-1}(y)$ is finite for each $y \in Y$.

THEOREM 1.1. (Stone's representation theorem for distributive lattices). The category DL is dual to the category CS. The duality above induces respectively dualities between the categories SL and SS, QBL and QBS, BA and BS, DBL and DS, FBA and FS.

PROOF. The duality sends a distributive lattice L to the Stone-Zariski prime spectrum S(L) = Spec(L) of proper prime ideals of L; its open sets may be identified

with arbitrary ideals of L, a point $p \in S(L)$ being in an open set $a \in Id(L)$ iff $a \notin p$. Conversely, the duality sends a coherent space X to the distributive lattice K(X) of compact open subsets of X. For details see [2] Ch.II, where the slightly more particular case of distributive lattices with a maximal element 1 is considered. ∎

The next result is immediate.

LEMMA 1.2. i) The categories DL, SL, QBL and BA are closed under directed inductive limits. The embeddings BA→QBL→SL→DL preserve such limits.

ii) The dual statement.

LEMMA 1.3. i) QBL (BA) is the closure of DSL(FBA) under directed inductive limits.

ii) QBS(BS) is the closure of DS(FS) under directed projective limits.

Proof. Let X be a quasi-boolean space and P be the set of partitions of X into compact open non-empty subsets of X. By assumption P is non-empty. P is partially ordered by refinement relation $\leq$ and $(P,\leq)$ is directed. For $p \in P$ let X_p be the factor set of X w.r.t. the equivalence relation induced by p, and $f_p : X \to X_p$ be the canonic surjection. For $p,q \in P$, $p \leq q$, let $f_{p,q} : X_q \to X_p$ be the canonic surjection. Obviously, $\{X_p; f_{p,q}\}_{p \in P, p \leq q}$ is a directed projective system in DS and $X \cong \varprojlim X_p$. Of course we may replace P by an arbitrary cofinal subset of P. ∎

2. DISTRIBUTIONS ON DISTRIBUTIVE LATTICES

Let L be a distributive lattice with 0 and A be an abelian group. As defined already in Introduction, a distribution on L with values in A is a map $\varphi : L \to A$ such that $\varphi(a \vee b) = \varphi(a) + \varphi(b)$ for $a,b \in L$ with $a \wedge b = 0$. It follows $\varphi(0) = 0$.

Assuming L is a Stone lattice, one checks easily that a necessary and sufficient condition for a map $\varphi : L \to A$ to be a distribution is that $\varphi(0) = 0$ and $\varphi(a \vee b) + \varphi(a \wedge b) = \varphi(a) + \varphi(b)$ for $a,b \in L$.

Now assume L is a quasi-boolean lattice, and let $X = S(L)$ be the quasi-boolean space assigned to L by Stone duality according to Theorem 1.1. By Lemma

1.3, L is the inductive limit in QBL of a directed inductive system $\{L_i; f_{ij} : L_i \to L_j\}_{i\in I, i\leq j}$ of discrete Stone sublattices of L. For each $i \in I$, the discrete space $X_i = S(L_i)$ is identified with the set of atoms of L_i. The space X is the projective limit of the directed projective system $\{X_i; p_{ij} : X_j \to X_i\}_{i\in I, i\leq j}$ where the $p_{ij} : X_j \to X_i$ for $i \leq j$ are the canonic surjections with finite fibers induced by the embeddings $f_{ij} : L_i \to L_j$. Given a map $\varphi : L \to A$, denote by $\varphi_i : X_i \to A$ for $i \in I$ the restriction of φ to $X_i \subset L_i$. The correspondence $\varphi \mapsto (\varphi_i)_{i\in I}$ maps bijectively the set of distributions on L with values in the abelian group A onto the set of those families $(\varphi_i)_{i\in I}$ subject to the condition $\varphi_i(x) = \sum_{p_{ij}(y)=x} \varphi_j(y)$ for $i \leq j$, $x \in X_i$. Call such families distributions on the quasi-boolean space X. For a boolean space X we get Mazur's distributions.

Let us give some examples of distributions on certain topological groups. In order to do this we need the next immediate lemma.

LEMMA 3.1. The following statements are equivalent for a topological group G:

i) There exists a directed projective system $\{G_i; p_{ij} : G_j \to G_i\}_{i\in I}$ of discrete groups with Ker p_{ij} finite for $i \leq j$, such that G is isomorphic as topological group to $\varprojlim G_i$;

ii) G is a Hausdorff space and the family of compact open normal subgroups of G is a fundamental system of neighbourhoods of 1.

Call a topological group G satisfying the equivalent conditions above a quasi-boolean group. Obviously, the quasi-boolean groups are quasi-boolean spaces. A quasi-boolean group G is compact iff G is profinite iff G is group in the category QBS (and hence in BS); the canonic projections $p_i : G \times G \to G$, $i = 1,2$, are coherent maps iff G is compact.

Assume G is a quasi-boolean group and let N(G) be the set of compact open normal subgroups of G partially ordered by the relation $U \leq V$ iff $V \subseteq U$. N(G) is directed and $G \simeq \varprojlim_{U\in N(G)} G/U$. A distribution $\varphi = (\varphi_U : G/U \to A)_{U\in N(G)}$ on G with

values in an abelian group A is invariant if the maps φ_U are constant maps. Consider the directed projective system $\{A_U; f_{U,V} : A_V \to A_U\}_{U\in N(G), U\leq V}$, where $A_U = A$ for each $U \in N(G)$ and $f_{U,V}(a) = (U : V)a$ for $U \leq V$, $a \in A$. Then the set of invariant distributions on G with values in A is identified with $\varprojlim A_U$. In particular, if A is uniquely divisible (i.e. A is a vector space over $\underline{Q}$) then $\varprojlim A_U \cong A$; an invariant distribution φ is completely determined by $\varphi_U(1)$ for a fixed $U \in N(G)$.

For instance, let G be the underlying additive topological group of a local field K, i.e., a Cauchy complete valued field with discrete valuation v and finite residue field k. Let $\underline{O}$ be the valuation ring and π be a prime element, i.e., $v(\pi) = 1$, the smallest positive element of the value group $vK = \underline{Z}$. Then $G = K^+ \cong \varprojlim_{n\in N} K/\pi^n\underline{O}$ is an abelian quasi-boolean group with $N(G) = \{\pi^n\underline{O} : n \in \underline{Z}\}$. An invariant distribution $\varphi = (\varphi_n : K/\pi^n\underline{O} \to A)_{n\in \underline{Z}}$ with values in an uniquely divisible abelian group A is completely determined by $\varphi_o(\underline{O})$.

Another special way to obtain distributions is the following one. Let $\{X_i; p_{ij} : X_j \to X_i\}_{i\in I,\ i\leq j}$ be a directed projective system, where $X_i = M$ is a discrete space for each $i \in I$, and $p_{ij} : M \to M$ are surjections wih finite fibers. A distribution $(\varphi_i : M \to A)_{i\in I}$ on the quasi-boolean space $X = \varprojlim_{i\in I} X_i$ is called an ordinary distribution if the maps φ_i for $i \in I$ coincide, say $\varphi_i = \varphi$. In other words, an ordinary distribution on X is a map $\varphi : M \to A$ subject to $\varphi(x) = \sum_{p_{i,j}(y)=x} \varphi(y)$ for $i,j \in I$, $i \leq j$, $x \in M$.

For instance, let $I = \underline{Z}_{\geq 1}$ with the partial order given by the divisibility relation, $X_n = \underline{Q}/\underline{Z}$ for $n \geq 1$, $p_{n,m} : X_m \to X_n$ for $n|m$ be defined by $p_{n,m}(x) = (m/n)x$. An ordinary distribution on the quasi-boolean space $X = \varprojlim_{n\geq 1} X_n$ is an ordinary distribution in the usual sense [11] Ch.2, § 8, i.e., a map $\varphi : \underline{Q}/\underline{Z} \to A$ subject to $\varphi(x) = \sum_{ny=x} \varphi(y)$ for $n \geq 1$, $x \in \underline{Q}/\underline{Z}$. A typical example is the first Bernoulli distribution $\underline{Q}/\underline{Z} \to \underline{Q} : x \mapsto B_1(\langle x\rangle) = \langle x\rangle - \frac{1}{2}$, where $\langle x\rangle$ is the smallest rational number ≥ 0 in the residue class of x mod $\underline{Z}$.

3. UNIVERSAL DISTRIBUTIONS

Given a distsributive lattice L with 0 and a commutative ring R with unit, consider the category $\underline{\underline{D}}_{L,R}$ whose objects are the distributions $\varphi : L \to A$ with values in R-modules; if $\varphi_i : L \to A_i$, $i = 1,2$, are objects of $\underline{\underline{D}}_{L,R}$, the set of morphisms $\underline{\underline{D}}_{L,R}(\varphi_1, \varphi_2)$ consists of those morphisms $h : A_1 \to A_2$ of R-modules for which $\varphi_2 = h \circ \varphi_1$.

Let $R^{(L)}$ be the free R-module generated by the underlying set of L and let D be the submodule of $R^{(L)}$ generated by the subset $\{(a \vee b) - (a) - (b) : a,b \in L, a \wedge b = 0\}$. Let $U = U_{L,R}$ be the factor R-module $R^{(L)}/D$. Then obviously the map $u : L \to U : a \mapsto (a) \bmod D$ is the initial object of the category $\underline{\underline{D}}_{L,R}$, i.e., for each distribution φ on L with values in an R-module A there is a unique morphism $d\varphi : U \to A$ of R-modules such $\varphi = d\varphi \circ u$. Call $u : L \to U$ the <u>universal distribution</u> on L w.r.t. the ring R.

The universal distribution $u : L \to U$ admits a natural equivalent description whther L is a Stone lattice (in particular a quasi-boolean lattice). Let $X = S(L)$ be the Stone space assigned to L by Stone duality and denote by $C(X,R)$ the set of those continuous functions $f : X \to R$ defined on X with values in the discrete space R for which the support $\mathrm{Supp}(f) = \{x \in X : f(x) \neq 0\}$ is compact; $C(X,R)$ has a natural structure of R-module. Let $u' : L \to C(X,R)$ be the map assigning to each $a \in L$ the function $u'(a)$ defined as follows: if $p \in X = \mathrm{Spec}(L) = \mathrm{Max}(L)$ then $u'(a)(p) = \begin{cases} 1 & \text{if } a \notin p \\ 0 & \text{otherwise} \end{cases}$. One checks easily that u' is a distribution and u,u' are canonically isomorphic in $\underline{\underline{D}}_{L,R}$; we identify them in the following. The correspondence $\varphi : L \to A \mapsto d\varphi : U = C(X,R) \to A$ maps bijectively the R-module of distributions on L with values in the R-module A onto the R-module of morphisms of R-modules from U to A. For $f \in C(X,R)$ denote the image $(d\varphi)(f)$ by $\int_X f d\varphi$ and call it the <u>integral</u> of f on the Stone space X w.r.t. the distribution (measure) φ. Assuming X is quasi-boolean, write X as a directed projective limit $\varprojlim_{i \in I} X_i$ of discrete spaces, where the connecting maps $p_{ij} : X_j \to X_i$ for $i \leq j$ are onto with finite fibers. Consider the family $(\varphi_i : X_i \to A)_{i \in I}$ induced by φ. As Supp (f) is a compact

open subset of X there exist $i \in I$ and $f' : X_i \to A$ with Supp(f') finite such that $f = f'p_i$, where $p_i : X \to X_i$ is the canonic projection. Then $\int_X f d\varphi = \sum_{x \in X_j} f'(p_{ij}(x))\, \varphi_j(x)$ for each $j \in I$, $j \geq i$.

4. STICKELBERGER DISTRIBUTIONS

Given a profinite group G, a G-lattice is a distributive lattice L with 0, together with a continuous action $G \times L \to L : (g,a) \mapsto ga$ of G on the discrete distributive lattice L, i.e., $(gh)a = g(ha)$, $g(a \vee b) = ga \vee gb$, $g(a \wedge b) = ga \wedge gb$, $g \cdot 0 = 0$ for $g,h \in G$, $a,b \in L$, and $L = \bigcup_{U \in N(G)} L^U$, where N(G) is the directed system of open normal subgroup of G and $L^U = \{a \in L : ga = a \text{ for } g \in U\}$; note that the distributive sublattice L^U is cofinal in L for each $U \in N(G)$ and hence L is the inductive limit $\varinjlim L^U$ in $\underline{DL}$.

Given a G-lattice L and a discrete G-module A, a distribution $\varphi : L \to A$ is called a G-distribution if $\varphi(ga) = g\varphi(a)$ for $g \in G$, $a \in L$.

Let us denote by $\underline{D}_{G,L}$ the category whose objects are the G-distributions on the G-lattice L; if $\varphi_i : L \to A_i$, $i = 1,2$, are G-distributions, the set of morphisms $\underline{D}_{G,L}(\varphi_1, \varphi_2)$ consists of those morphisms $f : A_1 \to A_2$ of G-modules for which $f\varphi_1 = \varphi_2$.

Let H be a closed subgroup of G. The continuous action of G on L induces a natural structure of H-lattice on L. We define a covariant functor $St_G^H : \underline{D}_{H,L} \to \underline{D}_{G,L}$ as follows. Let $\varphi : L \to A$ be a H-distribution. The induced module $M_G^H(A)$ admits as underlying set the set of those continuous functions $f : G \to A$ for which $f(hg) = hf(g)$, where $h \in H$, $g \in G$. The profinite group G acts continuously on the discrete abelian group $M_G^H(A)$ according to the rule $(gf)(g') = f(g'g)$. If $U \in N(G)$ then $M_G^H(A)^U \simeq M_{G/U}^{H/H \cap U}(A^{H \cap U})$. Define $St_G^H(\varphi) : L \to M_G^H(A)$ by $St_G^H(\varphi)(a)(g) = \varphi(ga)$ for $a \in L$, $g \in G$. Call the G-distribution $St_G^H(\varphi)$ the Stickelberger G-distribution induced by the H-distribution φ. If $\varphi_i : L \to A_i$, $i = 1,2$, are H-distributions and $s \in \underline{D}_{H,L}(\varphi_1, \varphi_2)$, define $St_G^H(s) : M_G^H(A_1) \to M_G^H(A_2)$ by $St_G^H(s)(f) = s \circ f$ for $f \in M_G^H(A_1)$; obviously

$St_G^H(s) \in \underline{\underline{D}}_{G,L}(St_G^H(\varphi_1), St_G^H(\varphi_2))$. Call the functor $St_G^H : \underline{\underline{D}}_{H,L} \to \underline{\underline{D}}_{G,L}$ Stickelberger functor assigned to the data G, H, L.

PROPOSITION 4.1. $St_G^H : \underline{\underline{D}}_{H,L} \to \underline{\underline{D}}_{G,L}$ is a right-adjoint of the forgetfull functor $\underline{\underline{D}}_{G,L} \to \underline{\underline{D}}_{H,L}$.

Proof. Let $\varphi : L \to A$, $\psi : L \to B$ be objects respectively of $\underline{\underline{D}}_{H,L}$ and $\underline{\underline{D}}_{G,L}$. The functorial bijection $\Theta : \underline{\underline{D}}_{H,L}(\psi, \varphi) \to \underline{\underline{D}}_{G,L}(\psi, St_G^H(\varphi))$ is given by $\Theta(s)(b)(g) = s(gb)$ for $s \in \underline{\underline{D}}_{H,L}(\psi, \varphi)$, $b \in B$, $g \in G$; $\Theta^{-1}(t)(b) = t(b)(1)$ for $t \in \underline{\underline{D}}_{G,L}(\psi, St_G^H(\varphi))$, $b \in B$.

In particular, taking $H = \{1\}$, we get the right-adjoint $St_G : \underline{\underline{D}}_L \to \underline{\underline{D}}_{G,L}$ of the fogetfull functor $\underline{\underline{D}}_{G,L} \to \underline{\underline{D}}_L$.

5. PROFINITE GROUPS AND LATTICE ORDERED MONOIDS

In this section we consider a natural connection between certain profinite groups and certain lattice ordered monoids, which will play a basic role in the rest of the paper.

Given a set I, let us denote by S_I the free abelian group $Z^{(I)}$ generated by I. Each element of S_I admits a unique representation $s = \sum_{i \in I} s(i)i$ with $s(i) \in Z$ and $Supp(s) = \{ i \in I : s(i) \neq 0 \}$ a finite set; put $s_i = s(i)i$, the i-th component of s. S_I has a canonic structure of lattice ordered group w.r.t. the partial order $s \leq t \Leftrightarrow s(i) \leq t(i)$ for all $i \in I$, having the lattice operations given by $(s \wedge t)(i) = \min(s(i), t(i))$, $(s \vee t)(i) = \max(s(i), t(i))$. For $s \in S_I$, let $s^+ = s \vee 0$, $s^- = -s \vee 0$, $|s| = s \vee -s$. Denote by S_I^+ the lattice ordered monoid with universe $\{ s \in S_I : s \geq 0 \}$. Note that for each $s \in S_I^+$ the set $\{ t \in S_I^+ : t \leq s \}$ is finte.

Given a profinite group G, the set N(G) of open normal subgroups of G, partially ordered by the relation $U \leq V \Leftrightarrow V \subseteq U$, is a lattice with $U \vee V = U \cap V$, $U \wedge V = U.V$ and a smallest element G. N(G) satisfies the same finiteness condition as S_I^+, so it is quite natural to consider lattice morphisms from S_I^+ to N(G).

DEFINITION. A map $U : S_I^+ \to N(G)$ is compatible if the following conditions

are satisfied:

i) U is a morphism of lattices with a smallest element, i.e., $U(s \vee t) = U(s) \cap U(t)$, $U(s \wedge t) = U(s) \cdot U(t)$ for $s,t \in S_I^+$, and $U(0) = G$;

(ii) $U(S_I^+) = \{U(s) : s \in S_I^+\}$ is cofinal with N(G), i.e., for each $V \in N(G)$ there exists $s \in S_I^+$ such that $U(s) \subseteq V$;

(iii) for each subset J of I such that $J \neq I$, $V_J = \bigcap_{Supp(s) \subseteq J} U(s) \neq \{1\}$.

Condition i) is equivalent with the fact that $G(s) := G/U(s) \simeq \prod_{i \in I} G(s_i)$ for each $s \in S_I^+$. Condition ii) is equivalent with the fact that $G \simeq \varprojlim_{s \in S_I^+} G(s)$. Condition iii) states that the canonic epi $G \to G_J$, where $G_J = G/V_J \simeq \varprojlim_{Supp(s) \subseteq J} G(s)$ is an isomorphism iff J = I. Obviously, condition iii) is satisfied whether U is injective.

REMARK. There exist compatible maps defined on non-isomorphic S_I^+'s with values in the same lattice N(G). For instance, let G be the procyclic group $\hat{Z} = \varprojlim Z/nZ$, I be the set of all prime numbers and I' be a singleton. Then S_I^+ is isomorphic tothe l.o. multiplicative monoid of positive integers (with order given by divisibility), while $S_{I'}^+$ is isomorphic to the linearly ordered additive monoid of natural numbers. The maps $U : S_I^+ \to N(G) : n \mapsto n\hat{Z}$ and $U' : S_{I'}^+ \to N(G) : n \mapsto (n+1)!\hat{Z}$ are compatible maps.

If $J \subseteq I$, the map $U_J : S_J^+ \to G_J : s \mapsto U(s)/V_J$ induced by a compatible map U is compatible too. In particular, for $i \in I$, we get compatible maps $U_i = U_{\{i\}} : S_{\{i\}}^+ \simeq N \to G_i \simeq \varprojlim_{n \in N} G(ni)$. The U_i's determine completely U as follows from the next immediate lemma.

LEMMA 5.1. a) Assume that $U : S_I^+ \to N(G)$ is a compatible map. Then $G \simeq \prod_{i \in I} G_i$.

b) Conversely, let $\{G_i\}_{i \in I}$ be a family of profinite groups and $U_i : N \to N(G_i)$ be compatible maps. Let $G \simeq \prod_{i \in I} G_i$ and $U : S_I^+ \to N(G)$ be the map given by $U(s) = \bigcap_{i \in I} \ker(G \to G_i/U_i(s(i)))$. Then U Is a compatible map.

COROLLARY 5.2. Let I be a set and G be a profinite group. The necessary and sufficient condition for the existence of compatible map $U : S_I^+ \to N(G)$ is that there

exists a family $\{G_i\}_{i\in I}$ of profinite groups such that $G_i \neq \{1\}$, $|N(G_i)| \leq \aleph_o$ and $G \simeq \prod_{i\in I} G_i$.

DEFINITION. A special profinite group is a triple $\underline{\underline{G}} = (G,I,U)$ where G is a profinite group, I is a set and $U : S_I^+ \to N(G)$ is a compatible map.

DEFINITION. Given a special profinite group $\underline{\underline{G}} = (G,I,U)$, a $\underline{\underline{G}}$-set is a discrete G-set with the action $G \times X \to X : (g,x) \mapsto gx$ together with a partition $X = \bigcup_{s\in S_I^+} X_s$ and an action $S_I^+ \times X \to X : (s,x) \mapsto {}^s x$ of the commutative monoid S_I^+ on the G-set X such that the following conditions are satisfied:

i) $GX_s = X_s$ and $\mathrm{Inv}(x) = \{g\in G : gx = x\} = U(s)$ for $s\in S_I^+$, $x \in X_s$;

ii) ${}^s X_t = \{{}^s x : x \in X_t\} = X_{(t-s)^+}$ for $s,t \in S_I^+$;

iii) for $s,t\in S_I^+$, $s \geq t$, $x,y\in X_t$, ${}^s x = {}^s y \Rightarrow y \in Gx$.

It follows ${}^s X_{s+t} = X_t$, so the G-maps $X \xrightarrow{s} X : x \mapsto {}^s x$ are surjections with finite fibers. For $s,t\in S_I^+$ such that $s \wedge t = 0$, the G-map $X_t \xrightarrow{s} X_t$ is bijective and hence the subgroup $S_{I\setminus \mathrm{Supp}(t)}$ of S_I acts on X_t; in particular X_o is an S_I - set.

The $\underline{\underline{G}}$ - sets forms a category $\underline{\underline{G}}$ - Sets having as morphisms the maps $f : X \to X'$ commuting with the actions of G and S_I^+ such that $f(X_s) \subseteq X'_s$ for $s \in S_I^+$. This category admits the following interpretation in cohomological terms.

If Y is an S_I - set then the profinite group $G^Y = \{f : Y \to G\}$, with the law group defined componentwise, has a canonic structure of S_I - group with the action $S_I \times G^Y \to G^Y : (s,f) \mapsto {}^s f$ given by ${}^s f(y) = f({}^s y)$ for $y \in Y$. Let $J \subseteq I$ and $G_J = \prod_{i\in J} G_i \simeq \varprojlim_{\mathrm{Supp}(s)\subseteq J} G/U(s)$. As G_J^Y is in particular $S_{I\setminus J}$ - group, we may consider the set of 1-cocycles $Z^1(S_{I\setminus J}, G_J^Y)$ consisting of the maps $u : S_{I\setminus J} \to G_J^Y : s \mapsto u_s$ subject to $u_s \cdot {}^s u_t = u_{s+t}$ for $s,t \in S_{I\setminus J}$. Now consider the set $Z^1_{\underline{\underline{G}}}(S_I, G^Y)$ of $\underline{\underline{G}}$-1-cocycles consisting of the families $(u_J)_{J\subseteq I}$ where $u_J \in Z^1(S_{I\setminus J}, G_J^Y)$ such that the next diagram commutes

$$\begin{array}{ccc} S_{I\setminus J} & \xrightarrow{u_J} & G_J^Y \\ \downarrow & & \downarrow \\ S_{I\setminus J'} & \xrightarrow{u_{J'}} & G_{J'}^Y \end{array}$$

for $J' \subseteq J$, where the left vertical arrow is the canonic embedding while the right vertical arrow is the epi induced by the projection $G_J \to G_{J'}$. One checks easily that the set $Z^1_G(S_I, G^Y)$ may be identified with the set of the families $u = (u_i)_{i \in I}$ ~~with the set of the families $u = (u_i)_{i \in I}$~~ with $u_i \in G^Y_{I \setminus \{i\}}$ such that $\overrightarrow{u_i} \cdot {}^i\overrightarrow{u_j} = \overrightarrow{u_j} \cdot {}^j\overrightarrow{u_i}$ for $i,j \in I$, $i \neq j$, where $\overrightarrow{u_i}, \overrightarrow{u_j}$ are the projections of u_i, u_j in $G^Y_{I \setminus \{i,j\}}$. Now consider the category G-Cocycles having as objects the pairs (Y,u) where Y is an S^I-set and $u \in Z^1_G(S_I, G^Y)$. As morphisms $(Y,u) \to (Y',u')$ consider the pairs (f,h) where $f : Y \to Y'$ is an S_I-map and $h \in G^Y$ such that $h_i^{-1} \cdot u_i \cdot {}^i h_i = u'_i \circ f$ for $i \in I$, where h_i is the projection of h in $G^Y_{I \setminus \{i\}}$; the composition rule is $(f',h') \circ (f,h) = (f' \circ f,\ h \cdot (h' \circ f))$.

PROPOSITION 5.3. The categories G-Sets and G-Cocycles are equivalent.

Proof. Define a covariant functor F : G-Cocycles $\to$ G-Sets as follows. For an object (Y,u) of G-Cocycles define $X_s = Y \times G(s)$, $s \in S_I^+$, and let $F(Y,u) = X$ be the disjoint union $\overset{\circ}{\bigcup}_{s \in S_I^+} X_s$. G acts continuously from the left on the discrete set X according to the rule

$$g'(y,g) = (y, p_t(g') \cdot g) \in X_t \quad \text{for } t \in S_I^+,\ (y,g) \in X_t,\ g' \in G .$$

In order to define an action of the monoid S_I^+ on the G-set X it suffices to define commuting G-maps $X \xrightarrow{i} X : x \mapsto {}^i x$ for $i \in I$. Let $t \in S_I^+$, $(y,g) \in X_t$, $i \in I$. For $i \leq t$ define ${}^i(y,g) = (y, p_{t-i,t}(g)) \in X_{t-i}$, while for $t \wedge i = 0$, define ${}^i(y,g) = ({}^i y, g \cdot p_t(u_i(y))) \in X_t$. One checks that X is a G-set. To a morphism $(f,h) : (Y,u) \to (Y',u')$ in G-Cocycles one assigns the morphism $F(f,h) : F(Y,u) \to F(Y',u')$ of G-sets given by $F(f,h)(y,g) = (f(y), g p_t(h(y))) \in F(Y',u')_t$ for $(y,g) \in F(Y,u)_t$, $t \in S_I^+$. Obviously, the map $\mathrm{Hom}((Y,u),(Y',u')) \to \mathrm{Hom}(X,X')$, where $X = F(Y,u)$, $X' = F(Y',u')$, is injective. Moreover, it is bijective. Indeed, let $\eta : X \to X'$ be a morphism of G-sets. Let $f : Y \to Y'$ be the restriction of η at $X_0 \equiv Y$. For $s \in S_I^+$, $(y,g) \in X_s$, one gets $\eta(y,g) = (f(y), g h_{y,s}) \in X'_s$ with $h_{y,s} \in G(s)$. One checks that $h(y) = (h_{y,s})_{s \in S_I^+} \in G$, $(f,h) : (Y,u) \to (Y',u')$ is a morphism and $F(f,h) = \eta$.

In order to conclude that F is an equivalence of categories it remains to show that for every G-set X there is an object (Y,u) of G-Cocycles such that

$X \simeq F(Y,u)$. Consider the directed projective system of discrete G-spaces $\{X_s, X_s \xrightarrow{s-t} X_t : x \mapsto {}^{s-t}x\}_{s \in S_I^+, t \leq s}$ and let $\hat{X}$ be the quasi-boolean G-space $\varprojlim X_s$ with the coherent G-epis $\beta_s : \hat{X} \to X_s$. Let $\mu : X_o \to \hat{X}$ be a section of β_o, i.e., $\beta_o \circ \mu = 1_{X_o}$. For each $y \in X_o$ the map $G \to \mu^{-1}(y) : g \mapsto g\mu(y)$ is an isomorphism of G-spaces. For $i \in I$, we get a coherent epi $\lambda_i : \hat{X} \to \hat{X}_i = \varprojlim_{\lambda i : s \to i} X_s$ of quasi-boolean G-spaces. Then $\mu_i = \lambda_i \circ \mu$ is a section of the coherent epi $\hat{X}_i \to X_o$. $\hat{X}_i$ is an $G_{I \setminus \{i\}}$-space and the maps $G_{I \setminus \{i\}} \to \mu_i^{-1}(y) : g \mapsto g\mu_i(y)$ for $y \in X_o$ are isomorphisms of $G_{I \setminus \{i\}}$-spaces. Note also that the G-map $X \xrightarrow{i} X$ induces an automorphism $\hat{X}_i \xrightarrow{i} \hat{X}_i$ of the $G_{I \setminus \{i\}}$-space $\hat{X}_i$. Define the object (Y,u) of G-cocycles by $Y = X_o$, ${}^i\mu_i(y) = u_i(y)\mu_i({}^i y)$ for $i \in I$, $y \in X_o$ and the isomorphism $\eta : F(Y,u) \to X$ by $\eta(y,g) = g\beta_s(\mu(y)) \in X_s$ for $s \in S_I^+$, $(y,g) \in F(Y,u)_s$. ∎

Given the S_I-set Y, two G-1-cocycles $u,u' \in Z^1_G(S_I, G^Y)$ are called cohomologous if there exists $h \in G^Y$ such that $h_i^{-1} u_i {}^i h_i = u'_i$ for $i \in I$, where h_i is the projection of h in $G_{I \setminus \{i\}}$. The condition above defines an equivalence relation $\sim$ on $Z^1_G(S_I, G^Y)$; let $H^1_G(S_I, G^Y)$ be the factor set $Z^1_G(S_I, G^Y)/\sim$.

COROLLARY 5.4. Given an S_I-set Y, the equivalence F : G-cocycles → G-sets induces a bijection from $H^1_G(S_I, G^Y)$ onto the set of isomorphism classes of the G-sets X with $X_o \simeq Y$.

To a G-set X one assigns in a functorial manner three quasi-boolean G-spaces $\hat{X}$, X^* and ${}^*X$ as follows. $\hat{X} \simeq X_o \times G$ was already considered in the proof of Proposition 5.3. In order to define X^* let us consider the directed projective system of discrete G-spaces $\{X_s^*, X_s^* \xrightarrow{s-t} X_t^* : x \mapsto {}^{s-t}x\}_{s \in S_I^+, t \leq s}$ with $X_s^* = X$. Let $X^* = \varprojlim X_s^*$. Next consider the directed projective system of discrete G-spaces $\{{}^*X_s, {}^*X_s \xrightarrow{s-t} {}^*X_t\}_{s \in S_I^+, t \leq s}$, where ${}^*X_s$ is the disjoint union $\dot{\bigcup}_{t \leq s} X_t$, and let ${}^*X = \varprojlim {}^*X_s$. Thus we get the coherent embeddings of quasi-boolean G-spaces $\hat{X} \to {}^*X \to X^*$. Note that $\hat{X}$ and ${}^*X$ are boolean spaces whether X_o is finite.

DEFINITION. The G-set X is called a G-module if X has an additional structure of abelian group which is compatible with the actions of G and S_I^+, i.e. X

is a discrete G-module on which the commutative monoid S_I^+ acts, and moreover, ${}^*X_s = \overset{\circ}{\bigcup}_{t<s} X_t$ is a submodule of X for each $s \in S_I^+$. [The last condition is superfluous whether the compatible map $U : S_I^+ \to N(G)$ is injective].

The G-modules forms a category G-Mod having a as morphisms the group morphisms $f : X \to X'$ commuting with the actions of G and S_I^+ such that $f(X_s) \subseteq X'_s$ for $s \in S_I^+$.

For a G-module X, the quasi-boolean spaces X^* and ${}^*X$ have canonic structures of topological G-modules.

A typical example: Let $G = \hat{Z}^X = \varprojlim (Z/nZ)^X \simeq \prod Z_p^X$ be the group of units of the commutative ring $\hat{Z} = \varprojlim Z/nZ \simeq \prod Z_p$, I be the set P of all prime numbers and $U : S_P^+ \simeq (Z_{\geq 1}, \cdot) \to N(\hat{Z}^X)$ be the map $n \mapsto U(n) = \mathrm{Ker}\,(\hat{Z}^X \to (Z/nZ)^X)$. Then $G = (\hat{Z}^X, P, U)$ is a special profinite group. $\hat{Z}^X$ acts continuously on the discrete abelian group $X = Q/Z$ by multiplication, $(Q/Z)^{U(n)} = (1/n)Z/Z$ and

$$X_n = \{x \in X : Inv(x) = U(n)\} = \{\tfrac{a}{n} \in \tfrac{1}{n}Z/Z : (a,n) = 1\}$$

for $n \geq 1$. The commutative monoid $S_I^+ \simeq (Z_{\geq 1}, \cdot)$ acts on the discrete $\hat{Z}^X$-module X according to the rule $(n,x) \mapsto nx$ for $n \geq 1$. Thus $X = Q/Z$ has a canonic structure of G-module. One checks easily that $\hat{X} \simeq \hat{Z}^X$, ${}^*X \simeq \varprojlim (1/n)Z/Z \simeq \hat{Z}$ (as topological $\hat{Z}^X$-module) and $X^* \simeq \varprojlim Q/nZ$ is canonically isomorphic to the underlying topological $\hat{Z}^X$-module of the ring of non-archimedean adeles of Q, $Ad(Q) = \{(a_p)_{p \in P} \in \prod Q_p : \{p \in P : v_p(a_p) < 0\}$ finite$\}$.

The previous example is the paradigm of the following general procedure for construction of abelian special profinite groups. Consider a family $\{K_i\}_{i \in I}$ of local fields and denote resepctiely by v_i, O_i, m_i, k_i, O_i^x the discrete valuation, the maximal ideal, the finite residue field and the group of units. Such families occur naturally by considering a global field, a set of non-archimedean primes on it and the completions w.r.t. these primes.

Let $K = \prod_{i \in I} K_i$, $O = \prod_{i \in I} O_i$, $G = O^* \simeq \prod_{i \in I} O_i^*$ and let Ad(K) be the

intermediate ring between $\mathcal{O}$ and K consisting of those $(a_i)_{i\in I} \in K$ for which the set $\{i\in I : v_i(a_i)<0\}$ is finite. Call Ad(K) the ring of adeles assigned to the family $\{K_i\}_{i\in I}$. For each $i\in I$, choose an element $\pi_i \in \mathcal{O}$ such that its i-th component $\pi_{i,i}$ is a prime element of $\mathcal{O}_i$, i.e., $m_i = \pi_{i,i}\mathcal{O}_i$, and its j-th component $\pi_{i,j} \in \mathcal{O}_j^x$ for $j\neq i$. The map $I\to\mathcal{O} : i\mapsto\pi_i$ extends to a group monomorphism $S_I \to Ad(K)^x : s\mapsto\pi_s$. It follows $\mathcal{O} \simeq \varprojlim_{s\in S_I^+} \mathcal{O}/\pi_s\mathcal{O}$, $G \simeq \varprojlim_{s\in S_I^+} (\mathcal{O}/\pi_s\mathcal{O})^x$ and Ad(K) is the ring of fractions of $\mathcal{O}$ w.r.t. the multiplicative monoid $\{\pi_s : s\in S_I^+\}$.

Defining the map $U : S_I^+ \to N(G)$ by $U(s) = Ker(p_s : G \to (\mathcal{O}/\pi_s\mathcal{O})^x)$ we get the special profinite group $G = (G,I,U)$. Let $X = Ad(K)/\mathcal{O} \simeq \bigoplus_{i\in I} K_i/\mathcal{O}_i \simeq \varinjlim_{s\in S_I^+} \pi_{-s}\mathcal{O}/\mathcal{O}$. X has a canonic structure of discrete G-module with the action of G induces by multiplication. For $s\in S_I^+$, $X^{U(s)} = \pi_{-s}\mathcal{O}/\mathcal{O}$ and $X_s = \{x\in X : Inv(x) = U(s)\}$ is the image of $(\mathcal{O}/\pi_s\mathcal{O})^x$ through the isomorphism of $\mathcal{O}/\pi_s\mathcal{O}$-modules $\mathcal{O}/\pi_s\mathcal{O} \to \pi_{-s}\mathcal{O}/\mathcal{O}$ given by multiplication with π_{-s}. S_I^+ acts on the discrete G-module X according to the rule $(s,x) \mapsto \pi_s x$. Thus X becomes a G-module and $\hat{X} \simeq G$ as G-space. The G-morphisms $Ad(K) \to X : a \mapsto \pi_{-s}\, a \bmod \mathcal{O}$ for $s\in S_I^+$ induce the isomorphisms of topological G-modules $Ad(K) \simeq \varprojlim_{s\in S_I^+} Ad(K)/\pi_s\mathcal{O} \xrightarrow{\sim} X^*$ and $\mathcal{O} \xrightarrow{\sim} {}^*X$.

The G-module X defined above admits an equivalent description by appealing to Lubin-Tate theory of formal groups [13], [18], [11] Ch.8. Given the map $I\to\mathcal{O} : i\mapsto\pi_i$, as above, let $\pi\in\mathcal{O}$ be the diagonal element $(\pi_{i,i})_{i\in I}$ and $f\in\mathcal{O}[[X]]$ be such that $f(X)\equiv \pi X \bmod X^2\mathcal{O}[[X]]$ and $f(X)\equiv X^{q_i} \bmod \pi_i\mathcal{O}[[X]]$ for $i\in I$, where q_i is the cardinality of the residue field k_i. Thus the power series f is identified with a family of power series $(f_i)_{i\in I} \in \prod_{i\in I} \mathcal{O}_i[[X]]$ such that $f_i(X)\equiv \pi_{i,i}X \bmod X^2\mathcal{O}_i[[X]]$, $f_i(X)\equiv X^{q_i} \bmod \pi_{i,i}\mathcal{O}_i[[X]]$ for $i\in I$. The power series $f(X) = (\pi_{i,i}X + X^{q_i})_{i\in I}$, called the basic Frobenius power series associated with π is the simplest power series of the type above. Given such an $f\in\mathcal{O}[[X]]$, let $F = F_f \in \mathcal{O}[[X,Y]]$ be the unique formal group defined over $\mathcal{O}$ such that f is an endomorphism of F. F is called the Lubin-Tate group associated to f. F admits $\mathcal{O}$ as

a ring of endomorphisms in a natural way $\mathcal{O} \to \mathrm{End}(F) : a \mapsto [a]$ such that $[\pi] = f$. Let $\bar{m}_i$ be the maximal ideal of the valuation ring of the algebraic closure of K_i, for $i \in I$, and let $\bar{m} = \coprod_{i \in I} \bar{m}_i$. F induces on $\bar{m}$ a natural structure of $\mathcal{O}$-module $\bar{m}_f$ with the module operations given by $(x,y) \mapsto F(x,y)$, $(a,x) \mapsto [a](x)$. Let X_f be the submodule of $\bar{m}_f$ consisting of those $x \in \bar{m}$ for which there exists $s \in S_I^+$ such that $[\pi_s](x) = 0$. X_f is a discrete G-module on which the monoid S_I^+ acts according to the rule $(s,x) \mapsto [\pi_s](x)$. Given $x \in X_f$ the smallest $s \in S_I^+$ for which $[\pi_s](x) = 0$ is called the order of x. For $s \in S_I^+$ denote by $X_{f,s}$ the finite subset of X_f consisting of the elements of order s, and let $\hat{X}_f = \varprojlim X_{f,s}$ w.r.t. the connecting G-epis $X_s \to X_t : x \mapsto [\pi_{s-t}](x)$ for $t \leq s$. Let $(x_s)_{s \in S_I^+}$ be an element of $\hat{X}_f$. It follows by Lubin-Tate theory that the maps $X_s \to X_{f,s} : a \mapsto [\pi_s a](x_s)$ for $s \in S_I^+$ induce an isomorphism of G-modules $X \xrightarrow{\sim} X_f$.

6. ORDINARY DISTRIBUTIONS ON G-SETS

Let $G = (G,I,U)$ be a special profinite group, X be a G-set and $\hat{X}$, X^*, ${}^*X$ be the quasi-boolean G-spaces associated to X as in Section 5. Let A be an S_I^+-module, i.e., A is an abelian group together with an action $S_I^+ \times A \to A : (s,a) \mapsto {}^s a$ of the commutative monoid S_I^+ on A. A map $\varphi : X \to A$ is an ordinary distribution if $\varphi(x) = {}^s(\sum_{sy=x} \varphi(y))$ for $x \in X$, $s \in S_I^+$. Such a map φ induces a distribution $(\varphi_s)_{s \in S_I^+}$ on the quasi-boolean space X^* with $\varphi_s : X \to A$ given by $\varphi_s(x) = {}^s\varphi(x)$. It also induces a distribution $(\varphi'_s)_{s \in S_I^+}$ on the quasi-boolean space ${}^*X$, where $\varphi'_s : {}^*X_s = \bigcup_{t \leq s} X_t \to A$ is the restriction of φ_s at ${}^*X_s$ for $s \in S_I^+$. If the S_I^+-module A is torsion-free, i.e., ${}^s a = 0 \Rightarrow a = 0$ for all $s \in S_I^+$, in particular, if A is an S_I-module, then the correspondences $\varphi \mapsto (\varphi_s)$, $\varphi \mapsto (\varphi'_s)$ are injective.

Examples: a) Consider the typical example $G = (\hat{Z}^X, P, U)$, $X = Q/Z$ described in Section 5, and let A be an abelian group on which S_I^+ acts trivially, i.e. ${}^s a = a$ for $a \in A$, $s \in S_I^+$. An ordinary distribution $\varphi : X \to A$ on the G-module X is a customary ordinary distribution [11], Ch.2, §8, i.e., $\varphi(a) = \sum_{nb=a} \varphi(b)$ for arbitrary

$a \in \underline{Q}/\underline{Z}$, $n \geq 1$.

b) Let k, n be positive integers and A be an abelian group. A map $\varphi : \underline{Q}^k/\underline{Z}^k \to A$ is called by Kubert [4] a distribution of weight n if for each positive integer m and for each $a \in \underline{Q}^k/\underline{Z}^k$, $\varphi(ma) = m^n \sum_{mb=ma} \varphi(b)$. For instance, for k = 1, the Bernoulli polynomial $B_n(X)$ yields a distribution of weight n - 1 [11] Ch.2, § 2. Kubert's weighted distributions are particular examples of ordinary distributions on G-sets. Indeed, let I be the set P of all prime numbers. For $p \in P$, let K_p be the unique unramified extension of degree k over the field $\underline{Q}_p$ of p-adic numbers and $\underline{O}_p$ be its valuation ring. Let $K = \prod_{p \in P} K_p$, $\underline{O} = \prod_{p \in P} \underline{O}_p$, $G = \underline{O}^x \simeq \prod_{p \in P} \underline{O}_p^x$. The diagonal embedding $P \to K : p \mapsto \vec{\pi}_p = p$ maps isomorphically S_P onto $\underline{Q}^x_{>0}$. We get $\underline{O} \simeq \hat{\underline{Z}}^k \simeq \prod_{p \in P} \underline{Z}_p^k$ as $\hat{\underline{Z}}$-modules, $Ad(K) = \bigcup_{m \geq 1} (1/m)\underline{O}$, $X = Ad(K)/\underline{O} \simeq \underline{Q}^k/\underline{Z}^k \simeq \bigoplus_{p \in P} \underline{Q}_p^k/\underline{Z}_p^k$ as $\hat{\underline{Z}}$-modules. Defining $U : S_P^+ \simeq (\underline{Z}_{\geq 1}, \cdot) \to N(G)$ by $U(m) = \mathrm{Ker}(G \to (\underline{O}/m\underline{O})^X)$ we get the special profinite group $\underline{G} = (G,P,U)$ and X becomes a G-module as described in Section 5. Now consider an abelian group A on which $S_P^+ \simeq (\underline{Z}_{\geq 1}, \cdot)$ acts according to the rule $^m a := m^n a = \sum_{j=1}^{m^n} a_j$ with $a_j = a$, for $a \in A$, $m \in \underline{Z}_{\geq 1}$. Then a distribution $\varphi : \underline{Q}^k/\underline{Z}^k \to A$ of weight n is identified with an ordinary distribution on the G-module X with values in the S_I^+-module A.

The next lemma provides a criterion for a map $\varphi : X \to A$ to be an ordinary distribution on the G-set X.

LEMMA 6.1. Let $\varphi : X \to A$ be a map defined on a G-set X with values in an S_I^+-module A. The necessary and sufficient condition for φ to be an ordinary distribution is that $\varphi(x) = {}^i(\sum_{iy=x} \varphi(y))$ for arbitrary $i \in I$, $x \in X$.

Proof. Let $x \in X$, $s \in S_I^+$. We have to show that $\varphi(x) = {}^s(\sum_{sy=x} \varphi(y))$. The verification is immediate by induction on $n_s = \sum_{i \in I} s(i)$. ∎

Given the ordinary distribution $\varphi : X \to A$, let us denote by $\langle \varphi \rangle$ the S_I^+-submodule of A generated by the image of φ. We represent $\langle \varphi \rangle$ as a directed union of finitely generated S_I^+-submodules as follows. For $s \in S_I^+$ and a finite subset D

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of X_o, let $X_{s,D}$ be the finite G-set $\{x \in X_s : {}^s x \in D\}$, ${}^s D = \{{}^s y : y \in D\}$, $\bar{s} = \sum_{i \in \mathrm{Supp}(s)} i$ and denote by ${}^* X_{s,D}$ the finite G-set $\bigcup_{t \le s} X_{t, \bigcup_{\bar{t} \le k \le \bar{s}} {}^k D}$. Consider the partial order $(s,D) \le (s',D')$ iff $s \le s'$ and $D \subseteq D'$ on the set of all pairs (s,D). Then the G-set X is the directed union $\bigcup {}^* X_{s,D}$. For $x \in X$, denote by $o(x)$ the element $s \in S_I^+$ for which $x \in X_s$; $o(x)$ is the smallest $s \in S_I^+$ for which $\hat{x} \in X_o$; call $o(x)$ the order of x. For every $s \in S_I^+$, define the map $\bar{\varphi}_s : {}^* X_s \to A$ by $\bar{\varphi}_s(x) = \sum_{s - o(z) \, z = x} \varphi(z)$; then $\varphi(x) = {}^{s - o(x)} \bar{\varphi}_s(x)$ and $\bar{\varphi}_s(x) = {}^{t-s} \bar{\varphi}_t(x)$ for $s,t \in S_I^+$, $s \le t$, $x \in {}^* X_s$; in particular, $\bar{\varphi}_s$ coincides with φ on ${}^* X_s$ whether S_I^+ acts trivially on A. For a pair (s,D) as above let $\langle \varphi \rangle_{s,D}$ be the S_I^+-submodule of $\langle \varphi \rangle$ generated by $\{\bar{\varphi}_s(x) : x \in {}^* X_{s,D}\}$. It follows that $\langle \varphi \rangle_{s,D} \subseteq \langle \varphi \rangle_{s',D'}$ for $(s,D) \le (s',D')$ and hence $\langle \varphi \rangle$ is the directed union $\bigcup \langle \varphi \rangle_{s,D}$ of finitely generated S_I^+-modules. In the following we investigate the S_I^+-modules $\langle \varphi \rangle_{s,D}$.

Denote by $[s]$ the finite set $\{t \in S_I^+ : t \le s,\ t \wedge (s-t) = 0\}$ and by $\bar{X}_{s,D}$ the subset $\{x \in {}^* X_{s,D} : o(x) \in [s]\}$.

LEMMA 6.2. The S_I^+-module $\langle \varphi \rangle_{s,D}$ is generated by $\{\bar{\varphi}_s(z) : z \in \bar{X}_{s,D}\}$.

Proof. Let $x \in X_{t, k_D}$, $t \le s$, $\bar{t} \le k \le \bar{s}$, and let $t' = \sum_{i \in \mathrm{Supp}(t)} s_i$. Then $t' \in [s]$, $\bar{t}' = \bar{t}$ and $\{z \in X : {}^{t'-t} z = x\} \subseteq X_{t', k_D} \subseteq \bar{X}_{s,D}$. Therefore $\bar{\varphi}_s(x) = \sum_{t'-t \, z = x} \sum_{s-t' \, y = z} \varphi(y) = \sum_{t'-t \, z = x} \bar{\varphi}_s(z)$ as required. ∎

Now consider the quasi-boolean G-space $\hat{X} = \varprojlim X_s$ with the coherent epis $\beta_s : \hat{X} \to X_s$ for $s \in S_I^+$, and let us fix a section $\mu : X_o \to \hat{X}$ of β_o. For a pair (s,D), let $Z_{s,D} = \{x \in \bar{X}_{s,D} : o(x) - s_{i_x} \ne \beta_{s_i}(\mu({}^{o(x)} x)) \text{ for } i \in \mathrm{Supp}\,(o(x))\}$.

PROPOSITION 6.3. For every pair (s,D) the S_I^+-module $\langle \varphi \rangle_{s,D}$ is generated by $\{\bar{\varphi}_s(z) : z \in Z_{s,D}\}$.

Proof. Let $B_{s,D}$ be the S_I^+-submodule of $\langle \varphi \rangle_{s,D}$ generated by $\{\bar{\varphi}_s(z) : z \in Z_{s,D}\}$. According to Lemma 6.2, we have to show that $\bar{\varphi}_s(x) \in B_{s,D}$ for every $x \in \bar{X}_{s,D}$. Let $x \in X_{t, k_D}$, $t \in [s]$, $\bar{t} \le k \le \bar{s}$, $m = m(t)$ be the cardinality of $\mathrm{Supp}\,(t)$ and $n = n(x)$ be the cardinality of the set $E_x = \{i \in \mathrm{Supp}(t) : {}^{t-s_i} x = \beta_{s_i}(\mu({}^t x))\}$.

We proceed be double induction on m and n. The case $n = 0$ is trivial, so assume $1 \leq n \leq m$ and let $i \in E_x$, $a = {}^{s_i}x \in X_{t-s_i, kD}$, $b = {}^{-i}a \in X_{t-s_i,\, k-i_D} \subsetneq \overline{X}_{s,D}$; note that i acts as an automorphism on the G-set X_{t-s_i} since $i \wedge (t-s_i) = 0$. Note also that $\{z \in X : {}^{s_i-i}z = b\} = \{z \in {}^{*}X_{t-i} : {}^{s_i}z = a\}$. By induction hypothesis w.r.t. the parameter m we get $\overline{\varphi}_s(a), \overline{\varphi}_s(b) \in B_{s,D}$. On the other hand, we have $\overline{\varphi}_s(a) =$

$$\sum_{s-t+s_i\, y=a} \varphi(y) = \sum_{z \in X_t,\ {}^{s_i}z = a} \overline{\varphi}_s(z) + \sum_{{}^{s_i-i}z = b} \sum_{s-t\, y = z} \varphi(y) = \sum_{z \in X_t,\ {}^{s_i}z = a} \overline{\varphi}_s(z) + {}^{i}\overline{\varphi}_s(b).$$

Consequently, $\overline{\varphi}_s(x) = \overline{\varphi}_s(a) - {}^{i}\overline{\varphi}_s(b) - \sum_{z \in L} \overline{\varphi}_s(z)$, where $L = \{ z \in X_{t,kD} : {}^{s_i}z = a,\ {}^{t-s_j}z = {}^{t-s_j}x \text{ for } j \in \mathrm{Supp}(t),\ j \neq i.$ As ${}^{s_i}z = {}^{s_i}x$, $z \neq x \} \subsetneq \overline{X}_{s,D}$. If $z \in L$ then ${}^{t}z = {}^{t}x$, $z \neq x$ and $\mathrm{Inv}(x) = U(t)$, there is $g \in G$ uniquely determined modulo $U(t)$ such that $z = gx$, $p_{t-s_i}(g) = 1$ and $p_t(g) \neq 1$. Since $G(t) \simeq G(t - s_i) \times G(S_i)$ it follows $p_{s_i}(g) \neq 1$ and hence ${}^{t-s_i}z \neq {}^{t-s_i}x$ as $\mathrm{Inv}({}^{t-s_i}x) = U(s_i)$. Thus $E_z = E_x \setminus \{i\}$ for every $z \in L$, therefore $\overline{\varphi}_s(z) \in B_{s,D}$ for $z \in L$ by induction hypothesis on the parameter n. We conclude that $\overline{\varphi}_s(x) \in B_{s,D}$. ∎

COROLLARY 6.4. For every pair (s,D), the rank of the S_I^+-module $\langle \varphi \rangle_{s,D}$ is

$$\leq \sum_{t \in [s]} \Big| \bigcup_{0 \leq k \leq s-t} \overline{\quad}^{k} D \Big| \prod_{i \in \mathrm{Supp}(t)} (\phi(s_i) - 1), \text{ where } \phi(k) = (G : U(k)).$$

Now assume that the S_I^+-module A is equipped also with a structure of discrete G-module in such a way that the actions of S_I^+ and G on A commute. An ordinary distribution $\varphi : X \to A$ is called an ordinary G-distribution if $\varphi(gx) = g\varphi(x)$ for $g \in G$, $x \in X$.

Proceeding as in Section 4 one assigns to an ordinary distribution $\varphi : X \to A$ an ordinary G-distribution $\mathrm{St}(\varphi) : X \to M_G(A)$, called the Stickellerger ordinary G-distribution associated to φ. Note that the action of S_I^+ on A induces an action of S_I^+ on the induced module $M_G(A)$ commuting with the action of G, according to the rule $({}^{s}f)(g) = {}^{s}(f(g))$ for $f \in M_G(A)$, $g \in G$, $s \in S_I^+$. $\mathrm{St}(\varphi)$ is defined by $\mathrm{St}(\varphi)(x)(g) = \varphi(gx)$ for $x \in X$, $g \in G$.

7. ORDINARY DISTRIBUTIONS WITH VALUES IN VECTOR SPACES

Consider a special profinite group $\underline{G} = (G,I,U)$, a $\underline{G}$-set X, a field K of characteristic zero and a map $w : I \to K^X$; denote also by w its extension to a group morphism $S_I \to K^X$. For a vector space E over K, S_I acts on E according to the rule ${}^s a = w(s)a$ for $s \in S_I$, $a \in E$; call this action the action induced by the weight w. In this section we investigate ordinary distributions defined on the $\underline{G}$-set X with values in a K-space E with the action of S_I induced by the weight w, i.e., maps $\varphi : X \to E$ such that $\varphi(x) = w(s) \sum_{z \cdot z = x} \varphi(z)$ for $x \in X$, $s \in S_I^+$; call them ordinary distributions of weight w. The study is reduced to the case E = K thanks to the immediate lemma.

LEMMA 7.1. Let E be a K-space, B be a basis of E and $\pi_b : E \to K$ for $b \in B$ be the corresponding projections. The correspondence $\varphi \mapsto (\pi_b \circ \varphi)_{b \in B}$ maps isomorphically the K-space of ordinary distributions of weight w on X with values in E onto the K-space consisting of the families $(\varphi_b)_{b \in B}$, where φ_b is an ordinary distribution of weight w on X with values in K for every $b \in B$, such that the set $\{b \in B : \varphi_b(x) \neq 0\}$ is finite for each $x \in X$.

For $s \in S_I^+$ denote by Γ_s the group K-algebra K[G(s)] and by Λ_s its underlying left Γ_s-module. The canonic epis $p_{t,s} : G(s) \to G(t)$ for $t \leq s$ extend to epis $p_{t,s} : \Gamma_s \to \Gamma_t$ of K-algebras. The group K-algebra K[G] has a canonic structure of topological K-algebra for which the family of two-sided ideals $\{\mathrm{Ker}(K[G] \to \Gamma_s\}_{s \in S_I^+}$ is a fundamental system of open neighbourhoods of 0. Let $\Gamma = \varprojlim \Gamma_s$ be the completion of K[G], and $p_s : \Gamma \to \Gamma_s$ be the canonic continuous epis.

For $s \in S_I^+$, the correspondence $f \mapsto \sum_{g \in G(s)} f(g) g^{-1}$ is an isomorphism of left Γ_s-modules $M_{G(s)}(K) = K^{G(s)} \to \Lambda_s$; to the compatible pair $(p_{t,s} : \Gamma_s \to \Gamma_t,\ M_{G(t)}(K) \hookrightarrow M_{G(s)}(K))$ for $t \leq s$, corresponds the compatible pair $(p_{t,s}, \Lambda_t \to \Lambda_s : h \in G(t) \mapsto \sum_{p_{t,s}(g) = h} g)$. Thus we get a canonic isomorphism of discrete left Γ-modules from the induced module $M_G(K)$ onto $\Lambda = \varinjlim \Lambda_s$; in the following we identify them.

If $\varphi : X \to K$ is an ordinary distribution of weight w then $St(\varphi) = \varinjlim St(\varphi)_s$, where $St(\varphi)_s : {}^*X_s \to \Lambda_s$ is given by $St(\varphi)_s(x) = \sum_{g \in G(s)} \varphi(gx)g^{-1}$ for $s \in S_I^+$, $x \in {}^*X_s$. Let M_φ be the K-subspace of Λ generated by the image of $St(\varphi)$. M_φ is a discrete left Γ-submodule of Λ and $M_\varphi = \varinjlim M_{\varphi,s}$ where $M_{\varphi,s}$ is the left Γ_s-module $\sum_{x \in {}^*X_s} KSt(\varphi)(x)$. As the dimension of the K-space $M_{\varphi,s}$ is preserved by a field base change we may assume in the following that K is algebraically closed.

A finite representation of G over K is a pair (F, ρ) where F is a finitely dimensional K-space and $\rho : G \to Aut_K(F)$ is a group morphism with $Ker(\rho) \in N(G)$. ρ extends to a continuous morphism $\rho : \Gamma \to End_K(F)$ of K-algebras. The character χ_ρ assigned to ρ is the continuous map $\Gamma \to K : a \mapsto tr(\rho(a))$. Define the conductor of χ_ρ as the smallest $s \in S_I^+$ for which $U(s) \subseteq Ker(\rho)$. Denote by G^* the set of the characters of all irreducible finite representations of G over K. For $\chi \in G^*$ let f_χ be its conductor and d_χ be the dimension of the corresponding K-space.

For $s \in S_I^+$, the semisimple ring Γ_s is a direct product $\prod \Gamma_{s,\chi}$ of simple rings where χ ranges over those $\chi \in G^*$ for which $f_\chi \leq s$; $\Gamma_{s,\chi}$ is a matrix algebra over K and the center $Z(\Gamma_s)$ of Γ_s is the direct produt of the centers $Z(\Gamma_{s,\chi}) = Ke_{s,\chi}$, where the idempotent $e_{s,\chi} = (d_\chi / |G(s)|) \sum_{g \in G(s)} \chi(g^{-1})g$ is the unit of $\Gamma_{s,\chi}$. Setting $e_{s,\chi} = 0$ and $\Gamma_{s,\chi} = 0$ for $f_\chi \not\leq s$, we get $e_\chi = (e_{s,\chi})_{s \in S_I^+} \in \Gamma$ for $\chi \in G^*$ and $\Gamma \simeq \prod_{\chi \in G^*} \Gamma_\chi$, where $\Gamma_\chi = \Gamma e_\chi = e_\chi \Gamma = \varprojlim \Gamma_{s,\chi}$ with the connecting epis $\Gamma_{s,\chi} \to \Gamma_{t,\chi}$ induces by $p_{t,s} : \Gamma_s \to \Gamma_t$ for $t \leq s$, $\chi \in G^*$. Note also that each family $\{\gamma_\chi\}_{\chi \in G^*}$ with $\gamma_\chi \in \Gamma_\chi$ is summable in the topological ring Γ and $\gamma_\chi = e_\chi \gamma$, where $\gamma = \sum_{\chi \in G^*} \gamma_\chi$; in particular $\sum_{\chi \in G^*} e_\chi = 1$ and $N = \bigoplus_{\chi \in G^*} e_\chi N$ for every discrete left Γ-module N.

According to Proposition 5.3 we may assume that $X = F(Y,u)$, where F is the equivalence of categories described in the proposition mentioned above, $Y = X_o$ and $u \in Z^1_{\underline{G}}(S^I, G^Y)$. The abelian group S_I acts on the K-algebra $\Gamma^Y = \{f : Y \to \Gamma\}$ according to the rule $({}^s f)(y) = f({}^s y)$ for $s \in S_I$, $f \in \Gamma^Y$, $y \in Y$. In particular, S_I acts on the multiplicative monoid Γ^Y so we may consider the set of 1-cocycles $Z^1(S_I, \Gamma^Y)$

consisting of the maps $h : S_I \to \Gamma^Y : s \mapsto h_s$ subject to $h_s \cdot {}^s h_t = h_{s+t}$ for $s,t \in S_I$. The G-1-cocycle u induces a 1-cocycle $\Theta \in Z^1(S_I, \Gamma^Y)$ defined by $p_t(\Theta_s(y)) = (1/\phi(\sum_{i \le |s|} t_i)) p_{t - \sum_{i \le |s|} t_i}(u_s(y)) \in \Lambda_{t - \sum_{i \le |s|} t_i} \subset \Lambda_t \equiv \Gamma_t$ for $s \in S_I$, $t \in S_I^+$, $y \in Y$, where $\phi(k) = (G : U(k))$ for $k \in S_I^+$. The 1-cocycle Θ plays a key role in the description of the K-space $\mathrm{Dist}_G(X, \Lambda)$ of ordinary G-distributions of weight w taking values in Λ.

LEMMA 7.2. Let $\chi \in G^*$ and $s \in S_I$ be such that $s \wedge f_\chi \neq 0$. Then $e_\chi \Theta_s = 0$.

Proof. We have to show that $p_t(e_\chi \Theta_s(y)) = 0$ for $y \in Y$, $t \in S_I^+$ such that $f \le t$. We get: $p_t(e_\chi \Theta_s(y)) = \frac{1}{\phi(\sum_{i \le |s|} t_i)} p_t(e_\chi) \cdot p_{t - \sum_{i \le |s|} t_i}(u_s(y)) =$

$$\frac{1}{\phi(\sum_{i \le |s|} t_i)^2} p_{t - \sum_{i \le |s|} t_i}(e_\chi) \cdot p_{t - \sum_{i \le |s|} t_i}(u_s(y)) = 0$$

since $p_{t - \sum_{i \le |s|} t_i}(e_\chi) = 0$ as $f_\chi \not\le t - \sum_{i \le |s|} t_i$. ∎

Consider the canonic continuous action from the right $\Lambda \times \Gamma \to \Lambda : (a, \gamma) \mapsto a^\gamma$ of Γ on Λ. Let us define a morphism of right Γ-modules $\eta : \Gamma^Y \to \Lambda^X$ as follows. For $\delta \in \Gamma^Y$, $(y,g) \in X_s = Y \times G(s)$, $s \in S_I^+$, set $\eta(\delta)(y,g) = w(-s) \sum_{t \in [s]} (-1)^{m(t)} {}_{w(\bar{t})g}(\Theta_{-\bar{t}} \cdot {}^{-\bar{t}}\delta)(y)$, where $m(t)$ is the cardinality of $\mathrm{Supp}(t)$ and $\bar{t} = \sum_{i \in \mathrm{Supp}(t)} i$.

LEMMA 7.3. η takes values in $\mathrm{Dist}_G(X, \Lambda)$.

Proof. Let $\delta \in \Gamma^Y$. Obviously $\eta(\delta)$ is a left G-map, so it remains to check that it is an ordinary distribution of weight w. According to Lemma 6.1. we have to show that $\eta(\delta)(y,g) = w(i) \sum_{{}^i x = (y,g)} \eta(\delta)(x)$ for $(y,g) \in X_s$, $s \in S_I^+$, $i \in I$.

We distinguish two cases.

Case 1: $i \le s$. Immediate since $\{x \in X : {}^i x = (y,g)\} = \{(y, g') \in X_{s+i} : p_{s, s+i}(g') = g\}$.

Case 2: $i \wedge s = 0$. Immediate since $\{x \in X : {}^i x = (y,g)\} = \{(y,g') \in X_{s+i} : \rho_{s,s+i}(g') = g\} \cup \{({}^{-i}y, g\rho_s(u_{-i}(y))) \in X_s\}$, $[s+i] = [s] \cup \{t+i : t \in [s]\}$ and $\Theta_{-i} \cdot {}^{-i}\Theta_{-\bar{t}} = \Theta_{-\bar{t}-i}$ for $t \in [s]$. ∎

Moreover we get

PROPOSITION 7.4. The map $\eta : \Gamma^Y \to \mathrm{Dist}_G(X,\Lambda)$ is bijective.

Proof. Let us define a map $\omega : \mathrm{Dist}_G(X,\Lambda) \to \Gamma^Y$ by $\omega(\Psi)(y) = \sum_{\chi \in G^*} w(f_\chi) e_\chi \omega_{\Psi,y,f_\chi}$ where $\omega_{\Psi,y,s} \in \Gamma$ for $s \in S_I^+$ is determined by its projections $p_t(\omega_{\Psi,y,s}) = \frac{1}{(U(s):U(t))} \Psi(y, \rho_{s,t}(g)) \in \Lambda_s \subset \Lambda_t \equiv \Gamma_t$ for $t \geq s$.

First let us show that $\eta(\omega(\Psi)) = \Psi$ for every $\Psi \in \mathrm{Dist}_G(X,\Lambda)$. Let $\Psi \in \mathrm{Dist}_G(X,\Lambda)$, (y,g), $(y,g) \in X_s$, $s \in S_I^+$. We have to show that

$$(*)\quad w(s-f)\Psi(y,g)^{e_\chi} = \sum_{t \in [s]} (-1)^{m(t)} w(\bar{t})\, g^{e_\chi} \Theta_{-\bar{t}}(y)\, \omega_{\Psi, -\bar{t}y, f}$$

for each $\chi \in G^*$ such that $f = f_\chi \leq s$. We proceed by induction on $n = n_{s,\chi}$, the cardinality of the set $\mathrm{Supp}(s) \setminus \mathrm{Supp}(f)$. First assume $n = 0$, i.e., $\mathrm{Supp}(s) = \mathrm{Supp}(f)$. By Lemma 7.1., $e_\chi \Theta_{-\bar{t}} = 0$ for $t \in [s]$, $t \neq 0$, so the right member R of (*) is $g^{\omega_{\Psi,y,f} e_\chi} = (1/(U(f):U(s)))\, \Psi(y, \rho_{f,s}(g))^{e_\chi}$. On the other hand, $\Psi(y, \rho_{f,s}(g)) = w(s-f) \sum_{\rho_{f,s}(h) = \rho_f(1)} \Psi(y,hg) = w(s-f)\rho_f(1)\Psi(y,g) = w(s-f)\Psi(y,g)\rho_f(1)$. It follows $R = w(s-f)\Psi(y,g)^{e_\chi} = L$ as required.

Next assume $n \geq 1$ and let $i \in \mathrm{Supp}(s) \setminus \mathrm{Supp}(f)$, i.e., $f \leq k = s - s_i$. Applying the distribution relation to $(y, \rho_{k,s}(g)) \in X_k$ we get $\Psi(y, \rho_{k,s}(g)) = w(s_i) \sum_{{}^{s_i}x = (y, \rho_{k,s}(g))} \Psi(x)$

$$= w(s_i)\Psi(y,g)\rho_k(1) + w(i)\, g \cdot \Psi({}^{-i}y, \rho_k(u_{-i}(y))).$$

By induction hypothesis it follows $L = w(s-f)\Psi(y,g)^{e_\chi} =$

$$\frac{w(s-f)}{\phi(s_i)} \Psi(y,g)\rho_k(1)^{e_\chi} = \frac{1}{\phi(s_i)} \sum_{t \in [k]} (-1)^{m(t)} w(\bar{t}) \rho_{k,s}(g)^{e_\chi} \Theta_{-\bar{t}}(y)\, \omega_{\Psi, -\bar{t}y, f} -$$

$$- \frac{1}{\phi(s_i)} \sum_{t \in [k]} (-1)^{m(t)} w(\bar{t}+i)\, g\, \rho_k(u_{-i}(y))^{e_\chi} \Theta_{-\bar{t}}({}^{-i}y)\, \omega_{\Psi, -\bar{t}-i y, f} = R,$$

since $[s] = [k] \cup \{t+s_i : t \in [k]\}$, $\frac{1}{\phi(s_i)} p_{k,s}(g)^{e_\chi} = g^{e_\chi}$ and

$\frac{1}{\phi(s_i)} g\, p_k(u_{-i}(y))^{\Theta_{-\bar{t}}(-^i y)} = g^{\Theta_{-\bar{t}-i}(y)}$ for $t \in [k]$.

In order to conclude that η and ω are inverse each to other it remains to verify that $\omega(\eta(\delta)) = \delta$ for each $\delta \in \Gamma^Y$. Let $\delta \in \Gamma^Y$, $y \in Y$, $\chi \in G^*$, $f = f_\chi$, $t \in S_I^+$, $t \geq f$. Applying Lemma 7.2, we get $p_t(e_\chi\, \omega(\eta(\delta))(y)) =$

$$w(f)\, p_t(e_\chi) \frac{1}{(U(f):U(t))} \eta(\delta)(y, p_f(1)) = \frac{1}{(U(f):U(t))} p_t(e_\chi) \sum_{s \in [f]} (-1)^{m(s)} w(\bar{s})\, p_f(1)^{(\Theta_{-\bar{s}} \cdot {}^{-\bar{s}}\delta)(y)}$$

$$= \frac{1}{(U(f):U(t))} p_t(e_\chi)\, p_f(1)^{\delta(y)} = p_t(1)^{e_\chi \delta(y)} = p_t(e_\chi\, \delta(y)),$$

as contended. ∎

COROLLARY 7.5. The right Γ-module $\mathrm{Dist}_G(X,\Lambda)$ is free of rank $|X_o|$.

The isomorphism $\eta = \omega^{-1} : \Gamma^Y \to \mathrm{Dist}_G(X,\Lambda)$ of right Γ-modules induces a structure of K-algebra on $\mathrm{Dist}_G(X,\Lambda)$ with the multiplication $*$, given by $\Psi * \Psi' = \eta(\omega(\Psi).\omega(\Psi'))$; the unit $\eta(1)$ is completely determined by the 1-cocycle Θ.

The composite map $\mathrm{Dist}(X,K) \xrightarrow{St} \mathrm{Dist}_G(X,\Lambda) \xrightarrow{\omega} \Gamma^Y$ is an isomorphism of K-spaces. Its inverse $\eta' = St^{-1} \circ \eta$ is explicitely described by the next easy lemma.

LEMMA 7.6. Let $\delta \in \Gamma^Y$ and $(y,g) \in X_s$, $s \in S_I^+$. For $t \in S_I^+$, write $p_t(\delta(y))$ in the form $p_t(\delta(y)) = \sum_{h \in G(t)} \delta_{t,h,y} h$ with $\delta_{t,h,y} \in K$. Then

$$\eta'(\delta)(y,g) = w(-s) \sum_{t \in [s]} \frac{(-1)^{m(t)} w(\bar{t})}{\phi(t)}\, \delta_{s-t,\, p_{s-t}(u_{\bar{t}}(-^{\bar{t}}y)) p_{s-t,s}(g^{-1}),\, -^{\bar{t}}y}.$$

In particular, $\eta'(1)(y,g) = w(-s) \sum_{t \in [s]} \frac{(-1)^{m(t)} w(\bar{t})}{\phi(t)}\, \varepsilon_s(-^{\bar{t}}y, g, t)$, where

$$\varepsilon_s(y,g,t) = \begin{cases} 1 & \text{if } p_{s-t,s}(g) = p_{s-t}(u_{\bar{t}}(y)) \\ 0 & \text{otherwise.} \end{cases}$$

Considering the structure of K-algebra on Dist(X,K) induced by η' with multiplication given by $\varphi * \varphi' = \eta'(\omega(St(\varphi)).\omega(St(\varphi')))$, the map $St : Dist(X,K) \to Dist_G(X,\Lambda)$ becomes an isomorphism of K-algebras.

PROPOSITION 7.7. Given $\Psi \in Dist_G(X,\Lambda)$, $s \in S_I^+$, the K-subspace $M_{\Psi,s}$ of Λ_s generated by $\{\Psi(x) : x \in {}^*X_s\}$ is the left Γ_s-module $\sum_{y \in Y} \Lambda_s^{\omega(\Psi)(y)}$. The K-subspace M_Ψ of Λ generated by $\{\Psi(x) : x \in X\}$ is the discrete Γ-module $\sum_{y \in Y} \Lambda^{\omega(\Psi)(y)}$.

Proof. Let $M'_{\Psi,s} = \sum_{y \in Y} \Lambda_s^{\omega(\Psi)(y)}$. The inclusion $M_{\Psi,s} \subseteq M'_{\Psi,s}$ is immediate by Proposition 7.4. Conversely, as $M_{\Psi,s}$ is a left Γ_s-module, we have to show that $e_{s,\chi}^{\omega(\Psi)(y)} \in M_{\Psi,s}$ for arbitrary $y \in Y$, $\chi \in G^*$ such that $f = f_\chi \leq s$. Consider the element $\lambda = \frac{w(f)}{(U(f):U(s))}\, e_{s,\chi}\, \Psi(y, p_f(1)) \in M_{\Psi,s}$.

By Lemma 7.2, $e_\chi \Theta_{-t} = 0$ for $t \in [f]$, $t \neq 0$, and hence

$$\lambda = \frac{1}{(U(f):U(s))}\, e_{s,\chi}\, p_f(1)^{\omega(\Psi)(y)} = e_{s,\chi}^{\omega(\Psi)(y)}. \qquad \blacksquare$$

REMARKS. a) Though we used in the proofs above the assumption that K is algebraically closed, it is quite obvious that the respective statements remain true without this assumption, requiring only that K is of characteristic zero.

b) Call $\Psi = St(\varphi)$ <u>central</u> if $\omega(\Psi)$ belongs to the center $Z(\Gamma^Y)$ of Γ^Y. For such a Ψ, $\omega(\Psi) = \sum_{\chi \in G^*} w(f_\chi) S(\chi,\varphi) e_\chi$, where

$$S(\chi,\varphi)(y) = \sum_{g \in G(f_\chi)} \chi(g^{-1})\, \varphi(y,g) \in K \text{ for } y \in Y.$$

Assuming K algebraically closed, it follows by Proposition 7.7 that for every $s \in S_I^+$, $M_{\Psi,s}$ is the direct sum $\bigoplus_{\chi \in C_{\varphi,s}} \Gamma_s . e_{s,\chi}$, with $C_{\varphi,s} = \{\chi \in G^* : f_\chi \leq s,\ S(\chi,\varphi) \neq 0\}$, and $M_\Psi = \bigoplus_{\chi \in C_\varphi} \Gamma\, e_{f_\chi,\chi}$ with $C_\varphi = \{\chi \in G^* : S(\chi,\varphi) \neq 0\}$. Consequently, for K not necessarily algebrically closed, $\dim_K(M_{\Psi,s}) = \sum_{\chi \in C_{\varphi,s}} d_\chi^2$ for $s \in S_I^+$.

c) In the particular case $|Y| = 1$, the isomorphism of right Γ-modules $\eta : \Gamma \to Dist_G(X,\Lambda)$ is given by $\eta(\gamma)(g) = w(-s) g^{\alpha_s \gamma}$ for $\gamma \in \Gamma$, $g \in X_s = G(s)$, $s \in S_I^+$, where $\alpha_s = \prod_{i \in Supp(s)} (1 - w(i)\Theta_i)$; note that $\Theta_i . \Theta_j = \Theta_j . \Theta_i$ for $i,j \in I$.

We end this section with a structure theorem for the K-space Dist(X,E) of

ordinary distributions of weight w defined on the G-set X with values in an arbitrary K-space E. Considering a basis B of E, the K-space Dist(X,E) is identified according to Lemma 7.1 with the subspace of $Dist(X,K)^B$ consisting of the families $(\varphi_b)_{b\in B}$ for which $\{b\in B : \varphi_b(x)\neq 0\}$ is finite for every $x\in X$. On the other hand consider the two-sided ideal J_B of the K-algebra $\Gamma^{Y\times B}=\{\delta : B\to\Gamma^Y : b\mapsto\delta_b\}$ consisting of those δ subject to the following finiteness condition: for arbitrary $y\in Y$, $s\in S_I^+$, the set $\{b\in B : p_s(\delta_b(y))\neq 0\}$ is finite. The isomorphism of K-algebras $\eta' : \Gamma^Y \to Dist(X,K)$ provided by Lemma 7.6 extends component-wise to an isomorphism of K-algebras $\eta'_B : \Gamma^{Y\times B}\to Dist(X,K)^B$. By composing with St^B we get the isomorphism $\eta_B : \Gamma^{Y\times B}\to Dist_G(X,\Lambda)^B$.

THEOREM 7.8. η'_B and η_B maps isomorphically the two-sided ideal J_B onto Dist(X,E) and respectively $Dist_G(X,M_G(E))\simeq Dist_G(X,\Lambda^{(B)})$.

Proof. Immediate by Lemma 7.6 and induction on the cardinality of Supp(s) for $s\in S_I^+$. ∎

8. UNIVERSAL ORDINARY DISTRIBUTIONS

Let G = (G,I,U) be a special profinite group, X be a G-set, R be an integral domain and K be its field of fractions. Assume K is of characteristic zero. Let $w : I\to R\setminus\{0\}$ be a map and extend it canonically to a morphism of monoids from S_I^+ to the multiplicative monoid R. The map w induces an action of the monoid S_I^+ on every R-module A according to the rule ${}^s a = w(s)a$ for $a\in A, s\in S_I^+$. Consider the category $D_{X,R,w}$ whose objects are ordinary distributions of weight w defined on X with values in R-modules, i.e., maps $\varphi : X\to A$, where A is an R-module, such that $\varphi(x) = w(s)\sum_{z=x^s}\varphi(z)$ for $x\in X$, $s\in S_I^+$. For $\varphi_i : X\to A_i$ in $D_{X,R,w}$, i = 1,2, consider as morphism $\lambda : \varphi_1\to\varphi_2$ the morphisms of R-modules $\lambda : A_1\to A_2$ for which $\lambda\circ\varphi_1 = \varphi_2$.

LEMMA 8.1. The category $D_{X,R,w}$ has an initial object.

Proof. Let $R^{(X)}$ be the free R-module generated by the set $\{(x) : x \in X\}$ and D be its submodule generated by the subset $\{(x) - w(i) \sum_{z=x} (z) : i \in I,\ x \in X\}$. It follows by Lemma 6.1 that the map $\varphi_U : X \to \underline{U}_{X,R,w} = R^{(X)}/D : x \mapsto (x) \bmod D$ is the initial object of $\underline{D}_{X,R,w}$. ∎

Call the initial object φ_U of $\underline{D}_{X,R,w}$, uniquely determined up to a canonic isomorphism, the <u>universal ordinary distribution</u> assigned to the triple (X,R,w). The main goal of this section is to prove a structure theorem for the R-module $\underline{U} = \underline{U}_{X,R,w}$, which extends Kubert's [4] Theorem 4.14 devoted to the special case $X = \underline{Q}/\underline{Z}$, $R = \underline{Z}$.

Let $\Gamma = \varprojlim \Gamma_s$, $\Lambda = \varinjlim \Lambda_s$, etc., be as in Section 7. For $s \in S_I^+$, let T_s be the group R-algebra $R[G(s)]$ and N_s be its underlying left T_s-module. Proceeding as in Section 7 we get the topological R-subalgebra $T = \varprojlim T_s$ of Γ and the discrete left T-submodule $N = \varinjlim N_s \simeq M_G(R)$ of $\Lambda \simeq M_G(K)$.

For $s \in S_I^+$, let $G(s)' = \{g \in G(s) : p_{s_i,s}(g) \neq p_{s_i}(1)$ for $i \in \mathrm{Supp}(s)\}$, and $\overline{G(s)} = \bigcup_{t \in [s]} G(t)'$; note that $|\overline{G(s)}| = \phi(s)$ and $\overline{G(s)}$ is identified with a subset of N_s.

LEMMA 8.2. For $s \in S_I^+$, the set $\overline{G(s)}$ is a basis of the free R-module $N_s = R[G(s)]$. More precisely, let $g \in G(s)$ and $l = l_g = \sum_{p_{s_i,s}(g) \neq p_{s_i}(1)} s_i \in [s]$. Then $g = \sum_{t \in [s-l]} (-1)^{m(t)} \sum_{h \in G(t+l)',\, p_{l,t+l}(h) = p_{l,s}(g)} h$, where $m(t)$ is the cardinality of $\mathrm{Supp}(t)$.

Proof. We proceed by induction on $n_g = m(s-l)$. If $n_g = 0$, i.e. $l = s$, then $g \in G(s)' \subseteq \overline{G(s)}$, so we have nothing to prove. Assume $n_g \geq 1$ and let $i \in \mathrm{Supp}(s-l)$ and $a = p_{s-s_i,s}(g) \in G(s-s_i)$. We get $g = a - \sum_{p_{s-s_i,s}(b) = a,\ b \neq g} b$, $l_a = l$, $n_a = n_g - 1$, $l_b = l + s_i$, $n_b = n_g - 1$ for all b occuring under the sum above. By induction hypothesis, it follows

$$g = \sum_{t \in [s-s_i-l]} (-1)^{m(t)} \Big[\sum_{h \in G(t+l)',\ p_{l,t+l}(h) = p_{l,s}(g)} h - \sum_{p_{s-s_i,s}(b) = a,\ b \neq g} \ \sum_{h \in G(t+l+s_i)',\ p_{l+s_i,t+l+s_i}(h) = p_{l+s_i,s}(b)} h \Big]$$

$$= \sum_{t \in [s-l]} (-1)^{m(t)} \sum_{h \in G(t+l)',\ p_{l,t+l}(h) = p_{l,s}(g)} h,$$

as contended. ∎

Now consider the K-space $E = K^{(Y)}$, where $Y = X_o$, with the canonic basis identified with Y. Then $M_G(E) \simeq \bigoplus_{y \in Y} \Lambda^{(y)}$, $M_G(R^{(Y)}) = \bigoplus_{y \in Y} N^{(y)}$ with $\Lambda^{(y)} \simeq \Lambda$, $N^{(y)} \simeq N$. By Proposition 5.3, we may assume that $X = F(Y,u)$ where $u \in Z^1_G(S_I, G^Y)$. Consider the map $\xi : X \to M_G(R^{(Y)}) : (y,g) \in X_s \mapsto g \in N^{(y)}_s$. For $s \in S_I^+$ and a finite subset D of Y, let $X_{s,D} = \{(g,y) \in X_s : y \in D\}$, ${}^*X_{s,D} = \bigcup_{t \leq s} X_{t, \cup^k D}$, $Z_{s,D} = \{(y,g) \in {}^*X_{s,D} : g \in \overline{G(s)}\}$, and $M_G(R^{(Y)})_{s,D}$ be the R-submodule of $M_G(R^{(Y)})$ generated by $\xi({}^*X_{s,D})$. Then $M_G(R^{(Y)})$ is the directed inductive limit of the finitely generated submodules $M_G(R^{(Y)})_{s,D}$.

LEMMA 8.3. $M_G R^{(Y)})_{s,D}$ is a free R-module with basis $\xi(Z_{s,D})$.

Proof. Immediate by Lemma 8.2.

Now let $\eta_Y : J_Y \to \mathrm{Dist}_G(X, M_G(E))$ be the bijection provided by Theorem 7.8. Consider the map $\delta : Y \to \Gamma^Y : y \mapsto \delta_y$ defined by $\delta_y(z) = \begin{cases} 1 & \text{if } z = y \\ 0 & \text{otherwise.} \end{cases}$ Obviously, $\delta \in J_Y$ and hence $\Psi = \eta_Y(\delta)$ is an ordinary G-distribution of weight w defined on X with values in the discrete G-module $M_G(E)$. Let $\langle \Psi \rangle$ be the R-submodule of $M_G(E)$ generated by the image of Ψ and $V = V_\Psi \simeq \langle \Psi \rangle \otimes_R K$ be the corresponding K-subspace of $M_G(E)$; $\langle \Psi \rangle$ has a natural structure of discrete left T-submodule of $M_G(E)$ while V is a discrete left Γ-submodule of $M_G(E)$. For a pair (s,D), let $\langle \Psi \rangle_{s,D}$ be the R-submodule of $\langle \Psi \rangle$ generated by $\{\bar{\Psi}_s(x) = w(o(x) - s)\Psi(x) : x \in {}^*X_{s,D}\}$, and let $V_{s,D} = \langle \Psi \rangle_{s,D} \otimes_R K$; $V_{s,D}$ is the K-subspace of V generated by $\{\Psi(x) : x \in {}^*X_{s,D}\}$. Thus $\langle \Psi \rangle = \varinjlim \langle \Psi \rangle_{s,D}$ and $V = \varinjlim V_{s,D}$.

LEMMA 8.4. For every pair (s,D), $V_{s,D} = M_G(R^{(Y)})_{s,D} \otimes_R K$. Consequently, $V = M_G(E)$.

Proof. Immediate by the definition of Ψ:

$$\Psi(y,g) = w(-s) \sum_{t \in [s]} \frac{(-1)^{m(t)} w(\bar{t})}{\phi(t)} \xi(-\bar{t}y, h_t)$$

for

$(y,g)\in X_s$, where $h_t = p_{s-t,s}(g)p_{s-t}(u_{-t}(y)) \in G(s-t)$. ∎

COROLLARY 8.5. For every pair (s,D), $\langle\Psi\rangle_{s,D}$ and $M_G(R^{(Y)})_{s,D}$ are isomorphic as R-modules. Consequently, $\langle\Psi\rangle$ and $M_G(R^{(Y)})$ are isomorphic R-modules.

Proof. Immediate by Lemma 8.4. Moreover we have a canonic isomorphism $\beta_{s,D} : M_G(R^{(Y)})_{s,D} \to \langle\Psi\rangle_{s,D}$ of free R-modules given by the substitution $\xi(x) \mapsto w(o(x)-s)\Psi(x)$ for $x\in Z_{s,D}$, acording to Lemma 8.3 and Proposition 6.3. The isomorphisms $\beta_{s,D}$ are compatible. Indeed let $(s,D) \leq (s',D')$, $x = (y,g)\in Z_{s,D}$, $o(x) = t\in[s]$, $t' = \sum_{i\in \mathrm{Supp}(t)} s'_i \in [s']$. We have to show that $\beta_{s,D}(\xi(x)) = \beta_{s',D'}(\xi(x))$. As $\xi(x) = \sum_{p_{t,t'}(g')=g} \xi(y,g')$ it follows $(y,g')\in Z_{s',D'}$ for all g' occuring in the sum above and hence $\beta_{s',D'}(\xi(x)) = \sum_{p_{t,t'}(g')=g} \beta_{s',D'}(\xi(y,g')) = w(t'-s') \sum_{p_{t,t'}(g')=g} \Psi(y,g') =$

$= w(t'-s') \sum_{t'-t_{z=x}} \Psi(z) = w(t'-s')w(t-t')\Psi(x) = w(t-s')\Psi(x) = \beta_{s,D}(\xi(x))$.

Consequently the isomorphisms $\beta_{s,D}$ induce a canonic isomorphism $\beta : M_G(R^{(Y)}) \to \langle\Psi\rangle$. ∎

THEOREM 8.6. The R-module $\underline{U} = \underline{U}_{X,R,x} = \langle\varphi_{\underline{U}}\rangle$ is isomorphic to the underlying R-module of the induced module $M_G(R^{(Y)})$.

Proof. As $\varphi_{\underline{U}}$ is universal there exists a unique morphism of R-modules $\lambda : \underline{U} \to \langle\Psi\rangle$ such that $\Psi = \lambda \circ \varphi_{\underline{U}}$. For every pair (s,D), λ induces an epimorphism of R-modules $\lambda_{s,D} : \langle\varphi_{\underline{U}}\rangle_{s,D} \to \langle\Psi\rangle_{s,D}$, therefore $\mathrm{rank}_R(\langle\varphi_{\underline{U}}\rangle_{s,D}) \geq \mathrm{rank}_R(\langle\Psi\rangle)$. By Corollary 6.4, $\mathrm{rank}_R(\langle\varphi_{\underline{U}}\rangle_{s,D}) \leq |Z_{s,D}|$. According to Lemma 8.3 and Corollary 8.5, the R-module $\langle\Psi\rangle_{s,D}$ is free of rank $|Z_{s,D}|$. Consequently, the $\lambda_{s,D}$ are isomorphisms and hence λ is an isomorphism of R-modules, i.e., Ψ is universal. By composing λ with the isomorphism $\beta : \langle\Psi\rangle \to M_G(R^{(Y)})$ from Corollary 8.5 we get the required isomorphism of R-modules $\beta\circ\lambda : \underline{U} \to M_G(R^{(Y)})$. ∎

REMARK. While $\langle\Psi\rangle$ is isomorphic as R-module with $M_G(R^{(Y)})$, its structure of discrete left G-module is more intricate than the structure of the

induced module $M_G(R^{(Y)})$. The G-module structure of $\langle\Psi\rangle$ will be the object of a further paper, where some results of Sinott [19] and Kubert [5] devoted to the special case $\underline{X} = \underline{Q}/\underline{Z}$ will be extended to the general framework developed in the present paper.

REFERENCES

1. K. Iwasawa, A class number formula for cyclotomic fields, Ann. Math. 76(1962), 171 - 179.
2. P. T. Johnstone, Stone Spaces, Cambridge Univ. Press 1982.
3. D. Kubert, A system of free generators for the universal even ordinary $\underline{Z}_{(2)}$ - distribution on $\underline{Q}^{2k}/\underline{Z}^{2k}$, Math. Ann. 224(1976), 21-31.
4. D. Kubert, The universal ordinary distribution, Bull.Soc.Math.France 107(1979), 179 - 202.
5. D. Kubert, The $\underline{Z}/2\underline{Z}$ cohomology of the universal ordinary distribution, Bull.Soc.Math.France 107(1979), 203-224.
6. D. Kubert and S.Lang, Units in the modular function field I, Diophantine applications, Math.Ann. 218(1975), 67-96.
7. D. Kubert and S. Lang, Idem II, A full set of units, 175-189.
8. D. Kubert and S. Lang, Idem III, Distribution relations, 273-285.
9. D. Kubert and S. Lang, Idem IV,The Siegel functions are generators, Math. Ann.227(1977), 223-242.
10. D. Kubert and S. Lang, Distributions on toroidal groups, Math.Zeit. 148(1976), 33-51.
11. S. Lang, Cyclotomic Fields, Springer 1978.
12. S. Lang, Idem II, Springer 1980.
13. J. Lubin and J.Tate, Formal complex multiplication in local fields, Ann.Math. 81(1965), 380-387.
14. B.Mazur, Analyse p-adique, Bourbaki report, 1972.
15. B. Mazur and H.Swinnerton - Dyer, Arithmetic on Weil curves, Invent. Math. 25(1974), 1-61.

16. L. R. McCulloh, A class number formula for elementary-abelian-groups rings, J. Algebra 68(1981), 443-452.

17. L. R. McCulloh, Galois module structure of elementary abelian extensions, J.Algebra 82(1983), 102-134.

18. J. P. Serre, Local class field theory, in "Algebraic Number Theory", ed. by J.W.S.Cassels and A.Frohlich, Academic Press 1967.

19. W. Sinnott, On the Stickelberger ideal and the circular units of a cyclotomic field, Ann. Math. 108(1978), 107-134.