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APPROXIMATION AND EXISTENCE OF PERIODIC SOLUTIONS FOR CONTROLLED DIFFUSION EQUATION

by

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§1. INTRODUCTION

The existence of periodic solution in probability for diffusion equations

$$dx=f(t,x)dt+\sum_{i=1}^{m} g_i(t,x)dw_i(t), \quad x\in R^n, \quad t\geq 0,$$

where $w(\cdot)$ is a standard m-dimensional Wiener process over the probability space $\{\Omega, \mathcal{F}, P; F_t\}$, and f, g_i are periodic with respect to the variable t, is obeying the same general scheme as it is done for the existence of stationary distributions; sufficient conditions of the stability type as in [1], [2] and [3] could be used to achieve the goal.

Generally, the existence of a moment function $g: R^n \to [0,\infty)$ (see [1]) such that $\sup_{t\geq 0} Eg(\mathcal{X}(t))\leq M<\infty$ ensures the existence of a periodic solution but a nondegeneracy condition on the diffusion part makes it difficult and requires a strong dissipativity property for the drift part $f(t,x)$. Under some nondegeneracy condition on the diffusion part we prove the existence of bounded periodic controls $u_i(t,x)$, $i=1,\ldots,m$, acting in the same directions as the Wiener process such that the following equation

$$(*) \quad dx=\left[f(t,x)+\sum_{i=1}^{m} u_i(t,x)g_i(t,x)\right]dt+\sum_{i=1}^{m} g_i(t,x)dw_i(t) \quad , \quad t\geq 0 \ ,$$

has a periodic solution in distribution.

The main tool is an approximation theorem for solutions in (*) which gives the advantage to check sufficient stability conditions on a more suitable differential equation.

Roughly speaking it can be stated as follows. We are given a finite set of smooth functions $g_i:[0,T]\times R^n \to R^n$, $i=1,\ldots,m$, and denote $\mathcal{L}(g_1,\ldots,g_m)$ the Lie algebra over reals generated by them, where $[g_i,g_j](t,x)=((\partial g_j/\partial x)g_i-(\partial g_i/\partial x)g_j)(t,x)$. Take $h_1,\ldots,h_l\in\mathcal{L}(g_1,\ldots,g_m)$ and denote $y(\cdot)$ the solution in

$$(**) \quad dy=\left[f(t,y)+\sum_{i=1}^{\ell} u_i(t,y)h_i(t,y)\right]dt+\sum_{k=1}^{d} \sigma_k(t,y)dw_k(t), \quad t\in[0,T],$$

$$y(0)=x_o\in L_2(\Omega,P) \ ,$$

where f, $\sigma_k:[0,T]\times R^n \to R^n$ are fixed.

Along with (**) we consider

$$(***) \quad dx=\left[f(t,x)+\sum_{i=1}^{m} v_i(t,x)g_i(t,x)\right]dt+\sum_{k=1}^{d} \sigma_k(t,x)dw_k(t) \ ,$$

$$x(0)=y(0), \qquad t\in[0,T],$$

where f, g_i , σ_k are the given functions.

The first two theorems we give here state the existence of a sequence of solutions in (***) which approximate the solution in (**) using the usual metric $d(x(\cdot),y(\cdot))=(E \max_{t\in[0,T]} |x(t)-y(t)|^2)^{1/2}$. This result is connected with the controllability properties of deterministic control systems as it appears in [4] and [5] but the technique used in [4] to treat the case $f(t,x)\neq 0$, cannot be useful here because of the perturbation generated by the diffusion part. The procedure we use here originate in [6] and in stochastic case it completes the results in [7].

A similar but somehow weaker result than that given here regarding the approximation will appear in [8]. The result in Theorems 1 and 2 remains the same in the case the Wiener process $w(\cdot)$ in (**) and (***) is replaced by a continuous square integrable martingale for which the quadratic variation matrix $V(t)=\langle M(t),M(t)\rangle$ has the form $V(t)=\int_0^t H(s,w)ds$ with H a bounded measurable matrix valued process. The existence of periodic solutions in distribution for (*) is stated in Theorem 3 and Theorem 4 contains the existence of periodic solutions for controlled diffusion equations in (***).

In Theorem 4 we disregard specific properties of the diffusion part in (***) and give the result for any σ_k fulfilling some linear growth condition.

2. FORMULATION OF THE APPROXIMATE PROBLEM AND MAIN RESULTS

Denote $C_b^{\ell,p}([0,T]xR^n)$ the space consisting of real functions which are continuously differentiable up to order ℓ with respect to $t\in[0,T]$, up to order p with respect to $x\in R^n$ and are bounded along with all their derivatives; if the boundedness condition is omitted we denote it by $C^{\ell,p}([0,T]xR^n)$. We are given f,g_i,σ_k : $[0,T]xR^n \longrightarrow R^n$ which are continuous and $g_i\in C^{1,\infty}([0,T]xR^n)$, $f,\sigma_k\in C^{0,2}([0,T]xR^n)$, $i=1,\dots,m$, $k=1,\dots,d$. For $i_o,i_1\in\{1,\dots,m\}$ and $I=\{i_o,i_1\}$ define $|I|=2$, $g_I(t,x)=[g_{i_o},g_{i_1}](t,x)$, where $[g_i,g_j](t,x)$ denote the Lie bracket with respect to $x\in R^n$; generally for i_o, $i_1,\dots,i_L\in\{1,\dots,m\}$ and $I=\{i_o, i_1,\dots,i_L\}$, define $|I|=L+1$ and $g_I(t,x)=[g_{i_o},g_{I_1}](t,x)$, where $I_1=\{i_1,\dots,i_L\}$, $L\in N$. For each

$u_i \in C_b^{0,1}([0,T]xR^n)$, $i=1,\dots,m$, $u_I \in C_b^{1,2}([0,T]xR^n)$, $2 \leq |I| \leq L+1$, we consider the following stochastic differential equation

$$dy=\Big[f(t,y)+\sum_{i=1}^{m} u_i(t,y)g_i(t,y)+\sum_{|I|=2}^{L+1} u_I(t,y)g_I(t,y)\Big]dt+\sum_{k=1}^{d}\sigma_k(t,y)dw_k(t) \tag{1}$$

$$y(0)=x_o\ ,\quad t\in[0,T],$$

where $w(t)$, $t\in[0,T]$, is a standard Wiener process over the filtered probability space $\{\Omega,\mathcal{F},P;\mathcal{F}_t\}$ and $x_o\in L_2(\Omega,P)$ is independent of $\mathcal{F}_t$, $t>0$. We associate with (1) the following stochastic differential equation

$$dx=\Big[f(t,x)+\sum_{i=1}^{m} u_i(t,x)g_i(t,x)+\sum_{i=1}^{m} v_i(t,x)g_i(t,x)\Big]dt+\sigma(t,x)dw(t) \tag{2}$$

$$x(0)=x_o\ ,\quad t\in[0,T],$$

where $\sigma=(\sigma_1,\dots,\sigma_d)$, f,g_i,σ_k,u_i are as in (1) and $v_i\in C_b^{1,1}([0,T]xR^n)$. Denote $G=(g_1,\dots,g_m)$. We need the following conditions to be fulfilled

C_1) $\partial f/\partial x_j$, $\partial\sigma_k/\partial x_j\in C_b^{0,1}([0,T]\times R^n)$, $G\in C_b^{1,L+2}([0,T]xR^n)$

C_2) $(\partial g_I/\partial x)f^*$, $(\partial g_I/\partial x)\sigma_k^*\in C_b([0,T]\times R^n)$, $(\partial^2 g_I/\partial x_i\partial x_j)A_{ij}\in C_b([0,T]xR^n)$

for any $1\leq|I|\leq L$, $k=1,\dots,d$, $i,j\in\{1,\dots,n\}$, where $A=\sigma\sigma^*$, "v^*" is the transposed of v, and a vector or a matrix belongs to $C_b^{\ell,p}$ if all their components fulfil it.

THEOREM 1. Assume that (C_1) and (C_2) are fulfilled for (1) and let $y(\cdot)$ be the solution in (1) corresponding to $u_i,u_I,2\leq|I|\leq L+1$, and $y(0)=x_o$. Then there exists a sequence $\{v_i^h\}_h\subseteq C_b^{1,1}$, $h>0$, such that the corresponding solutions x^h in (2) fulfil

$$E\max_{t\in[0,T]}|x^h(t)-y(t)|^2\leq Ch, \text{ uniformly with respect to}$$

x_o, u_i and u_I in bounded sets, where $C>0$ is a constant.

In addition if $u_I(0,y)=u_I(T,y)$, $|I|=2,\dots,L+1$, in (1), then $v_i^h(0,x)=v_i^h(T,x)$, $i=1,\dots,m$, in (2).//

If we relax the hypotheses in the Theorem by neglecting (C_2) and replacing $G\in C_b^{1,L+2}([0,T]xR^n)$ in (C_1) by $\partial g_j/\partial x_k \in C_b^{1,L+1}([0,T]xR^n)$, $j=1,\dots,m$, then using a standard argument of truncation it follows that there exists a sequence $\{x^h(\cdot)\}_{h>0}$ of solutions in (2) such that

$$\lim_{h\to 0} E \max_{t\in[0,T]} |x^h(t)-y(t)|^2=0$$

uniformly with respect to x_o in bounded sets.

It can be stated more precisely as follows.

We replace (C_1) and (C_2) by the following

$\tilde{C}_1$) $\partial f/\partial x_j$, $\partial \sigma_k/\partial x_j \in C_b^{0,1}$, $\partial g_i/\partial x_j \in C_b^{1,L+1}$

$\tilde{C}_2$) $u_i\in C^{0,1}$, $u_I\in C^{1,2}$ and $\partial(\sum_{|I|=1}^{L+1} u_I g_I)/\partial x_j \in C_b$ for $i=1,\dots,m$, $j=1,\dots,n$

THEOREM 2. Assume that $(\tilde{C}_1)$ and $(\tilde{C}_2)$ are fulfilled for (1) and let $y(\cdot)$ be the solution in (1) corresponding to u_i, u_I and x_o. Then there exist sequences $(u_i^h, v_i^h)_{h>0}$, $i=1,\dots,m$, $\{u_i^h\}\subseteq C_b^{0,1}$, $\{v_i^h\}\subseteq C_b^{1,1}$, such that the corresponding solution $x^h(\cdot)$ in (2) fulfil

$$\lim_{h\to 0} E \max_{t\in[0,T]} |x^h(t)-y(t)|^2=0,$$

uniformly with respect to x_o in bounded sets.

In addition, if $u_I(0,y)=u_I(T,y)$, $|I|=1,\dots,L+1$, in (1) then $u_i^h(0,x)=u_i^h(T,x)$ and $v_i^h(0,x)=v_i^h(T,x)$, $i=1,\dots,m$, in (2).

REMARK 1. Since v_i^h and u_i in (2) are continuous the theorems have an analogous version for the corresponding backward parabolic equations associated with (1) and (2).

§ 3. SOME AUXILIARY RESULTS AND PROOF OF THE THEOREMS

We associate with (1) the maximal number L of the Lie brackets contained in (1) and call it the order of the system. To prove Theorems we need to approximate the solution in (1) by one determined by a system which has an order less than L.

It is done in the next Lemma by using the following approximate equation.

Denote $\tilde{f}(t,y)=f(t,y)+\sum_{i=1}^{m} u_i(t,y)g_i(t,y)+\sum_{|I|=2}^{L} u_I(t,y)g_I(t,y)$

and (1) is rewritten as

$$dy=\left\{\tilde{f}(t,y)+\sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}} u_{ij}(t,y)\,[g_i,b_j](t,y)\right\}dt+\sigma(t,y)dw(t)\ , \qquad (3)$$

$y(0)=x_o$, $t\in[0,T]$, where the second term in the right hand side of (3) is the same with the third term in (1).

Let N be a natural number. We consider a partition π_o of $[0,T]$ determined by the intervals $[kh,(k+1)h)$, $k=0,1,\dots,N-1$, with $|\pi_o|=h=T/N$. For each $k\in\{0,1,\dots,N-1\}$, let A_{ij}^k , $i=1,\dots,m$, $j=1,\dots,\tilde{m}$ be a partition of $[kh,(k+1)h)$ with $|A_{ij}^k|=h_1=h/m\tilde{m}$. Denote P^o the space consisting of scalar polynomial functions defined on $[0,1]$ and fulfilling $\int_0^1 p(t)dt=0$. Let $p_1(\cdot)$, $p_2(\cdot)\in P^o$ be such that $p_i(0)=p_i(1)=0$, $dp_i/dt(0)=dp_i/dt(1)=0$ and

$\int_0^1 p_2(t)\tilde{p}_1(t)dt=1$, where $\tilde{p}(t)=\int_0^t p(s)ds$. These functions will be fixed in the sequel and they could be chosen as polynomials of sixth and fifth degree respectively. Let $p_i^k(t,h):[kh,(k+1)h]\to R$, $i=1,2$, be defined by

$$p_i^k(t,h)=p_i(t-kh/h_1) \quad t\in A_{11}^k=[kh,\ kh+h_1],\ldots,$$

$$p_i^k(t,h)=p_i(t-(kh+(m\tilde{m}-1)h_1)/h_1), \quad t\in A_{m\tilde{m}}^k=[kh,\ (k+1)h)$$

$k=0,1,\ldots,N-1.$

Obviously $p_i^k(\cdot)\in C^1([kh,(k+1)h];R)$ and $p_i^k(kh)=p_i^k((k+1)h)=0$. With (3) and partition π_o we associate the following differential equation of order L-1

$$dx=\Big\{\tilde{f}(t,x)+\sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}}\langle(\partial u_{ij}/\partial x),\ b_j\rangle(t,x)g_i(t,x)+ \tag{4}$$
$$+\sum_{k=0}^{N-1}\sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}}(m\tilde{m}/\sqrt{h_1})\big[p_1^k(t,h)(u_{ij}g_i)(t,x)+$$
$$+p_2^k(t,h)b_j(t,x)\big]C_{A_{ij}^k}(t)\Big\}dt+\sigma(t,x)dw(t), \quad x(0)=x_o\ , \quad t\in[0,T],$$

where $C_A(t)=0$, $t\notin A$, and $C_A(t)=1$, $t\in A$.
Let $x^h(t)$, $t\in[0,T]$, be the solution in (4) and denote

$$M^h(t)=\int_0^t\Big\{\sigma(s,x^h(s))-\sigma(s,y(s))+\sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}}\big[\tilde{p}_1(s,h)(\partial/\partial x)(u_{ij}g_i)\ (s,x^h(s)) \tag{5}$$
$$+\tilde{p}_2(s,h)(\partial b_j/\partial x)\sigma(s,x^h(s))\big]\Big\}dw(s), \quad t\in[0,T]$$

Let $\eta(r)$ be a random or a deterministic vector fulfilling $(E|\eta(r)|^2)^{1/2}\le Cr$ for some fixed constant $C>0$.

LEMMA 1. Assume that (C_1) and (C_2) are fulfilled. Let $x^h(\cdot)$ be the solution in (4) and $y(\cdot)$ fulfils (3). Then there exists a martingale $M^h(t)$, $t\in[0,T]$ (see (5)) such that

$$x^h(t'')-x^h(t')=y(t'')-y(t')+M^h(t'')-M^h(t')+(t''-t')\eta(\sqrt{h})$$

$$t',t''\in\{0,h,2h,\dots,(N-1)h=T\},\quad t'<t'',\quad \text{where}$$

$$(E|M^h(t'')-M^h(t')|^2)^{1/2}\leq\sqrt{(t''-t')}\;\eta(\sqrt{h})$$

uniformly with respect of x_o, $u_i(\cdot)$, $u_I(\cdot)$, $2\leq|I|\leq L+1$ in bounded sets.

<u>Proof</u>

Denote $f_1(t,x)=\tilde{f}(t,x)+\sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}}\langle\frac{\partial u_{ij}}{\partial x}, b_j\rangle(t,x)g_i(t,x)$

By definition

$$x^h(h_1)=x_o+\int_0^{h_1}f_1(t,x^h(t))dt+(m\tilde{m}/\sqrt{h_1})\int_0^{h_1}[p_1^o(t,h)u_{11}(t)g_1(t,x^h(t))+$$

$$+p_2^o(t,h)b_1(t,x^h(t)]dt+\int_0^{h_1}\sigma(t,x^h(t))dw(t)=x_o+T_1+T_2+T_3\;.$$

By hypothesis $\tilde{f}$ and σ are Lipschitz continuous with respect to $x\in R^n$ and computation shows

$$(E\max_{t\in[0,h]}|x^h(t)-y(t)|^2)^{1/2}\leq(1-Ch)^{-1}\eta(\sqrt{h})\tag{6}$$

where $C>0$ is the Lipschitz constant for $\tilde{f}$ and σ. Using (6) we get

$$T_1=\int_0^{h1}f_1(t,y(t))dt+\int_0^{h1}[f_1(t,x^h(t))-f_1(t,y(t))]dt=\tag{7}$$

$$=\int_0^{h1}f_1(t,y(t))dt+h_1\,\eta(\sqrt{h})$$

$$T_3 = \int_0^{h1} \sigma(t,y(t))dw(t) + \int_0^{h1} [\sigma(t,x^h(t)) - \sigma(t,y(t))]dw(t) = \tag{8}$$

$$= \int_0^{h1} \sigma(t,y(t))dw(t) + M_1'(h_1), \text{ where } E|M_1'(h_1)|^2 \leq h_1 \eta(h).$$

Denote $\tilde{x}(s) = x^h(sh_1)$, $s \in [0,1]$, $A = \sigma\sigma^*$,

$$(\mathcal{L}u)(t,x) = \left[(\partial/\partial t) + \sum_{i=1}^{n} \tilde{f}_i(t,x)(\partial/\partial x_i) + 1/2 \sum_{i,j=1}^{n} A_{ij}(t,x)(\partial^2/\partial x_i \partial x_j)\right] u(t,x)$$

With these notations and using $p_i(\cdot) \in P^o$, $i=1,2$ we get

$$T_2 = m\tilde{m}\sqrt{h_1} \int_0^1 [p_1(s)u_{11}g_1(sh_1,\tilde{x}(s)) + p_2(s)b_1(sh_1,\tilde{x}(s))\, ds = \tag{9}$$

$$= m\tilde{m}h_1^{3/2} \Big[\int_0^1 p_1(s)ds \int_0^s \mathcal{L}(u_{11}g_1)(s_1h_1,\tilde{x}(s_1))ds_1 +$$

$$+ \int_0^1 p_2(s)ds \int_0^s (\mathcal{L}b_1)(s_1h_1,\tilde{x}(s_1))ds_1\Big] +$$

$$+ m\tilde{m}h_1 \Big[\int_0^1 p_1(s)ds \int_0^s (p_1(s_1)(\partial/\partial x)(u_{11}g_1)(s_1h_1,\tilde{x}(s_1)) +$$

$$+ p_2(s_1)(\partial/\partial x)(u_{11}g_1)b_1(s_1h_1,\tilde{x}(s_1)))ds_1 +$$

$$+ \int_0^1 p_2(s)ds \int_0^s (p_1(s_1)(\partial b_1/\partial x)(u_{11}g_1)(s_1h_1,\tilde{x}(s_1)) +$$

$$+ p_2(s_1)(\partial b_1/\partial x)b_1(s_1h_1,\tilde{x}(s_1)))ds_1\Big] +$$

$$+ (m\tilde{m})/(\sqrt{h_1}) \int_0^{h1} [\tilde{p}_1^o(t,h)(\partial/\partial x)(u_{11}g_1)\sigma(t,x^h(t)) + \tilde{p}_2^o(t,h)(\partial b_1/\partial x)\sigma(t,x^h(t))]dw(t)$$

$$= T_2' + T_2'' + M_1''(h_1)$$

By hypothesis (see (C_2)) we have

$$|\mathcal{L}(u_{11}g_1)(t,x^h(t))| + |\mathcal{L}b_1(t,x^h(t))| \leq C_1, \quad t \in [0,T], \tag{10}$$

where $C_1 > 0$ is a constant which doesn't depend on h, and using (10) in (9) we obtain

$$T_2' = h_1\eta(\sqrt{h}), E|M_1''(h_1)|^2 \leq h_1\eta(h) \tag{11}$$

Since $\int_0^1 p_2(s)\tilde{p}_1(s)ds = \int_0^1 p_1(s)\tilde{\tilde{p}}_2(s)ds = 1$ and

$\int_0^1 p_i(s)(\tilde{p}_i(s))^j ds = 0$, $i,j=1,2$ we get

$$T_2'' = m\tilde{m}h_1\left[u_{11}g_1, b_1\right](0, x_o) + h_1\eta(\sqrt{h}) = \tag{12}$$

$$= \int_0^h \left[u_{11}g_1, b_1\right](t, y(t))dt + h_1\eta(\sqrt{h})$$

Using (11) and (12) in (9) it follows

$$T_2 = \int_0^h \left[u_{11}g_1, b_1\right](t, y(t))dt + h_1\eta(\sqrt{h}) + M_1''(h_1) \tag{13}$$

and from (7), (8) and (13) we get

$$x^h(h_1) = x_o + \int_0^{h1} f_1(t, y(t))dt + \int_0^h \left[u_{11}g_1, b_1\right](t, y(t))dt + \tag{14}$$

$$+ \int_0^{h1} \sigma(t, y(t))dw(t) + h_1\eta(\sqrt{h}) + M_1(h_1)$$

where

$$M_1(h_1) = M_1'(h_1) + M_1''(h_1) \quad \text{fulfils} \quad E\,|M_1(h)|^2 \leq h_1\eta(h)$$

On the next interval $[h_1, 2h_1]$ we repeat the computations for $[0, h_1]$. By definition

$$x^h(2h_1) = x^h(h_1) + \int_{h_1}^{2h1} f_1(t, x^h(t))dt + (m\tilde{m})/(\sqrt{h_1}) \int_{h_1}^{2h_1}\left[p_1^o(t,h)u_{12}g_1(t, x^h(t)) + \right. \tag{1}$$

$$\left. + p_2^o(t,h)b_2(t, x^h(t))\right)dt + \int_{h_1}^{2h_1} \sigma(t, x^h(t))dw(t) = x^h(h_1) + \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3$$

and we get easily

$$\tilde{T}_1 = \int_{h_1}^{2h_1} f_1(t, y(t))dt + \int_{h_1}^{2h_1}\left[f_1(t, x^h(t)) - f_1(t, y(t))\right]dt = \tag{16}$$

$$= \int_{h_1}^{2h_1} f_1(t, y(t))dt + h_1\eta(\sqrt{h})$$

$$\tilde{T}_3 = \int_{h_1}^{2h_1} \sigma(t,y(t))dw(t) + \int_{h_1}^{2h_1} [\sigma(t,x^h(t)) - \sigma(t,y(t))]\, dw(t) = \tag{17}$$

$$= \int_{h_1}^{2h_1} \sigma(t,y(t))dw(t) + M_2'(h_1)$$

where

$$E|M_2'(h_1)|^2 \le h_1\eta(h), \quad M_2'(h_1) = \int_{h_1}^{2h_1} [\sigma(t,x^h(t)) - \sigma(t,y(t))]\, dw(t)$$

Similarly, repeating the computations in (9)-(11) we get

$$\tilde{T}_2 = \tilde{T}_2' + \tilde{T}_2'' + M_2''(h_1) \tag{18}$$

where

$$\tilde{T}_2' = h_1\eta(h) \quad , \tag{19}$$

$$M_2''(h_1) = (m\tilde{m})/(\sqrt{h_1}) \int_{h_1}^{2h_1} [\tilde{p}_1^o(t,h)(\partial/\partial x)(u_{11}g_1)\sigma(t,x^h(t)) +$$

$$+ \tilde{p}_2^o(t,h)(\partial b_2/\partial x)\sigma(t,x^h(t))dw(t)$$

and

$$E|M_2''(h_1)|^2 \le h_1\eta(h) \; .$$

Also, we have

$$\tilde{T}_2'' = m\tilde{m}h_1[u_{12}g_1,\ b_2](h_1,x^h(h_1)) + h_1\eta(\sqrt{h}) = \tag{20}$$

$$= m\tilde{m}h_1[u_{12}g_1,\ b_2](0,\ x_o) + h_1\eta(\sqrt{h}) =$$

$$= \int_o^h [u_{12}g_1,\ b_2](t,y(t))dt + h_1\eta(\sqrt{h})$$

Denote $M_2(h_1) = M_2'(h_1) + M_2''(h_1)$ and using (16)-(20) in (15) we get

$$x^h(2h_1) = x_o + \int_{h_1}^{2h_1} f_1(t,y(t))dt + \sum_{j=1}^{2} \int_o^h [u_{1j}g_1,\ b_j](t,y(t))dt + \tag{21}$$

$$+ \int_o^{2h_1} \sigma(t,y(t))dw(t) + 2h_1\eta(\sqrt{h}) + M_1(h_1) + M_2(h_1)$$

where $M_1(h_1)$ is defined in (14), and

$$E|M_1(h_1)+M_2(h_1)|^2 = E|M_1(h_1)|^2 + E|M_2(h_1)|^2 \leq 2h_1\eta(h) \quad . \tag{22}$$

Finally, for $t=h$, we get $M_i(h_1)$, $i=1,\ldots,m\tilde{m}$, such that

$$x^h(h)=x^h(m\tilde{m}h_1)=x_o+\int_o^h f_1(t,y(t))dt+\sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}}\int_o^h [u_{ij}g_i,b_j](t,y(t))dt+ \tag{23}$$

$$+\int_o^h \sigma(t,y(t))dw(t)+h\eta(\sqrt{h})+M_1(h)=y(h)+h\eta(\sqrt{h})+M_1(h)$$

where

$$f_1(t,y)+\sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}}[u_{ij}g_i,b_j](t,y)= \tilde{f}(t,y)+\sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}}u_{ij}[g_i,b_j](t,y)$$

and

$$M_1(h)=\sum_{i=1}^{m\tilde{m}}M_i(h_1)=\int_o^h[\sigma(t,x^h(t))-\sigma(t,y(t))]dw(t)+ \tag{24}$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{\tilde{m}}(m\tilde{m}/\sqrt{h_1})\int_o^h[\tilde{p}_1^o(t,h)(\partial/\partial x)(u_{ij}g_i)\sigma(t,x^h)(t))+$$

$$+\tilde{p}_2^o(t,h)(\partial b_j/\partial x)\sigma(t,x^h(t))]dw(t)$$

and $E|M_1(h)|^2\leq h\eta(h)$.

Lemma was proved for $t''=h$, $t'=0$.

For the next interval $[h,2h]$ we have to repeat the computations done on $[0,h]$. Using (23) we get

$$(E\max_{t\in[h,2h]}|x^h(t)-y(t)|^2)^{1/2}\leq (E|x^h(h)-y(h)|^2)^{1/2}+\eta(\sqrt{h})(1-Ch)^{-1}\leq \tag{25}$$

$$\eta(\sqrt{h})(1+h+\sqrt{h})(1-Ch)^{-1}$$

where $C>0$ is the Lipschitz constant for $\tilde{f}$ and σ.

For $t=2h$ we get a similar representation as in $t=h$. Namely

$$x^h(2h)=x^h(h)+\int_h^{2h} f_1(t,y(t))dt+\sum_{i=1}^{m}\sum_{j=1}^{\tilde m}\int_h^{2h}[u_{ij}g_i,\ b_j](t,y(t))dt+ \tag{26}$$

$$+\int_h^{2h}\sigma(t,y(t))dw(t)+h\eta(\sqrt{h})+M_2(h)=y(2h)+2h\eta(\sqrt{h})+M_1(h)+M_2(h)$$

where $M_1(h)$ is defined in (24) and

$$M_2(h)=\int_h^{2h}[\sigma(t,x^h(t))-\sigma(t,y(t))]dw(t)+ \tag{27}$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{m}(m\tilde m)/(\sqrt{h_1})\int_h^{2h}[\tilde p_1^1(t,h)(\partial/\partial x)(u_{ij}g_j)\sigma(t,x^h(t))+$$

$$+\tilde p_2^1(t,h)(\partial b_j\ /\partial x)\sigma(t,x^h(t))]dw(t)$$

fulfil

$$E|M_2(h)|^2=h\eta(h),E|M_1(h)+M_2(h)|^2=E|M_1(h)|^2+E|M_2(h)|^2=2h\eta(h) \tag{28}$$

By using an induction argument we get (see (25), (26))

$$(E\max_{t\in[kh,(k+1)h]}|x^h(t)-y(t)|^2)^{1/2}\leq\eta(\sqrt{h})(1+kh+\sqrt{kh})(1-Ch)^{-1}=\eta(\sqrt{h}) \tag{2}$$

and $M_1(h),\dots,M_k(h)$ such that

$$x^h(kh)=y(kh)+kh\eta(\sqrt{h})+\sum_{i=1}^{k}M_i(h)=y(kh)+kh\eta(\sqrt{h})+M^h(kh),\quad k=0,1,\dots,N-1\ , \tag{30}$$

where $M^h(t)$ is defined by

$$M^h(t)=\int_0^t[\sigma(s,x^h(s))-\sigma(s,y(s))]dw(s)+\sum_{i=1}^{m}\sum_{j=1}^{\tilde m}(m\tilde m)/(\sqrt{h_1})\cdot \tag{31}$$

$$\int_0^t[\tilde p_1(s,h)(\partial/\partial x)(u_{ij}g_j)\sigma(s,x^h(s))+\tilde p_2(s,h)(\partial b_j\ /\partial x)\sigma(s,x^h(s))]dw(s))$$

and fulfil

$$E\left|M^h(k''h)-M^h(k'h)\right|^2=E\left|\sum_{i=k'}^{k''} M_i(h)\right|^2=\sum_{i=k'}^{k''} E\left|M_i(h)\right|^2= \tag{31'}$$

$$=(k''-k')h\eta(h),\ k'<k'',\ k',\ k''\in\{0,1,\dots,N\}.$$

From (30)-(31') we get the conclusion. The proof is complete.

The approximation equation (4) has some coefficients $u_I^h(t,x)$ depending on h being unbounded with respcet to h. These functions $u_I^h(t,x)$ are of class C^1 and with respect to h they fulfil the following condition

$$hu_I^h(t,x)=\eta(\sqrt{h})\,u_I^h(t,x),\ h^2(\partial u_I^h/\partial t)(t,x)=\eta(\sqrt{h})\,\bar{v}_I^h(t,x)$$

where u_I^h, $\bar{v}_I^h$ are uniformly bounded with respect to h.

These properties are essential in order to reduce the order of a system which has unbounded coefficients with respect to the parameter h. Now we consider the following stochastic equation

$$\text{S)}\qquad dy=\left[f(t,y)+\sum_{i=1}^{m} u_i^h(t,y)g_i(t,y)+\sum_{|I|=2}^{L+1} u_I^h(t,y)g_I(t,y)\right]dt+\sigma(t,y)dw(t)$$

$$y(0)=x_o\ ,\ t\in[0,T],$$

where x_o, f, g_i, are as in (1) and $u_i^h\in C_b^{0,1}$, $u_I^h\in C_b^{1,2}$. With respect to the parameter h we assume that there exist $r(h)>o\,(r(h)=\eta(h))$ and a partition π_r of $[0,T]$ with intervals of the length r such that

a) $\quad ru_I^h(t,x)=\eta(\sqrt{h})\,v_I^h(t,x),\ 1\leq|I|\leq L+1,$

b) $\quad r^2(\partial u_I^h/\partial t)(t,x)=\eta(\sqrt{h})\,\bar{v}_I^h(t,x),\ 2\leq|I|\leq L+1$ where $v_I^h(\cdot)$, $\bar{v}_I^h(\cdot)$ are uniformly bounded with respect to h.

DEFINITION. A system S of order L for which there exists $r(h)>0$ such that $u_I(\cdot)$, $1\leq|I|\leq L+1$, fulfil (a) and (b) is called of index (L,r).

LEMMA 2. Assume (C_1) and (C_2) fulfilled. Let (S) be a system of index (L,r), where $r=\eta(h)$. Then there exists a system (S_1) of index $(L-1,\ r_1)$ with $r_1=T/MK^4$, $M=\text{card}\{I:|I|=L+1\}$, $K=T/r$, such that the corresponding solutions $y(\cdot)$ in (S) and $y_1(\cdot)$ in (S_1) with $y(0)=y_1(0)=x_o$, fulfil

*) $$y_1(t'')-y_1(t')=y(t'')-y(t')+(t''-t')\eta(\sqrt{h})+M^h(t'')-M^h(t')$$

for $t'<t''$, $t',t''\in\{0,\tilde r_1,2\tilde r_1,\ldots,K^3\tilde r_1,\ldots,2K^3\tilde r_1,\ldots,K^4\tilde r_1=T\}$,

$\tilde r_1=T/K^4=Mr_1$

**) $M^h(t)$ is a martingale and $E|M^h(t'')-M^h(t')|^2=(t''-t')\eta(h)$

***) coefficients $u_I(\cdot)$ in (S_1) fulfil (a) and (b) with r replaced by r_1.

The proof of Lemma 2 repeats the same general scheme as in Lemma 1 and it is omitted.

Proof of the Theorem 1

Suppose L=1. By hypothesis the conditions in Lemma 1 are fulfilled and denote $x^h(\cdot)$ the solution in the approximate system (4). By definition we have

$$x^h(t)-y(t)=\int_0^t[\bar f(x,x^h(s))-\bar f(s,y(s))]ds+\int_0^t[\sigma(s,x^h(s))-\sigma(s,x^h(s))]\cdot dw(s)+$$

$$+\sum_{i=1}^m\int_0^t(v_i^hg_i)(s,x^h(s))ds-\sum_{|I|=2}\int_0^t(u_Ig_I)(s,y(s))ds$$

where $\bar{f}(t,x)=f(t,x)+\sum_{i=1}^{m} u_i(t,x)g_i(t,x)$, and

$$u_t \overset{\Delta}{=} \sum_{i=1}^{m}\int_0^t (v_i^h g_i)(s,x^h(s))ds=\sum_{i=1}^{m}\int_0^{m_1\tilde{r}_1}(v_i^h g_i)(s,x^h(s))ds+\sum_{i=1}^{m}\int_{m_1\tilde{r}_1}^{t}(v_i^h g_i)(s,x^h(s))ds$$

where $m_1\tilde{r}_1$ is the nearest node to t which is smaller than t in the partition π_0.

Using Lemma 1 we get

$$u_t=x^h(m_1\tilde{r}_1)-x_0-\int_0^{m_1\tilde{r}_1}\bar{f}(s,x^h(s))ds-\int_0^{m_1\tilde{r}_1}\sigma(s,x^h(s))dw(s)+$$

$$+\eta(\sqrt{h})=y(m_1\tilde{r}_1)-x_0+m_1\tilde{r}_1\eta(\sqrt{h})-\int_0^{m_1\tilde{r}_1}\bar{f}(s,x^h(s))ds-$$

$$-\int_0^{m_1\tilde{r}_1}\sigma(s,x^h(s))dw(s)+\eta(\sqrt{h})=\int_0^{m_1\tilde{r}_1}\left[\bar{f}(s,y(s))-\bar{f}(s,x^h(s))\right]ds+$$

$$+\int_0^{m_1\tilde{r}_1}\left[\sigma(s,y(s))-\ (s,x^h(s))\right]dw(s)+\sum_{|I|=2}\int_0^t(u_I g_I(s,y(s))ds+4\eta(\sqrt{h})$$

Since

$$\left|\int_0^{m_1\tilde{r}_1}\left[\sigma(s,y(s))-\sigma(s,x^h(s))\right]dw(s)\right|^2\leq \max_{\tau\leq t}\left|\int_0^{\tau}\left[\sigma(s,y(s))-\sigma(s,x^h(s))\right]dw(s)\right|^2$$

it follows

$$E\max_{t\leq v}|x^h(t)-y(t)|^2\leq C\int_0^v E(\max_{s\leq t}|x^h(s)-y(s)|^2)dt+4\eta(\sqrt{h})$$

and $E\max_{t\in[0,T]}|x^h(t)-y(t)|^2\leq C_1 h$ for some constant $C_1>0$.

Generally, for $L\geq 1$, applying Lemma 1 to (1) we get a system (S) of index $(L-1,\ h/M)$, where $h=T/N$, $M=\text{card}\{I:|I|=L+1\}$.

In order to finish the proof we need to know that the previous estimate holds true in the case (1) is replaced by (S). It is enough to consider only the case when the order of the system

(S) is equal to one.

By hypothesis (S) fulfils the conditions in Lemma 2, and applying Lemma 2 we get $y_1^h(\cdot)$ the solution of a system of order zero such that

$$y_1^h(t)-y^h(t)=\int_0^t\left[f^h(s,y_1^h(s))-f^h(s,y^h(s))\right]ds+\int_0^t\left[\sigma(s,y_1^h(s))-\sigma(s,y^h(s))\right]dw(s)+$$

$$+\sum_{i=1}^m \int_0^t (v_i^h g_i)(s,y_1^h(s))ds-\sum_{|I|=2}\int_0^t (u_I^h g_1)(s,y^h(s))ds$$

where $y^h(\cdot)$ is the solution in (S). In addition

$$U_t \triangleq \sum_{i=1}^m \int_0^t (v_i^h g_i)(s,y_1^h(s))ds=\sum_{i=1}^m \int_0^{m_1\tilde{r}_1} (v_i^h g_i)(s,y_1^h(s))ds+$$

$+\sum_{i=1}^m \int_{m_1\tilde{r}_1}^t (v_i^h g_i)(s,y_1^h(s))ds$, where $m_1\tilde{r}_1\leq t$ is as before, fulfils

$$U_t=y_1^h(m_1\tilde{r}_1)-x_o+m_1\tilde{r}_1\,\eta(\sqrt{h})-\int_0^{m_1\tilde{r}_1} f^h(s,y_1^h(s))ds-$$

$$-\int_0^{m_1\tilde{r}_1}\sigma(s,y_1^h(s))dw(s))+\eta(\sqrt{h})=\int_0^{m_1\tilde{r}_1}\left[f^h(s,y^h(s))-f^h(s,y_1^h(s))\right]ds+$$

$$+\int_0^{m_1\tilde{r}_1}\left[\sigma(s,y^h(s))-\sigma(s,y_1^h(s))\right]dw(s)+\sum_{|I|=2}\int_0^t (u_I^h g_I)(s,y^h(s))ds+4\eta(\sqrt{h})\ .$$

It follows

$$|y_1^h(t)-y^h(t)|^2\leq C\Big\{\Big|\int_{m_1\tilde{r}_1}^t\left[f^h(s,y_1^h(s))-f^h(s,y^h(s))\right]ds\Big|^2$$

$$+\max_{\tau\leq t}\Big|\int_0^\tau\left[\sigma(s,y_1^h(s))-\sigma(s,y^h(s))\right]dw(s)\Big|^2+\eta(h)\Big\}$$

for some constant $C>0$.

Since $t\in\left[m_1\tilde{r}_1,(m_1+1)r_1\right)$ and $\tilde{r}_1|f^h(s,y_1)-f^h(s,y)|\leq$ $\eta(\sqrt{h})|y_1-y|$ for any $s\in[0,T]$, $y_1,y\in R^n$, we obtain

$$E \max_{t\in[0,T]} \left|y_1^h(t)-y^h(t)\right|^2 \leq C_2 h, \text{ for some constant } C_2>0 \text{ and}$$

the proof is complete.

Proof of the Theorem 2

By hypothesis there is a unique strong solution $y(\cdot)$ in (1). Multiply $u_I g_I$ in (1) by a $C^\infty(R^n)$ function $(\rho_N)^{|I|+1}$, $|I|=1,\ldots,L+1$, where $\rho_N : R^n \to [0,1]$ has its compact support in the ball $B_{2N}=\{x\in R^n;\ |x|\leq 2N\}$ and $\rho_N(x)=1$, for $x\in B_N$. Denote $\tilde{u}_I=\rho_N u_I$, $\tilde{g}_i=\rho_N g_i$, $i=1,\ldots,m$. Denote by (1_N) the modified equation and y_N the corresponding solution in (1_N). Each $(\rho_N)^{|I|+1} u_I g_I$ can be rewritten in the form $\tilde{u}_I\tilde{g}_I$ plus some terms containing Lie brackets of lower degree than $|I|$ multiplied by C_b^∞ functions, where $\tilde{g}_I$ is defined as g_I using $\tilde{g}_i$, $i=1,\ldots,m$. Therefore the equation (1_N) can be written in a form satisfying the conditions (C_1) and (C_2) in Theorem 1, and for each N fixed we get $\{x_N^h\}$ solutions in (2_N) such that $E \max\limits_{t\in[0,T]} |x_N^h(t)-y_N(t)|^2 \leq C_N h$

On the other hand the coefficients in (1) and (1_N) agree on $[0,T]\times B_N$ and they are global Lipschitz with respect to $x\in R^n$ which gives by a direct computation that $\lim\limits_{N\to\infty} E \max\limits_{t\in[0,T]} |y(t)-y_N(t)|^2=0$

uniformly with respect to x_0 in bounded sets.

This allow one to substract a subsequence in $\{x_N^h\}_{h,N}$ also denoted by $\{x^h\}_{h\downarrow 0}$ such that $\lim\limits_{h\downarrow 0} E \max\limits_{t\in[0,T]} |x^h(t)-y(t)|^2=0$

which complete the proof.

§4. THE EXISTENCE OF PERIODIC SOLUTIONS

The controlled diffusion equation we are adressing here is defined in (*) (see §1) and we assume that f, g_i have a common period T with respect to the variable t. We are looking for bounded and periodic controls $u_i(t+T,x)=u_i(t,x)$, $t\geq 0$, $x\in R^n$, such that (*) has a periodic solution in distribution which require the existence of $x_o \in L_2(\Omega,P)$ independent of $\mathcal{F}_t$, $t>0$, such that the measure $\mu(t)$ on R^n generated by the solution $x(t)$ in (*) fulfil $\mu(kT)=\mu(0)$, $k=1,2,\ldots$. Apart from the conditions $(\tilde{C}_1)$ and $(\tilde{C}_2)$ in Theorem 2 we need to assume the following

C_3) $f\in C^{1,2}$, $g_i \in C^{1,\infty}$ and $\partial f/\partial x_j \in C_b^{1,1}$, $\partial g_i/\partial x_j \in C_b^{1,\infty}$

C_4) there exist $h_1,\ldots,h_n \in \mathcal{L}(g_1,\ldots,g_m)$ and $K>0$ such that

$$\lambda^* H(t,x)H^*(t,x)\lambda \geq K|\lambda|^2 \qquad (\forall)\ \lambda\in R^n,\ t\in[0,T],\ x\in R^n,$$

where $H(t,x)=(h_1(t,x)\ldots h_n(t,x))$.

THEOREM 3. Assume that (C_3) and (C_4) are fulfilled for f, g_i in (*). Then there exist $u_i \in C_b^{0,1}$ and periodic $(u_i(t+T,x)=u_i(t,x))$ such that (*) has a periodic solution in distribution with the period T.

Proof

The controlled equation in (*) is a particular case of the equation (2) where $\sigma_i = g_i$, $d=m$, and the coefficients are periodic with respect to the variable t. Using (C_4) we associate to (*) the corresponding enlarged system (1) including the given

$h_1,\dots,h_n \in \mathcal{L}(g_1,\dots,g_m)$, among the vector fields g_I.

By hypotheses g_i fulfil a linear growth condition and as a consequence we have

$$|\text{trace } A(t,y)| \leq C(1+|y|^2), \quad \text{where } A=GG^*,\ G=(g_1,\dots,g_m)$$

We look for u_I, $|I|=1,\dots,L+1$, such that $u_I(t+T,y)=u_I(t,y)$, and

$$\sum_{|I|=1}^{L+1} (u_I g_I)(t,y)+f(t,y)=-Ky, \quad (\forall)\ t\geq 0,\ y\in R^n$$

where $K>C$, K a constant.

By definition $\partial(\sum u_I g_I)/\partial x \in C_b$ and the conditions in Theorem 2 are fulfilled.

Denote $\varphi(t)=E|y(t)|^2$, where y is the corresponding solution in (1). Using Ito's differential rule we get $\frac{d\varphi}{dt}\leq -K\varphi(t)+C$ and it follows that an $\varepsilon>0$ will exist such that for any $\varphi_o=E|x_o|^2\leq r_o$, with $-K(r_o/2)+C<0$, $\varphi(t)\leq r_o-\varepsilon$, for $t\geq T$.

Using Theorem 2 we approximate the solution y on $[0,T]$ by the solutions $x^h(\cdot)$ in (2) and it gives that for h sufficiently small there exist $u_i^h\in C_b^{0,1}$ such that $u_i^h(0,x)=u_i^h(T,x)$ and

$$E \max_{t\in[0,T]} |x^h(t)-y(t)|^2\leq \varepsilon/2, \quad E|x^h(T)|^2\leq r_o$$

uniformly with respect to the initial condition $y(0)=x^h(0)$ fulfilling $\varphi_o\leq r_o$. Using $x^h(T)$ as the new initial condition for (1) on the interval $[T, 2T]$, we repeat the above approximation by using the same controls u_i^h determined on $[0,T]$ and it follows $E|x^h(2T)|^2\leq r_o$, uniformly with respect to $x(0)=x_o$, $E|x_o|^2\leq r_o$, and

$$u_i^h(t+T,x)=u_i^h(t,x),\ t\in[0,T],\ x\in R^n.$$

Finally we get $u_i^h:[0,\infty)xR^n \to R$, bounded and verifying

$$u_i^h(t+T,x)=u_i^h(t,x),\quad (\forall)\ t\geq 0,\ x\in R^n,$$

such that the corresponding solutions in (2) fulfil

$$E|x^h(kT)|^2\leq r_o,\quad (\forall)\ k=0,1,2,\ldots$$

if the initial condition is taken such that $x_o\in R^n$, $|x_o|^2\leq r_o$.

It shows that the sequence of the probability measures

$\mu_N^h=\frac{1}{N}\sum_{k=1}^{N} P^h(kT,0,x_o)$, $N\geq 1$, is weakly compact, where $P^h(kT,0,x_o)(A)=$ $=P\{x^h(kT)\in A/x^h(0)=x_o\}$; it is equivalent with the existence of the periodic solution for (*) (see [2]) and the proof is complete.

REMARK 2. Assuming that the Wiener process $w(\cdot)$ in (*), (**) and (***) is replaced by a continuous square integrable martingale for which the quadratic variation matrix $V(t)=\langle M(t),M(t)\rangle$ has the form $V(t)=\int_o^t H(s,w)ds$, with H a bounded measurable matrix valued process, then the result in Theorem 3 is not anymore true but we get that there exist $u_i\in C_b^{0,1}$ and periodic such that (*) has a bounded solution in the mean square, i.e. $E|x(t)|^2\leq M$, $(\forall)\ t\geq 0$, for each $x_o\in L_2(\Omega,P)$.

In the following we need to combine $(\tilde{C}_1)$ in Theorem 2 with (C_4) in Theorem 3.

THEOREM 4. Assume that $(\tilde{C}_1)$ and (C_4) are fulfilled for f,g_i,σ_k in (***). Then there exist $v_i\in C_b^{o,1}$ and periodic

$(v_i(t+T,x)=v_i(t,x))$ which do not depend on σ_k and such that (***) has a periodic solution in distribution with the fixed period T.

The proof of this theorem repeats the same arguments as in Theorem 3 and it is omitted.

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