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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS
No.16/1987

BUCURESTI

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April 1987

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The aim of the present paper is to give a precise formula for the index homomorphism associated to certain smooth extensions of an odd sphere. Our index result is parallel to that of Douglas and Voiculescu in [8] and applies to a certain class of extensions which contains the Toeplitz extensions. Though this kind of results are available by Connes`cyclic cohomology, our formula contains less terms than are expected by Connes method applied to the particular class of extensions which appear in our paper.

These cancellations are obtained by pure combinatorial tools.

Let H be an infinite dimensional separable Hilbert space. We denote by $\mathcal{L}(H)$ and $\mathcal{K}(H)$ the algebra of all bounded operators and respectively compact operators. Let Q(H) denote the Calkin algebra (that is the quotient $\mathcal{L}(H)/\mathcal{K}(H)$ and let π be the canonical surjection onto Q(H). Let $\mathcal{C}_p(H)$ be the ideal of Schatten-von Neumann bounded linear operators in $\mathcal{L}(H)$,for $p\geq 1$. For a compact metric space X, a unital * morphism $\varphi:C(X) \longrightarrow Q(H)$ will be called an extension of $\mathbb{C}(X)$ by $\mathcal{K}(H)$.

In this paper, we will be concerned only with extensions of odd dimensional spheres.

Let $\varphi:C(S^{2n-1}) \longrightarrow Q(H)$ be an extension of $\mathbb{C}(S^{2n-1})$

by $\mathcal{K}(H)$, and let $\varphi(z_i) \in Q(H)$ be the images by φ of the canonical coordinate functions z_i on $S^{2n-1} \subseteq \mathbb{C}^n$. Such an extension is called $\mathcal{C}_p$-smooth if there exist an n-tuple $T_1, T_2, \ldots, T_n$ in $\mathcal{L}(H)$ such that $\varphi(z_i) = \pi(T_i)$ and all the commutators $[T_i, T_j], [T_i^*, T_j]$ belong to $\mathcal{C}_p(H)$ for all i, j.

Recall that the reduced topological K-theory is vanishing in dimension zero and $K^1(S^{2n-1}) \cong \mathbb{Z}$ has a canonical generator α, which is a unitary $2^{n-1} \times 2^{n-1}$ matrix over $C(S^{2n-1})$ and which has an easy recursively description (see[1],[2]).

Moreover, the Brown-Douglas-Fillmore extension group of S^{2n-1} is $\mathrm{Ext}(S^{2n-1}) \cong \mathbb{Z}$, the isomorphism beeing given by the homomorphism $\mathrm{Ext}(S^{2n-1}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K^1(S^{2n-1}), \mathbb{Z})$ associated to the index map. Therefore extensions of $C(S^{2n-1})$ by $\mathcal{K}(H)$ could be classified by the integer ind $\varphi(\alpha)$, where we still denote by φ the obvious lift of the extension φ to $2^{n-1} \times 2^{n-1}$ matrices over $C(S^{2n-1})$.

In their paper [8], Douglas and Voiculescu were concerned with $\mathcal{C}_n$ - smooth extensions of the 2n-1 dimensional sphere, which have the additional property that $T_1^*T_1 + \ldots T_n^*T_n - 1 \in \mathcal{C}_n(H)$. They proved the beautiful formula:

$$(1.1) \qquad \operatorname{ind} \varphi(\alpha) = \operatorname{tr} \; [T_1, T_1^*, \ldots, T_n, T_n^*]$$

where the term in the right side involves the 2n-complete antysimmetric commutator in $T_1^*, T_1, \ldots, T_n^*, T_n$ (i.e.:

$$[x_1, x_2, \ldots, x_{2n}] = \sum_{\sigma \in S_{2n}} \varepsilon(\sigma) x_{\sigma(1)} \ldots x_{\sigma(2n)} .$$

Moreover they proved this way that all the $\mathcal{C}_{n-1}$ smooth extensions of S^{2n-1} are trivial. Unfortunately, as it is pointed out at the end of their paper, there is not known any example of a nontrivial (that is ind $\varphi(\alpha) \neq 0$) $\mathcal{C}_n$-smooth extension of S^{2n-1} with $T_1^*T_1 + \ldots T_n^*T_n - 1 \in \mathcal{C}_n(H)$. As they say the obvious candidate should be the Toeplitz operators on the Hardy space $H^2(\partial B_n)$ (where B_n is the unit ball of $\mathbb{C}^n$) with symbols $z_1, z_2, \ldots, z_n$, but this n-tuple satisfies

$[T_i, T_j^*] \in \mathcal{C}_p(H)$ only for $p>n$. (see [4]).

The starting point of this paper is the remark that , there is another class of extensions of S^{2n-1} for which the formula (1.1) still holds and which contains the Toeplitz extension.

We consider extensions φ of S^{2n-1} determined by the n-tuple $(T_1,\dots,T_n)$ in $\mathcal{L}(H)$, by $\varphi(z_i) = \pi(T_i)$, with the property that $T_1, T_2,\dots,T_n$ generate a cryptointegral algebra A of dimension n (in the sense of Helton and Howe [9], [10]) and $T_1^* T_1 + \dots T_n^* T_n - 1$ belongs to $A_1 + \mathcal{C}_1(H)$ where A_1 is the commutator ideal of A (see part 2 for a brief recall of definitions concerning such algebras). Such algebras have the property that A_1 is contained in $\mathcal{C}_{n+1}(H) \subseteq \subseteq \mathcal{K}(H)$, so that $T_1,\dots,T_n$ provide indeed an extension of S^{2n-1}.

In particular the Toeplitz operators on $H^2(\partial B_n)$ with symbols $z_1,\dots,z_n$, provide such an example, as proved by Helton and Howe in [10]. (Quite specifically in [10] is proved only that the algebra of Toeplitz operators on the Bergmann space $H^2(B^n)$ is a cryptointegral algebra of dimension n, but an easy inspection of the proof allows one to conclude that also the algebra of Toeplitz operators $H^2(\partial B_n)$ is cryptointegral of dimension n and that formula 7.2 from [10] still holds in this context; see also [9], [10] , [5].

Our main result asserts that formula (1.1) still holds if one considers extensions of S^{2n-1} which have the property that $T_1,\dots,T_n$ generate a cryptointegral algebra of dimension n and such that $T_1^*T_1 + \dots T_n^*T_n - 1$ belongs to $A_1 + + \mathcal{C}_1(H)$. In particular, for the Toeplitz extension formula (1.1) combined with formula 7.2 from [10] is just the statement of Venugopalkrishna's index theorem. Let us give some explanations about the proof of our main result which is rather combinatorial.

If A is a cryptointegral algebra of dimension n generated by n elements $T_1 ,\dots,T_n$ with $T_1^*T_1 + \dots T_n^*T_n - 1 \in \in A_1$ then the C^∞ functional calculus of Helton and Howe provides a linear map $e_T : C^\infty(S^{2n-1}) \longrightarrow A$ such that $e_T(fg) - e_T(f)e_T(g)$ belongs to $\mathcal{C}_{n+1}(H)$. Connes' theory (see [6]) then applies to yield an element τ_{2n+1} in the odd cyclic cohomology group $H_\lambda^{2n+1}(C^\infty(S^{2n-1}))$ which computes the index by means of the natural pairing between $K^1(S^{2n-1})$ and this last group.

On the other hand the fundamental trace form of Helton and Howe yields an element τ_{2n-1} in $H^{2n-1}_{\lambda}(C^{\infty}(S^{2n-1}))$ (in fact in the homology group $H_{2n-1}(S^{2n-1})$).

The only difficult point in the proof of formula (1.1) for this class of extensions, is the determination of a universal constant such that these two elements differ modulo a canonical isomorphism by it.

This is the content of our combinatorial lemma 3.2. Once we have determined this universal constant,the proof of equality (1.1) can be easy deduced from the fact that the index is computed from τ_{2n+1} by means of the coupling of $K^1(S^{2n-1})$ with $H^{2n+1}(S^{2n-1})$,(see Propposition 17, [6]).

Instead of doing this,since it would imply computations with many universal constants,we prefere to repeat the arguments from the paper of Douglas and Voiculescu.

Now returning to the general case of a n-dimensional cryptointegral algebra A ,generated by k selfadjoint elements with essential joint spectrum E,one cannot further expect that the index of a Fredholm element in $M_n(A)$ can be computed only in terms of the fundamental trace form. The reason of this phenomenon is that (as observed by Helton and Howe in [9],[10] for dimensions 1 or 2)the fundamental trace form determines an element l_T in the (de Rham) homology group $H_{2n}(R^k,E)$and hence by means of the coboundary map $\partial:H_{2n}(R^k,E)\longrightarrow H_{2n-1}(E)$ it determines an element in $H_{2n-1}(E)$.

When the cryptointegral algebra is generated by n elements $T_1,\ldots,T_n$ with essential spectrum $E \subseteq \mathbb{C}^n \subseteq \mathbb{R}^{2n}$, the functional l_T yields a distribution u_T on $C_0^{\infty}(\mathbb{R}^{2n})$ which is constant on $\mathbb{R}^{2n}\setminus E$.

However,some information about the index of the elements in A,can be recuperated only in terms of l_T,as seen from the following equality:

$$\left(\frac{1}{(-2i)^n \mathrm{vol}(B_{2n})}\ \mathrm{ind}\,(\lambda-T_1,\ldots,\lambda-T_n)\right)\mu\Big|_U = u_T\Big|_U$$

where U is any connected component of $R^{2n}\setminus E$, $\lambda=(\lambda_1,\ldots,\lambda_n)$ is any point in U and μ is the Lebesque measure on $\mathbb{R}^{2n}$, $vol(B_{2n})$ is the volume of the unit ball in $\mathbb{R}^{2n}$ and the index is the index of a Fredholm n-tuple as introduced in [14],[7].

Such an equality in dimension 1 already appeared in the work of [9],[10].

Acknowledgements. I would like to express thanks to M.Putinar for suggesting me the subject of this paper and for invaluable assistance and helpful comments at various stages of this paper.

I am much indebted to M.Pimsner for many enlighting and stimulating disscutions.

Thanks are also due to A.Negru for aiding me to put this work in an appropriate graphical form.

2. A (brief) recall of definitions and results

In this section we recall some facts from [9],and [10] about cryptointegral algebras. First, let us recall the definition of the commutator filtration.

Definition 2.1 Given an arbitrary (abstract) ring $\mathcal{A}$ one defines recursively a sequence of ideals $\{\mathcal{A}_j\}_{j\geq 0}$ - called the commutator filtration of $\mathcal{A}$ -as follows: $\mathcal{A}_0$ is $\mathcal{A}$ and if $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{j-1}$ are defined then $\mathcal{A}_j$ is the ideal generated by all commutators of elements from $\mathcal{A}_p$ with elements from $\mathcal{A}_{j-p-1}$, $p \in \{0,1,\ldots,j-1\}$.

Second, we recall the definition of the complete antisymmetric sum.

Definition 2.2 If $a_1, a_2, \ldots, a_p$ are arbitrary elements of $\mathcal{A}$, $p \geq 1$ then the complete antisymmetric sum of $a_1, \ldots, a_p$ is:

$$[a_1, a_2, \ldots, a_p] = \sum_{\sigma \in S_p} \varepsilon(\sigma)\, a_{\sigma(1)} \ldots a_{\sigma(p)}$$

where S_p is the group of permutations of $\{1,2,\ldots,p\}$ and $\varepsilon(\sigma)$ is the signature of the permutation σ. Obviously, if $p=2$ then $[a_1, a_2] = a_1 a_2 - a_2 a_1$ is the usual commutator.

It is proved in [10],(page 277), that for any positive integers i,j one has the following inclusions:

$$\mathcal{A}_j \mathcal{A}_k \subseteq \mathcal{A}_{j+k} \qquad (2.1)$$

$$[\mathcal{A}_j, \mathcal{A}_k] \subseteq \mathcal{A}_{j+k+1} \qquad (2.2)$$

The following formulas will be used in the sequel(for a proof see [10], Proposition 1.1):

$$2[a_1,a_2,\dots,a_{2r}]=[a_1,[a_2,\dots,a_{2r}]]-[a_2,[a_1,a_3,\dots]]+\dots-$$

$$-[a_{2r},[a_1,\dots,a_{2r-1}]] \qquad (2.3)$$

$$[a_1,a_2,\dots,a_{2r}]=\sum_{\tau}\varepsilon(\tau)[a_{\tau(1)},a_{\tau(2)}]\dots[a_{\tau(2r-1)},a_{\tau(2r)}] \qquad (2.4)$$

where $a_1,a_2,\dots,a_{2r}$ are arbitrary elements of $\mathcal{A}$, $r\geq 1$ and in the second equality, τ runs over a sequence of coset representatives of N in S_{2p}, where N is the subgroup of S_{2p} generated by the transposition $\{2j-1,2j\}$, $j=\overline{1,p}$.

Definition 2.3 ([10]). Let H be a Hilbert space, and n a strictly positive integer. When $n>1$, a selfadjoint subalgebra A of $\mathcal{L}(H)$ is called cryptointegral of dimension n if A_{n+1} (the $n+1$ term of the commutator filtration of A) is contained in $\mathcal{C}_1(H)$ (the trace class operators) and any antisymmetric sum of $2n$ elements of A is contained in $\mathcal{C}_1(H)$. When $n=1$, one requires only that $[x,y]\in \mathcal{C}_1(H)$ for any x,y in A.

An easy consequence of (2.1) and of the above definition is that the elements of a cryptointegral algebra always commute modulo the compact operators. (In fact they commute modulo $\mathcal{C}_{n+1}(H)$).

The $2n$-multilinear alternate functional on A, given by $(x_1,x_2,\dots,x_{2n})\longrightarrow \mathrm{tr}[x_1,x_2,\dots,x_{2n}]$ is called the fundamental trace form.

The following vannishing results are proved in [10], Lemma 1.3, and 1.4. For later use we state them separately.

Lemma 2.4. If A is a cryptointegral algebra of dimension n, then :

a. If i is any integer such that $0\leq i\leq n$, x is an element of A_i and y an element of A_{n-i} then $[x,y]$ is trace class and $\mathrm{tr}[x,y]=0$.

b. For any elements $x_1,x_2,\dots,x_{2n}$ of A, one has:

$$\mathrm{tr}\,[x_1,x_2,\dots,x_{2n}]=0$$

whenever $x_i \in A_1 + \mathcal{C}_1(H)$, for some $i \in \{1,2,\ldots,2n\}$.

An important property of cryptointegral algebras is their closure to a C^∞ functional calculus. More precisely, given selfadjoint elements $X_1, X_2, \ldots, X_k$ in $\mathcal{L}(H)$ that generate a selfadjoint cryptointegral algebra A`, one can find a maximal cryptointegral algebra A containing A`, so that there is a map e_X: $C^\infty(R^k) \longrightarrow A$ with the following properties([10],Proposition 3.4):

(i) $e_X(f)^* - e_X(\bar{f}) \in A_1 \cap \mathcal{C}_{m+1}(H)$

(ii) $e_X(f)e_X(g) - e_X(fg) \in A_1 \cap \mathcal{C}_{m+1}(H)$

(iii) $[\, e_X(f),\, e_X(g)] \in A_1 \cap \mathcal{C}_{m+1}(H)$.

Here, f, g are elements of $C^\infty(R^n)$, $\bar{f}$ is the complex conjugate of f. Moreover, if p is any polynomial with complex coefficients in n variables:

$$p(x_1, x_2, \ldots, x_n) = \sum a_{\alpha_1, \ldots, \alpha_m} \cdot x_1^{\alpha_1} \ldots x_n^{\alpha_m}$$

then:

$$e_X(p) = \sum a_{\alpha_1, \ldots, \alpha_m} \cdot X_1^{\alpha_1} \ldots X_n^{\alpha_m}.$$

Given a cryptointegral algebra A of dimension n generated by selfadjoint elements $X_1, X_2, \ldots, X_k$ with joint essential spectrum $E \subseteq \mathbb{R}^k$, Helton and Howe have introduced a 2n-linear continuous functional T_X on $C^\infty(R^k)$ as follows. If $f_1, f_2, \ldots, f_{2n}$ are functions in $C^\infty(R^k)$, then T_X is defined by:

$$T_X(f_1, f_2, \ldots, f_{2n}) =$$

$$= \operatorname{tr}\, [e_X(f_1), \ldots, e_X(f_{2n})].$$

This linear functional has the following properties (see Propositions 3.5, 3.6 in [10]) :

a. If one of the functions f_i vanishes in the neighbourhood of E, then $T_X(f_1,f_2,\ldots,f_{2n})=0$.

b. If $g_1,g_2,\ldots,g_m$, are real valued functions in $C^{\infty}(R^k)$ and $Y_i=\frac{1}{2}(e_X(g_i)+e_X(g_i)^*)$, $i=1,m$, then denoting by $\theta : R^k \longrightarrow R^m$ the map with entries g_i, one finds that $Y_1,Y_2,\ldots,Y_m$ still generate a cryptointegral algebra, and $T_Y=T_X\circ\theta$.

In the proof of a., the following result appears:

Lemma 2.5. If f vanishes in the neighbourhood of E then $e_X(f)\in \mathcal{C}_1(H)$.

3. The index formula

In this section we prove formula (1.1) for the following particular class of extensions of $C(S^{2n-1})$ by $\mathcal{K}(H)$. We consider extensions $\varphi: C(S^{2n-1}) \longrightarrow Q(H)$ of $C(S^{2n-1})$ by $\mathcal{K}(H)$ such that there exist an n-tuple $(T_1, T_2, \ldots, T_n)$ in $\mathcal{L}(H)$ with $\varphi(z_i) = \pi(T_i)$ in $Q(H)$ and such that $T_1, \ldots, T_n$ generate a cryptointegral algebra A of dimension n, with the property that $T_1^*T_1 + \ldots T_n^*T_n - 1$ belongs to $A_1 + \mathcal{C}_1(H)$.

First we want to describe the canonical generator α of $K^1(S^{2n-1})$ (the topological K-theory in dimension 1) as a unitary $2^{n-1} \times 2^{n-1}$ matrix over $C(S^{2n-1})$.

Let $\Lambda(\mathbb{C}^n)$ be the exterior algebra of $\mathbb{C}^n$. Then:

$$\Lambda(\mathbb{C}^n) = \Lambda^e(\mathbb{C}^n) \oplus \Lambda^o(\mathbb{C}^n)$$

where by $\Lambda^e(C^n)$ (respectively by $\Lambda^o(\mathbb{C}^n)$) we denote the forms of even(respectively odd)degree. $\Lambda(\mathbb{C}^n)$ has a natural Hilbert space structure coresponding to the orthonormal basis $(e_J)_{J \subseteq \{1,2,\ldots,n\}}$, where $e_\emptyset = 1$ and $e_J = e_{j_1} \wedge \ldots \wedge e_{j_k}$ if $J = \{j_1, j_2, \ldots, j_k\}$, with $j_1 < \ldots < j_k$. On $\Lambda(C^n)$ we define as usual the operators a_i by $a_i h = h \wedge e_i$, for h in $\Lambda(C^n)$. It is known (see [14], Lemma III.6.5) that they satisfy the anticommutation relations:

(3.1) $a_i a_j + a_j a_i = 0$ for all i,j.

(3.2) $a_i^* a_j + a_j a_i^* = 0$ for $i \neq j$.

(3.3) $a_i^* a_i + a_i a_i^* = 1$ for all i.

We denote by $f_i^{\pm}$ the idempotents:

$$f_i^+ = a_i^* a_i; \quad f_i^- = a_i a_i^*.$$

The description of α is as follows:

For a fixed element $z=(z_1,\dots,z_n)$ in $S^{2n-1}\subseteq \mathbb{C}^n$, $\alpha(z)$ is the linear operator $\sum_i z_i a_i + \sum_i \bar{z}_i a_i^*$ in $\mathcal{L}(\Lambda^e(\mathbb{C}^n),\Lambda^o(\mathbb{C}^n))$. We want also to describe the image of α by φ . Therefore, let d be the bounded linear operator in $\mathcal{L}(H\otimes\mathbb{C}^n)$ defined by $d=\sum T_i\otimes a_i$. Let $d^*=\ \ T_i^*\otimes a_i^*$ be the Hilbert space adjoint of d and denote by A the restriction of $(d+d^*)$ to $H\otimes\Lambda^e(\mathbb{C}^n)$ so that A belongs to $\mathcal{L}(H\otimes\Lambda^e(\mathbb{C}^n)), H\otimes\Lambda^o(\mathbb{C}^n))$ and $\varphi(\alpha)-A$ is compact.

Taking into account the description of α given below, and since $\varphi(z_i)=\pi(T_i)$ in $\mathcal{L}(H)/\mathcal{K}(H)$, we obtain that:

$$\operatorname{ind}\varphi(\alpha)=\operatorname{ind}A \tag{3.4}$$

We observe that , by Theorem III.7.1 in [14], if the n-tuple $(T_1,T_2,\dots,T_n)$ is a Fredholm n-tuple then :

$$\operatorname{ind}(T_1,T_2,\dots,T_n)=\operatorname{ind}A,$$

(see [14] or [7], for the definition of a Fredholm n-tuple).

As in [8], given $\mathcal{A}\subseteq\mathcal{L}(H)$ an arbitrary unital subalgebra, one defines the operator valuated trace $\tau:\mathcal{A}\otimes\mathcal{L}(\Lambda(\mathbb{C}^n))\mapsto\mathcal{A}$, by:

$$\tau((x_{J,K})_{J,K\subseteq\{1,2,\dots,n\}})=\sum_{J\subseteq\{1,\dots,n\}} x_{J,J}$$

$J=\{j_1<\dots<j_p\}$.

In [8] it is proved that :

$$\tau((d+d^*)^k(1\otimes p_e-1\otimes p_o))$$

vanishes if $k<2n$, and is equal to $[T_1^*,T_1,\dots,T_n^*,T_n]$ if $k=2n$, where p_e(respectively p_o) are the orthogonal projections of $\Lambda(\mathbb{C}^n)$onto $\Lambda^e(\mathbb{C}^n)$(respectively $\Lambda^o(\mathbb{C}^n)$).

In what follows, we also need the 2(n+1) term,denoted M_{n+1}, ; The next lemma contains the description of M_{n+1}.

Lemma 3.1. Keeping all the notations defined above :

$$M_{n+1}=\tau((d+d^*)^{2n+2}(1\otimes p_e-1\otimes p_o))=$$

$$\sum_{i=1}^{n}\Big(\sum_{\sigma\in S_{+,i}}\varepsilon(\sigma)X^i_{\sigma(-1)}X^i_{\sigma(0)}\cdots X^i_{\sigma(2n)}-\sum_{\sigma\in S_{-,i}}\varepsilon(\sigma)X^i_{\sigma(-1)}X^i_{\sigma(0)}\cdots X^i_{\sigma(2n)}\Big)$$

where $X^i_{-1}=T^*_i$; $X^i_0=T_i$; $X^i_{2s-1}=T^*_s$ and $X^i_{2s}=T_s$, for i,s in $\{1,\dots n\}$, and $S_{+,i}$, (respectively $S_{-,i}$) are those permutations of the set $\{-1,0,\dots,2n\}$ which have the property:

$$\sigma^{-1}(-1)<\sigma^{-1}(0)<\sigma^{-1}(2i-1)<\sigma^{-1}(2i)$$

and, respectively:

$$\sigma^{-1}(0)<\sigma^{-1}(-1)<\sigma^{-1}(2i)<\sigma^{-1}(2i-1)$$

for $i\in\{1,2,\dots,n\}$.

Proof. We start by inspecting $(b(1\otimes p_e-1\otimes p_o))$, where $b=X\otimes E$ is one of the monomials from the expansion of :

$$(T_1\otimes a_1+\dots+T_n\otimes a_n+T^*_1\otimes a^*_1+\dots+T^*_n\otimes a^*_n)^{2n+2}.$$

As observed in the proof of Proposition 1, in [8]:

$$\tau(b(1\otimes p_e-1\otimes p_o))=0$$

unless :

$$E=\pm f^+_{i_1}\dots f^+_{i_s}f^-_{j_1}\dots f^-_{j_t}$$

where $s+t=n$ and $\{i_1,\dots i_s,j_1,\dots,j_t\}=\{1,2,\dots,n\}$. In this case $f^+_{i_1}\dots f^+_{i_s}f^-_{j_1}\dots f^-_{j_1}$ is the projection of $\Lambda(\mathbb{C}^n)$ onto the subspace $\mathbb{C}e_J$ where $J=\{j_1,\dots j_t\}, j_1<j_2<\dots<j_t$.

In this case b is a product in $T_1,T^*_1,\dots,T_n,T^*_n$ in which excepting some $i\in\{1,2,\dots,n\}$ all T_j, T^*_j appear exactly once for $j\in\{1,2,\dots,n\}$ $j\neq i$ and T_i, T_i appear twice, in the order $\dots T_i^*\dots T_i\dots T_i^*\dots T_i\dots$, if $i\notin J$ or in the order $\dots T_i\dots T_i^*\dots T_i\dots T_i^*\dots$, if $i\in J$.

Letting S_J be the set of all permutations of the set $\{-1,0,1,\dots,2n\}$ such that $\sigma^{-1}(2i_l-1)<\sigma^{-1}(2i_l)$, and $\sigma^{-1}(2j_k-1)>\sigma^{-1}(2j_k)$, for $l=1,2,\dots,s$; $k=1,2,\dots,t$, it follows that there is a permutation σ in $S_{+,i}\bigcap S_J$ (respectively in $S_{-,i}\bigcap S_J$) corresponding to the case $i\in J$ (respectively $i\notin J$) such that:

$$b=X^i_{\sigma(-1)}X^i_{\sigma(0)}\dots X^i_{\sigma(2n)}.$$

Moreover:

$$E=(-1)^{|J|}\varepsilon(\sigma)f^+_{i_1}\dots f^+_{i_s}f^-_{j_1}\dots f^-_{j_t}\quad,\text{ if } i\notin J$$

and :

$$E=(-1)^{|J|+1}\varepsilon(\sigma)f^+_{i_1}\dots f^+_{i_s}f^-_{j_1}\dots f^-_{j_t},\text{ if } i\in J.$$

Hence:

$$\tau((d+d^*)^{2n+2}(1\otimes p_e-1\otimes p_\sigma))=$$

$$\sum_J(-1)^{|J|}\Big(\sum_{i\notin J}\sum_{\sigma\in S_{+,i}}(-1)^{|J|}\varepsilon(\sigma)X^i_{\sigma(-1)}X^i_{\sigma(0)}\dots X^i_{\sigma(2n)}+$$
$$+\sum_{i\in J}\sum_{\sigma\in S_{-,i}}(-1)^{|J|+1}\varepsilon(\sigma)X^i_{\sigma(-1)}X^i_{\sigma(0)}\dots X^i_{\sigma(2n)}\Big),$$

where J runs over all ordered subsets of $\{1,2,\dots,n\}$

Since the sets $S_{+,i}\bigcap S_J$ (respectively $S_{-,i}\bigcap S_J$) form a partition of $S_{+,i}$ (respectively $S_{-,i}$), by changing the order of sumation in the last expression, one obtains the formula in the statement of the lemma.

The next lemma proves that modulo a universal constant, the trace of the term M_{n+1} can be computed only in terms of the fundamental trace form. Its proof relies on a purely combinatorial lemma, which we state separately in an Apendix, at the end of the paper.

Lemma 3.2. With the notations before:

$$\operatorname{tr} M_{n+1} = n \operatorname{tr}[T_1^*, T_1, \ldots, T_n^*, T_n]$$

Proof. We compute $2\operatorname{tr}M_{n+1} = \operatorname{tr}M_{n+1} + \operatorname{tr}M_{n+1}$. By combining terms in the first summand starting with a T_s with terms in the second summand ending with the same T_s and terms in the second summand starting with a T_s^* with terms in the first summand ending with the same T_s^*, one obtains:

$$2M_{n+1} = \sum_{i=1}^{n} R_i$$

where:

$$2R_i = \sum_{s\neq i}\Big(\sum_{\sigma\in S_{+,i}} \varepsilon(\sigma)[X^i_{\sigma(-1)}, X^i_{\sigma(0)}\cdot\ldots\cdot X^i_{\sigma(2n)}] - \sum_{\sigma\in S_{-,i}} \varepsilon(\sigma)[X^i_{\sigma(-1)}, X^i_{\sigma(0)}\cdot\ldots\cdot X^i_{\sigma(2n)}]\Big) +$$

$$+ \sum_{s\neq i}\Big(\sum_{\sigma\in S_{+,i}} \varepsilon(\sigma)[X^i_{\sigma(-1)}, X^i_{\sigma(0)}\cdot\ldots\cdot X^i_{\sigma(2n)}] - \sum_{\sigma\in S_{-,i}} \varepsilon(\sigma)[X^i_{\sigma(-1)}, X^i_{\sigma(0)}\cdot\ldots\cdot X^i_{\sigma(2n)}]\Big)$$

$$+ \sum_{\sigma\in S_{+,i}} \varepsilon(\sigma)[X^i_{\sigma(-1)}, X^i_{\sigma(0)}\cdot\ldots\cdot X^i_{\sigma(2n)}] - \sum_{\sigma\in S_{-,i}} \varepsilon(\sigma)[X^i_{\sigma(-1)}, X^i_{\sigma(0)}\cdot\ldots\cdot X^i_{\sigma(2n)}].$$

where in the first two terms, σ runs over all permutations in $S_{+,i}$ (respectively in $S_{-,i}$) with the additional property that $\sigma(-1) = 2s-1$; in the third and in the forth term σ runs over all permutations in $S_{+,i}$ (respectively $S_{-,i}$) with the additional property that $\sigma(-1)=2s$, and in the fifth (respectively sixth) term σ runs over all permutations in $S_{+,i}$ (respectively $S_{-,i}$) with the additional property that $\sigma(-1)=-1$ (respectively $\sigma(-1)=0$).

Applying points (ii),(iii), from the lemma in the appendix, and since terms of the form $[x,y]$ with $x\in A, y\in A_n$ are trace class with null trace, we obtain that, modulo terms of null trace:

$$\frac{1}{n}(2R_i) = \sum_{s\neq i} [X^i_{2s-1}, X^i_{-1}X^i_0[X^i_1,\ldots,\widehat{X^i_{2s-1}},\ldots X^i_{2n}]] -$$

$$- \sum_{s\neq i} [X_{2s}, X^i_{-1}X^i_0[X^i_1,\ldots,\widehat{X^i_{2s}},\ldots,X^i_{2n}]] +$$

$$+ [X^i_{-1}, X^i_0 X^i_{2i-1}[X^i_{2i}, X^i_1,\ldots,\widehat{X^i_{2i-1}},\widehat{X^i_{2i}},\ldots,X^i_{2n}]]$$

$$-[X_0^i, X_{-1}^i X_{2i}^i [X_{2i-1}^i, X_1^i, \ldots, \widehat{X_{2i-1}^i}, \widehat{X_{2i}^i}, \ldots X_{2n}^1]]$$

where for the first two sums we applied point (iii) of the lemma in the Appendix and for the last two sums we applied point (ii) of the same lemma. (As usually a mark ^ over a symbol in a sequence means that this symbol is missing in that sequence).

Finally, we deduce that modulo terms of null trace :

$$\frac{1}{n}(2R_i)= \sum_{s\neq i} [T_s^*, T_i^* T_i [T_1^*, \ldots, \widehat{T_s^*}, T_s, \ldots, T_n]]-$$

$$- \sum_{s\neq i} [T_s, T_i^* T_i [T_1^*, \ldots, T_s^*, \widehat{T_s}, \ldots, T_n]]+$$

$$+ [T_i^*, T_i^* T_i [T_i, T_1^*, T_1, \ldots, \widehat{T_i^*}, \widehat{T_i}, \ldots, T_n^*, T_n]]-$$

$$- [T_i, T_i^* T_i [T_i^*, T_1^*, T_1, \ldots, \widehat{T_i^*}, \widehat{T_i}, \ldots, T_n^*, T_n]].$$

By Lemma 2.4.(b) and since $T_1^* T_1 + \ldots T_n^* T_n - 1 \in A_1 + \mathcal{C}_1(H)$ it follows, after performing the sum over i and applying Lemma 2.3., that:

$$\operatorname{tr}(2R_i)=(2n) \operatorname{tr}[T_1^*, T_1, \ldots, T_n^*, T_n].$$

The next statement determines the index homomorphism for the class of smooth extensions of S^{2n-1} we considered.

The only point in which its proof differs from that given to Proposition 2 in [8], is, that:

$$A= (d+d^*): \ H \otimes \Lambda^e (\mathbb{C}^n) \longrightarrow H \otimes \Lambda^o (\mathbb{C}^n)$$

is no longer a unitary modulo the ideal $\mathcal{C}_n(H)$, but it is certainly a unitary modulo the ideal $\mathcal{C}_{n+1}(H)$ so that the formula:

$$\operatorname{ind} A = \operatorname{tr}((1-A^*A)^p - (1-AA^*)^p)$$

is valid only for $p \geqslant n+1$. To compute the right hand term from the last formula we must therefore use the expression for the term M_{n+1} given in the last lemma.

Theorem 3.3. Let $(T_1, T_2, \ldots, T_n)$ be an n-tuple in $\mathcal{L}(H)$ such that $T_1, T_2, \ldots, T_n$ generate a cryptointegral algebra A of dimension n and $T_1^* T_1 + \ldots + T_n^* T_n - 1 \in \mathcal{C}_1(H) + A_1$. Then $(T_1, T_2, \ldots, T_n)$ determine an extension φ of $C^\infty(S^{2n-1})$ by $\mathcal{K}(H)$ and :

$$\operatorname{ind} \varphi(\alpha) = (-1)^n \operatorname{tr}[T_1^*, T_1, \ldots, T_n^*, T_n]$$

where α is the canonical generator of $K^1(S^{2n-1})$ described at the beginning of this section.

Proof. We proceed as in [8]. Let η be a unitary in $\mathcal{L}(H \otimes \Lambda^e(\mathbb{C}^n), H \otimes \Lambda^o(\mathbb{C}^n))$. Then the space:

$$H_1 = (H \otimes \Lambda^e(\mathbb{C}^n)) \oplus (H \otimes \Lambda^e(\mathbb{C}^n))$$

is unitary maped by $\Phi = \mathrm{Id}_{H \otimes \Lambda^e(\mathbb{C}^n)} \oplus \eta$ onto $H \otimes \Lambda(\mathbb{C}^n)$. Let, as before:

$$A = (d+d^*) : H \otimes \Lambda^e(\mathbb{C}^n) \longrightarrow H \otimes \Lambda^o(\mathbb{C}^n).$$

Then by (3.4):

$$\operatorname{ind} A = \operatorname{ind}(\eta \circ A) = \operatorname{ind} \varphi(\alpha).$$

The matrix of the operator $B = \Phi^*((d+d^*)\Phi$ is therefore:

$$\begin{pmatrix} 0 & A^* \cdot \eta^* \\ \eta \circ A & 0 \end{pmatrix}$$

so that:

$$B^{2p} = \begin{pmatrix} (A^*A)^p & 0 \\ 0 & \eta\,(AA^*)^p\,\eta^* \end{pmatrix}$$

Denoting by $\tau_e: \mathcal{L}(H \otimes \Lambda^e(\mathbb{C}^n)) \longrightarrow \mathcal{L}(H)$ the operatorial trace (the restriction of τ defined before to $\mathcal{L}(H \otimes \Lambda^e(\mathbb{C}^n))$ we have:

$$\tau_e((A^*A)^p - (\eta AA^*\eta^*)^p) = (\tau_e \oplus \tau_e)(B^{2p}(p_1 - p_2)) =$$

$$= \tau((d+d^*)^{2p}(1\otimes p_e - 1\otimes p_o)$$

where p_i is the orthogonal projection onto the i-th component of H_1 and $\tau_e \oplus \tau_e$ is the obvious extension of τ_e to $\mathcal{L}(H_1)$.

Let us remark that $(A^*A-I) \in \mathcal{C}_{n+1}(H)$, $(AA^*-I) \in \mathcal{C}_{n+1}(H)$.

Indeed , this statement is equivalent to $(d+d^*)^2 - 1 \in \mathcal{C}_{n+1}(H)$ and this follows from:

$$(d+d^*)^2 = \sum_{i \neq j} (T_i^* T_j \otimes a_i^* a_j + T_j T_i^* \otimes a_j a_i^* + T_i^* T_j^* \otimes a_i^* a_j^* + T_i T_j \otimes a_i a_j) + \sum_i (T_i^* T_i \otimes a_i^* a_i + T_i T_i^* \otimes a_i a_i^*) + \sum_i (T_i^2 \otimes a_i^2) + \sum_i (T_i^*)^2 \otimes (a_i^*)^2.$$

The last two sums vanish by the first commutation relation (3.1).Hence by using again the commutation relations one finds:

$$(d+d^*)^2 = \sum_{i \neq j} [T_i^*, T_j] \otimes a_i^* a_j + \sum_{i<j} [T_i^*, T_j^*] \otimes a_i^* a_j^* + \sum_{i<j} [T_i, T_j] \otimes a_i a_j + \sum_i (T_i^* T_i - 1) \otimes 1 + \sum_i [T_i^*, T_i] \otimes a_i a_i^* + I \otimes 1.$$

By hypothesis and since the commutator of any two elements of A is in $\mathcal{C}_{n+1}(H)$ one gets that $(d+d^*)^2 - 1 \in A_1 + \mathcal{C}_1(H) \subseteq \mathcal{C}_{n+1}(H)$.

Because of lemma 7.1 in [11]:

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$$\operatorname{ind} A = \operatorname{ind}(\eta \circ A) = \operatorname{Tr}\left((I - A^*A)^{n+1} - (I - \eta AA^* \eta^*)^{n+1}\right) =$$

$$= \operatorname{Tr}\left(\tau_e((I-A^*A)^{n+1} - (I - \eta AA^* \eta^*)^{n+1}\right) =$$

$$= \operatorname{Tr}, \tau_e\left(\sum_{p=0}^{n+1} C_n^p (-1)^p ((A^*A)^p - (\eta AA^* \eta^*)^p)\right) =$$

$$= \operatorname{Tr}\left(\sum_{p=0}^{n+1} C_n^p (-1)^p \, \tau((d+d^*)^{2p}(1 \otimes p_e - 1 \otimes p_o))\right) =$$

$$= (n+1)(-1)^n \operatorname{tr}(\tau((d+d\;)^{2n}(1 \otimes p_e - 1 \otimes p_o))) +$$

$$+ (-1)^{n+1} \operatorname{tr} \tau((d+d^*)^{2n+2}(1 \otimes p_e - 1 \otimes p_o))) =$$

$$= (-1)^n((n+1)\operatorname{tr}([T_1^*, T_1, \ldots, T_n^*, T_n]) -$$

$$- \operatorname{tr}(M_{n+1})) = (-1)^n \operatorname{tr}([T_1^*, T_1, \ldots, T_n^*, T_n])$$

where we used the lemma mentioned above, and denoted $C_n^p = \dfrac{n!}{p!(n-p)!}$

Corrolary 3.4. Assuming the conditions from the hypothesis of Theorem 3.3 one has:

$$\operatorname{ind}(T_1, T_2, \ldots, T_n) = (-1)^n \operatorname{tr}[T_1^*, T_1, \ldots, T_n^*, T_n].$$

If we let in Theorem 3.3, φ be the extension of $C(S^{2n-1})$ by $\mathcal{K}(H)$ obtained by assigning $T_i = T_{z_i}$, the Toeplitz operator with symbol z_i acting on $H^2(\partial B^n)$, then the computations from Theorem 7.2 in [10], yield the Venugopalkrishna's index theorem :

Theorem: If α is the generator of $K^1(S^{2n-1})$ described at the beginning of this section, then:

$$\operatorname{ind} \varphi(\alpha) = \operatorname{tr}[T_{z_1}, T_{z_1}^*, \ldots, T_{z_n}, T_{z_n}^*] =$$

$$= \frac{1}{\operatorname{vol}(B^n)} \int_{B_n} dz_1 \wedge d\bar{z}_1 \wedge \ldots dz_n \wedge d\bar{z}_n = (-1)^n.$$

where by $\operatorname{vol}(B^n)$ we denote this time the volume of the unit ball in C^n, with respect to the volume form : $d\bar{z}_1 \wedge dz_1 \wedge \ldots \wedge d\bar{z}_n \wedge dz_n$.

4. The fundamental form determines an element in $H_{2n}(R^k,E)$

In this section we are going to prove that the fundamental trace form determines a homology class.
Let us denote as in [10], for an open subset D of R^k, by $\Omega^j(D)$ the C^∞-exterior j forms with compact support in D, by $C\Omega^j(D)$ the closed forms in $\Omega^j(D)$ and by $E\Omega^j(D)$ the exact forms in $\Omega^j(D)$ (that is the image under exterior differentiation of $\Omega^{j-1}(D)$). Then the following theorem holds for any $n \in \mathbb{N}$.

Theorem 4.1. If $X_1, X_2, \ldots, X_k$ are selfadjoint generators of a cryptointegral algebra **A** of dimension n, with essential joint spectrum $E \subsetneq \mathbb{R}^k$, then there is a continuos linear functional l on $C\Omega^{2n}(\mathbb{R}^k)$ which vanishes on $E\Omega^{2n}(\mathbb{R}^k\setminus E)$, so that the trace form T_X satisfies :

$$T_X(f_1,f_2,\ldots,f_{2n}) = l(df_1 \wedge df_2 \ldots \wedge df_{2n})$$

for $f_1,f_2,\ldots,f_{2n}$ in $C^\infty(\mathbb{R}^k)$. (Therefore l determines an element in $H_{2n}(\mathbb{R}^k,E)$, and by means of the coboundary operator $\partial: H_{2n}(\mathbb{R}^k,E) \longrightarrow H_{2n-1}(E)$, an element in $H_{2n-1}(E)$.

This theorem was proved in [10] in dimension 1 and 2. The proof of this theorem is based on an algebriac lemma due to N.Wallach, which asserts that a continuos p- linear functional φ on $C^\infty(\mathbb{R}^k)$ with compact support is of the form $l \circ S_p$, where l is a continuos linear functional with compact support on the closed p forms on $\mathbb{R}^k$, and :

$$S_p(f_1,f_2,\ldots f_p) = df_1 \wedge \ldots \wedge df_p,$$

if and only if φ has the p-fold collapsing property.(Recall that φ it is said to have the p-fold collapsing property if $\varphi(f_1,\ldots,f_p)=0$, whenever $f_1,\ldots,f_p$ are polynomials on $\mathbb{R}^n$ and there exist $i \neq j$, such that f_i,f_j,are both polynomials of a third polynomial).

Hence, the technical difficulty is that the associeted fundamental trace form has the 2n-fold collapsing property. Helton and Howe proved that this property holds if the fundamental trace form has the property that, for any $x_1,x_2,\ldots,x_{2n-2} \in A$, $a \in A$,we have:

$$tr[a^2,a,x_1,x_2,\ldots,x_{2n-2}]=0.$$

Furthermore, they proved that this identity holds if n=1 or n=2, so that the proof of the theorem was accomplished in dimension 1 or 2.

Taking all of these facts into account, it follows that the proof of Theorem 4.1 is reduced to the following technical fact:

Proposition 4.2. Let A be a cryptointegral algebra of dimension n, $n \geqslant 3$. For every $a,x_1,x_2,\ldots,x_{2n-2} \in A$ we have:

$$tr[a^2,a,x_1,x_2,\ldots,x_{2n-2}]=0.$$

Before proceeding in the rather long but exclusively proof of this proposition, we introduce a few notations and state some remarks.

Definition 4.1. Given a p-tuple $\alpha=(y_1,y_2,\ldots,y_p)$, $p \geqslant 1$ of elements of A, we denote by $[\alpha]$ the antisymmetric sum $[\alpha]=[y_1,y_2,\ldots,y_p]$. If σ is any permutation of $\{1,2,\ldots,p\}$ and $\sigma(\alpha)$ is the p-tuple $(y_{\sigma(1)},y_{\sigma(2)},\ldots,y_{\sigma(p)})$, then it is obvious that :

$$[\sigma(\alpha)] = \varepsilon(\sigma)[\alpha] \qquad (4.1).$$

We denote the cardinality of α by $|\alpha| = p$ and we put by convention $[\alpha]=1$ if $|\alpha| = 0$. The following remark contains another way of writing the antisymmetric sum.

Lemma 4.3. Let α be as before a p-tuple $(y_1, y_2, \ldots, y_p)$, $p \geq 1$ and let k be an arbitrary element of $\{0,1,\ldots,p\}$. Then the following equality holds:

$$[\alpha] = \sum{}' \{\varepsilon(\beta\gamma)[\beta][\gamma] : |\beta| = k; \{\beta, \gamma\} \text{partition of } \alpha\}.$$

Here β (respectively γ) runs over k-tuples (respectively (p-k)-tuples) of elements of the set $\{y_1, y_2, \ldots, y_p\}$, which considered as sets have the property that $\{\beta, \gamma\}$ is a partition of α (i.e. of the set $\{y_1, \ldots, y_p\}$). To each such β and γ we can associate a permutation $\sigma_{\beta\gamma}$ of $\{1,2,\ldots,p\}$ obtained by concatenation of β with γ. By $\varepsilon(\beta\gamma)$ we denote the signature of this permutation.

The $'$ in the sum before has the meaning that we perform the sum only over a family of β and γ such that $\{\sigma_{\beta\gamma}\}$ form a sequence of coset representatives of $N_{k,p-k}$ in S_p, where $N_{k,p-k}$ is the subgroup of S_p generated by the permutations σ in S_p which preserve the subsets $\{1,2,\ldots,k\}$ and $\{k+1,\ldots,p\}$. Equality (4.1) ensures that it does not matter what choice of these coset representatives we perform.

More generally, from the same reasons, the following statement holds:

Lemma 4.3$'$. If $k_1, k_2, \ldots, k_s$, are positive integers with $k_1+k_2+\ldots+k_s=p$, then:

$$[\alpha] = \sum{}' \{\varepsilon(\alpha_1\alpha_2\cdots\alpha_s)[\alpha_1][\alpha_2]\ldots[\alpha_s] : |\alpha_i| = k_i, i=\overline{1,s};$$

$$\{\alpha_1, \alpha_2, \ldots, \alpha_s\} \text{ is a partition of } \{1,2,\ldots,p\}\},$$

where the $'$ has an analoguous meaning. For instance, equality (2.4) is of this type.

Taking into account equality (4.1) we can prove the next equality which is used in the proof of Proposition 4.2.

Lemma 4.4. Let $\alpha=(y_1,y_2,\dots,y_{2p})$ be a 2p-tuple of elements of A ,$p\geq 1$, and a an arbitrary element of A.Then the following equality holds:

$$[a,[\alpha]]= \sum_{1\le r<s\le 2p}{}^{'} \{ \varepsilon(\beta_1 rs\beta_2)[\beta_1][a,[y_r,y_s]][\beta_2] :$$
$$: |\beta_1|,|\beta_2| \text{ even}, \{\beta_1,\beta_2\}\text{partition of } \alpha\setminus\{y_r,y_s\}\}.$$

The meaning of the ` above in the second sum is as in Lemma 4.3.:for fixed r,s,and for a fixed k in $\{0,1,2,\dots,p-1\}$ we take the sum over a family of 2k-tuples(respectively 2((p--1)-k)-tuples) β_1 (respectively β_2) such that the permutations σ_{β_1,β_2} of $\{1,2,\dots,2p\}\setminus\{r,s\}$ determined by β_1 and β_2 form a sequence of coset representatives of $N_{2k,2(p-1-k)}$ in S_{2p-2}. (Here, we identify the ordered set $\{1,2,\dots,2p\}\setminus\{r,s\}$ with $\{1,2,\dots,2p-2\}$,from the ordering point of view).

Again, $\varepsilon(\beta_1 rs\beta_2)$ is the signature of the permutation of $\{1,2,\dots,2p\}$,determined by the concatenation of β_1,r,s,β_2.

Proof. (of Lemma 4.4.) By formula (4.1) one has :

$$[\alpha]=\sum_{\tau\in A}{}^{'}\varepsilon(\tau)[y_{\tau(1)},y_{\tau(2)}]\dots[y_{\tau(2p-1)},y_{\tau(2p)}]$$

where τ runs over a sequence A of coset representatives of N in S_{2p}(here N is the subgroup of S_{2p} generated by interchanges of the pairs $(\{2l-1,2l\}, l=\overline{1,p})$. We may also assume that the permutations A are so chosen that:

$$\tau(2l-1) < \tau(2l),\ l=\overline{1,p}.$$

Making use of the formula :

$$[a, x_1x_2\dots x_p]=\sum_{k=0}^{p} x_1\dots x_{k-1}[a,x_k]x_{k+1}\dots x_p ,$$

which is valid for all $x_1,x_2,\dots,x_p$ in A , and where a product over an empty set of indices is assumed to be 1, one obtaines:

$$[a,[\alpha]] = \sum_{\tau\in A}\sum_{k=1}^{p}\varepsilon(\tau)[y_{\tau(1)},y_{\tau(2)}]\ldots[y_{\tau(2k-3)},y_{\tau(2k-2)}]\cdot$$

$$\cdot[a,[y_{\tau(2k-1)},y_{\tau(2k)}]]\ldots[y_{\tau(2p-1)},y_{\tau(2p)}].$$

From Definition 4.1.and by a suitable rearrangement of the terms in the sum before , one gets the desired equality. In the proof of the Proposition we need the following rewritting of Lemma 4.3.:

Lemma 4.5. Let $\alpha=(y_1,y_2,\ldots,y_{2p})$ be a 2p-tuple of elements in $A, p\geq 1$. Then, with the notations from Lemma 4.3. we have :

$$(p+1)[\alpha] = \sum_{k=0}^{p}\sum{}'\{\varepsilon(\beta\gamma)[\beta][\gamma] : |\beta|=2k,\ \{\beta,\gamma\}\text{partition of }\alpha\} =$$

$$= \sum{}'\{\varepsilon(\beta\gamma)[\beta][\gamma] : |\beta|\text{ even}, \{\beta,\gamma\}\text{ partition of }\alpha\}$$

where the meaning of ` in the first term, is that for a fixed $k, k=\overline{0,p}$ we take the sum over a sequence of coset representatives.

The next remark is a straightforward consequence of relations (2.1) and (2.2)and of Lemma 2.4a.

Lemma 4.6. If $a_1,a_2,\ldots,a_r$ are elements of $\mathbb{A}$ such that $a_i\in A_{j_i}$, where j_i are positive integers with:

$$j_1+j_2+\ldots+j_r=n,$$

then the difference :

$$a_1a_2\ldots a_r-a_2\ldots a_ra_1$$

is an element in $A_{n+1}\subseteq \mathcal{C}_1(H)$, with null trace.

Now we are able to proceed to the proof of Proposition 4.2.. From (2.4), and by a suitable rearrangement of the terms, one obtains :

$$[a^2,a,x_1,x_2,\ldots,x_{2n-2}] =$$

$$= \sum_{1\leq i+j\leq 2n-2}\sum{}'\{s_{ij\alpha\beta\gamma}\cdot([\alpha][a,x_j][\beta][a^2,x_i][\gamma] +$$

$+[\alpha][a^2,x_i][\beta][a,x_j][\gamma])\ :\ |\alpha|,\ |\beta|\ ,\ |\gamma|$ even;

$\{\alpha,\beta,\gamma\}$ partition of $\{1,2,\ldots,2n-2\}\backslash\{i,j\}\}$.

The meaning of ` is the same as in Lemma 4.4(i.e. keeping the lenght of α,β,γ fixed, the permutations determined by α,β,γ of $\{1,2,\ldots,2n-2\}\backslash\{i,j\}$ run trough a sequence of coset representatives), and $s_{ij\alpha\beta\gamma}$ is the signature of the permutation :

$$\begin{pmatrix} a^2 & a & x_1 & x_2 \ldots\ldots & x_{2n-2} \\ \alpha & a & x_j & \beta & a^2\ x_i\ \gamma \end{pmatrix}$$

which is the same as the signature of the permutation obtained by intertwining above the pair (a,x_i) with the pair (a^2,x_j), and both are equal to $-\varepsilon(ij\alpha\beta\gamma)$, where $\varepsilon(ij\alpha\beta\gamma)$ is the signature of the permutation:

$$\begin{pmatrix} 1 & 2 & \ldots\ldots & 2n-2 \\ i & j & \alpha\beta\gamma \end{pmatrix}$$

Since:

$$[a^2,x]=a[a,x]+[a,x]a$$

for x in A, and by Lemma 4.6 we obtain that, modulo terms of null trace , and keeping i,j,α,β,γ fixed:

$$[\alpha][a^2,x_i][\beta][a,x_j][\gamma]+[\alpha][a,x_j][\beta][a^2,x_i][\gamma]=$$

$$=[a,x_i]a[\beta][a,x_j][\gamma][\alpha]+[a,x_i]a[\gamma][\alpha][a,x_j]\ [\beta]+$$

$$+[a,x_j]\ [\beta]a\ [a,\ x_i]\ [\gamma][\alpha]+[a,x_j][\gamma][\alpha]a[a,x_i][\beta].$$

By combining this terms with the analoguous terms obtained by interchanging i with j, and since :

$$\varepsilon(ij\alpha\beta\gamma) = -\varepsilon(ji\alpha\beta\gamma)$$

one finds, modulo terms of null trace:

$$[a^2,a,x_1,x_2,\ldots,x_{2n-2}]=$$

$$= \sum_{i\neq j}\sum{}' \{-\varepsilon(ij\alpha\beta\gamma)([a,x_i][a,[\beta]][a,x_j][\gamma][\alpha] +$$

$$+ [a,x_i][a,[\gamma][\alpha]][a,x_j][\beta]) : |\alpha|,|\beta|,|\gamma| \text{ even},$$

$$\{\alpha,\beta,\gamma\} \text{ partition of } \{1,\ldots,2n-2\}\setminus\{i,j\}\}=$$

$$=(n+1)\sum_{i\neq j}\sum{}' \{-\varepsilon(ij\beta'\beta''))[a,x_i][a,[\beta']][a,x_j][\beta''] :$$

$$: |\beta'|,|\beta''| \text{ even}, \{\beta',\beta''\} \text{partition of } \{1,\ldots,2n-2\}\setminus\{i,j\}\}.$$

The last equality is obtained from Lemma 4.5, by combining terms from the first sum with terms from the second sum such that $\beta = \gamma\cup\alpha$ .The constant $(n+1)$ is obtained as:

$$(p+1) + [(n-1)-p+1] \quad , \; p=\overline{0,\, n-1}.$$

The terms in which β' is empty are vanishing, so that, applying Lemma 4.4, it follows that modulo terms of null trace, we have :

$$[a^2,a,x_1,\ldots,x_{2n-2}]=$$

$$=(n+1)\sum_{i\neq j}\sum{}' \{-\varepsilon(ij\beta_1' rs\beta_2'\beta'')([a,x_i][\beta_1'][a,[x_r,x_s]][\beta_2'][a,x_j]$$

$$[\beta'']) : |\beta_1'|,|\beta_2'| \text{ even}, r<s, \{\beta_1',\beta_2',\beta''\}\text{partition of } \{1,\ldots,2n-2\}\setminus$$

$$\setminus\{i,j,r,s\}\}=$$

$$=(n+1)\sum_{i,j,r,s}\sum{}' \{-\varepsilon(ij\beta_1' rs\beta_2'\beta'')[a,x_i][\beta_1'][x_r,[a,x_s]][\beta_2'][a,x_j][\beta'']:$$

$: |\beta_1'|, |\beta_2'|, |\beta''|$ even, $\{\beta_1', \beta_2', \beta''\}$ partition of $\{1,2,\ldots,2n-2\}\backslash$

$\backslash\{i,j,r,s\}\}$

where the second sum runs over all distinct $i,j,r,s=1,2,\ldots,2n-2$.

The last identity follows from Jacobi identity:

$$[a,[x_r,x_s]]=[x_r,[a,x_s]]-[x_s,[a,x_r]]$$

taking into account that:

$$\varepsilon(ij\beta_1'rs\beta_2'\beta'')= -\varepsilon(ij\beta_1'sr\beta_2'\beta'').$$

Since:

$$[x_r,[a,x_s]]=x_r[a,x_s]-[a,x_s]x_r$$

and using again Lemma 4.6. we obtain that, modulo terms of null trace, the following identity holds:

$$[a^2,a,x_1,\ldots,x_{2n-2}]=$$

$$=(n+1)\sum_{i,j,r,s}\sum{}'\{-\varepsilon(ij\beta_1'rs\beta_2'\beta'')[\beta_1']x_r[a,x_s][\beta_2'][a,x_j][\beta''][a,x_i]:$$

$$: |\beta_1'|, |\beta_2'|, |\beta''| \text{ even}; \{\beta_1', \beta_2', \beta''\} \text{ partition of } \{1,\ldots,2n-2\}\backslash\{i,j,r,s,\}\}$$

$$-(n+1)\sum_{i,j,r,s}\sum{}'\{-\varepsilon(ij\beta_1'rs\beta_2'\beta'')x_r[\beta_2'][a,x_j][\beta''][a,x_i][\beta_1'][a,x_s]:$$

$$: |\beta_1'|, |\beta_2'|, |\beta''| \text{ even}; \{\beta_1', \beta_2', \beta''\} \text{ partition of } \{1,\ldots,2n-2\}\backslash\{i,j,r,s\}\}$$

where the sum runs over all distinct i,j,r,s in $\{1,2,..,2n-2\}$

With Lemma 4.3., we group together the terms in which appears $[\beta_1']x_r$ (respectively $x_r[\beta_2']$) in the first sum (respectively the second sum). Therefore, by changing the indices of sumation in a suitable way, and since:

$$\varepsilon(ij\beta_1' rs\beta_2'\beta'')=-\varepsilon(ijs\beta_1' r\beta_2'\beta'')=-\varepsilon(ijsr\beta_2'\beta_1'\beta''),$$

one finds that, modulo terms of null trace:

$$[a^2,x_1,x_2,\dots,x_{2n-2}]=$$

$$=(n+1)\sum_{i,j,s}\sum{}'\{\varepsilon(ijs\alpha_1\alpha_2\alpha_3)[\alpha_1][a,x_s][\alpha_2][a,x_j][\alpha_3][a,x_i]:$$

$$:|\alpha_1|\text{odd}; |\alpha_2|,|\alpha_3|\text{even}; \{\alpha_1,\alpha_2,\alpha_3\}\text{partition of } \{1,\dots,2n-2\}\setminus\{i,j,s\}\}-$$

$$-(n+1)\sum_{i,j,s}\sum{}'\{\varepsilon(ijs\alpha_1\alpha_2\alpha_3)[\alpha_1][a,x_j][\alpha_2][a,x_i][\alpha_3][a,x_s]:$$

$$:|\alpha_1|\text{odd}, |\alpha_2|,|\alpha_3|\text{even}; \{\alpha_1,\alpha_2,\alpha_3\}\text{partition of } \{1,\dots,2n-2\}\setminus\{i,j,s\}\}$$

where as before, i,j,s take only distinct values. Since the signature of the permutation:

$$\begin{pmatrix} s & j & i \\ j & i & s \end{pmatrix}$$

is 1, it follows that this two last terms cancell, so that:

$$\operatorname{tr}[a^2,a,x_1,x_2,\dots,x_{2n-2}]=0.$$

5. The index of certain essentialy commuting n-tuples

Let A be a cryptointegral algebra of dimension n, generated by the operators $T_1, T_2, \ldots, T_n$ in $\mathcal{L}(H)$. Then the essential joint spectrum $\sigma_e(T_1, T_2 \ldots, T_n)$ is a compact subset E of $\mathbb{C}^n$. If T_k has the decomposition $X_k + iY_k$, with selfadjoint $X_k, Y_k \in A, k=\overline{1,n}$ then X_i, Y_i, $i=\overline{1,n}$,are essentialy commuting and the essential spectrum of $X_1, Y_1, \ldots, X_n, Y_n$ as a subset of R^{2n} coincides with E if one identifies $\mathbb{R}^{2n}$ with $\mathbb{C}^k$.

If T_T is the 2n-linear functional associeted to the real and imaginary parts $X_1, Y_1, \ldots, X_n, Y_n$ of $T_1, \ldots, T_n$, then by Theorem 3.1, there is a continuos linear functional l_T on $C\Omega^{2n}(\mathbb{R}^{2n})$ which vanishes on $E\Omega^{2n}(\mathbb{R}^{2n}-E)$ such that for any $f_1, \ldots, f_{2n}$ in $C\Omega^{2n}(\mathbb{C}^n)$, one has :

$$\mathrm{tr}([e_T(f_1), \ldots, e_T(f_{2n})] = T_T(f_1, \ldots, f_{2n}) = l_T(df_1 \wedge \ldots \wedge df_{2n})$$

where by e_T we denote the Helton and Howe C^∞ functional calculus associated with the selfadjoint operators $X_1, Y_1, \ldots, X_n, Y_n$ in A.

Since any form in $\Omega^{2n}(\mathbb{R}^{2n})$ is closed,and by the canonical identification of $\Omega^{2n}(\mathbb{R}^{2n})$ with $C_0^\infty(R^{2n})$

$$g dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n \longrightarrow g \in C_0^\infty(\mathbb{R}^{2n})$$

it follows that l_T corresponds to a distribution u_T on $C_0^\infty(\mathbb{R}^{2n})$ (or on $C_0^\infty(\mathbb{C}^n)$),(when $n=1, u_T$ is a measure absolutely continuous with respect to the Lesbegue measure,([3],[9])) with the properties stated in the following lemma:

Lemma 5.1. If $A, T_1, \ldots, T_n$ and T_T, l_T, u_T are as before, then:

(5a). If $f_1, \ldots, f_{2n} \in C^\infty(\mathbb{C}^n)$ and $df_1 \wedge \ldots \wedge df_{2n} = g\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \ldots \wedge dx_n \wedge dy_n$, then :

$$\operatorname{tr}\,[\, e_T(f_1), \ldots, e_T(f_{2n})] = T_T(f_1, \ldots, f_{2n}) = u_T(g).$$

(5b). If U is any open set in $\mathbb{C}^n$, with $U \cap E = \emptyset$, then $u_T \mid C_0^\infty(U) = c\,\mu|_U$, where μ is the Lesbegue measure on $\mathbb{C}^n$, and c is a constant.

(5c). If $\theta : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $\theta = (g_1, g_2, \ldots, g_n)$ is a C^∞ map, then, let $S_i = g_i(T_1, \ldots, T_n)$ be the operators obtained by the Helton and Howe C^∞ functional calculus :

$$S_i = e_T(g_i)).$$

By regarding g_i as a function in $x_1, y_1, \ldots, x_n, y_n$, with the identification $z_k = x_k + iy_k$, $k = 1, 2, \ldots, n$, for any f in $C_0^\infty(\mathbb{C}^n)$ one has :

$$u_S(f) = u_T(f \circ \theta\ \mathrm{Jac}(\theta)),$$

where by $\mathrm{Jac}(\theta)$ we denote the jacobian of the transformation θ from R^{2n} into R^{2n}.

(5d). With the notations in (5c), if U is an open set in C^n such that $u_T | C_0^\infty(\theta^{-1}(U)) = c\mu$, $u_S|_{C_0^\infty(U)} = c`\mu$ and $\theta$ is a diffeomorphism from $\theta^{-1}(U)$ onto $U$ then $c = c`$.

Proof. (5a). This is an easy consequence of the definition of u_T.

(5b). This is a consequence of the fact that:

$$l_T \,\big|\, E\Omega^{2n}(\mathbb{C}^n - E) = 0.$$

(5d). Follows from (5c).

(5c). By Proposition 3.6 in [10] $T_S = T_T \circ \theta$, hence, for any functions $f_1, f_2, \dots, f_{2n}$ in $C_0^\infty(\mathbb{R}^{2n})$, one has :

$$1_S(df_1 \wedge df_2 \dots \wedge df_{2n}) = 1_T(d(f_1 \circ \theta) \wedge d(f_2 \circ \theta) \dots \wedge d(f_{2n} \circ \theta)).$$

After rather easy computations, if:

$$df_1 \wedge df_2 \dots \wedge df_{2n} = g dx_1 \wedge dy_1 \dots dx_n \wedge dy_n,$$

then:

$$d(f_1 \circ \theta) \wedge \dots \wedge d(f_{2n} \circ \theta) = (g \circ \theta)(\operatorname{Jac}\theta) dx_1 \wedge \dots \wedge dy_n.$$

Since the space $\Omega^{2n}(\mathbb{R}^{2n})$ is the linear span of forms of the type $df_1 \wedge \dots \wedge df_{2n}$, it follows that (5c) holds.

For the rest of this section we shall be mainly concerned with the determination of the constant c appearing in (5b). If n=1, Helton and Howe proved in [9] that $c = (-1/2\pi i)\operatorname{Ind}(\lambda - T_1)$, for any λ in U. As expected, this result is still true for a general n, and the index of a single operator is replaced by the index of the essentially commuting n-tuple $T_1, T_2, \dots, T_n$. We refer to [14] or [7] for definitions and results involving Fredholm n-tuples.

Proposition 5.5. Let $(T_1, T_2, \dots, T_n) \subseteq \mathcal{L}(H)$ be a system of operators generating a cryptointegral algebra A of dimension n and suppose that $(T_1^* T_1 + \dots + T_n^* T_n) - 1 \in A_1 + \mathcal{C}_1(H)$
If u_T is the distribution associeted to $(T_1, T_2, \dots, T_n)$ then:

$$u_T \Big|_{C_0^\infty(B^{2n})} = \frac{1}{(-2i)^n \operatorname{vol}(B^{2n})} \operatorname{ind}(T_1, T_2, \dots, T_n)\, \mu \Big|_{B^{2n}}$$

where B^{2n} is the unit ball in R^{2n}, and $\operatorname{vol}(B^{2n})$ is the Lesbegue measure of B^{2n}.

Proof. Remark that, since $\sum T_i^* T_i - 1 \in A_1 + \mathcal{C}_1(H)$ it follows that $\sigma_e(T_1, T_2, \dots, T_n) \subseteq S^{2n-1}$. Let $\delta < 1$ be fixed, and $\varepsilon < \min(\delta, 1-\delta)$. Let f_ε be a function in $C^\infty[[0,\infty))$ such that $f_\varepsilon(r) = r/\delta$ if $r \leq \delta - \varepsilon$ and $f_\varepsilon(r) = 1$ if $r > \delta + \varepsilon$. From the developement in Taylor series one obtaines :

$$f_\varepsilon(r)- f_\varepsilon(1)= (r-1)h_\varepsilon(r)$$

for some $h_\varepsilon \in C_0^\infty([0,\infty))$. Hence:

$$f_\varepsilon(r)-r = (r^2-1)g_\varepsilon(r) \qquad (5.1)$$

for some $g_\varepsilon \in C^\infty([0,\infty))$.
Moreover, if $0\leqslant r <\delta-\varepsilon<1$, then:

$$g_\varepsilon(r)= \frac{(\frac{1}{\delta}-1)r}{r^2-1} \qquad (5.2)$$

Let $\gamma_i(z)= \frac{z_i}{\|z\|}$ for $\|z\| \neq 0$, $z \in \mathbb{C}^n$ (where $\|\cdot\|$ is the euclidian norm on $\mathbb{C}^n$), and:

$$\varphi_i^\varepsilon(z)= f_\varepsilon(\|z\|)\gamma_i(z)$$

$$\psi_i^\varepsilon(z)= g_\varepsilon(\|z\|)\gamma_i(z).$$

The definition of f_ε and (5.2) shows that $\varphi_i^\varepsilon, \psi_i^\varepsilon \in C^\infty(\mathbb{C}^n)$ and from (5.1), by multiplication with γ_i we deduce that:

$$\varphi_i^\varepsilon(z)-z_i= \left(\sum |z_i|^2 - 1\right) \psi_i^\varepsilon(z).$$

Therefore, by the properties (i),(ii) of the Helton and Howe functional calculus, recalled in paragraph 2,one obtaines that :

$$e_T(\varphi_i^\varepsilon)-T_i = \left(\sum T_i^* T_i-1\right) e_T(\psi_i^\varepsilon)$$

modulo terms from A_1.By hypothesis and since A_1 and

$\mathcal{C}_1(H)$ are ideals, one gets:

$$e_T(\varphi_i^{\varepsilon}) - T_i \in A_1 + \mathcal{C}_1(H).$$

Hence by lemma 2.4(b), by property i) of the Helton and Howe C^{∞} functional calculus, and by corrolary 3.4:

$$\operatorname{ind}(T_1,T_2,\dots,T_n) = (-1)^n \operatorname{Tr}[T_1^*,T_1,\dots,T_n^*,T_n] =$$

$$= (-1)^n \operatorname{tr}([e_T(\bar{\varphi}_1^{\varepsilon}), e_T(\varphi_1^{\varepsilon}),\dots, e_T(\bar{\varphi}_n^{\varepsilon}), e_T(\varphi_n^{\varepsilon})] =$$

$$= (-1)^n\, l_T(d\bar{\varphi}_1^{\varepsilon} \wedge d\varphi_1^{\varepsilon} \wedge \dots \wedge d\bar{\varphi}_n^{\varepsilon} \wedge d\varphi_n^{\varepsilon}) =$$

$$= (-1)^n\, l_T(J_{\mathbb{C}}(\bar{\varphi}_1^{\varepsilon}, \varphi_1^{\varepsilon},\dots,\bar{\varphi}_n^{\varepsilon},\varphi_n^{\varepsilon}) d\bar{z}_1 \wedge dz_1 \wedge \dots d\bar{z}_n \wedge dz_n)) =$$

$$= (-1)^n (+2i)^n\, u_T(J_{\mathbb{C}}(\bar{\varphi}_1^{\varepsilon}, \varphi_1^{\varepsilon},\dots,\bar{\varphi}_n^{\varepsilon},\varphi_n^{\varepsilon}))$$

$$= (-2i)^n\, c \int_{\bar{B}^{2n}(0,\delta+\varepsilon)} J_{\mathbb{C}}(\bar{\varphi}_1^{\varepsilon},\dots,\varphi_n^{\varepsilon})\, d\mu$$

where by $J_{\mathbb{C}}$ we denote the Jacobian with respect to $\frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial \bar{z}_n}, \frac{\partial}{\partial z_n}$ and c is the constant defined by $u_T \mid C_0^{\infty}(B^{2n}(0,1)) = c\,\mu$, (where we denote $B^{2n}(0,a) = \{x \in \mathbb{R}^{2n} : \|x\| < a\}$).
By approximating the function f defined by $f(r) = \frac{r}{\delta}$ if $r \leq \delta$ and $f(r) = 1$ if $r > \frac{1}{\delta}$, with C^{∞} functions f_{ε} such that the sequence of the derivatives is uniformely bounded so that $J_{\mathbb{R}}(\bar{\varphi}_1^{\varepsilon}, \varphi_1^{\varepsilon},\dots,\bar{\varphi}_n^{\varepsilon},\varphi_n^{\varepsilon})$ converges uniformely bounded to $\frac{1}{\delta^{2n}} \chi_{B^{2n}(0,\delta)}$ and since $J_{\mathbb{C}}(\bar{z}_1,z_1,\dots,\bar{z}_n,z_n) = 1$, it follows that:

$$\operatorname{ind}(T_1,T_2,\dots,T_n) = (-2i)^n\, c\, \frac{1}{\delta^{2n}}\, \delta^{2n} c_{2n}$$

where c_{2n} is the volume of the unit ball in $\mathbb{R}^{2n}$.

Therefore:

$$c = \frac{1}{(-2i)^n c_{2n}} \operatorname{ind}(T_1,T_2,\dots T_n)$$

(As usual, for a subset $A \subseteq \mathbb{C}^n$, we denote by χ_A his characteristic function)

Theorem 5.6. Let $T_1, T_2, \ldots, T_n$ generate a cryptointegral algebra A of dimension n. Let $E \subseteq \mathbb{C}^n$ be the essential joint spectrum of $T_1, T_2, \ldots, T_n$, let U be a connected open set such that $U \cap E = \emptyset$, and let u_T be the distribution associated to $T_1, T_2, \ldots, T_n$. If $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is any point in U, then:

$$u_T\big|_U = \frac{1}{(-2i)^n c_{2n}} \text{ ind } (\lambda_1 - T_1, \ldots, \lambda_n - T_n)\, \mu\big|_U$$

Proof. The ideea of proof is to reduce the problem to the foregoing case.The proof is an easy extension of that given in [9], section I.7, in the case n=1. We assume that $0 \in U$ and that $\lambda_i = 0, i = \overline{1,n}$.

Let $1 > \delta > 0$ be such that the closed ball $\bar{B}(0, \delta)$ with radius δ is contained in U. As observed in Lemma 5.1:

$$u_T\big|_{C_0^\infty(U)} = c\, \mu\big|_U \qquad (5.3)$$

where c is a constant.

Let f be a positive C^∞ function such that $f(r)=1$ for $r < \frac{\delta}{2}$, and $f(r) = \frac{1}{r}$ for $r \geq \delta$ and such that $rf(r)$ is monotone increasing.

Let $g: \mathbb{C}^n \to \mathbb{C}^n$ be defined by :

$$g_i(z) = f(\|z\|) z_i, \quad g = (g_1, \ldots, g_n)$$

Then on $B(0, \frac{\delta}{2})$ one has that $g(z) = z$, $z \in B(0, \frac{\delta}{2})$, and $\sum |g_i(z)|^2 = 1$ in a neighbourhood of E. Let $S_i = e_T(g_i)$. By properties i, ii, of the functional calculus, we obtain that: $\sum S_i^* S_i - 1 \in A_1$.
In particular $\sigma_e(S_1, \ldots, S_n) \subseteq S^{2n-1}$. Since :

$$g(z) = f(\|z\|) z$$

and $f(r)r$ is monotone increasing, it follows that:

$$g(z) < \frac{\delta}{2} \Leftrightarrow z < \frac{\delta}{2} \qquad (5.4)$$

Let u_S be the distribution associeted to the operators $S_1, S_2, \ldots, S_n$, which still generate a cryptointegral algebra of dimension n. Since $B^{2n}(0,1) \cap \cap \sigma_e(S_1, \ldots, S_n) = \emptyset$ it follows that:

$$u_S \mid B^{2n}(0,\tfrac{\delta}{2}) = c`\mu|B^{2n}(0,\tfrac{\delta}{2}); u_S|B^{2n}(0,1) = c`\mu|B^{2n}(0,1) \qquad (5.5)$$

for some constant c\`. Since by (5.4) $g^{-1}(B^{2n}(0,\frac{\delta}{2})) = B^{2n}(0,\frac{\delta}{2})$ it follows by lemma 5.1 (d), that c=c\`.
The only thing that remains to be proved is that ind T=ind S (because of the previous proposition). But one easily constructs a homotopy of Fredholm operators between S and T.

Consider $g_t(z) = z((1-t)+tg(\|z\|))$. Then:

$$S_i^t = e_T(g_{t,i}), \quad i=\overline{1,n}.$$

is a Fredholm n-tuple, and gives a path of Fredholm n-tuples connecting S and T. Hence by [7]:

$$\text{ind } S = \text{ind } T.$$

Therefore by Proposition 5.5 (since $S_1^*S_1 + \ldots + S_n^*S_n - 1 \in A_1$), and by comparing equalities (5.3), (5.5) and $c=c'$ with the preceding equality, the Theorem follows.

Appendix

Lemma. Let A be a ring. Let $a_1,a_2,\dots,a_{2m+1}$, $m\geqslant 1$ be elements of A. Let S_2 be the set of permutations of $\{1,2,\dots,2m+1\}$ with $\sigma^{-1}(1) < \sigma^{-1}(2)$; let S_3 be the set of permutations with $\sigma^{-1}(1) < \sigma^{-1}(2) < \sigma^{-1}(3)$; let S_4^+ be the set of permutations with :

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \sigma^{-1}(3) < \sigma^{-1}(4);$$

and let S_4^- be the set of permutations with:

$$\sigma^{-1}(2) < \sigma^{-1}(1) < \sigma^{-1}(4) < \sigma^{-1}(3).$$

Let:

$$\Sigma_i = \sum_{\sigma\in S_i}{}' \varepsilon(\sigma)\, a_{\sigma(1)}\cdots a_{\sigma(2m+1)} \qquad ,i=2,3$$

and:

$$\Sigma_4 = \sum_{\sigma\in S_4^+}{}' \varepsilon(\sigma)\, a_{\sigma(1)}\cdots a_{\sigma(2m+1)} - \sum_{\sigma\in S_4^-}{}' \varepsilon(\sigma)\, a_{\sigma(1)}\cdots a_{\sigma(2m+1)}$$

Then, modulo elements from A_m (the m-th ideal of the commutator filtration of A defined in paragraph 2):

(i) $\qquad \Sigma_2 = m\, a_1a_2[a_3,\dots,a_{2m+1}]$

(ii) $\quad$ If $a_1=a_3$ then $\Sigma_3 = m a_1a_2[a_3,\dots,a_{2m+1}]$

(iii) If $a_1=a_3, a_2=a_4$ then $\sum_4 = m a_1 a_2 [a_3, a_4 \dots a_{2m+1}], m \geq 2$

Proof. The proof will be done by induction, simultaneously. We abreviate by (mod A_j), after an equality, the fact that this equality is only valid modulo terms from A_j. If $m=1$, then (i),(ii) are trivial, and there is no meaning for (iii). The general induction step shows that (iii), for $m=2$, follows from (i),(ii) in step $m=1$. Hence, we can start our induction. Assume (i),(ii),(respectively (iii)) are proved for $k=\overline{1,m-1}$, (respectively $k=\overline{2,m-1}$ for (iii)). As already mentioned before , we use the following conventions : if the integers $j_1, j_2, \dots, j_{2m+1}$ define a permutation of $1,2,\dots,2m+1$, then $\varepsilon(j_1,\dots,j_{2m+1})$ will be the sign of this permutation. The proof will be divided into three parts, corresponding to (i),(ii) and (iii).

Proof of (i). The terms appearing in $\sum_2$ can be grouped into :

(a). Terms starting with $a_r a_s$, where $r,s \notin \{1,2\}$. By the induction hypothesis for $\sum_2$, these terms may be grouped in order to give terms of the form:

$$(m-1)[a_r,a_s]a_1a_2[a_{j_1},\dots,a_{j_{2m-3}}]\varepsilon(r,s,1,2,j_1,\dots,j_{2m-3}) =$$

$$= (m-1)\varepsilon(r,s,1,2,j_1,\dots)a_1a_2[a_r,a_s][a_{j_1},\dots,a_{j_{2m-3}}] \pmod{A_m},$$

where $r<s$ and $j_1<\dots<j_{2m-3}$ are defined by $\{r,s,1,2,j_1,\dots,j_{2m-3}\} = \{1,2,\dots,2m+1\}$.
Performing the sum after all $r,s \notin \{1,2\}$, $r<s$, we obtain:

$$(m-1)a_1a_2[a_3,\dots,a_{2m+1}].$$

(b). Terms starting with a_1a_s or a_sa_1 with $s \neq 2$. These terms are directly groupped into:

$$[a_1,a_s][a_2,\dots,\hat{a}_s,\dots,a_{2m+1}] \in A_m$$

(c). Terms starting with a_1a_2. Obviously they are grouped into $a_1a_2[a_3,\ldots,a_{2m+1}]$.

Therefore, the sum of all terms in a,b,c is equal (mod A_m) to $ma_1a_2[a_3,\ldots,a_{2m+1}]$.

Proof of (ii). The terms may be grouped into:

(a`).Terms starting with a_1a_2.As in c their sum is:

$$a_1a_2[a_3,\ldots,a_{2m+1}].$$

(b`).Terms starting with a_ra_s or a_sa_r with $r,s\notin\{1,2\}$ (and a priori $r,s\neq 3$, since the associated permutations lie in S_3), $r<s$. By the (m-1) induction step for Σ_3, for fixed r<s, the sum of these terms is:

$$\varepsilon(r,s,1,2,3\ldots,\hat{r}\ldots,\hat{s}\ldots)(m-1)[a_r,a_s]a_1a_2[a_3,\ldots,\hat{a}_r,\ldots,\hat{a}_s,\ldots a_{2m+1}]$$

for all r<s.

Further, this last terms are equal (mod A_m):

$$\varepsilon(1,2,r,s,3,\ldots,\hat{r}\ldots,\hat{s}\ldots)(m-1)a_1a_2[a_r,a_s][a_3,\ldots,\hat{a}_r\ldots\hat{a}_s,\ldots a_{2m+1}]$$

(c`).Terms starting with a_1a_s or a_sa_1 with $s\neq 2$ (and therefore $s\neq 3$).By the induction hypothesis on Σ_2 these terms together give :

$$\varepsilon(1,s,2,3,4,\ldots,\hat{s},\ldots)(m-1)[a_1,a_s]a_2a_3[a_4,\ldots,\hat{a}_s\ldots]\ (\mathrm{mod}\ A_m)$$

Since $a_1=a_3$ and the permutation $\begin{pmatrix}1 & s & 2 & r\\ 1 & 2 & 3 & s\end{pmatrix}$ is even, for fixed s we obtain:

$$\varepsilon(1,2,3,s,4,\ldots,\hat{s},\ldots)(m-1)a_1a_2[a_3,a_s][a_4,\ldots\hat{a}_s,\ldots]$$

By reasoning as in Lemma 4.3, the sum of the terms in (b`),(c`) is (mod A_m):

$$(m-1)a_1a_2[a_3,\ldots,a_{2m+1}].$$

Adding the terms from a` we obtain (the coefficient) m.

Proof of (iii). The terms in $\sum_4$ are grouped into:

a``. Terms starting with a_1a_s or a_sa_1 with $s \notin \notin \{2,3,4\}$ (these are terms in S_4^+). We will say that a term is in $S_4^{\pm}$ if the associeted permutation is $S_4^{\pm}$. Since $a_3=a_1$, by the (m-1) induction step for $\sum_3$, for a fixed s, the sum of these terms is(mod A_m):

$$(m-1)\ \varepsilon(1,s,2,3,4,\ldots\hat{s},\ldots)[a_1,a_s]a_2a_3[a_4,\ldots,\hat{a}_s,\ldots]=$$

$$=(m-1)\varepsilon(1,2,3,s,4,\ldots\hat{s},\ldots)a_1a_2[a_3,a_s][a_4,\ldots\hat{a}_s,\ldots](\mathrm{mod}\ A_m).$$

b``. Terms starting with a_2a_r or a_ra_2 with $r \notin \notin \{1,2,3,4\}$. These terms are in S_4^-. By m-1 induction step for $\sum_3$ they can be grouped, modulo terms in A_m in :

$$-\ \varepsilon(2,r,1,4,3,5,\ldots\hat{r},\ldots)[a_2,a_r]a_1a_4[a_3,a_5,\ldots\hat{a}_r,\ldots,a_{2m+1}].$$

Since the signature of the permutation:

$$\begin{pmatrix} 2 & r & 1 & 4 & 3 & 5 & 6\ldots\hat{r}\ldots \\ 1 & 2 & 4 & r & 3 & 5 & 6\ldots\hat{r}\ldots \end{pmatrix}$$

is negative, and since $a_2=a_4$, modulo terms from A_m, we obtain:

$$(m-1)\ (1,2,4,r,3,5,\ldots,\hat{r},\ldots)a_1a_2[a_4,a_r][a_3,a_5,\ldots,\hat{a}_r\ldots]$$

c``. Terms starting with a_ra_s, or a_sa_r $r<s, r,s \notin \notin \{1,2,3,4\}$. By the induction step for $\sum_4$, it follows that (mod A_m) they are equal to :

$$\varepsilon(r,s,1,2,3,\ldots\hat{r},\ldots\hat{s},\ldots)(m-1)[a_r,a_s]a_1a_2[a_3,\ldots\hat{a}_r,\ldots,\hat{a}_s\ldots a_{2m+1}]=$$

$$\varepsilon(1,2,r,s,3,\ldots,\hat{r},\ldots\hat{s},\ldots)(m-1)a_1a_2[a_r,a_s][a_3,\ldots\hat{a}_r,\ldots,\hat{a}_s,\ldots,a_{2m+1}]$$

d_1. Terms starting with a_1a_2 in S_4^+;

d_2. Terms starting with a_2a_1 in S_4^-.

Summing all the terms of the type d_2 and using the identity $a_2a_1=[a_2,a_1]+a_1a_2$, we obtain:

$$-\sum_{\alpha}\varepsilon(2,1,\alpha)[a_2,a_1]\tilde{\alpha}+\sum_{\alpha}\varepsilon(1,2,\alpha)a_1a_2\tilde{\alpha} \qquad (6.1)$$

where α runs over all permutations of the set $\{3,4,\ldots,2m+1\}$, with the property that $\alpha^{-1}(4)<\alpha^{-1}(3)$, and for such an α, $\tilde{\alpha}$ is the product $a_{\alpha(1)}\cdots a_{\alpha(2m+1)}$.

By the (m-1) induction step for $\sum_2$, the first sum in (6.1) is equal (mod A_m) to:

$$-(m-1)\varepsilon(2,1,4,3,5,..2m+1)[a_2,a_1]a_4a_3[a_5,\ldots a_{2m+1}].$$

Since $\varepsilon(2,1,4,3,5,\ldots 2m+1)=1$ and since $a_1=a_3, a_2=a_4$ we obtain further :

$$(m-1)a_1a_2[a_3,a_4][a_5,\ldots,a_{2m+1}]$$

Reasoning as in Lemma 4.3` the sum of the terms in a``,b``,c`` with this last term is equal (mod A_m) with:

$$(m-1)a_1a_2[a_3,\ldots,a_{2m+1}]$$

The remaining terms from $\sum_4$ are those from the second sum in (6.1) and those of type d_1. By the definition of the antisymmetric commutator, the sum of all terms of this type is:

$$a_1a_2[a_3,\ldots,a_{2m+1}].$$

This ends the proof of the lemma.

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