CANONICAL FORMS OF UNBOUNDED UNITARY
OPERATORS IN KREIN SPACES

by

Aurelian GHEONDEA

Mea 23785

CANONICAL FORMS OF UNBOUNDED UNITARY OPERATORS IN KREIN SPACES

by

Aurelian GHEONDEA*)

May 1987

*) Department of Mathematics, The National Institute for Scientifical and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.

CANONICAL FORMS OF UNBOUNDED

UNITARY OPERATORS IN KREIN SPACES

by Aurelian GHEONDEA

§ 0. INTRODUCTION

Unbounded unitary operators in Krein spaces have been con-
sidered in [13] in connection with maximal extensions of iso-
metric operators and in [14] in connection with unbounded
selfpolar norms. (however, the definition in [4] of a unitary
operator is too large for our setting). On the other hand,
bounded unitary operators in Krein spaces have certain canonical
forms with respect to fundamental decompositions (cf. [8] ,
[9] , [1] , [2]). The aim of this paper it to show that
these kind of canonical forms can be carried over, with appro-
priate modifications, to this general setting (see Theorem 5.5
and Corollary 5.6). Let us briefly present our approach.

First, by reformulating the Cartan decompositions of a
unitary operator [14] one can reduce the problem to find ca-
nonical forms of unbounded positive symmetries. So, a study of
these unbounded symmetries is needed. Further, from these we
reach unbounded selfadjoint projections in a Krein space (these
operators were formally introduced in [14]).

This article is divided in five sections. Apart from re-
sults concerning the geometry of selfadjoint projections, posi-
tive symmetries and unitary operators, which we need in order
to solve our problem, we have considered in the third section a
characterization of the non-degeneracy of the closed linear
span of a non-decreasing sequence of non-degenerate subspaces

of a Krein space in terms of the convergence of the correspon-
ding sequence of selfadjoint projections. Also, in the first sec-
tion there is a specification of notation and terminology of Krein
spaces used in this paper.

§1. NOTATION AND TERMINOLOGY

Let $\mathcal{K}$ be a Krein space and denote by $[.,.]$ the inner
product of $\mathcal{K}$. If $\perp$ stands for the orthogonality with res-
pect to this inner product then a fundamental decomposition (in
brief f.d.) of $\mathcal{K}$ is a decomposition

$$\mathcal{K} = \mathcal{K}^+ + \mathcal{K}^-$$

where $\mathcal{K}^+$, $\mathcal{K}^-$ are linear submanifolds of $\mathcal{K}$ such that
$\mathcal{K}^+ \perp \mathcal{K}^-$ and $(\mathcal{K}^+, [.,.])$, $(\mathcal{K}^-, -[.,.])$ are Hilbert
spaces. Usually we write

$$\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$$

in order to mark the orthogonality of the components.
Also one denotes $\varkappa^+(\mathcal{K}) = \dim \mathcal{K}^+$ and $\varkappa^-(\mathcal{K}) = \dim \mathcal{K}^-$.
The cardinal number $\varkappa(\mathcal{K}) = \min(\varkappa^-(\mathcal{K}), \varkappa^+(\mathcal{K}))$ is
called the rank of indefiniteness of the Krein space .

A fundamental symmetry (in brief f.s.) of $\mathcal{K}$ is a linear
operator J on $\mathcal{K}$ such that $J^2 = I$ and the identity

$$(x,y)_J = [Jx,y] , \qquad x,y \in \mathcal{K}$$

defines a positive definite inner product on $\mathcal{K}$ such that
$(\mathcal{K}, (.,.)_J)$ is a Hilbert space. The corresponding norm is cal-
led a unitary norm. Any two unitary norms on $\mathcal{K}$ are equivalent.

If we denote by $\mathcal{L}(\mathcal{K})$ the algebra of bounded linear
operators in $\mathcal{K}$ then uniform, strong operator and weak operator
topologies have the usual meaning, with respect to an arbitrary
unitary norm on $\mathcal{K}$.

A subspace of $\mathcal{K}$ is a closed linear submanifold of $\mathcal{K}$.
A subspace $\mathcal{L}$ of $\mathcal{K}$ is non-negative (positive) if $[x,x] \geqslant 0$, $x \in \mathcal{L}$.

$([x,x] > 0, \quad x \in \mathcal{L} \setminus \{0\})$. $\mathcal{L}$ is uniformly positive if for some (equivalently for any) unitary norm $\|\cdot\|$ there exists $\alpha > 0$ such that

$$[x,x] \geqslant \alpha \|x\|^2, \qquad x \in \mathcal{L}.$$

Let $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$ be a f.d. of $\mathcal{K}$. Then the associated f.s. is $J = J^+ - J^-$ where $J^{\pm}$ is the projection of $\mathcal{K}$ onto $\mathcal{K}^{\pm}$ along $\mathcal{K}^{\mp}$. Also let $\|\cdot\|$ denote the corresponding unitary norm. If $\mathcal{L}$ is a non-negative subspace of $\mathcal{K}$ then $\mathcal{L}_+ = J^+\mathcal{L}$ is a subspace of $\mathcal{K}^+$, the linear mapping

$$K : J^+x \longmapsto J^-x, \qquad x \in \mathcal{L}$$

is well defined, K is a contraction, i.e.

$$\|Kx\| \leqslant \|x\|, \qquad x \in \mathcal{L}_+,$$

and $\mathcal{L}$ is the graph of K

$$\mathcal{L} = G(K) = \{x+Kx \mid \qquad x \in \mathcal{L}_+\}.$$

K is called the angular operator of $\mathcal{L}$ with respect to the f.d. $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$. Moreover, $\mathcal{L}$ is a positive subspace if and only if K is a strict contraction, i.e.

$$\|Kx\| < \|x\|, \qquad x \in \mathcal{L}_+ \setminus \{0\}.$$

$\mathcal{L}$ is a uniformly positive subspace if and only if K is a uniform contraction i.e. $\|K\| < 1$. $\mathcal{L}$ is a maximal non-negative subspace if and only if $J^+\mathcal{L} = \mathcal{K}^+$.

If $\mathcal{M}$ is an arbitrary subspace of $\mathcal{K}$ then $\mathcal{M}^{\perp} = \{x \in \mathcal{K} \mid x \perp \mathcal{M}\}$ denotes its orthogonal companion and $\mathcal{M}^{\circ} = \mathcal{M} \cap \mathcal{M}^{\perp}$ its isotropic subspace. $\mathcal{M}$ is non-degenerate if $\mathcal{M}^{\circ} = \{0\}$. $\mathcal{M}$ is regular if $\mathcal{K} = \mathcal{M} + \mathcal{M}^{\perp}$.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Krein spaces. If T is a densely defined operator $\mathcal{D}(T) \subseteq \mathcal{K}_1$ and $\mathcal{R}(T) \subseteq \mathcal{K}_2$ then we let T^* denote its adjoint

$$[Tx,y] = [x,T^*y], \qquad x \in \mathcal{D}(T), \quad y \in \mathcal{D}(T^*).$$

If J_i is a f.s. on $\mathcal{K}_i$, i=1,2, then T^* denotes the (J_1, J_2)-adjoint operator of T, i.e.

$$(Tx,y)_{J_2} = (x,T^*y)_{J_1}, \quad x \in \mathcal{D}(T), \quad y \in \mathcal{D}(T^*).$$

Then

$$T^{\#} = J_1 T^* J_2$$

also holds. As a rule, positive operators, selfadjoint operators etc. on a

Krein space $\mathcal{K}$ are understood with respect to the indefinite inner product of $\mathcal{K}$. If J is a f.s. on $\mathcal{K}$ then J-positive operators, J-selfadjoint operators etc. are referring to the positive definite inner product $(.,.)_J$.

§2. SELFADJOINT PROJECTIONS

Let $\mathcal{K}$ be a Krein space. A linear operator P in $\mathcal{K}$ is called a selfadjoint projection if it is selfadjoint and idempotent, i.e.

$$P^{\#} = P = P^2 .$$

Observe that with this definition a selfadjoint projection can be unbounded (the meaning of the equality $P^2 = P$ is as follows: $\mathcal{R}(P) \subseteq \mathcal{D}(P)$ and $P^2 x = Px$ for all $x \in \mathcal{D}(P)$).

2.1. PROPOSITION. <u>A subspace $\mathcal{L}$ of the Krein space $\mathcal{K}$ is the range of a selfadjoint projection if and only if $\mathcal{L}$ is a non-degenerate subspace.</u>

<u>Proof.</u> Let $\mathcal{L}$ be a non-degenerate subspace of $\mathcal{K}$. Then $\mathcal{L} + \mathcal{L}^{\perp}$ is a dense linear manifold in $\mathcal{K}$. We define a linear operator P in $\mathcal{K}$ as follows: $\mathcal{D}(P) = \mathcal{L} + \mathcal{L}^{\perp}$ and

$$P(x_1 + x_2) = x_1, \qquad x_1 \in \mathcal{L} , \quad x_2 \in \mathcal{L}^{\perp}.$$

Then P is correctly defined, $\mathcal{R}(P) = \mathcal{L}$ and $P^2 = P$. Observing that

$$[P(x_1 + x_2), y_1 + y_2] = [x_1, y_1 + y_2] = [x_1, y_1] = [x_1 + x_2, y_1] = [(x_1 + x_2), P(y_1 + y_2)] ,$$

$$x_1, y_1 \in \mathcal{L} , \quad x_2, y_2 \in \mathcal{L}^{\perp}$$

it follows $P \subseteq P^{\#}$. In order to prove the converse inclusion let y be a vector in $\mathcal{D}(P^{\#})$. Then

$$[Px, y] = [x, P^{\#}y] , \qquad x \in \mathcal{D}(P)$$

Taking $x \in \mathcal{L}$ it follows $y - P^{\#}y \in \mathcal{L}^{\perp}$ while letting $x \in \mathcal{L}^{\perp}$ we get $P^{\#}y \in \mathcal{L}$. Therefore

$$y = (y - P^{\#}y) + P^{\#}y \in \mathcal{L}^{\perp} + \mathcal{L} = \mathcal{D}(P).$$

Conversly, let P be a selfadjoint projection in $\mathcal{K}$. Then I-P is also a selfadjoint projection and $\mathcal{R}(P) = \ker(I-P)$, in particular $\mathcal{R}(P)$ is closed. In order to prove that $\mathcal{R}(P)$ is non-degenerate let

$x \in \mathcal{R}(P)^0$. Then $x \in \mathcal{R}(P)$ and

$$0 = [x,Py] = [Px,y] = [x,y] \quad , \qquad y \in \mathcal{D}(P),$$

hence $x = 0$ follows since $\mathcal{D}(P)$ is dense in $\mathcal{K}$. ▪

2.2 REMARKS. a) Considering the following correspondence: to each self-adjoint projection we let correspond the subspace determined by its range, then this is a bijective correspondence between non-degenerate subspaces and selfadjoint projections. Also, by the closed graph principle, it follows that in this correspondence the regular spaces are precisely the ranges of bounded selfadjoint projections.

b) If $\mathcal{L}$ is a non-degenerate subspace of $\mathcal{K}$ and P denotes the corresponding selfadjoint projection onto $\mathcal{L}$ then I -P is the selfadjoint projection onto $\mathcal{L}^\perp$. $\mathcal{L}$ is a positive (negative) subspace if and only if P is a positive (negative) operator. $\mathcal{L}$ is maximal positive if and only if P is positive and I - P is negative.

c) The existence of unbounded selfadjoint projections in $\mathcal{K}$ depends on wether $\varkappa(\mathcal{K})$ is finite or not, more presisely $\varkappa(\mathcal{K})$ is finite (i.e. $\mathcal{K}$ is a Pontryagin space) if and only if any selfadjoint projection in $\mathcal{K}$ is bounded. This follows from the well-known fact that Pontryagin spaces are characterized within Krein spaces by the condition that any non-degenerate subspace is regular (see e.g. [2] , [3])

d) Let us assume that $\varkappa(\mathcal{K})$ is infinite. Then, it was proved in [10] that there exist two subspaces $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{K}$ such that $\mathcal{M}$ is positive, $\mathcal{N}$ is negative, $\mathcal{M} \perp \mathcal{N}$ and the linear manifold $\mathcal{M} + \mathcal{N}$ is dense in $\mathcal{K}$, but neither $\mathcal{M}$ is maximal positive nor $\mathcal{N}$ is maximal negative. If we let Q denote the linear operator in $\mathcal{K}$ defined as follows: $\mathcal{D}(Q) = \mathcal{M} + \mathcal{N}$ and

$$Q(x_1 + x_2) = x_1 , \quad x_1 \in \mathcal{M} , \quad x_2 \in \mathcal{N} ,$$

then Q is a positive, closed, densely defined projection in $\mathcal{K}$ which is not selfadjoint.

e) Let P be an unbounded selfadjoint projection in $\mathcal{K}$ (from c) we neces-

sarily need $x(\mathcal{K}) = \infty$). Then $\sigma_p(P) = \{0,1\}$ and $\sigma_c(P) = \mathbb{C} \setminus \{0,1\}$, in particular $\sigma(P)$ covers the whole complex plane.

For a maximal uniformly definite subspace $\mathcal{L}$ of the Krein space $\mathcal{K}$ there is an explicit formula of the corresponding bounded selfadjoint projection onto $\mathcal{L}$ in terms of its angular operator with respect to a certain f.d. of $\mathcal{K}$ (cf. [5] , see also [10],[2]). The following result is a generalization of this fact to the case of maximal definite subspace, dropping the assumption on uniformity.

2.3. PROPOSITION. Let $\mathcal{L}$ be a maximal positive subspace of $\mathcal{K}$ and K its angular operator with respect to a f.d. $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$. We denote

$$(2.1) \qquad \mathcal{D}_+ = \mathcal{R}(I_+ - K^*K) \subseteq \mathcal{K}^+, \qquad \mathcal{D}_- = \mathcal{R}(I_- - KK^*) \subseteq \mathcal{K}^-,$$

where I_+ is the identity operator on $\mathcal{K}^\pm$. Then the selfadjoint projection onto $\mathcal{L}$ is the closure of the following linear operator

$$P_0 = \begin{bmatrix} (I_+ - K^*K)^{-1} & -K^*(I_- - KK^*)^{-1} \\ \\ K(I_+ - K^*K)^{-1} & -KK^*(I_- - KK^*)^{-1} \end{bmatrix} \quad \text{w.r.t. } \mathcal{D}(P_0) = \mathcal{D}_+ + \mathcal{D}_- .$$

Proof. If $\mathcal{L}$ is a maximal positive subspace of $\mathcal{K}$ then $\mathcal{L}^\perp$ is a maximal negative subspace, hence K and K^* are strict contractions, equivalently $I_+ - K^*K$ and $I_- - KK^*$ are one-to-one. So the block-matrix operator P_0 makes sense.

Let z be an arbitrary vector in $\mathcal{D}(P_0)$, i.e.

$$z = (I_- - K^*K)x + (I_- - KK^*)y$$

for certain $x \in \mathcal{K}^+$ and $y \in \mathcal{K}^-$. Observing

$$z = (x - K^*y) + K(x - K^*y) + (y - Kx) + K^*(y - Kx) \in \mathcal{L} + \mathcal{L}^\perp = \mathcal{D}(P),$$

where P denotes the selfadjoint projection onto $\mathcal{L}$, it follows

$$P_0 z = (x - K^*y) + K(x - K^*y) = Pz,$$

hence $P_0 \subseteq P$. Therefore $\overline{P_0} \subseteq P$ holds.

In order to prove the converse inclusion let z denote an arbitrary

vector in $\mathcal{D}(P)$. Then $z = x+y$ for some $x \in \mathcal{L}$ and $y \in \mathcal{L}^{\perp}$ hence the representation

$$z = x^+ + Kx^+ + y^- + K^*y^- \; ,$$

for certain $x^+ \in \mathcal{K}^+$ and $y^- \in \mathcal{K}^-$, follows. We consider now the operator

$$T = \begin{bmatrix} I_+ & -K^* \\ \\ -K & I_- \end{bmatrix} \qquad \text{w.r.t.} \quad \mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^- \; .$$

Then T is bounded and J-selfadjoint, where J denotes the f.s. determined by the considered f.d.. Making use of the well-known factorization

$$T = \begin{bmatrix} I_+ & -K^* \\ \\ 0 & I_- \end{bmatrix} \begin{bmatrix} I_+ - K^*K & 0 \\ \\ 0 & I_- \end{bmatrix} \begin{bmatrix} I_+ & 0 \\ \\ -K & I_- \end{bmatrix} ,$$

and observing that the extremal operators in the right side are invertible, it follows that T has dense range in $\mathcal{K}$. In particular there exist two sequences $(x_n)_{n \in \mathbb{N}} \subset \mathcal{K}^+$ and $(y_n)_{n \in \mathbb{N}} \subset \mathcal{K}^-$ such that

$$x_n - K^*y_n \longrightarrow x^+ \; , \qquad y_n - Kx_n \longrightarrow y^- \qquad (n \to \infty) .$$

Then take the sequence $(z_n)_{n \in \mathbb{N}} \subset \mathcal{D}(P_o)$,

$$z_n = (I_+ - K^*K)x_n + (I_- - KK^*)y_n \; , \qquad n \in \mathbb{N}$$

and notice that

$$z_n = (x_n - K^*y_n) + K(x_n - K^*y_n) + (y_n - Kx_n) + K^*(y_n - Kx_n) \longrightarrow$$

$$\longrightarrow x^+ + Kx^+ + y^- + Ky^- = z \qquad (n \to \infty)$$

and

$$P_o z_n = (x_n - K^*y_n) + K(x_n - K^*y_n) \longrightarrow x^+ + Kx^+ \qquad (n \to \infty) ,$$

therefore $P \subseteq \overline{P}_o$ also holds.

§ 3. MONOTONE SEQUENCES OF SELFADJOINT PROJECTIONS

Let $(\mathcal{L}_n)_{n \in \mathbb{N}}$ be a non-decreasing (i.e. $\mathcal{L}_n \subseteq \mathcal{L}_{n+1}$, $n \in \mathbb{N}$) sequence of subspaces of the Krein space $\mathcal{K}$ and let $\mathcal{L}$ denote the subspace spanned by $\mathcal{L}_n$, $n \in \mathbb{N}$. The problem is to decide wether $\mathcal{L}$ is non-degenerate

or regular. (Also, let us observe that if the sequence of subspaces $(\mathcal{L}_n)_{n \in \mathbb{N}}$ is non-increasing, the problem of wether $\bigcap_{n \in \mathbb{N}} \mathcal{L}_n$ is a non--degenerate or a regular subspace can be reduced to the above case by considering the orthogonal companions). It can be shown by examples that even when all the subspaces $\mathcal{L}_n$ are regular (or, more restictively, $\mathcal{L}_n$ are all non-degenerate and $\mathcal{L}$ is a Pontryagin space) the subspace $\mathcal{L}$ can be degenerate (see [2], [3], [9]).

In this section we will give an equivalent characterization for the non-degeneracy of $\mathcal{L}$ when all the subspaces $\mathcal{L}_n$ are assumed non-degenerate. In order to do this we recall first the definitions of strong and weak graph convergences (cf. [6], see also [12]).

Let $(C_n)_{n \in \mathbb{N}}$ be a sequence of linear operators in $\mathcal{H}$. Then one can define two linear submanifolds $G_s((C_n)_{n \in \mathbb{N}})$ and $G_w((C_n)_{n \in \mathbb{N}})$ of $\mathcal{H} \times \mathcal{H}$ as follow: a pair of vectors $(x,y) \in \mathcal{H} \times \mathcal{H}$ belongs to $G_s((C_n)_{n \in \mathbb{N}})$ (respectively to $G_w((C_n)_{n \in \mathbb{N}})$) if there exists a squence $(x_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ such that for any $n \in \mathbb{N}$, $x_n \in \mathcal{D}(C_n)$ and

$$x_n \longrightarrow x, \quad C_n x_n \longrightarrow y \quad (n \rightarrow \infty)$$

strongly (weakly). In general these linear manifolds are not graphs of operators. If there exists an operator C in $\mathcal{H}$ such that $G_s((C_n)_{n \in \mathbb{N}}) = G(C)$ - the graph of the operator C - (respectively, $G_w((C_n)_{n \in \mathbb{N}}) = G(C)$) one says that the sequence $(C_n)_{n \in \mathbb{N}}$ converges in the strong graph sense (in the weak graph sense, respectively) to the operator C. Clearly, if the strong graph limit (or the weak graph limit) of the sequence $(C_n)_{n \in \mathbb{N}}$ exists then it is uniquely determined.

In the following we shall consider the order relation on selfadjoint projections in $\mathcal{H}$ determined by the inclusion of ranges, more precisely if P and Q are two selfadjoint projections in $\mathcal{H}$ then $P \prec Q$ if $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$.

3.1. LEMMA. Let $(P_n)_{n \in \mathbb{N}}$ be a non-decreasing sequence of selfad-joint projections in $\mathcal{K}$ and $\mathcal{L}$ the subspace generated by $\mathcal{R}(P_n)$, $n \in \mathbb{N}$,

$$\mathcal{L} = \bigvee_{n \in \mathbb{N}} \mathcal{R}(P_n).$$

Then

$$G_s((P_n)_{n \in \mathbb{N}}) = G_w((P_n)_{n \in \mathbb{N}}) = \{(x + y, x) \mid x \in \mathcal{L}, y \in \mathcal{L}^\perp\}.$$

Proof. It is sufficient to prove the inclusions

$$G_s((P_n)_{n \in \mathbb{N}}) \subseteq G_w((P_n)_{n \in \mathbb{N}}) \subseteq \{(x + y, x) \mid x \in \mathcal{L}, y \in \mathcal{L}^\perp\} \subseteq G_s((P_n)_{n \in \mathbb{N}}).$$

The first inclusion is obvious. For the second, let $(z,x) \in G_w((P_n)_{n \in \mathbb{N}})$, hence there exists a sequence of vectors $(z_n)_{n \in \mathbb{N}}$ such that $z_n \in \mathcal{D}(P_n)$, $n \in \mathbb{N}$, and the weak convergences

$$z_n \longrightarrow z, \quad P_n z_n \longrightarrow x \quad (n \to \infty)$$

hold. In particular $x \in \mathcal{L}$. If $m \in \mathbb{N}$ is fixed then for any $t \in \mathcal{R}(P_m)$

$$[z_n - P_n z_n, t] \longrightarrow [z - x, t] \quad (n \to \infty).$$

But considering only $n \geqslant m$ we have

$$[z_n - P_n z_n, t] = [(I-P_n)z_n, P_m t] = 0, \quad t \in \mathcal{R}(P_m),$$

hence

$$[z - x, t] = 0, \quad t \in \mathcal{R}(P_m).$$

Since $m \in N$ is arbitrary we get

$$z - x \in \bigcap_{m \in \mathbb{N}} \mathcal{R}(P_m)^\perp = \mathcal{L}^\perp,$$

i.e. $z = x + y$ for some $y \in \mathcal{L}^\perp$.

In order to prove the last inclusion let $x \in \mathcal{L}$ and $y \in \mathcal{L}^\perp$.

Then

$$y \in \bigcap_{n \in \mathbb{N}} \mathcal{R}(P_n)^\perp = \bigcap_{n \in \mathbb{N}} \ker(P_n).$$

Also, there exists a sequence of vectors $(x_n)_{n \in \mathbb{N}}$, $x_n \in \mathcal{R}(P_n)$, $n \in \mathbb{N}$ such that $x_n \longrightarrow x \quad (n \to \infty)$

strongly, hence the strong convergences

$$x_n + y \longrightarrow x + y, \quad P_n(x_n + y) = P_n x_n = x_n \longrightarrow x \quad (n \to \infty)$$

hold, i.e. $(x + y, x) \in G_s((P_n)_{n \in \mathbb{N}})$. ∎

3.2. PROPOSITION. Let the assumption of Lemma 3.1 hold. The following assertion are equivalent:

(i) The subspace $\mathcal{L}$ is non-degenerate.

(ii) The sequence $(P_n)_{n \in \mathbb{N}}$ converges in the strong graph sense.

(iii) The sequence $(P_n)_n$ $_\mathbb{N}$ converges in the weak graph sense.

Moreover, if one (hence all) of these assertions holds then the limits from (ii) and (ii) coincide with the selfadjoint projection onto the non--degenerate subspace $\mathcal{L}$.

Proof. (i) $\Rightarrow$ (ii) If $\mathcal{L}$ is non-degenerate let P be the selfadjoint projection onto $\mathcal{L}$. Then

$$G(P) = \{(x + y, x) \mid x \in \mathcal{L} , y \in \mathcal{L}^{\perp}\}$$

hence, by Lemma 3.1, the sequence $(P_n)_{n \in \mathbb{N}}$ converges in the strong graph to P.

(ii) $\Rightarrow$ (iii). Obvious, also by Lemma 3.1.

(iii) $\Rightarrow$ (i). If $(P_n)_{n \in \mathbb{N}}$ converges in the weak graph sense then Lemma 3.1 says that the linear manifold

$$\{(x + y, x) \mid x \in \mathcal{L} , y \in \mathcal{L}^{\perp}\}$$

is the graph of an operator, i.e. $x + y = 0$, $x \in \mathcal{L}$ and $y \in \mathcal{L}^{\perp}$ implies $x = 0$, hence $\mathcal{L}^0 = \mathcal{L} \cap \mathcal{L}^{\perp} = \{0\}$, i.e. $\mathcal{L}$ is a non-degenerate subspace. ∎

3.3. COROLLARY. Let $(P_n)_{n \in \mathbb{N}}$ be a non-decreasing sequence of bounded selfadjoint projections in $\mathcal{K}$ and denote

$$\mathcal{L} = \bigvee_{n \in \mathbb{N}} \mathcal{R}(P_n).$$

Then $\mathcal{L}$ is regular if and only the sequence converges in the strong graph (equivalently, in the weak graph) to an every where defined linear operator.

We end this section by showing that in the situation from Corollary 3.3 one cannot use, in general, either the strong operator or the weak operator topology.

3.4. LEMMA. Let $(P_n)_{n \in \mathbb{N}}$ be a non-decreasing sequence of bounded self-adjoint projections in $\mathcal{K}$. The following assertions are equivalent:

(i) $(P_n)_{n \in \mathbb{N}}$ is uniformly bounded.

(ii) $(P_n)_{n \in \mathbb{N}}$ converges in the strong operator topology.

(iii) $(P_n)_{n \in \mathbb{N}}$ converges in the weak operator topology.

Proof. (i) $\Rightarrow$ (ii) Let $\|\cdot\|$ denote a unitary norm on $\mathcal{K}$ and assume

$$M = \sup_{n \in \mathbb{N}} \|P_n\| < +\infty .$$

We prove first that the sequence $(P_n)_{n \in \mathbb{N}}$ converges in the strong graph. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of vectors in $\mathcal{K}$ such that

$$\|x_n\| \longrightarrow 0, \quad \|P_n x_n - y\| \longrightarrow 0 \ (n \to \infty),$$

for some $y \in \mathcal{K}$. Then, for any $n \in \mathbb{N}$ we have

$$\|y\| \leqslant \|y - P_n x_n\| + \|P_n x_n\| \leqslant \|y - P_n x_n\| + M\|x_n\| ,$$

and letting $n \to \infty$ we get $y = 0$. Hence, by Proposition 3.2 it follows that the subspace $\bigvee_{n \in \mathbb{N}} P_n \mathcal{K}$ is non-degenerate, in particular the linear manifold

$$\mathcal{D} = \bigcup_{n \in \mathbb{N}} P_n \mathcal{K} + \bigcap_{n \in \mathbb{N}} (I - P_n) \mathcal{K}$$

is dense in $\mathcal{K}$. Observing that for any $x \in \mathcal{D}$ the sequence of vectors $(P_n x)_{n \in \mathbb{N}}$ converges and taking account of the uniform boundedness of $(P_n)_{n \in \mathbb{N}}$ it follows that $(P_n)_{n \in \mathbb{N}}$ converges in the strong operator topology.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) This is a consequence of the uniform boundedness principle in Hilbert space. ▪

3.5 REMARK. The proof of the implication (i) $\Rightarrow$ (ii) in Lemma 3.4 can be done also by means of Alaoglu Theorem but we prefered this very elementary way. ▪

3.6. EXAMPLE. Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and $\{g_k\}_{k \in \mathbb{N}}$ an orthogonal basis of $\mathcal{H}$. Take $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ and the symmetry J defined as follows:

$$J(x_1 \oplus x_2) = x_1 \oplus - x_2, \qquad x_1, x_2 \in \mathcal{H} .$$

Then, defining the inner product

$$[x, y] = (Jx, y) , \qquad x, y \in \mathcal{K} ,$$

$(\mathcal{K}, [.,.])$ is a Krein space and J a f.s. on $\mathcal{K}$. We consider two sequences of vectors of

$$e_k = g_k \oplus \frac{k}{k+1} \, g_k, \qquad k \in \mathbb{N},$$

$$f_k = \frac{k}{k+1} \, g_k \oplus g_k, \qquad k \in \mathbb{N},$$

and the non-degenerate subspaces

$$\mathcal{L}_1 = \langle e_1 \rangle,$$

$$\mathcal{L}_k = \langle e_1, \ldots, e_k \, ; f_1, \ldots, f_{k-1} \rangle, \qquad k \in \mathbb{N}.$$

It is easy to see that the linear manifold

$$\bigcup_{k \geqslant 1} \mathcal{L}_k$$

is dense in $\mathcal{K}$, hence, if we let P_k denote the selfadjoint projection onto $\mathcal{L}_k$, $k \in \mathbb{N}$, $(P_k)_{k \in \mathbb{N}}$ converges in the strong graph (and in the weak graph, too) to the identity operator on $\mathcal{K}$. Let us remark that the subspaces $\mathcal{L}_k$ are all regular, hence the operators P_k are bounded.

On the other hand, it is easy to see that

$$P_k(x g_k + y g_k) = \frac{(k+1)((k+1)x - ky)}{2k + 1} \, e_k, \quad x, y \in \mathcal{H} \quad, k \in \mathbb{N},$$

hence

$$P_k(g_k \oplus - g_k) = (k+1) e_k, \qquad k \in \mathbb{N},$$

therefore

$$\| P_k \| \geqslant \frac{\| P_k(g_k \oplus - g_k) \|}{\| g_k \oplus - g_k \|} = \sqrt{k^2 + k + \frac{1}{2}} \longrightarrow \infty \qquad (k \rightarrow \infty).$$

By means of Lemma 3.4 this means that $(P_k)_{k \in \mathbb{N}}$ cannot converge in the weak operator topology.

4 . POSITIVE SYMMETRY OPERATORS

A densely defined linear operator U from $\mathcal{K}_1$ into $\mathcal{K}_2$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are Krein spaces, is called unitary if it is one-to-one and

$$U^\# = U^{-1}$$

(cf. [13] and [14]). A unitary operator is always closed but in general unbounded.

A linear operator S in a Krein space $\mathcal{K}$ is called a symmetry operator if it is selfadjoint and unitary, i.e.

$$S = S^{\#} = S^{-1}.$$

4.1. REMARKS a) A linear operator S in the Krein space $\mathcal{K}$ is a symmetry operator if and only if the operator

$$P = \frac{1}{2}(S + I)$$

is a selfadjoint projection. In this case

$$S = 2P - I$$

also holds and this relation defines a bijective correspondence between symmetry operators and selfadjoint projections. Moreover, denoting $\mathcal{L} = \mathcal{R}(P)$ then $\mathcal{D}(S) = \mathcal{L} + \mathcal{L}^{\perp}$ and

$$S(x_1 + x_2) = x_1 - x_2, \quad x_1 \in \mathcal{L}, \quad x_2 \in \mathcal{L}^{\perp}.$$

Also we have

$$\mathcal{L} = \ker(S-I), \quad \mathcal{L}^{\perp} = \ker(S+I)$$

b) The symmetry S is bounded if and only if the subspace $\ker(S-I)$ is regular.

c) The symmetry S is positive if and only if the subspace $\ker(S-I)$ is maximal positive.

d) A linear operator is a fundamental symmetry of the Krein space $\mathcal{K}$ if and only if it is a bounded positive symmerty operator in $\mathcal{K}$.

e) Let $\mathcal{K}$ be a Krein space. The following statements are equivalent:

(i) $\varkappa(\mathcal{K})$ is finite.

(ii) each symmetry operator in $\mathcal{K}$ is bounded.

(iii) each positive symmetry operator in $\mathcal{K}$ is bounded.

f) Assume $\varkappa(\mathcal{K})$ infinte and let S be an unbounded symmetry in $\mathcal{K}$. Then $\sigma_p(S) = \{-1,1\}$ and $\sigma_c(S) = \mathbb{C} \setminus \{-1,1\}$.

Positive symmetries were also considered in [14] (in that paper they are introduced under the name of $\#$-positive $\#$-unitary operators) in connection with some other geometrial aspects of Krein spaces (see also [7] for related ideas).

In the remaining part of this section we focus on producing some canonical forms of positive symmetries in terms of angular operators.

4.2. LEMMA Let $\mathcal{L}$ be a maximal positive subspace of $\mathcal{K}$, K its angular operator with respect to a f.d. $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$ and the linear manifolds $\mathcal{D}_+$ and $\mathcal{D}_-$ defined by (2.1). Then the positive symmetry S which corresponds to $\mathcal{L}$ (i.e. such that $\mathcal{L} = \ker(S-I)$) is the closure of the following linear operator

$$(4.1) \quad S_0 = \begin{bmatrix} (I_+ + K^*K)(I_+ - K^*K)^{-1} & -2K^*(I_- - KK^*)^{-1} \\ 2K(I_+ - K^*K)^{-1} & -(I_- + KK^*)(I_- - KK^*)^{-1} \end{bmatrix} \quad \text{w.r.t.} \ \mathcal{D}(S_0) = \mathcal{D}_+ + \mathcal{D}_-.$$

Proof. Let P denote the positive selfadjoint projection onto $\mathcal{L}$. By Proposition 2.3 P is the closure of the operator P_0 defined at (2.2). It remains to notice that $S_0 = 2P_0 - I$.

4.3. LEMMA Let S be a positive symmetry in the Krein space $\mathcal{K}$. Then, for any maximal uniformly positive (maximal uniformly negative) subspace $\mathcal{M}$ of $\mathcal{K}$, $\mathcal{D}(S) \cap \mathcal{M}$ is dense in $\mathcal{M}$ and the subspace $\overline{S(\mathcal{D}(S) \cap \mathcal{M})}$ is maximal positive (maximal negative, respectively).

Proof. Let $\mathcal{M}$ be a maximal uniformly positive subspace of $\mathcal{K}$ and denote $\mathcal{K}^+ = \mathcal{M}$ and $\mathcal{K}^- = \mathcal{M}^\perp$. Then $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$ is a f.d. of $\mathcal{K}$. Taking $\mathcal{L} = \ker(S - I)$ it follows that $\mathcal{L}$ is a maximal positive subspace of $\mathcal{K}$. Let K be its angular operator with respect to the f.d. $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$. Then K is a strict contraction. According to Lemma 4.2 S is the closure of the operator S_0 defined in (4.1). Since

$$\mathcal{D}(S) \cap \mathcal{K}^+ \supseteq \mathcal{D}(S_0) \cap \mathcal{K}^+ = \mathcal{D}_+,$$

(recall that $\mathcal{D}_+ = \mathcal{R}(I_+ - K^*K)$ is dense in $\mathcal{K}^+$) the first part of the lemma is proved.

Further, it is easy to see that

$$S_0 \mathcal{D}_+ = \left\{ x + 2K(I_+ + K^*K)^{-1}x) \mid x \in \mathcal{D}_+ \right\}$$

and

$$(4.2) \quad \overline{S(\mathcal{D}(S) \cap \mathcal{K}^+)} = \overline{S_0 \mathcal{D}_+} = \left\{ x + 2K(I_+ + K^*K)^{-1}x) \mid x \in \mathcal{K}^+ \right\},$$

i.e. the regular operator of the subspace $\overline{S(\mathcal{D}(S) \cap \mathcal{K}^+)}$ is the operator $2K(I_+ + K^*K)^{-1}$. It remains to prove that this operator is a strict contraction, i.e.

(4.3) $\| 2Kx \| < \|(I_+ + K^*K)x\|$, $x \in \mathcal{K}^+ \smallsetminus \{0\}$,

($\| \cdot \|$ is the unitary norm associated to the f.d. $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$). It

is easy to see that this is equivalent with

(4.4) $\|(I_+ - K^*K)x\| > 0$, $x \in \mathcal{K}^+ \smallsetminus \{0\}$,

which is evidently true since K is a strict contraction.   ▪

  4.4. REMARK It will be proved in Lemma 5.4 that the above fact holds for

any unitary operator.

  4.5. PROPOSITION Let $\mathcal{S}$ be a maximal positive subspace of the Krein

space $\mathcal{K}$ and T its angular operators with respect to the f.d. $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$.

Then there exists a unique positive symmetry S in $\mathcal{K}$ such that

$\overline{S(\mathcal{D}(S) \cap \mathcal{K}^+)} = \mathcal{S}$ and this is the closure of the following linear

operator:

$$(4.5) \quad S_0 = \begin{bmatrix} (I_+ - T\,T)^{-\frac{1}{2}} & -T\,(I_- - TT)^{-\frac{1}{2}} \\ T(I_+ - T\,T)^{-\frac{1}{2}} & -(I_- - TT)^{-\frac{1}{2}} \end{bmatrix} \quad \underline{\text{w.r.t.}} \; \mathcal{D}(S_0) = \mathcal{D}_+ + \mathcal{D}_-$$

where we have denoted $\mathcal{D}_+ = \mathcal{R}(I_+ - T^*T)^{\frac{1}{2}} \subseteq \mathcal{K}^+$ and $\mathcal{D}_- = \mathcal{R}((I_- - TT^*)^{\frac{1}{2}}) \subseteq \mathcal{K}^-$.

  Proof. Let S be a positive symmetry in $\mathcal{K}$ such that $S(\mathcal{D}(S) \cap \mathcal{K}^+) = \mathcal{S}$

holds. Then, representing S as the closure of the operator S_0 defined in

(4.1) with a strict contraction K, it follows from (4.2) that

(4.6) $T = 2K(I_+ + K^*K)^{-1}$.

So we are led to prove that there exists a unique strict contraction K which

satisfies (4.6). To this end we first observe that if K satisfies (4.6) then

$|T| = 2|K|(I_+ + |K|^2)^{-1}$,

where, as usually, $T = (T^*T)^{\frac{1}{2}}$. Considering the function $\varphi : [0,1] \to [0,1]$

$\varphi(k) = \dfrac{2k}{1 + k^2}$, $k \in [0,1]$,

we have

$|T| = \varphi(|K|)$,

(e.g. by continuous functional calculus). But φ is invertible, more preci-

sely, φ^{-1} is continuous and

$$\varphi^{-1}(t) = \begin{cases} \dfrac{1-\sqrt{1-t^2}}{t}, & t \in (0,1], \\ \\ 0, & t = 0, \end{cases}$$

hence, also by continuous functional calculus, we have

$$|K| = \varphi^{-1}(|T|).$$

On the other hand, (4.6) implies that $\ker T = \ker K$ and $\mathcal{R}(T) = \mathcal{R}(K)$ hence it is easy to see that the partial isometries which correspond by left polar decompositions to T and K respectively must coincide. If X denotes this partial isometry then

$$K = X\varphi^{-1}(|T|)$$

is the unique strict contraction which satisfies (4.6). Indeed, K is clearly a contraction and it is strict since (4.3) is equivalent to (4.4).

It remains to show that if (4.6) holds then the operator S_0 has the block - matrix representation (4.5). Indeed

$$I_+ - T^*T = I_+ - 4K^*K(I_+ + K^*K)^{-2} = (I_+ - K^*K)^2 (I_+ + K^*K)^{-2}$$

hence

$$(I_+ - T^*T)^{-\frac{1}{2}} = (I_+ + K^*K)(I_+ - K^*K)^{-1}$$

and then

$$T(I_+ - T^*T)^{-\frac{1}{2}} = 2K(I_+ - K^*K)^{-1}.$$

Similary we get

$$(I_- - TT^*)^{-\frac{1}{2}} = (I_- + KK^*)(I_- - KK^*)^{-1}$$

and

$$T^*(I_- - TT^*)^{-\frac{1}{2}} = 2K^*(I_- - KK^*)^{-1},$$

therefore S_0 has the representation (4.5) in terms of T.

5. UNITARY OPERATORS

In this section we let $\mathcal{K}_1$ and $\mathcal{K}_2$ denote Krein spaces.

At the beginning of the preceding section we have specified what we mean by a unitary operator in Krein spaces.

We state first, for the readers' convenience, a result which was proved essentially in [14] (the so-called Cartan decompositions).

5.1. THEOREM Let U be a unitary operator from $\mathcal{K}_1$ into $\mathcal{K}_2$ and J_i f.s. of $\mathcal{K}_i$, $i = 1,2$. Then U admits the following representations

$$U = VA_1 = A_2V$$

where V, A_1, A_2 are uniquely determined by the following properties:

(a) V is a unitary and (J_1, J_2) - unitary operator from $\mathcal{K}_1$ into $\mathcal{K}_2$ (in particular V is bounded).

(b) A_i is a J_i - positive, J_i -selfadjoint, unitary operator in $\mathcal{K}_i$, $i = 1,2$.

Proof. Consider U as a closed, densely defined operator from the Hilbert space $(\mathcal{K}_1, (.,.)_{J_1})$ into the Hilbert space $(\mathcal{K}_2, (.,.)_{J_2})$.

We factor U according to the polar decompositions

$$U = VA_1 = A_2V$$

where

$$A_1 = (U^*U)^{\frac{1}{2}}, \quad A_2 = (UU^*)^{\frac{1}{2}}$$

and V is a (J_1, J_2) - unitary operator. Since U is unitary,

$$U^{-1} = U^* = J_1 U^* J_2,$$

it is easy to verify the following equalities

$$J_1(U^*U) J_1 = (J_1 U^* J_2)(J_2 U J_1) = U^{-1}U^{-1*} = U^{-1}U^{*-1} = (U^*U)^{-1},$$

i.e. $J_1 A_1^2 J_1 = (A_1^{-1})^2$. Therefore, since

$$J_1 A_1^2 J_1 = (J_1 A_1 J_1)^2$$

holds and $J_1 A_1 J_1$ is J_1- positive J_1 - selfadjoint, from the uniqueness of the square root operator property we infer $A_1^{-1} = J_1 A_1 J_1$,i.e. A_1 is a unitary operator in $\mathcal{K}_1$. Similary one shows that A_2 is a unitary operator in $\mathcal{K}_2$. Then, for arbitrary $x,y \in \mathcal{K}_1$ we have

$$[A_1 x, A_1 y] = [x,y] = [Ux, Uy] = [V_1 x, VA_1 y],$$

hence $V | \mathcal{R}(A_1)$ is isometric. But $\mathcal{R}(A_1)$ is dense in $\mathcal{K}_1$ and V is bounded invertible (since it is (J_1, J_2) - unitary operator) hence V is also unitary. ■

For our purposes it is convenient to reformulate this theorem as follows:

5.2. LEMMA Let U, J_1 and J_2 be as in Theorem 5.1. Then U has the

representations

$$U = WS_1 = S_2W$$

where W, S_1, S_2 are uniquely determined by the following properties:

(α) W is a unitary and (J_1, J_2) - unitary operator from $\mathcal{K}_1$ into $\mathcal{K}_2$.

(β) S_i is a positive symmetry operator in $\mathcal{K}_i$, $i = 1,2$.

Proof. We consider the representations of U obtained in Theorem 5.1

and define

$$W = VJ_1 = J_2V, \quad S_1 = J_1A_1, \quad S_2 = A_2J_2$$

Then W is unitary and (J_1, J_2) - unitary. Also, it is easy to verify

$$(J_1A_1)^{\#} = J_1(J_1A_1)^*J_1 = J_1(A_1J_1)J_1 = J_1A_1$$

and

$$(J_1A_1)^{\#} = A_1^{\#}J_1 = A_1^{-1}J_1 = (J_1A_1)^{-1}$$

hence S_1 is a symmetry operator in $\mathcal{K}_1$. Moreover

$$[J_1A_1x, x] = (A_1x, x)_{J_1} \geqslant 0, \quad x \in \mathcal{D}(A_1)$$

therefore S_1 is a positive symmetry operator. Similary one proves that S_2 is

a positive symmetry operator in $\mathcal{K}_2$.

5.3. REMARKS a) In order to exist unitary operators from $\mathcal{K}_1$ into

$\mathcal{K}_2$ it is necessary and sufficient that $\varkappa^+(\mathcal{K}_1) = \varkappa^+(\mathcal{K}_2)$ and $\varkappa^-(\mathcal{K}_1) =$

$= \varkappa^-(\mathcal{K}_2)$.

b) Let $\mathcal{K}$ denote a Krein space. Then $\varkappa(\mathcal{K})$ is finite if and only if

any unitary operator in $\mathcal{K}$ is bounded.

c) A linear submanifold $\mathcal{D}$ of a Krein space $\mathcal{K}$ is the domain (equiva-

lently, the range) of some unitary operator if and only if $\mathcal{D} = \mathcal{L} + \mathcal{L}^{\perp}$

for some maximal positive subspace $\mathcal{L}$ of $\mathcal{K}$.

5.4. LEMMA Let U be a unitary operator from $\mathcal{K}_1$ into $\mathcal{K}_2$ and $\mathcal{M}$ a

maximal uniformly positive (maximal uniformly negative) subspace of $\mathcal{K}_1$. Then

$\mathcal{D}(U) \cap \mathcal{M}$ is dense in $\mathcal{M}$ and $\overline{U(\mathcal{D}(U) \cap \mathcal{M})}$ is a maximal positive

(maximal negative, respectively) subspace of $\mathcal{K}_2$.

Proof. Considering the representations obtained in Lemma 5.2 we

notice that $\mathcal{D}(U) = \mathcal{D}(S_1)$ and

$$\overline{U(\mathcal{D}(U) \cap \mathcal{M})} = W \ \overline{S_1(\mathcal{D}(S_1) \cap \mathcal{M})}.$$

Then use Lemma 4.3 and take account that W is bounded unitary operator. ▨

5 THEOREM Assume $\varkappa^{\pm}(\mathcal{K}_1) = \varkappa^{\pm}(\mathcal{K}_2)$ and consider two f.d. $\mathcal{K}_i = \mathcal{K}_i^+ [+] \mathcal{K}_i^-$

$i = 1,2$. Let $\mathcal{L}$ be a maximal positive subspace of $\mathcal{K}_2$, T its angular

operator with respect to the f.d. $\mathcal{K}_2 = \mathcal{K}_2^+ [+] \mathcal{K}_2^-$ and denote

$$\mathcal{D}_+ = ((I_+ - T^*T)^{\frac{1}{2}}) \subseteq \mathcal{K}_2^+, \quad \mathcal{D}_- = ((I_- - TT^*)^{\frac{1}{2}}) \subseteq \mathcal{K}_2^-.$$

Then, the following assertions are equivalent:

(i) The unitary operator U from $\mathcal{K}_1$ into $\mathcal{K}_2$ satisfies

$$\overline{U(\mathcal{D}(U) \cap \mathcal{K}_1^+)} = \mathcal{L}.$$

(ii) U is the closure of the linear operator U_0

$$U_0 = \begin{bmatrix} (I_+ - T^*T)^{-\frac{1}{2}} V_+ & T^*(I_- - TT^*)^{-\frac{1}{2}} V_- \\ T(I_+ - T^*T)^{-\frac{1}{2}} V_+ & (I_- - TT^*)^{-\frac{1}{2}} V_- \end{bmatrix} \quad \text{w.r.t.} \ \mathcal{D}(U_0) = V_+^{-1}\mathcal{D}_+ + V_-^{-1}\mathcal{D}_-$$

$V_+ \in \mathcal{L}(\mathcal{K}_1^+, \mathcal{K}_2^+)$, $V_- \in \mathcal{L}(\mathcal{K}_1^-, \mathcal{K}_2^-)$ unitary operators.

Proof. If U is an arbitrary unitary operator from $\mathcal{K}_1$ into $\mathcal{K}_2$

let
$$U = S_2 W$$

be the representation obtained in Lemma 5.2. Since W is unitary and (J_1, J_2)-

-unitary operator it follows

$$W = \begin{bmatrix} W_+ & 0 \\ 0 & W_{-1} \end{bmatrix} \quad \text{w.r.t.} \quad \mathcal{K}_i = \mathcal{K}_i^+ + \mathcal{K}_i^-, \quad i = 1,2.$$

Define $V_+ = W_+$ and $V_- = -W_-$. Observing that $\overline{U(\mathcal{D}(U) \cap \mathcal{K}_1^+)} = \mathcal{L}$

if and only if $\overline{S_2(\mathcal{D}(S_2) \cap \mathcal{K}_2^+)} = \mathcal{L}$, it remains only to apply

Proposition 4.5. ▨

5.6 COROLLARY Assume $\varkappa^\pm(\mathcal{K}_1) = \varkappa^\pm(\mathcal{K}_2)$ and consider two f.d. $\mathcal{K}_i = \mathcal{K}_i^+ [+] \mathcal{K}_i^-$ i=1,2. Let $\mathcal{L}$ be a maximal positive subspace of $\mathcal{K}_1$, T its angular operator with respect to the f.d. $\mathcal{K}_1 = \mathcal{K}_1^+ + \mathcal{K}_1^-$ and denote

$$\mathcal{D}_+ = \mathcal{R}((I - T^*T)^{\frac{1}{2}}) \subseteq \mathcal{K}_1^+ \quad , \quad \mathcal{D}_- = \mathcal{R}((I_- - TT^*)^{\frac{1}{2}}) \subseteq \mathcal{K}_1^- .$$

Then the following assertions are equivalent:

(i) The unitary operator U from $\mathcal{K}_1$ into $\mathcal{K}_2$ satisfies

$$U(\mathcal{D}(U) \cap \mathcal{L}) = \mathcal{K}_2^+$$

(ii) U is the closure of the linear operator

$$U_o = \begin{bmatrix} V_+(I_+ - T^*T)^{-\frac{1}{2}} & -V_+T^*(I_- - TT^*)^{-\frac{1}{2}} \\ -V_-T(I_+ - T^*T)^{-\frac{1}{2}} & V_-(I_- - TT^*)^{-\frac{1}{2}} \end{bmatrix} \quad \text{w.r.t.} \quad \mathcal{D}(U_o) = V_+^{-1}\mathcal{D}_+ + V_-^{-1}\mathcal{D}_-$$

$V_+ \in (\mathcal{K}_1^+, \mathcal{K}_2^+), V_- \in \mathcal{L}(\mathcal{K}_1^-, \mathcal{K}_2^-)$ are unitary operators.

5.7 REMARK The canonical forms obtained in Theorem 5.5 and Corollary 5.6 can be also regarded as parametrizing the class of unitary operators from $\mathcal{K}_1$ into $\mathcal{K}_2$, when $\varkappa^\pm(\mathcal{K}_1) = \varkappa^\pm(\mathcal{K}_2)$ are assumed (in [8] this was the original motivation to obtain them when only Pontryagin spaces are considered ; later it was observed that they hold for arbitrary bounded unitary operators in Krein spaces and their geometric interpretation was also added, [1]). If instead of using the canonical form of a positive symmetry in (4.5) we use (4.1) then different parametrization formulae of the class of unitary operators from $\mathcal{K}_1$ in $\mathcal{K}_2$ can be obtained (we leave to the reader to write down explicitly these statement). But the geometric interpretation is less clear in this case.

We end by some simple observations concering spectra of unitary operators.

5.8 REMARKS a) The symmetry of the spectrum of a bounded unitary (e.g. see [3]) is preserved also for unbounded unitary operators, i.e. if

$\mathcal{K}$ is a Krein space, U is a unitary operator in $\mathcal{K}$, $\lambda \neq 0$ is a complex number and $\lambda^* = \dfrac{1}{\overline{\lambda}}$ then:

$$\lambda \in \sigma_p(U) \text{ implies } \lambda^* \in \sigma_p(U) \cup \sigma_r(U).$$

$$\lambda \in \sigma_r(U) \text{ implies } \lambda^* \in \sigma_p(U).$$

$$\lambda \in \sigma_c(U) \text{ implies } \lambda^* \in \sigma_c(U).$$

$$\lambda \in \rho(U) \text{ implies } \lambda^* \in \rho(U).$$

  b) Let $\mathcal{K}$ be a Krein space, U and A linear operators in $\mathcal{K}$, $\varepsilon, \zeta \in \mathbb{C}$ such that $|\varepsilon| = 1$ and $\zeta \neq \overline{\zeta}$. Then the following relations are equivalent (e.g. see [3]):

  (i) $\ker(A - \overline{\zeta} I) = \{0\}$, $U = \varepsilon(A - \overline{\zeta} I)(A - \zeta I)^{-1}$, $\mathcal{D}(U) = \mathcal{R}(A - \zeta I)$.

  (ii) $\ker(U - \varepsilon I) = \{0\}$, $A = (\zeta U - \overline{\zeta}\varepsilon I)(U - \varepsilon I)^{-1}$, $\mathcal{D}(A) = \mathcal{R}(U - \varepsilon I)$.

Assuming these relations satisfied it follows that U is unitary and

$$\varepsilon \notin \sigma_p(U) \text{ if and only if A is selfadjoint and } \zeta \in \sigma_c(A) \cup \rho(A).$$

  c) In [3] it is constructed a bounded selfadjoint operator A in a Krein space such that it possesses a value $\zeta \in \sigma_c(A)$, $\zeta \neq \overline{\zeta}$. Then we let U denote the Cayley transformation of A corresponding to ζ and $\varepsilon = 1$, conform item b). Taking account of the behaviour of spectra under the Cayley transformation (see also [3]) it follows that U is an unbounded unitary operator in $\mathcal{K}$ with non-void resolvent set (compare with Remark 2.2.e) and Remark 4.1.f)).

REFERENCES

[1]   T. ANDO; Linear operators in Krein spaces, Lecture Notes, Hokkaido
         University 1979.

[2]   T.Ya. AZIZOV, I.S. IOHVIDOV; Foundations of the theory of linear
         operators in spaces with indefinite metric (Russian), Nauka,
         Moscow 1986.

[3]   J. BOGNAR; Indefinite inner product spaces, Springer Verlag,
         Berlin 1974.

[4]   Ch. DAVIS; J-unitary dilation of a general operator, Acta Sci. Math.,
         32(1970), 75-86.

[5]   Yu.P. GINZBURG; Projections in a Hilbert space with bilinear metric
         (Russian), Dokl. Akad. Nauk SSSR, 139(1961), 775-778.

[6]   J. GLIMM; A. JAFFE: Singular perturbations of selfadjoint operators,
          Comm. Pure Appl. Math. 22(1969), 401-414.

[7]   F. HANSEN; Selfpolar norms on an indefinite inner product space, Publ.
          RIMS, Kyoto Univ., 16(1980), 889-913.

[8]   I.S.IOHVIDOV; M.G.KREIN; Spectral theory of operators in spaces
          with an indefinite metric (Russian) I. Trudy Moskov. Mat.
          Obshch. 5 (1956), 367-432.

[9]   I.S.IOHVIDOV; M.G.KREIN, H.LANGER; Introduction to the spectral theory
          of operators in spaces with an indefinite metric,
          Akademie-Verlag, Berlin 1982.

[10] M.G. KREIN; Introduction to the geometry of indefinite J-spaces
          and to the theory of operators in those spaces (Russian)
          In: Second mathematical summer school, PartI, pp. 15-92,
          Naukova Dumka, Kiev 1965.

[11] H. LANGER; Maximal dual pairs of invariant subspaces of J-selfadjoint
          operators(Russian), Mat. Zametki 7 (1970), 443-447.

[12] M. REED, B.SIMON; Methods of modern mathematical physics, I Functional
          Analysis, Academic Press, New York 1972.

[13] Yu. L.SHMUL'YAN; Extension theory operators and spaces with an indefi-
          nite metric, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974).

[14] M. TOMITA; Operators and operator algebras in Krein spaces. I. Spectral
          analysis in Pontrjagin spaces. RIMS- Kukyuuroku 398 (1980),
          131-158.