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DUALITIES BETWEEN COMPLETE LATTICES

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We study dualities between two complete lattices E and F, i.e., mappings $\Delta : E \to F$ satisfying $\Delta(\inf_{i\in I} x_i) = \sup_{i\in I} \Delta(x_i)$ for all $\{x_i\}_{i\in I} \subseteq E$ and all index sets I, including the empty set $I=\emptyset$. We give characterizations and representations of dualities Δ, and some results on the dual $\Delta^* : F \to E$ of Δ and on the associated hull operator $\Delta^*\Delta : E \to E$, in the general case and in various particular cases. Among several applications, we devote special attention to Fenchel-Moreau conjugations.

Key Words: Dualities, Fenchel-Moreau conjugations, hull operators, coupling functions, complete lattices, infimal generators, fuzzy subsets, functions of 0-1 variables.

§ 0. INTRODUCTION

It is well-known that the theory of Fenchel-Moreau conjugations $f \in \overline{R}^X \to f^{c(\varphi)} \in \overline{R}^W$ (where X and W are two sets and $\varphi : X \times W \to \overline{R}$ is a coupling function) has important applications to duality in optimization (see e. g. [20] and the references therein). An axiomatic study of these conjugations has been started in [17], where it has been shown, among other results, that, for any sets X and W (throughout the sequel we assume, without special mention, that $X, W \neq \emptyset$), an operator $f \in \overline{R}^X \to f^c \in \overline{R}^W$ is the generalized Fenchel-Moreau conjugation operator $f \to f^{c(\varphi)}$ with respect to a (necessarily unique) coupling function $\varphi : X \times W \to \overline{R}$ if and only if for each index set I (including the empty set $I = \emptyset$) we have

$$(\inf_{i\in I} f_i)^c = \sup_{i\in I} f_i^c \qquad (\{f_i\}_{i\in I} \subseteq \overline{R}^X), \tag{0.1}$$

$$(f \dot{+} d)^c = f^c \underset{\cdot}{+} -d \qquad (f \in \bar{R}^X,\ d \in \bar{R}); \qquad (0.2)$$

therefore, the operators $f \to f^c$ satisfying (0.1), (0.2), have been called, in [17], "conjugation operators". We recall that here $\bar{R}^X$ denotes the family of all functions $f: X \to \bar{R} = [-\infty, +\infty]$, inf and sup on $\bar{R}^X$, $\bar{R}^W$ are defined pointwise, $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$, and $\dot{+}$, $\underset{\cdot}{+}$ are, respectively, the "upper addition" and "lower addition" on $\bar{R}$, defined ([10], [11]) by

$$a \dot{+} b = a \underset{\cdot}{+} b = a + b \qquad (a, b \in R), \qquad (0.3)$$

$$a \dot{+} (+\infty) = +\infty, \quad a \underset{\cdot}{+} (-\infty) = -\infty \qquad (a \in \bar{R}); \qquad (0.4)$$

also, in (0.2), we use the same notation for the elements of $\bar{R}$ and the constant functions with values in $\bar{R}$. The approach of [17] has been parallel, in a certain sense, to the "axiomatic approach to duality" for sets, of Evers and van Maaren [6], in which a "duality" between two sets X, W is defined as a mapping $\Delta: 2^X \to 2^W$ (where 2^X denotes the family of all subsets of X), satisfying, for each index set I (including $I = \emptyset$),

$$\Delta\Big(\bigcup_{i \in I} G_i\Big) = \bigcap_{i \in I} \Delta(G_i) \qquad (\{G_i\}_{i \in I} \subseteq 2^X), \qquad (0.5)$$

which is "similar" to (0.1). Note also that there exist operators $f \in \bar{R}^X \to f^c \in \bar{R}^W$, with useful applications to duality in optimization, which satisfy only (0.1), but not (0.2) (see [2] and § 4 below).

The aim of the present paper is to study the operators satisfying (0.1) and (0.5), in the framework of complete lattices, both in the general and in some particular cases. We recall that if $E = (E, \leq)$ and $F = (F, \leq)$ are two complete lattices, a mapping $\Delta: E \to F$ is called a *duality* (or, a "polarity" [3], [12], [13]), if for each index set I (including $I = \emptyset$) we have

$$\Delta\Big(\inf_{i \in I} x_i\Big) = \sup_{i \in I} \Delta(x_i) \qquad (\{x_i\}_{i \in I} \subseteq E); \qquad (0.6)$$

thus, for $E = (\bar{R}^X, \leq)$, $F = (\bar{R}^W, \leq)$ (with the usual pointwise order), (0.6) becomes (0.1), while for $E = (2^X, \supseteq)$, $F = (2^W, \supseteq)$, (0.6) becomes (0. 5).

In § 1 we shall give, for two complete lattices E and F, characterizations and representations of dualities $\Delta: E \to F$, with the aid of an arbitrary "family of infimal generators" of E, in the sense of Kutateladze-Rubinov [9]. Furthermore, we shall also consider the "dual operators" $\Delta^*: F \to E$ (defined, e.g., in [13]).

In § 2 we shall apply the results of § 1 to $E = (A^X, \leq)$ (with the usual pointwise order), where $(A, \leq)$ is a subset of $(\overline{R}, \leq)$, which is a complete lattice, and to a suitable family of infimal generators of E; thus, we shall obtain characterizations and representations of dualities $\Delta : A^X \to F$, where F is an arbitrary complete lattice. Also, we shall consider separately the particular case $A = \{0, +\infty\}$, i.e., of dualities $\Delta : \{0, +\infty\}^X \to F$.

In § 3 we shall obtain characterizations and representations of dualities $\Delta : A^X \to B^W$, where $(A, \leq) \subseteq (\overline{R}, \leq)$ and $(B, \leq) \subseteq (\overline{R}, \leq)$ are complete lattices, and of the dual operators $\Delta^* : B^W \to A^X$, and we shall apply them to the cases when i) $A = \{0, +\infty\}$; ii) $A = B = \{0, +\infty\}$; iii) $A = B = [0,1]$ (so the elements of A^X and B^W are the fuzzy subsets of X and W respectively); iv) $A = B = \overline{R}$; v) $A = \overline{R}$, $B = \{0, +\infty\}$ (this will yield a solution of the problem, raised in [20], of the existence of a duality Δ such that $f^{\Delta\Delta^*} = \sup \{w \in W \mid w \leq f\}$ for all $f \in \overline{R}^X$); vi) $A = \overline{R}$, $B = \{0, +\infty\}$, $X = \{0, 1\}^n$, $W = (R^n)^*|_X$ (which will yield, as a corollary, that every function $f : \{0, 1\}^n \to R$ satisfying $f = f^{\Delta\Delta^*}$ can be extended to a polyhedral convex positively homogeneous function $f : R^n \to R$).

In § 4 we shall introduce and study a new concept of "strict duality" $\Delta : R^X \to R^W$, closely related to the "$\mathcal{F}$-conjugation" of Ben Tal and Ben Israel [2].

In § 5 we shall first show that for conjugations $c : \overline{R}^X \to \overline{R}^W$, i.e., operators satisfying (0.1), (0.2), some of the results of [17] can be obtained as particular cases of the preceding results on dualities $\Delta : \overline{R}^X \to \overline{R}^W$. Furthermore, in the particular case when $X = \{0, 1\}^n$ and $W = C_1 \times \ldots \times C_n$, where $C_i \subseteq R$ $(i = 1, \ldots, n)$ are unbounded from above and from below, and $\varphi : X \times W \to R$ is the "natural coupling function" given by the scalar product, we shall characterize the extended functions of 0-1 variables satisfying $f = f^{c(\varphi)c(\varphi)^*}$. Also, for $X = \{0, 1\}^n$, $W = (R^n)^*$ (the family of all linear functions on R^n) and φ as before, we shall give necessary and sufficient conditions in order that $\partial_c f(x_0) \neq \emptyset$, where $\partial_c f(x_0)$ is the subdifferential of $f : X \to R$ at $x_0 \in X$, with respect to $c = c(\varphi)$. As consequences, we shall show that in a result of Fujishige ([7], theorem 3.1) on submodular functions $f : X \to R$, where X is a distributive sublattice of $\{0, 1\}^n$, the assumpution of submodularity of f is superfluous, and we shall obtain some results on the extension of arbitrary functions $f : \{0, 1\}^n \to R \cup \{+\infty\}$ and $f : \{0, 1\}^n \to R$ to certain proper convex functions $\hat{f}$ on R^n.

For some further results on dualities, conjugations and coupling functions, and on relations between them, see also [18], [21], [23].

Finally, let us mention some notations, which we shall use in the sequel. For a complete lattice E, whenever necessary, we shall denote by $\sup^E$ ($\inf^E$) the supremum (infimum) *in* E; also, we shall denote by $+\infty$ or $(+\infty)^E$ ($-\infty$ or $(-\infty)^E$) the greatest (smallest) element of E and we shall adopt the usual conventions

$$\inf \emptyset = +\infty, \quad \sup \emptyset = -\infty, \tag{0.7}$$

where $\emptyset$ denotes the empty set. For a function $f: X \to \overline{R}$, where X is a set, we shall use the notations

$$\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\}, \tag{0.8}$$

$$\zeta(f) = \{x \in X \mid f(x) = 0\}. \tag{0.9}$$

§ 1. DUALITIES BETWEEN GENERAL COMPLETE LATTICES

We recall that if $E = (E, \leq)$ and $F = (F, \leq)$ are two complete lattices, a mapping $\Delta: E \to F$ is called a *complete inf-anti-homomorphism*, if for every index set $I \neq \emptyset$ we have (0.6).

Remark 1.1. a) Condition (0.6) for $I = \emptyset$ means (by (0.7)) that

$$\Delta(+\infty) = -\infty. \tag{1.1}$$

Thus, *a duality* $\Delta: E \to F$ *is nothing else than a complete inf-anti-homomorphism satisfying* (1.1). In particular, by (1.2) below, a complete inf-anti-homomorphism Δ of E *onto* F is a duality.

b) Each inf-anti-homomorphism $\Delta: E \to F$ is antitone, whence

$$\Delta(+\infty) = \min_{x \in E} \Delta(x). \tag{1.2}$$

c) Considering the complete lattice $F^{-} = (F, \geq)$ or the complete lattice $E^{-} = (E, \geq)$, or both, our results on complete inf-anti-homomorphisms also yield results on complete inf-homomorphisms, complete sup-homomorphisms, and complete sup-anti-homomorphisms, respectively.

Example 1.1. For any sets X and W, let $E = (2^X, \supseteq)$, the lattice of all subsets of X, ordered by containment (i.e., $G_1 \leq G_2$ if and only if $G_1 \supseteq G_2$) and let $F = (2^W, \supseteq)$. Then, conditions (0.6) and (1.1) become, respectively,

$$\Delta\Big(\bigcup_{i\in I} G_i\Big) = \bigcap_{i\in I}\Delta(G_i) \qquad \big(\{G_i\}_{i\in I} \subseteq 2^X\big), \tag{1.3}$$

$$\Delta(\emptyset) = W. \tag{1.4}$$

Example 1.2. For any sets X, W and any subsets $(A, \leq)$, $(B, \leq)$ of $(\overline{R}, \leq)$, which are complete lattices, let $E = (A^X, \leq)$, $F = (B^W, \leq)$, with the usual pointwise order. Then, denoting $\Delta(f)$ by f^Δ, conditions (0.6) and (1.1) become, respectively,

$$\Big(\inf_{i\in I}{}^E f_i\Big)^\Delta (w) = \sup_{i\in I}{}^B f_i^\Delta (w) \qquad \big(\{f_i\}_{i\in I} \subseteq A^X,\ w\in W\big), \tag{1.5}$$

$$\big(+\infty^E\big)^\Delta (w) = -\infty^B \qquad (w\in W); \tag{1.6}$$

if $A \subseteq \overline{R}$ is closed for inf (i.e., $\inf M \in A$ for all $M \subseteq A$, or, in other words, $\inf M \in A$ for all $\emptyset \neq M \subseteq A$, and $+\infty \in A$), then $E = (A^X, \leq)$ is a complete inf-semi-lattice, and hence a complete lattice [3], and $\inf^E = \inf$, $+\infty^E = +\infty$ in (1.5), (1.6).

We recall that a complete inf-anti-homomorphism $c: \overline{R}^X \to \overline{R}^W$ is called [17] a *conjugation*, if

$$(f \dotplus d)^c = f^c \dotplus -d \qquad (f \in \overline{R}^X,\ d \in \overline{R}); \tag{1.7}$$

by (1.7) for $d = +\infty$, *every conjugation* $c: \overline{R}^X \to \overline{R}^W$ *is a duality* (but, the converse is not true).

We recall that a subset Y of a complete lattice E is called [9] a *family of infimal generators of* E, if for each $x \in E$ there exists $Y_x \subseteq Y$ such that

$$x = \inf Y_x; \tag{1.8}$$

a *family of supremal generators of* E is defined [9] similarly, with inf replaced by sup.

Example 1.3. For the complete lattice $E = (2^X, \supseteq)$, where X is a set, the family Y of all singletons $\{x\}$, where $x \in X$, is a family of infimal generators of E, since

$$G = \bigcup_{x\in G} \{x\} \qquad (G \subseteq X). \tag{1.9}$$

Example 1.4. For the complete lattice $E = (A^X, \leq)$ of example 1.2, let us define, by abuse of notation, the function

$$(\chi_{\{x\}} \dotplus a)(x') = \chi_{\{x\}}(x') \dotplus a = \begin{cases} a, & \text{if } x' = x \\ +\infty^A, & \text{if } x' \neq x. \end{cases} \tag{1.10}$$

Then, by [17], proof of lemma 3.1, we have

$$f = \inf_{x\in X}^{A^X} \{\chi_{\{x\}} \dot{+} f(x)\} \qquad (f\in A^X), \tag{1.11}$$

and hence

$$Y_1 = \{\chi_{\{x\}} \dot{+} a \mid x\in X,\ a\in A\} \tag{1.12}$$

is a family of infimal generators of E. If $(A, \leq)$ is closed for inf, then $(+\infty)^A = +\infty$ in (1.10) (and hence $\chi_{\{x\}} \dot{+} a$ coincides with the usual one) and $\inf^A = \inf$ in (1.11).

Proposition 1.1. *A subset* Y *of a complete lattice* E *is a family of infimal generators of* E, *if and only if*

$$x = \inf \{y\in Y \mid x \leq y\} \qquad (x\in E). \tag{1.13}$$

Proof. If we have (1.8), then $Y_x \subseteq \{y\in Y \mid x \leq y\}$, whence $x = \inf Y_x \geq \inf \{y\in Y \mid x \leq y\} \geq x$, so (1.13) holds. Conversely, if (1.13) holds, then, for $Y_x = \{y\in Y \mid x \leq y\}$, we have (1.8). ∎

Corollary 1.1. *If* Y *is a family of infimal generators of a complete lattice* E, *then*

$$\{y\in Y \mid x \leq y\} \neq \emptyset \qquad (x\in E,\ x < +\infty). \tag{1.14}$$

Proof. If $\{y\in Y \mid x \leq y\} = \emptyset$, then, by (1.13) and (0.7), $x = \inf \emptyset = +\infty$.

Theorem 1.1. *Let* E, F *be two complete lattices,* $\Delta: E \to F$ *a mapping, and* Y *a family of infimal generators of* E. *The following statements are equivalent:*

1°. *Δ is a duality.*

2°. *For every index set* I *(including* $I = \emptyset$*) we have*

$$\Delta\left(\inf_{i\in I} y_i\right) = \sup_{i\in I} \Delta(y_i) \qquad \left(\{y_i\}_{i\in I} \subseteq Y\right). \tag{1.15}$$

These statements imply

3°. *We have*

$$\Delta(x) = \sup \{\Delta(y) \mid y\in Y,\ \ x \leq y\} \qquad (x\in E). \tag{1.16}$$

Proof. The implication 1° ⇒ 2° is obvious.

2° ⇒ 3°. Assume 2° and let $x \in E$, $\{y_i\}_{i\in I} = \{y\in Y \mid x \leq y\} \subseteq Y$. Then, by (1.13) and (1.15), we obtain

$$\Delta(x) = \Delta\left(\inf\{y \in Y \mid x \le y\}\right) = \sup\{\Delta(y) \mid y \in Y, x \le y\}.$$

$2° \cap 3° \Rightarrow 1°$. By 3°, Δ is antitone, i.e.,

$$x', x'' \in E, \quad x' \le x'' \quad \Rightarrow \quad \Delta(x') \ge \Delta(x''). \tag{1.17}$$

Now let $\{x_i\}_{i \in I} \subseteq E$. Then, since $\inf_{j \in I} x_j \le x_i$ $(i \in I)$, we have, by (1.17), $\Delta(\inf_{j \in I} x_j) \ge \Delta(x_i)$ $(i \in I)$, whence

$$\Delta(\inf_{i \in I} x_i) \ge \sup_{i \in I} \Delta(x_i); \tag{1.18}$$

note that this holds for any $\Delta: E \to F$ satisfying (1.17) (we do not need Y). For the opposite inequality, observe that, by (1.13), we have

$$x_i = \inf Y_i \quad (i \in I), \quad \inf_{i \in I} x_i = \inf Y_0, \tag{1.19}$$

where

$$Y_i = \{y \in Y \mid x_i \le y\} \quad (i \in I), \quad Y_0 = \{y \in Y \mid \inf_{i \in I} x_i \le y\}. \tag{1.20}$$

We claim that

$$\inf_{i \in I} x_i = \inf \bigcup_{j \in I} Y_j = \inf\{y \in Y \mid \exists\ i \in I, x_i \le y\}. \tag{1.21}$$

Indeed, since $Y_i \subset \bigcup_{j \in I} Y_j \subset Y_0$ $(i \in I)$, we have

$$\inf Y_i \ge \inf \bigcup_{j \in I} Y_j \ge \inf Y_0 \quad (i \in I), \tag{1.22}$$

whence, by (1.19), we obtain

$$\inf_{i \in I} x_i = \inf_{i \in I} \inf Y_i \ge \inf \bigcup_{j \in I} Y_j \ge \inf Y_0 = \inf_{i \in I} x_i,$$

which proves the claim (1.21). Then, by (1.21), 2° and (1.17) (for $x' = x_i$, $x'' = y$),

$$\Delta(\inf_{i \in I} x_i) = \Delta\left(\inf\{y \in Y \mid \exists\ i \in I, x_i \le y\}\right) =$$

$$= \sup\{\Delta(y) \mid y \in Y, \exists\ i \in I, x_i \le y\} \le$$

$$\le \sup\{\Delta(y) \mid y \in Y, \exists\ i \in I, \Delta(y) \le \Delta(x_i)\} \le \sup_{i \in I} \Delta(x_i),$$

which, together with (1.18), yields (0.6). ∎

Remark 1.2. a) In general, $3° \not\Rightarrow 1°$. Indeed, if E is a complete lattice, then $Y = E$ is a family of infimal generators of E, and $\Delta: E \to F$ is antitone if and only if we have (1.16) with $Y = E$. However, there exist antitone mappings $\Delta: E \to F$ which are not dualities, e.g., any constant mapping

$$\Delta(x) \equiv y_0 \in F \setminus \{-\infty\} \qquad (x \in E). \qquad (1.23)$$

b) The above argument also yields that, *for* E, F *and* Y *as in theorem* 1.1, *and for* $\Delta: E \to F$, *the following statements are equivalent:*

1'. Δ *is a complete inf-anti-homomorphism.*

2'. *We have* (1.2), *and for every* $I \neq \emptyset$ *we have* (1.15).

These statements imply

3'. *We have* (1.16) *for all* $x \in E$, $x < +\infty$. *If* $+\infty \in Y$, *then we have* (1.16) *for all* $x \in E$ *(including* $x = +\infty$).

Indeed, the implication 1' $\Rightarrow$ 2' holds by remark 1.1 b). Now, assume 2' and let $x \in E$, $x < +\infty$, $\{y_i\}_{i \in I} = \{y \in Y \mid x \leq y\} \subseteq Y$. Then, by (1.14), we have $I \neq \phi$, whence, by (1.13) and (1.15), we obtain (1.16) for $x < +\infty$. If $+\infty \in Y$, then $\{y \in Y \mid +\infty \leq y\} = \{+\infty\}$, so (1.16) holds also for $x = +\infty$; thus, 2' $\Rightarrow$ 3'. Finally, the proof of the implication 2' $\cap$ 3' $\Rightarrow$ 1' is similar to that of the implication 2° $\cap$ 3° $\Rightarrow$ 1° of theorem 1.1. Note that here we need to assume (1.2) in 2', since otherwise we obtain (1.17) only for $x' \leq x'' < +\infty$ (if $+\infty \notin Y$) and (1.18) only for $x_i < +\infty$ $(i \in I)$; as an example, one can take $Y = E \setminus \{+\infty\}$ and

$$\Delta(x) = y_0 \in F \setminus \{+\infty\} \quad \text{if } x \in E, \quad x < +\infty$$
$$= +\infty, \quad \text{if } x = +\infty, \qquad (1.24)$$

which satisfy (1.15) for all $I \neq \emptyset$ and (1.16) for all $x < +\infty$, but not 1' (and Δ does not satisfy (1.2)). Note also that, again, 3' $\not\Rightarrow$ 1', even when $+\infty \in Y$, since there exist antitone mapppings $\Delta: E \to F$ which are not complete inf-anti-homomorphisms (see e. g. [15], Ch. II, § 18, example D).

c) Similarly, from the subsequent results on dualities one can obtain corresponding results on inf-anti-homomorphisms, which we shall omit (with the exception of remark 2.1 b) below).

We recall that if E and F are two complete lattices, the "dual" $\Delta^*: F \to E$ of any mapping $\Delta: E \to F$ is defined (see e. g. [13]) by

$$\Delta^*(z) = \inf \{x \in E \mid \Delta(x) \leq z\} \qquad (z \in F). \qquad (1.25)$$

It is well-known (see e. g. [13]) that if Δ is a duality, then Δ^* is a duality, too, and for any $x \in E$ and $z \in F$ we have the equivalence

$$\Delta(x) \leq z \Leftrightarrow \Delta^*(z) \leq x. \qquad (1.26)$$

In the sequel we shall use

Proposition 1.2. *Let* E, F *be two complete lattices,* $\Delta: E \to F$ *a duality, and* Y *a family of infimal generators of* E. *Then*

$$\Delta^*(z) = \inf \{y \in Y \mid \Delta(y) \leq z\} \qquad (z \in F). \qquad (1.27)$$

Proof. By (1.13) and (1.26), we have

$$\Delta^*(z) = \inf \{y \in Y \mid \Delta^*(z) \leq y\} = \inf \{y \in Y \mid \Delta(y) \leq z\} \qquad (z \in F). \quad ∎$$

Finally, we recall that if E and F are two complete lattices and $\Delta : E \to F$ is a duality, then $\Delta^*\Delta : E \to E$ is a "hull operator" (see e. g. [6], [23]).

§ 2. DUALITIES $\Delta : A^X \to F$

Lemma 2.1. *Let* $E = (A^X, \leq)$ *and* $Y_1 \subseteq E$ *be as in example* 1.4. *Then, for any* $f \in A^X$ *and* $\chi_{\{x\}} \dot{+} a \in Y_1$, *we have*

$$f \leq \chi_{\{x\}} \dot{+} a \Leftrightarrow f(x) \leq a. \qquad (2.1)$$

Proof. This is an obvious consequence of (1.10). ∎

Theorem 2.1. *Let* X *be a set and let* $(A, \leq) \subseteq (\bar{R}, \leq)$ *and* F *be complete lattices. For a mapping* $\Delta: A^X \to F$, *the following statements are equivalent:*

1°. Δ *is a duality.*

2°. *There exists a mapping* $\Gamma : X \times A \to F$, *satisfying, for every index set* I *(including* $I = \emptyset$*),*

$$\Gamma(x, \inf_{i \in I}{}^{A} a_i) = \sup_{i \in I} \Gamma(x, a_i) \qquad (x \in X, \{a_i\}_{i \in I} \subseteq A), \qquad (2.2)$$

and such that

$$f^{\Delta} = \sup_{x \in X} \Gamma(x, f(x)) \qquad (f \in A^X). \qquad (2.3)$$

Moreover, in this case, Γ *is uniquely determined by* Δ, *namely, we have*

$$\Gamma(x, a) = (\chi_{\{x\}} \dot{+} a)^{\Delta} \qquad (x \in X, \ a \in A). \qquad (2.4)$$

Proof. 1° ⇒ 2°. Let $E = (A^X, \leq)$ and Y_1 be as in example 1.4. Then, (1.15) becomes

$$\left(\inf_{i\in I}^{E}(\chi_{\{x_i\}}\dot{+}a_i)\right)^{\Delta}=\sup_{i\in I}(\chi_{\{x_i\}}\dot{+}a_i)^{\Delta}\qquad \left(\{(x_i,a_i)\}_{i\in I}\subseteq X\times A\right), \tag{2.5}$$

which, for $x_i = x\in X$ $(i\in I)$ and Γ of (2.4), yields

$$\Gamma(x,\inf_{i\in I}^{A}a_i)=(\chi_{\{x\}}\dot{+}\inf_{i\in I}^{A}a_i)^{\Delta}=(\inf_{i\in I}^{E}(\chi_{\{x\}}\dot{+}a_i))^{\Delta}=$$

$$=\sup_{i\in I}(\chi_{\{x\}}\dot{+}a_i)^{\Delta}=\sup_{i\in I}\Gamma(x,a_i)\qquad (x\in X,\ \{a_i\}_{i\in I}\subseteq A). \tag{2.6}$$

Finally, (1.16) and (2.1) yield

$$f^{\Delta}=\sup\{(\chi_{\{x\}}\dot{+}a)^{\Delta}\mid x\in X,\ a\in A,\ f\leq\chi_{\{x\}}\dot{+}a\}=$$

$$=\sup\{(\chi_{\{x\}}\dot{+}a)^{\Delta}\mid x\in X,\ a\in A,\ f(x)\leq a\}\qquad (f\in A^{X}),$$

whence, since Δ is antitone, we obtain (2.3).

$2^{\circ}\Rightarrow 1^{\circ}$. If 2° holds, then for any set I we have, by (2.3) and (2.2),

$$(\inf_{i\in I}^{E}f_i)^{\Delta}=\sup_{x\in X}\Gamma(x,\inf_{i\in I}^{A}f_i(x))=\sup_{x\in X}\sup_{i\in I}\Gamma(x,f_i(x))=$$

$$=\sup_{i\in I}\sup_{x\in X}\Gamma(x,f_i(x))=\sup_{i\in I}f_i^{\Delta}\qquad (\{f_i\}_{i\in I}\subseteq A^{X}).$$

Finally, to prove the last statement, assume that Γ is as in 2°. Then, by (2.2) for $I=\emptyset$, we have

$$\Gamma(x,+\infty^{A})=-\infty\qquad (x\in X), \tag{2.7}$$

whence, by (2.3) for $f=\chi_{\{x\}}\dot{+}a$ of (1.10),

$$(\chi_{\{x\}}\dot{+}a)^{\Delta}=\sup_{x'\in X}\Gamma(x',\chi_{\{x\}}(x')\dot{+}a)=\Gamma(x,a)\qquad (x\in X,\ a\in A).\quad ∎$$

Remark 2.1. a) Alternatively, one can also give the following direct proof of the implication $1^{\circ}\Rightarrow 2^{\circ}$: If 1° holds, then, for Γ defined by (2.4) we have (2.6). Also, by (1.11) and 1°, for Γ defined by (2.4) we obtain

$$f^{\Delta}=(\inf_{x\in X}^{E}(\chi_{\{x\}}\dot{+}f(x))^{\Delta}=\sup_{x\in X}(\chi_{\{x\}}\dot{+}f(x))^{\Delta}=\sup_{x\in X}\Gamma(x,f(x))\qquad (f\in A^{X}).$$

b) As in remark 1.2 b), one can prove again a corresponding result for complete inf-anti-homomorphisms Δ. However, in this case, if Δ is not a duality, then, by $a=\inf^{A}\{a,+\infty^{A}\}$ $(a\in A)$ and (2.2) (for $I\neq\emptyset$), we only have

$$\Gamma(x,a)=\sup\{\Gamma(x,a),\ \Gamma(x,+\infty)\}\qquad (x\in X,\ a\in A), \tag{2.8}$$

whence, by (2.3) for $f=\chi_{\{x\}}\dot{+}a$ of (1.10),

$$\Gamma(x, a) \leq (\chi_{\{x\}} \dot{+} a)^{\Delta} \leq \sup_{x' \in X} \Gamma(x', a) \qquad (x \in X,\ a \in A), \tag{2.9}$$

so we can only conclude that $\sup \Gamma(x, a)$ $(a \in A)$ are uniquely determined by Δ, namely,

$$\sup_{x \in X} \Gamma(x, a) = \sup_{x \in X} (\chi_{\{x\}} \dot{+} a)^{\Delta} \qquad (a \in A). \tag{2.10}$$

As an example, one can take Δ to be a constant mapping, say $f^{\Delta} \equiv y_0 \in F \setminus \{-\infty\}$ $(f \in A^X)$ and Γ to be any mapping such that each partial mapping $\Gamma(x, .) : a \to \Gamma(x, a)$ is constant on A, say $\Gamma(x, a) = \gamma(x)$ $(x \in X, a \in A)$, with $\gamma : X \to F$ satisfying $\sup_{x \in X} \gamma(x) = y_0$.

One can replace (2.2) by convenient equivalent conditions, using

Lemma 2.2. *Let* $(A, \leq) \subseteq (\overline{R}, \leq)$ *and* F *be complete lattices. For a mapping* $h : A \to F$, *the following statements are equivalent:*

1°. *For every index set* I *we have*

$$h(\inf_{i \in I}{}^{A} a_i) = \sup_{i \in I} h(a_i) \qquad (\{a_i\}_{i \in I} \subseteq A). \tag{2.11}$$

2°. h *is antitone and all "level sets"*

$$S_y(h) = \{a \in A \mid h(a) \leq y\} \qquad (y \in F) \tag{2.12}$$

are closed for $\inf^A$ (*i.e.*, $\inf^A M \in S_y(h)$ *for all* $\emptyset \neq M \subseteq S_y(h)$ and $+\infty^A = \inf^A \emptyset \in S_y(h)$).

If $A \subseteq \overline{R}$ *is closed for* inf *and closed* (*i.e., closed in* $\overline{R}$, *for the natural topology of* $\overline{R}$), *or equivalently, if* $A \subseteq \overline{R}$ *is closed and contains* $+\infty$, *these statements are equivalent to*

3°. h *is antitone and all* $S_y(h)$ $(y \in F)$ *are* non-empty and closed.

Proof. 1° ⇒ 2°. Assume 1°. Then, applying (2.11) to $\{a_1, a_2\} \subseteq A$, where $a_1 \leq a_2$, it follows that h is antitone.

Furthermore, let $y \in F$, $\emptyset \neq M \subseteq S_y(h)$. Then, by (2.11),

$$h(\inf{}^A M) = \sup_{a \in M} h(a) \leq y,$$

so $\inf^A M \in S_y(h)$. Finally, by (2.11) for $I = \emptyset$, we have $h(+\infty^A) = -\infty^F \leq y$ $(y \in F)$, whence $\inf^A \emptyset =$ $= +\infty^A \in S_y(h)$ $(y \in F)$.

2° ⇒ 1°. Assume 2°, and let $\{a_i\}_{i \in I} \subseteq A$, $I \neq \phi$. Then $\inf_{j \in I}^A a_j \leq a_i$ $(i \in I)$, whence, since h is antitone, we obtain

$$h(\inf^A_{i\in I} a_i) \geq \sup_{i\in I} h(a_i). \qquad (2.13)$$

In order to prove the opposite inequality, note that for $y_0 = \sup_{i\in I} h(a_i) \in F$ we have $a_i \in S_{y_0}(h)$ $(i\in I)$, whence, since $S_{y_0}(h)$ is closed for $\inf^A$, we obtain $\inf^A_{i\in I} a_i \in S_{y_0}(h)$, that is,

$$h(\inf^A_{i\in I} a_i) \leq y_0 = \sup_{i\in I} h(a_i). \qquad (2.14)$$

Furthermore, by 2° we have $+\infty^A = \inf^A \emptyset \in S_y(h)$ $(y\in F)$, whence $h(+\infty^A) = -\infty^F$, i.e., (2.11) holds also for $I = \emptyset$.

Finally, assume that $A \subseteq \overline{R}$ is closed for inf and closed.

2°⇒3°. Assume 2° and let $y\in F$, $\{a_n\} \subseteq S_y(h)$, $\lim_{n\to\infty} a_n = a \in A$. If there exists n_0 such that $a_{n_0} \leq a$, then, since $a_{n_0} \in S_y(h)$ and h is antitone, we have $y \geq h(a_{n_0}) \geq h(a)$, so $a\in S_y(h)$. On the other hand, if no such n_0 exists, then $a = \inf_n a_n$ and hence, by (2.11) (for $I = \{1,2,\ldots\}$), we obtain $h(a) = h(\inf_n a_n) = \sup_n h(a_n) \leq y$, so $a\in S_y(h)$.

3°⇒2°. Assume 3° and let $y\in F$, $\emptyset \neq M \subseteq S_y(h)$. Then, since $\inf M \in A$ and $S_y(h)$ is closed, we have $\inf^A M = \inf M \in S_y(h)$. Finally, since h is antitone and $S_y(h) \neq \emptyset$, say $a\in S_y(h)$, $a \leq +\infty^A$, we have $h(+\infty^A) \leq h(a) \leq y$, so $+\infty^A \in S_y(h)$. ∎

Combining theorem 2.1 and lemma 2.2, we obtain

Theorem 2.2. *For X, A and F as in theorem 2.1, and for a mapping* $\Delta : A^X \to F$, *the following statements are equivalent:*

1°. *Δ is a duality.*

2°. *There exists a mapping* $\Gamma : X \times A \to F$, *such that all "partial mappings"* $\Gamma(x,.) : a \to \Gamma(x,a)$ $(x\in X)$, *from A into F, are antitone, and all level sets*

$$S_y(\Gamma(x,.)) = \{a\in A \mid \Gamma(x,a) \leq y\} \qquad (x\in X, y\in Y) \qquad (2.15)$$

are closed for $\inf^A$, *and such that* (2.3) *holds.*

If $A \subseteq \overline{R}$ *is closed for* inf *and closed, these statements are equivalent to*

3°. *Same as 2°, with "closed for* $\inf^A$*" replaced by* "non-empty and closed".

Moreover, in these cases, Γ *is uniquely determined by* Δ, *namely, we have* (2.4).

Remark 2.2. By (2.7), formula (2.3) of theorems 2.1 and 2.2 can be replaced by the equivalent formula

$$f^{\Delta} = \sup_{x \in \mathrm{dom}\, f} \Gamma(x, f(x)) \qquad (f \in A^X). \tag{2.16}$$

Let us consider now the particular case when $A = \{0, +\infty\}$ (which is closed for inf and closed).

Theorem 2.3. *Let X be a set and F a complete lattice. For a mapping* $\Delta: \{0, +\infty\}^X \to F$, *the following statements are equivalent:*

1°. *Δ is a duality.*

2°. *There exists a mapping $\gamma: X \to F$, such that*

$$f^{\Delta} = \sup_{x \in \zeta(f)} \gamma(x) \qquad (f \in \{0, +\infty\}^X), \tag{2.17}$$

where $\zeta(f)$ is the set (0.9).

Moreover, in this case, γ is uniquely determined by Δ, namely, we have

$$\gamma(x) = (\chi_{\{x\}})^{\Delta} \qquad (f \in \{0, +\infty\}^X). \tag{2.18}$$

Proof. 1° ⇒ 2°. Assume 1° and let $\Gamma: X \times A \to F$ be as in theorem 2.1, with $A = \{0, +\infty\}$. Then, by (2.3), (2.7), (2.4) (with a=0) and (2.18), we obtain

$$f^{\Delta} = \sup_{x \in X} \Gamma(x, f(x)) = \sup_{x \in \zeta(f)} \Gamma(x, 0) = \sup_{x \in \zeta(f)} \gamma(x) \qquad (f \in \{0, +\infty\}^X). \tag{2.19}$$

2° ⇒ 1°. Note that, for any set I, we have

$$\zeta(\inf_{i \in I} f_i) = \bigcup_{i \in I} \zeta(f_i) \qquad (\{f_i\}_{i \in I} \subseteq \{0, +\infty\}^X). \tag{2.20}$$

Hence, if 2° holds, then, by (2.17), we obtain

$$(\inf_{i \in I} f_i)^{\Delta} = \sup_{x \in \zeta(\inf_{i \in I} f_i)} \gamma(x) = \sup_{x \in \bigcup_{i \in I} \zeta(f_i)} \gamma(x) =$$

$$= \sup_{i \in I} \sup_{x \in \zeta(f_i)} \gamma(x) = \sup_{i \in I} f_i^{\Delta} \qquad (\{f_i\}_{i \in I} \subseteq \{0, +\infty\}^X).$$

Finally, to prove the last statement, assume that γ is as in 2°. Then, by (2.17) for $f = \chi_{\{x\}}$, we obtain

$$(\chi_{\{x\}})^{\Delta} = \sup_{x' \in \zeta(\chi_{\{x\}})} \gamma(x') = \gamma(x) \qquad (x \in X). \qquad ■$$

Remark 2.3. One can also give the following direct proof of the implication $1^\circ \Rightarrow 2^\circ$, with γ of (2.18): We have

$$f = \chi_{\zeta(f)} \qquad (f \in \{0, +\infty\}^X), \tag{2.21}$$

where, for any set $M \subseteq X$,

$$\chi_M(x') = 0 \quad \text{if} \quad x' \in M, \tag{2.22}$$
$$= +\infty \quad \text{if} \quad x' \notin M.$$

Hence, by 1°, we obtain

$$f^{\Delta} = (\chi_{\zeta(f)})^{\Delta} = (\inf_{x \in \zeta(f)} \chi_{\{x\}})^{\Delta} = \sup_{x \in \zeta(f)} (\chi_{\{x\}})^{\Delta} \qquad (f \in \{0, +\infty\}^X). \tag{2.23}$$

§ 3. DUALITIES $\Delta : A^X \to B^W$.

Theorem 3.1. *Let* X *and* W *be two sets and let* $(A, \leq) \subseteq (\overline{R}, \leq)$ *and* $(B, \leq) \subseteq (\overline{R}, \leq)$ *be complete lattices. For a mapping* $\Delta : A^X \to B^W$, *the following statements are equivalent:*

1°. Δ *is a duality.*

2°. *There exists a mapping* $G : X \times W \times A \to B$, *satisfying, for every index set* I,

$$G(x, w, \inf^A_{i \in I} a_i) = \sup^B_{i \in I} G(x, w, a_i) \qquad (x \in X, w \in W, \{a_i\}_{i \in I} \subseteq A), \tag{3.1}$$

and such that

$$f^{\Delta}(w) = \sup^B_{x \in X} G(x, w, f(x)) \qquad (f \in A^X, w \in W). \tag{3.2}$$

Moreover, in this case, G *is uniquely determined by* Δ, *namely, we have*

$$G(x, w, a) = (\chi_{\{x\}} \dotplus a)^{\Delta}(w) \qquad (x \in X, w \in W, a \in A). \tag{3.3}$$

Proof. $1^\circ \Rightarrow 2^\circ$. If 1° holds, and $\Gamma : X \times A \to B^W = F$ is as in 2° of theorem 2.1, then for $G : X \times W \times A \to B$ defined by

$$G(x, w, a) = \Gamma(x, a)(w) \qquad (x \in X,\ w \in W,\ a \in A). \tag{3.4}$$

we have (3.1) and (3.2) (by (2.2), (2.3) and since the sup in B^W is defined pontwise).

$2^\circ \Rightarrow 1^\circ$. If $G : X \times W \times A \to B$ is as in 2°, then for $\Gamma : X \times A \to B^W = F$ defined by (3.4) we have (2.2) and (2.3), so we can apply theorem 2.1, implication $2^\circ \Rightarrow 1^\circ$.

Finally, (2.4) and (3.4) imply (3.3). ∎

Combining theorem 3.1 and lemma 2.2 (applied to $h_{x,w}(a) = G(x, w, a)$ and $F = (B, \leq) \subseteq (\overline{R}, \leq)$), we obtain

Theorem 3.2. *For* X, W, A *and* B *as in theorem* 3.1, *and for a mapping* $\Delta : A^X \to B^W$, *the following statements are equivalent:*

1°. Δ *is a duality.*

2°. *There exists a mapping* $G : X \times W \times A \to B$ *such that all partial mappings* $G(x, w, .) : a \to G(x,w,a)$ $(x \in X, w \in W)$, *from* A *into* B, *are non-increasing and all level sets*

$$S_y(G(x, w, .)) = \{a \in A \mid G(x, w, a) \leq y\} \qquad (y \in B) \tag{3.5}$$

are closed for $\inf^A$, *and such that we have* (3.2).

If $A \subseteq \overline{R}$ *is closed for inf and closed, these statements are equivalent to*

3°. *Same as* 2°, *with "all level sets* (3.5) *are closed for* $\inf^A$*" replaced by : all* $G(x, w, .)$ $(x \in X, w \in W)$ *are lower semi-continuous* and $G(x,w,+\infty^A) = -\infty^B$ $(x \in X,\ w \in W)$.

Moreover, in these cases, G *is uniquely determined by* Δ, *namely, we have* (3.3).

Remark 3.1. Similarly to remark 2.2, one can replace $\sup_{x \in X}$ by $\sup_{x \in \mathrm{dom}\, f}$ in formula (3.2) of theorems 3.1 and 3.2.

In the particular case when $A = \{0, +\infty\}$, we obtain

Theorem 3.3. *Let* X *and* W *be two sets, and let* $(B, \leq) \subseteq (\overline{R}, \leq)$ *be a comple lattice. For a mapping* $\Delta : \{0, +\infty\}^X \to B^W$, *the following statements are equivalent:*

1°. Δ *is a duality.*

2°. *There exists a mapping* $\varphi : X \times W \to B$, *such that*

$$f^{\Delta}(w) = \sup_{x \in \zeta(f)}^{B} \varphi(x, w) \qquad (f \in \{0, +\infty\}^X,\ w \in W), \tag{3.6}$$

Moreover, in this case, φ *is uniquely determined by* Δ, *namely, we have*

$$\varphi(x, w) = (\chi_{\{x\}})^{\Delta}(w) \qquad (x \in X,\ w \in W). \tag{3.7}$$

Proof. 1° ⇒ 2°. If 1° holds and $\gamma : X \to B^W = F$ is as in 2° of theorem 2.3, then for $\varphi : X \times W \to B$ defined by

$$\varphi(x, w) = \gamma(x)(w) \qquad (x \in X,\ w \in W), \tag{3.8}$$

we have (3.6) (by (2.17) and since the sup in B^W is defined pointwise).

2° ⇒ 1°. If $\varphi : X \times W \to B$ is as in 2°, then for $\gamma : X \to B^W = F$ defined by (3.8) we have (2.17), so we can apply theorem 2.3, implication 2° ⇒ 1°.

Finally, (2.18) and (3.8) imply (3.7). ∎

Remark 3.2. One can show that the particular case $B = \{0, +\infty\}$, of theorem 3.3, is equivalent to [18], theorem 1.1, which states that *if* X *and* W *are two sets, then for a mapping* $\Delta : (2^X, \supseteq) \to (2^W, \supseteq)$, *the following statements are equivalent:*

1°. Δ *is a duality.*

2°. *There exists a set* $\Omega \subseteq X \times W$ *such that*

$$\Delta(M) = \{w \in W \mid (x, w) \in \Omega\ (x \in M)\} \qquad (M \subseteq X). \tag{3.9}$$

Moreover, in this case, Ω *is uniquely determined by* Δ, *namely, we have*

$$\Omega = \{(x, w) \in X \times W \mid w \in \Delta(\{x\})\}. \tag{3.10}$$

Indeed, it is well known that the mapping $\zeta : f \to \zeta(f)$ is a complete lattice isomorphism of $\{0, +\infty\}^X$ onto $(2^X, \supseteq)$, with inverse $\zeta^{-1}(M) = \chi_M$ $(M \subseteq X)$. Hence, in one direction, to each duality $\Delta : (2^X, \supseteq) \to (2^W, \supseteq)$ there corresponds the duality $\delta : \{0, +\infty\}^X \to \{0, +\infty\}^W$ defined by $f^{\delta} = \chi_{\Delta(\zeta(f))}$ $(f \in \{0, +\infty\}^X)$ and to each $\varphi : X \times W \to \{0, +\infty\} = B$ there corresponds $\Omega = \zeta(\varphi) \subseteq X \times W$; in the converse direction, to each duality $\Delta : \{0, +\infty\}^X \to \{0, +\infty\}^W$ there corresponds the duality $\delta : (2^X, \supseteq) \to (2^W, \supseteq)$ defined by $\delta(M) = \zeta((\chi_M)^\Delta)$ $(M \subseteq X)$ and to each set $\Omega \subseteq X \times W$ there corresponds the mapping $\varphi = \chi_\Omega : X \times W \to \{0, +\infty\} = B$.

We shall use the notation $U = [0, 1]$. We recall that, given a set X, each $f \in U^X$ is called a *fuzzy subset* of X, and a fuzzy subset f_1 is said to contain a fuzzy subset f_2, if $f_1 \leq f_2$ in $(U^X, \leq)$, where U is endowed with the natural order $\leq$ induced by $(\overline{R}, \leq)$ and $\leq$ on U^X is the usual pointwise order (see e.g. [24]). Although the set $(U, \leq) \subseteq (\overline{R}, \leq)$ is neither closed for inf, nor closed for sup, it is a complete lattice, with $+\infty^U = 1, -\infty^U = 0$, and we have

$$\sup{}^U M = \sup M, \quad \inf{}^U M = \inf M \qquad (\emptyset \neq M \subseteq U); \tag{3.11}$$

also, U is closed. Hence, from theorems3.1 and 3.2, we obtain

Theorem 3.4. *Let* X *and* W *be two sets and let* $U = ([0, 1], \leq) \subseteq (\overline{R}, \leq)$. *For a mapping* $\Delta : U^X \to U^W$, *the following statements are equivalent:*

1°. Δ *is a duality.*

2°. *There exists a mapping* $G : X \times W \times U \to U$, *satisfying, for every index set* $I \neq \emptyset$,

$$G(x, w, \inf_{i \in I} a_i) = \sup_{i \in I} G(x, w, a_i) \qquad (x \in X, w \in W, \{a_i\}_{i \in I} \subseteq U), \tag{3.12}$$

$$G(x, w, 1) = 0, \tag{3.13}$$

and such that

$$f^\Delta(w) = \sup_{x \in X} G(x, w, f(x)) \qquad (f \in U^X, w \in W). \tag{3.14}$$

3°. *There exists a mapping* $G : X \times W \times U \to U$, *satisfying* (3.13), *such that all partial mappings* $G(x, w, .) : U \to U$ $(x \in X, w \in W)$ *are non-increasing and lower semi- continuous and such that we have* (3.14).

Moreover, in these cases, G *is uniquely determined by* Δ, *namely, we have* (3.3) *with* $A = U$ *(where* $\chi_{\{x\}} \dotplus a \in U^X$ *is defined by* (1.10) *with* $A = U, +\infty^A = 1$).

Remark 3.3. Theorem 3.4 can be also deduced from the particular case $A = B = \overline{R}$ of theorems 3.1 and 3.2, using e.g. the homeomorphism ψ of $[0, 1]$ onto $\overline{R} = [-\infty, +\infty]$ defined by

$$\psi(\alpha) = \operatorname{tg}\left(\pi\alpha - \frac{\pi}{2}\right) \text{ if } 0 < \alpha < 1, \quad \psi(0) = -\infty, \quad \psi(1) = +\infty, \tag{3.15}$$

and the observation that $\Delta : U^X \to U^W$ is a duality if and only if there exists a (unique) duality $\delta : \overline{R}^X \to \overline{R}^W$ such that

$$f^\Delta = \psi^{-1} \circ (\psi \circ f)^\delta \qquad (f \in U^X); \tag{3.16}$$

indeed, then, for $G' : X \times W \times \overline{R} \to \overline{R}$ corresponding to δ by theorems 3.1 and 3.2, the mapping $G : X \times W \times U \to$ $\to U$ defined by

$$G(x, w, a) = (\psi^{-1} \circ G')(x, w, \psi(a)) \qquad (x \in X, w \in W, a \in U) \tag{3.17}$$

has the properties stated in theorem 3.4.

Returning to general $(A,\leq), (B,\leq) \subseteq (\overline{R},\leq)$, let us consider now the *dual operator* $\Delta^* : B^W \to A^X$ defined by (1.25), that is,

$$g^{\Delta^*} = \inf^{A^X} \{f \in A^X \mid f^\Delta \leq g\} \qquad (g \in B^W). \tag{3.18}$$

Theorem 3.5. *Let* X *and* W *be two sets,* $(A, \leq) \subseteq (\overline{R},\leq)$ and $(B, \leq) \subseteq (\overline{R},\leq)$ *two complete lattices,* $\Delta : A^X \to B^W$ *a duality, with dual* $\Delta^* : B^W \to A^X$ (of (3.18)), *and* $G : X \times W \times A \to B$, $G^* : W \times X \times B \to A$ *the mappings corresponding to them by theorem* 3.1; *i.e., the unique mapping* G *satisfying* (3.1), (3.2), *given by* (3.3), *and the unique mapping* G^* *satisfying, for every index set* I,

$$G^*(w, x, \inf^B_{i \in I} b_i) = \sup^A_{i \in I} G^*(w, x, b_i) \qquad (w \in W, x \in X, \{b_i\}_{i \in I} \subseteq B), \tag{3.19}$$

$$g^{\Delta^*}(x) = \sup^A_{w \in W} G^*(w, x, g(w)) \qquad (g \in B^W, x \in X), \tag{3.20}$$

given by

$$G^*(w, x, b) = (\chi_{\{w\}} \dotplus b)^{\Delta^*}(x) \qquad (w \in W, x \in X, b \in B). \tag{3.21}$$

Then, we have

$$G^*(w, x, b) = \min \{a \in A \mid G(x, w, a) \leq b\} \qquad (w \in W, x \in X, b \in B). \tag{3.22}$$

Proof. Let Y_1 and T_1 be the families of infimal generators of $E = (A^X, \leq)$ and $F = (B^W, \leq)$ respectively, provided by example 1.4, that is, Y_1 of (1.12) and

$$T_1 = \{\chi_{\{w\}} \dotplus b \mid w \in W, b \in B\}. \tag{3.23}$$

Then, by (3.21), proposition 1.2 (for $Y = Y_1$), (1.10), lemma 2.1 (for $(B^W, \leq)$ and T_1), and (3.3), we have

$$G^*(w, x, b) = (\chi_{\{w\}} \dotplus b)^{\Delta^*}(x) =$$

$$= \inf^A \{\chi_{\{x'\}}(x) \dotplus a \mid x' \in X, a \in A, (\chi_{\{x'\}} \dotplus a)^\Delta \leq \chi_{\{w\}} \dotplus b\} =$$

$$= \inf^A \{a \in A \mid (\chi_{\{x\}} \dotplus a)^\Delta (w) \leq b\} =$$

$$= \inf^A \{a \in A \mid G(x, w, a) \leq b\} \qquad (w \in W, x \in X, b \in B). \tag{3.24}$$

But, from (3.24) and (3.1) we obtain

$$G\big(x, w, G^*(w, x, b)\big) = \sup^B \{G(x, w, a) \mid a \in A, G(x, w, a) \leq b\} \leq b,$$

so the last $\inf^A$ in (3.24) is attained for $G^*(w, x, b) \in A$, which proves (3.22). ∎

Remark 3.4. One can also give a direct proof of theorem 3.5 (which does not use proposition 1.2), by showing that for the mapping $H : W \times X \times B \to A$ defined by

$$H(w, x, b) = \inf^A \{a \in A \mid G(x, w, a) \leq b\} \qquad (w \in W, x \in X, b \in B), \tag{3.25}$$

the $\inf^A$ in (3.25) is attained for $G^*(w, x, b) \in A$ and we have $H = G^*$ of (3.19) - (3.21). We omit the details.

Let us consider now, for any duality $\Delta : A^X \to B^W$, the hull operator $\Delta^*\Delta : A^X \to A^X$. As usual, we shall denote $\Delta^*\Delta(f)$ by $f^{\Delta\Delta^*}$, for any $f \in A^X$.

Lemma 3.1. *Under the assumptions of theorem 3.5, for any* $f \in A^X$, $w \in W$ *and* $b \in B$ *we have the equivalence*

$$f^\Delta(w) \leq b \Leftrightarrow G^*(w, \cdot, b) \leq f. \tag{3.26}$$

Proof. By lemma 2.1, (3.21) and (1.26), we obtain

$$f^\Delta(w) \leq b \Leftrightarrow f^\Delta \leq \chi_{\{w\}} \dotplus b \Leftrightarrow G^*(w, \cdot, b) = (\chi_{\{w\}} \dotplus b)^{\Delta^*} \leq f.$$

Theorem 3.6. *Under the assumptions of theorem 3.5, we have*

$$f^{\Delta\Delta^*} = \sup^{A^X} \{G^*(w, \cdot, b) \mid w \in W, b \in B, G^*(w, \cdot, b) \leq f\} \qquad (f \in A^X). \tag{3.27}$$

Proof. By (3.20) (for $g = f^\Delta$), (3.19) and lemma 3.1, we obtain

$$f^{\Delta\Delta^*}(x) = \sup^A_{w \in W} G^*(w, x, f^\Delta(w)) = \sup^A_{w \in W} G^*(w, x, \inf^B \{b \in B \mid f^\Delta(w) \leq b\}) =$$

$$= \sup^A_{w \in W} \sup^A \{G^*(w, x, b) \mid b \in B, f^\Delta(w) \leq b\} =$$

$$= \sup^A \{G^*(w, x, b) \mid b \in B, w \in W, G^*(w, \cdot, b) \leq f\} \qquad (f \in A^X, x \in X). \quad ∎$$

The following problem has been raised in [20], section 3.3 : If X a set and $W \subseteq \overline{R}^X$, does there exist a duality $\Delta : \overline{R}^X \to \overline{R}^W$ such that

$$f^{\Delta\Delta^*} = f_{\mathfrak{X}(W)} \overset{df}{=} \sup \{w \in W \mid w \leq f\} \qquad (f \in \overline{R}^X), \tag{3.28}$$

where $f_{\mathfrak{X}(W)}$ is the "W-convex hull" of f, in the sense of [4]? It is well-known that if $W + R \subseteq W$, then the Fenchel conjugation $c: \overline{R}^X \to \overline{R}^W$ of [10], [11], [4], defined by

$$f^c(w) = \sup_{x \in X} \{w(x) \dot{+} - f(x)\} \qquad (f \in \overline{R}^X, w \in W), \tag{3.29}$$

satisfies (3.28); however, this assumption is not satisfied e.g. when X is a linear space and $W = X^{\#}$, the family of all linear functionals on X. Returning to the general case, in [20] it has been observed that if we define $\Delta : \overline{R}^X \to \overline{R}^W$ by

$$f^{\Delta} = \chi_{\{w' \in W \mid w' \leq f\}} \qquad (f \in \overline{R}^X), \tag{3.30}$$

then $f_{\mathfrak{X}(W)}$ is a "mixed" second dual of f, namely, we have

$$f_{\mathfrak{X}(W)} = (f^{\Delta})^{c^*} \qquad (f \in \overline{R}^X), \tag{3.31}$$

where $c^* : \overline{R}^W \to \overline{R}^X$ is the dual (in the sense (3.18)) of (3.29). A solution to the above problem can be obtained by regarding Δ of (3.30) as a mapping from $\overline{R}^X$ into $\{0, +\infty\}^W$ (instead of $\overline{R}^W$). Indeed, we shall prove

Theorem 3.7. *If X is a set and $W \subseteq \overline{R}^X$, then for $\Delta : \overline{R}^X \to \{0, +\infty\}^W$ defined by (3.30) we have (3.28).*

Proof. Let $A = \overline{R}$, $B = \{0, +\infty\}$. Then by (3.3), (3.30) and lemma 2.1, we have

$$G(x, w, a) = (\chi_{\{x\}} \dot{+} a)^{\Delta}(w) = \chi_{\{w' \in W \mid w' \leq \chi_{\{x\}} \dot{+} a\}}(w) =$$

$$= \chi_{\{w' \in W \mid w'(x) \leq a\}}(w) \qquad (x \in X, w \in W, a \in \overline{R}), \tag{3.32}$$

whence, by (3.22),

$$G^*(w,x,b) = \min \{a \in \overline{R} \mid \chi_{\{w' \in W \mid w'(x) \leq a\}}(w) \leq b\} =$$

$$= \begin{cases} \min \{a \in \overline{R} \mid w(x) \leq a\} = w(x) & \text{if } b = 0 \\ -\infty \quad \text{if } b = +\infty. \end{cases} \tag{3.33}$$

Therefore, by (3.20),

$$g^{\Delta^*} = \sup_{w \in W} G^*(w, ., g(w)) = \sup \{w \in W \mid g(w) = 0\} \qquad (g \in B^W), \tag{3.34}$$

whence, in particular, for $g = f^{\Delta}$ we obtain, by (3.30),

$$f^{\Delta\Delta^*} = \sup\{w \in W \mid f^{\Delta}(w) = 0\} = \sup\{w \in W \mid \chi_{\{w' \in W \mid w' \leq f\}}(w) = 0\} =$$

$$= \sup\{w \in W \mid w \leq f\} = f_{\mathfrak{X}(W)} \qquad (f \in \overline{R}^X). \quad \blacksquare$$

Proposition 3.1. *For* $\Delta : R^X \to B^W$ *of* (3.30), *where* $B = \{0, +\infty\}$, *and* $c : \overline{R}^X \to \overline{R}^W$ *of* (3.29), *we have*

$$f^{\Delta}(w) = \inf\{b \in B \mid f^c(w) \leq b\} \qquad (f \in \overline{R}^X, w \in W). \tag{3.35}$$

Proof. By $B = \{0, +\infty\}$ and (3.29), for each $f \in \overline{R}^X$ we have

$$\inf\{b \in B \mid f^c(w) \leq b\} =$$

$$= \inf\{b \in \{0, +\infty\} \mid \sup_{x \in X}\{w(x) \dotplus -f(x)\} \leq b\} =$$

$$= \begin{cases} 0 & \text{if } w \leq f \\ +\infty & \text{if } \exists\, x_0 \in X,\ w(x_0) > f(x_0), \end{cases}$$

which coincides with $f^{\Delta}(w)$, for f^{Δ} of (3.30). ■

As an application of theorem 3.7, let us prove

Theorem 3.8. *Let* $X = \{0, 1\}^n$, *let* $W = (R^n)^*|_X \subset R^X$, *where* $(R^n)^*$ *is the family of all linear functionals on* R^n, *and let* $\Delta : \overline{R}^X \to \{0, +\infty\}^W$ *be the duality* (3.30). *For a function* $f \in \overline{R}^X$, *denoting* $D = \operatorname{dom} f$, *the following statements are equivalent:*

1°. $f = f^{\Delta\Delta^*}$.

2°. *Either* $f \equiv -\infty$, *or* f *has the following properties:*

(i) $f \in (R \cup \{+\infty\})^X$;

(ii) $0 \leq f(0)$;

(iii) *for any* $\lambda_x \geq 0$ $(x \in D)$ *such that* $\sum_{x \in D} \lambda_x x \in X$, *we have* $\sum_{x \in D} \lambda_x x \in D$ *and*

$$f\Big(\sum_{x \in D} \lambda_x x\Big) \leq \sum_{x \in D} \lambda_x f(x)\,. \tag{3.36}$$

Proof. 1° ⇒ 2°. Assume 1° and assume that $f \not\equiv -\infty$, so there exists $x_0 \in X$ such that $f(x_0) > -\infty$. Then, by (3.28), we have

$$-\infty < f(x_0) = f^{\Delta\Delta^*}(x_0) = \sup\{w(x_0) \mid w \in W, w \leq f\},$$

whence

$$\{w \in W \mid w \leq f\} \neq \emptyset \tag{3.37}$$

and hence, again by 1° and (3.28), we obtain

$$f(x) = f^{\Delta\Delta^*}(x) = \sup\{w(x) \mid w \in W, w \leq f\} > -\infty \qquad (x \in X),$$

so (i) holds. Furthermore, by 1°, (3.28), $W = (R^n)^* |_X$ and (3.37), we have

$$f(0) = f^{\Delta\Delta^*}(0) = \sup\{w(0) \mid w \in W, w \leq f\} = 0,$$

that is, (ii). Finally, if $\lambda_x \geq 0$ $(x \in D)$ are such that $\sum_{x \in D} \lambda_x x \in X$, then, by 1°, (3.28) and $W = (R^n)^* |_X$ we obtain

$$f\Big(\sum_{x \in D} \lambda_x x\Big) = f^{\Delta\Delta^*}\Big(\sum_{x \in D} \lambda_x x\Big) = \sup\Big\{\sum_{x \in D} \lambda_x w(x) \mid w \in W, w \leq f\Big\} \leq$$

$$\leq \sum_{x \in D} \lambda_x \sup\{w(x) \mid w \in W, w \leq f\} = \sum_{x \in D} \lambda_x f^{\Delta\Delta^*}(x) = \sum_{x \in D} \lambda_x f(x),$$

so (iii) is satisfied.

2° ⟹ 1°. If $f \equiv -\infty$, then $f^{\Delta\Delta^*} \leq f \equiv -\infty$, whence $f = f^{\Delta\Delta^*}$.

Assume now that $f \in \overline{R}^X$ satisfies (i)-(iii) (then, by (ii), $f \not\equiv -\infty$). We claim that (3.37) holds. Indeed, by $X = \{0, 1\}^n$ and (i), for any sufficiently large k the function $w_k \in W$ defined by

$$w_k(x) = -k \sum_{i=1}^{n} \xi_i \qquad (x = \{\xi_j\}_1^n \in X) \tag{3.38}$$

satisfies $w_k(x) \leq f(x)$ $(x \in X \setminus \{0\})$, which, together with (ii), implies that $w_k \in \{w \in W \mid w \leq f\}$, proving the claim (3.37). Furthermore, by (iii) (for $\lambda_x = 0$, $x \in D$), we have $0 \in D$ and $f(0) \leq 0$; note also that, by (ii), this yields

$$f(0) = 0. \tag{3.39}$$

For any $x_0 \in X$ we have, by (3.28), (3.37) and linear programming duality theory,

$$f^{\Delta\Delta^*}(x_0) = \sup\{w(x_0) \mid w \in W, w(x) \leq f(x) \ (x \in D)\} =$$

$$= \inf\Big\{\sum_{x \in D} \lambda_x f(x) \mid \sum_{x \in D} \lambda_x x = x_0, \ \lambda_x \geq 0 \ (x \in D)\Big\}. \tag{3.40}$$

Indeed, the first program in (3.40) is feasible by (3.37). If $x_0 \in D$, then the dual program, i.e., the second one in (3.40), is also feasible, since for $\lambda_{x_0} = 1$, $\lambda_x = 0$ $(x \in D \setminus \{x_0\})$, we have $\sum_{x \in D} \lambda_x x = x_0$; hence, both programs admit optimal solutions and have the same value (e.g. by [22], theorem (1.7.13) (ii)). On the other hand, if $x_0 \notin D$, then, by (iii), we have

$$\Big\{(\lambda_x)_{x \in D} \mid \sum_{x \in D} \lambda_x x = x_0, \ \lambda_x \geq 0 \ (x \in D)\Big\} = \emptyset, \tag{3.41}$$

i.e., the second program in (3.40) is not feasible. Hence, we have again (3.40), with all terms being $= +\infty$ (by $\inf \emptyset = +\infty$ and e.g. [16], p. 114, theorem 5.1). This proves the claim (3.40).

Now, if $x_0 \in D$, then, by (3.40) and (iii), we obtain

$$f^{\Delta\Delta^*}(x_0) \geq \inf \left\{ f(\sum_{x\in D} \lambda_x x) \mid \sum_{x\in D} \lambda_x x = x_0,\ \lambda_x \geq 0\ (x\in D) \right\} =$$

$$= f(x_0) \geq f^{\Delta\Delta^*}(x_0), \tag{3.42}$$

whence $f^{\Delta\Delta^*}(x_0) = f(x_0)$. Finally, if $x_0 \notin D$, then, by (3.40) and the above proof of (3.41), we obtain $f^{\Delta\Delta^*}(x_0) = +\infty = f(x_0)$. ■

We recall (see [14], p. 172) that a function $f: X \to \bar{R}$, where $X \subseteq R^n$, is called *polyhedral convex*, if there exist affine functions $\psi_i = w_i + d_i : R^n \to R$ ($w_i \in (R^n)^*$, $d_i \in R$; $i = 1, \ldots, k$; $k < +\infty$), such that $g(x) = \max_{1\leq i\leq k} \psi_i(x)$ $(x\in X)$. A function $f: R^n \to \bar{R}$ is called a) *proper* ([14], p. 24) if $f(R^n) \subset R \cup \{+\infty\}$ and $f \not\equiv +\infty$ (i.e., $f(x) < +\infty$ for at least one x); b) *positively homogeneous*, if $f(\lambda x) = \lambda f(x)$ for all $x \in R^n$ and $\lambda \geq 0$.

Corollary 3.1. a) *Under the assumptions of theorem* 3.8, *for a mapping* $f \in \bar{R}^X$ *the following statements are equivalent:*

1°. $f = f^{\Delta\Delta^*}$.

2°. *Either* $f \equiv \pm\infty$, *or* f *can be extended to a proper lower semi-continuous convex positively homogeneous function* $\hat{f}$ *on* R^n.

b) *Every function* $f: X \to R$ *satisfying* 1° *can be extended to a polyhedral convex positively homogeneous function* $\hat{f}$ *on* R^n.

Proof. a) 1° ⇒ 2°. If 1° holds and $f \not\equiv \pm\infty$, let

$$\hat{f}(x) = \sup \{ \hat{w}(x) \mid \hat{w} \in (R^n)^*, \hat{w}|_X \leq f \} \qquad (x \in R^n). \tag{3.43}$$

Then, $\hat{f}$ is a lower semi-continuous convex positively homogeneous function, and, by $W = (R^n)^*|_X$, (3.28) and 1°, we have $\hat{f}|_X = f^{\Delta\Delta^*} = f$. Hence, by $f \not\equiv +\infty$, we have $\hat{f} \not\equiv +\infty$.

Finally, if there exists $x_0 \in R^n$ such that $\hat{f}(x_0) = -\infty$, then, by (3.43), $\{\hat{w} \in (R^n)^* \mid \hat{w}|_X \leq f\} = \emptyset$, whence, again by (3.43), $\hat{f} \equiv -\infty$, and hence $f = \hat{f}|_X \equiv -\infty$, in contradiction with our assumption. Thus, $\hat{f}$ is proper.

2° ⇒ 1°. If $f \equiv \pm\infty$, then, by theorem 3.8, $f^{\Delta\Delta^*} = f$. On the other hand, if $f \not\equiv \pm\infty$, then, by 2°, $f = \hat{f}|_X$ satisfies conditions (i)-(iii) of 2° of theorem 3.8, whence $f^{\Delta\Delta^*} = f$.

b) If $f: X \to R$ satisfies 1°, then, by linear programming duality theory (see the above proof of theorem 3.8), for each $x_0 \in X = D$ the sup in (3.40) is attained, for some $w_{x_0} \in W$. Let

$$\hat{f}(x) = \max_{x_0 \in X} \hat{w}_{x_0}(x) \qquad (x \in R^n), \qquad (3.44)$$

where $\hat{w}_{x_0} \in (R^n)^*$ is the unique linear functional such that $\hat{w}_{x_0}|_X = w_{x_0}$.

Then $\hat{f}$ is polyhedral convex and positively homogeneous and, since $w_{x_0} \leq f$, we have $\hat{f}|_X \leq f$; but, by (3.44), (3.40) and 1°, $\hat{f}(x) \geq w_x(x) = f(x)$ $(x \in X)$, whence $\hat{f}|_X = f$. ∎

§ 4. STRICT DUALITIES $\Delta : A^X \to B^W$.

Definition 4.1. Let X and W be two sets and $(A, \leq)$, $(B, \leq) \subseteq (R, \leq)$ two complete lattices. We shall say that a mapping $\Delta : A^X \to B^W$ is a *strict duality*, if Δ is a duality, such that for the mapping $G : X \times W \times A \to B$ defined by (3.3), and for each $(x, w) \in X \times W$, $G(x, w, .)$ is a strictly decreasing mapping of A onto $B(x, w) = \{G(x, w, a) \mid a \in A\} \subseteq B$.

Let us consider the case when $A = B = B(x, w) = \overline{R}$ $\quad ((x, w) \in X \times W)$.

Theorem 4.1. *Let* X *and* W *be two sets. For a mapping* $\Delta : \overline{R}^X \to \overline{R}^W$, *the following statements are equivalent:*

1°. Δ *is a strict duality, with* $B(x, w) = \overline{R}$ $\quad ((x, w) \in X \times W)$.

2°. *There exists a mapping* $F : W \times X \times \overline{R} \to \overline{R}$, *such that for each* $(w, x) \in W \times X$, $F(w, x, .)$ *is a strictly decreasing mapping of* $\overline{R}$ *onto* $\overline{R}$ (whence $F(w, x, \mp\infty) = \pm\infty$) and such that

$$f^{\Delta}(w) = \sup_{x \in \operatorname{dom} f} F^*(x, w, f(x)) \qquad (f \in \overline{R}^X, w \in W), \qquad (4.1)$$

where $F^* : X \times W \times \overline{R} \to \overline{R}$ *is the mapping defined by*

$$F^*(x, w, a) = F(w, x, .)^{-1}(a) \qquad (x \in X, w \in W; a \in \overline{R}). \qquad (4.2)$$

Moreover, in this case, F *is uniquely determined by* Δ, *namely,*

$$F(w, x, b) = G(x, w, .)^{-1}(b) \qquad (w \in W, x \in X, b \in \overline{R}), \qquad (4.3)$$

where $G : X \times W \times \overline{R} \to \overline{R}$ *is the mapping defined by* (3.3).

Proof. 1° ⇒ 2°. If 1° holds and $G : X \times W \times \overline{R} \to \overline{R}$ is as in definition 4.1, then for $F : W \times X \times \overline{R} \to \overline{R}$ defined by (4.3) and F^* of (4.2), we have $G = F^*$, whence, by (3.2), we obtain

$$f^{\Delta}(w) = \sup_{x \in X} F^*(x, w, f(x)) \qquad (f \in \bar{R}^X, w \in W). \tag{4.4}$$

But, for $x \in X \setminus \mathrm{dom}\, f$ we have $f(x) = +\infty$, whence, by (4.4) and (3.3), we obtain (4.1).

$2^\circ \Rightarrow 1^\circ$. If 2° holds, then all partial mappings $F^*(x, w, .)$ $(x \in X, w \in W)$, defined by (4.2), are non-increasing and continuous and satisfy $F^*(x,w,+\infty)=-\infty$, whence, by (4.1) and theorem 3.2, implication $3^\circ \Rightarrow 1^\circ$, applied to $G=F^*$, it follows that Δ is a duality.

Finally, again by theorem 3.2, $G = F^*$ is given by (3.3), whence, by (4.2), we obtain (4.3). ∎

Remark 4.1. Let us mention the following related concept of [2]. Let $X, W \subseteq R^n$, $A, B \subseteq R$ and let $F: W \times X \times B \to A$ be a mapping with the following properties: (i) the mappings $F(w, ., b): X \to A$ and $F(., x, .): W \times B \to A$ are continuous; (ii) $(w, b) \to F(w, ., b)$ is a one-to-one mapping of $W \times B$ onto $\mathcal{F} = \{F(w, ., b) | (w, b) \in W \times B\} \subset A^X$; (iii) for every convergent sequence $(w_n, b_n) \subset W \times B$, $(w_n, b_n) \to (w_o, b_o) \in R^n \times R$, we have either $\lim_{n \to \infty} F(w_n, x, b_n) = \pm\infty$ $(x \in X)$ or $(w_o, b_o) \in W \times B$; (iv) for each $(w, x) \in W \times X$, $F(w, x, .)$ is a strictly decreasing mapping of B onto A and the mapping $F^*: X \times W \times A \to B$ defined by (4.2) has the properties corresponding to (i)-(iii) (but, it is not assumed that $(A^X, \leq)$, $(B^W, \leq)$ are complete lattices; for example, if $A = B = R$, then $\pm\infty \notin A^X, B^W$). In this case the mapping $f^{\Delta}: B \to W$ of (4.1), (4.2), with $f \in A^X$, is called, in [2], "the $\mathcal{F}$-conjugate of f", and it is denoted by $f^{\mathcal{F}}$. Thus, if $A = B = R$, then, since each strictly decreasing mapping h of R onto R can be identified with its unique extension to a strictly decreasing mapping $\tilde{h}$ of $\bar{R}$ onto $\bar{R}$ $(\tilde{h}(\pm\infty) = \mp\infty)$, it follows that the mappings $f \to f^{\mathcal{F}}$ of [2], with $A = B = R$, are "particular cases" of strict dualities.

For G^* of (3.21), let $A(w,x) = \{G^*(w,x,b) | b \in B\}$ $((w,x) \in W \times X)$.

Theorem 4.2. *Let X and W be two sets, $\Delta: \bar{R}^X \to \bar{R}^W$ a strict duality, with $B(x,w)=\bar{R}$ $((x,w) \in X \times W)$, and $F: W \times X \times \bar{R} \to \bar{R}$ the mapping corresponding to Δ (by theorem 4.1). Then the dual $\Delta^*: \bar{R}^W \to \bar{R}^X$ (in the sense (3.18)) is a strict duality, with $A(w,x)=\bar{R}$ $((w,x) \in W \times X)$, and for the mapping $F': X \times W \times \bar{R} \to \bar{R}$ corresponding to Δ^* (by theorem 4.1), we have (where F^* is defined by* (4.2))

$$F' = F^*. \tag{4.5}$$

Proof. From (4.3) it follows that

$$G(x, w, a) = F(w, x, .)^{-1}(a) \qquad (x \in X, w \in W, a \in \bar{R}) \tag{4.6}$$

(where G is defined by (3.3)). Furthermore, Δ^* is a duality, and for G^* defined by (3.21) we have, by (3.22) and since each $F(w, x, .)$ $(w \in W, x \in X)$ is a stricly decreasing mapping of $\overline{R}$ onto $\overline{R}$,

$$G^*(w, x, b) = \min \{a \in R \mid F(w, x, .)^{-1}(a) \leq b\} =$$

$$= F(w, x, b) \qquad (w \in W, x \in X, b \in \overline{R}), \qquad (4.7)$$

whence, by (3.20), we obtain

$$g^{\Delta^*}(x) = \sup_{w \in W} F(w, x, g(w)) \qquad (g \in \overline{R}^W, x \in X). \qquad (4.8)$$

But, by (4.2) and our assumption on F, each $F^*(x, w, .)$ $(x \in X, w \in W)$ is a strictly decreasing mapping of $\overline{R}$ onto $\overline{R}$. Hence, by $(F^*)^* = F$, (4.8) and theorem 4.1, implication $2^\circ \Rightarrow 1^\circ$, it follows that Δ^* is a strict duality, with $A(w,x) = \overline{R}$ $((w,x) \in W \times X)$, and that the corresponding mapping F' satisfies (4.5). ∎

§ 5. CONJUGATIONS $c : \overline{R}^X \to \overline{R}^W$

In this section we shall consider conjugations $c : \overline{R}^X \to \overline{R}^W$, defined by (1.5) (with $A = B = \overline{R}$) and (1.7). Let us first show that some results of [17] can be deduced easily from the preceding results.

Theorem 5.1 ([17], theorem 3.1). *Let X and W be two sets. For a mapping $c : \overline{R}^X \to \overline{R}^W$, the following statements are equivalent:*

1°. *c is a conjugation.*

2°. *There exists a mapping $\varphi : X \times W \to \overline{R}$, such that*

$$f^c(w) = \sup_{x \in X} \{\varphi(x, w) \dotplus -f(x)\} \qquad (f \in \overline{R}^X, w \in W). \qquad (5.1)$$

Moreover, in this case, φ is uniquely determined by c, namely, we have

$$\varphi(x, w) = (\chi_{\{x\}})^c(w) \qquad (x \in X, w \in W). \qquad (5.2)$$

Proof. $1^\circ \Rightarrow 2^\circ$. If 1° holds, then c is a duality (by the remark made after formula (1.7)). Let $G : X \times W \times \overline{R} \to \overline{R}$ be as in theorem 3.2 (with $A = B = \overline{R}$ and $c = \Delta$). Then, by (3.3) and (1.7), we have

$$G(x, w, a) = (\chi_{\{x\}})^c(w) \dotplus -a \qquad (x \in X, w \in W, a \in \overline{R}), \qquad (5.3)$$

whence, by (3.2), we obtain (5.1), with $\varphi: X \times W \to \overline{R}$ defined by (5.2). Thus, $1^\circ \Rightarrow 2^\circ$.

The proofs of the implication 2° ⇒ 1° and of the last statement (on φ) are simple and can be found in [17]. ∎

Remark 5.1. a) By (5.2) and (5.3), we have

$$\varphi(x, w) = G(x, w, 0) \qquad (x \in X, w \in W). \tag{5.4}$$

b) By (5.3) and (5.2), we have

$$G(x, w, a) = \varphi(x, w) \dot{+} -a \qquad (x \in X, w \in W, a \in \overline{R}), \tag{5.5}$$

Hence, c *of* (5.1) *is a strict duality* (i.e., G satisfies the condition of definition 4.1) *if and only if* $\varphi(X \times W) \subset R$. This happens e.g. for $W \subset R^X$ and $\varphi(x, w) = w(x)$ $(x \in X, w \in W)$; in this case, if we replace $\overline{R}$ by $A = [0, 1]$, then $c: A^X \to \overline{R}^W$ is still a strict duality in the sense of definition 4.1, with $B(x, w) = w(x) + (-A)$ $(x \in X, w \in W)$.

We recall that if $c: \overline{R}^X \to \overline{R}^W$ is a conjugation, then, by [17], theorem 4.1, the dual operator $c^*: \overline{R}^W \to \overline{R}^X$ (in the sense of (3.18) above) is again a conjugation.

Theorem 5.2. ([17], theorem 4.2 and corollary 4.3). *Let* X *and* W *be two sets,* $c: \overline{R}^X \to \overline{R}^W$ *a conjugation, with dual conjugation* $c^*: \overline{R}^W \to \overline{R}^X$, *and let* $\varphi: X \times W \to \overline{R}$ *and* $\varphi^*: W \times X \to \overline{R}$ *be the mappings associated to* c *and* c^* *respectively, by theorem* 5.1. *Then*

$$(\chi_{\{w\}})^{c^*}(x) = \varphi^*(w, x) = \varphi(x, w) \qquad (x \in X, w \in W). \tag{5.6}$$

Proof. Applying (5.2) to X, W and c replaced by W, X and c^* respectively, we obtain the first equality in (5.6). Furthermore, if $G: X \times W \times \overline{R} \to \overline{R}$ and $G^*: W \times X \times \overline{R} \to \overline{R}$ are the mappings associated to c and c^* respectively, as in theorem 3.2, then, by (5.4), (3.22), (5.3), (5.2) and [11], corollary 3 c) and formula (2.1), we obtain

$$\varphi^*(w, x) = G^*(w, x, 0) = \min \{a \in \overline{R} \mid G(x, w, a) \leq 0\} =$$

$$= \min \{a \in \overline{R} \mid \varphi(x, w) \dot{+} -a \leq 0\} =$$

$$= \min \{a \in \overline{R} \mid \varphi(x, w) \leq a\} = \varphi(x, w) \qquad (x \in X, w \in W). \quad ∎$$

Remark 5.2. If $\varphi(X \times W) \subset R$, one can also give the following proof of theorem 5.2: By remark 5.1 b), c is a strict duality, whence, by (5.4), (4.7), (4.6) and (5.5), we obtain

$$\varphi^*(w,x) = G^*(w,x,0) = F(w,x,0) =$$

$$= G(x,w,.)^{-1}(0) = \varphi(x,w) \qquad (x \in X, w \in W).$$

Now we shall consider the particular case when $X = \{0,1\}^n$, $W = C_1 \times \ldots \times C_n$, where $C_i \subseteq R$ $(i = 1, \ldots, n)$ are unbounded from above and from below (e.g., $W = R^n$, or $W = Z^n$, the set of all points in R^n with integer coordinates) and φ is the "natural coupling function" on $X \times W$, given by the scalar product, i.e.,

$$\varphi(x,w) = w(x) = \sum_{i=1}^{n} \eta_i \xi_i \qquad (x = \{\xi_i\}_1^n \in X, w = \{\eta_i\}_1^n \in W). \tag{5.7}$$

Thus, the functions $f \in \overline{R}^X$ are now functions $f: \{0,1\}^n \to \overline{R}$ (i.e., extended functions of 0-1 variables, or extended "pseudo-boolean" functions [8]). Note also that (5.7) identifies W with a subset of $(R^n)^*$, the family of all linear functions on R^n, and that the conjugations c and c^* are now given by

$$f^c(w) = \max_{x \in X} (w(x) - f(x)) \qquad (f \in \overline{R}^X, w \in W), \tag{5.8}$$

$$g^{c^*}(x) = \sup_{w \in W} (w(x) - g(w)) \qquad (g \in \overline{R}^W, x \in X). \tag{5.9}$$

Remark 5.3. By (5.8) and (5.9), we have

$$f^c = \tilde{f}^*|_W \ , \tag{5.8'}$$

$$g^{c^*} = \tilde{g}^*|_X \ , \tag{5.9'}$$

where $\tilde{f}$ (respectively, $\tilde{g}$) denotes the extension of f (respectively, g) to R^n, such that $\tilde{f}|_{R^n \setminus X} \equiv +\infty$ ($\tilde{g}|_{R^n \setminus W} \equiv +\infty$), and * is the usual conjugation operator of convex analysis.

Theorem 5.3. *Let* $X = \{0,1\}^n$, $W \subseteq R^n$ *and* c *be as above. For a function,* $f \in \overline{R}^X$, *the following statements are equivalent:*

1°. $f = f^{cc^*}$.

2°. $f \in (R \cup \{+\infty\})^X \cup \{-\infty\}$.

Proof. 1° ⇒ 2°. If 1° holds and if there exists $x_0 \in X = \{0,1\}^n$ such that $f(x_0) = -\infty$, then, by (5.8) and 1°,

$$\sup_{w \in W} (w(x_0) \dotplus - f^c(w)) = f^{cc^*}(x_0) = f(x_0) = -\infty,$$

whence, since $w(x_0) \in R$ (by (5.7)), we obtain $f^c \equiv +\infty$. Hence, again by 1°, and (5.9), it follows that $f = f^{cc^*} = (+\infty)^{c^*} \equiv -\infty$.

2° ⇒ 1°. If $f \equiv -\infty$, then, by (5.8) and (5.9), $f^{cc^*} = f$.

Assume now that $f \in (R \cup \{+\infty\})^X$, so $f(X) \subseteq R \cup \{+\infty\}$, and let $x_0 = \{\xi_j^0\}_1^n \in X = \{0,1\}^n$. Then, for each $\lambda \in R$, $\lambda < f(x_0)$, there exists $w_\lambda = \{\eta_1^\lambda, \ldots, \eta_n^\lambda\} \in W$ such that

$$\eta_i^\lambda \leq \min_{\substack{x=\{\xi_j\}_1^n \not\geq x_o \\ f(x)<+\infty}} \frac{f(x)-\lambda}{\sum_{j=1}^n |\xi_j-\xi_j^o|} \qquad \text{if } \xi_i^o = 0, \tag{5.10}$$

$$\eta_i^\lambda \geq \max\left\{ \max_{\substack{x=\{\xi_j\}_1^n < x_o \\ f(x)<+\infty}} \frac{\lambda - f(x)}{\sum_{j=1}^n |\xi_j-\xi_j^o|}, 0 \right\} \qquad \text{if } \xi_i^o = 1, \tag{5.11}$$

where, in R^n, $\leq$ denotes the natural partial order and $<$ means: $\leq$ and $\neq$.

Let $x = \{\xi_j\}_1^n \in \{0,1\}^n$, $x \neq x_o$, $f(x) < +\infty$. If $x \not\leq x_o$, then, by (5.7), (5.10) and (5.11), we have

$$w_\lambda(x) - f(x) = \sum_{i=1}^n \eta_i^\lambda \xi_i - f(x) = \sum_{\xi_i=1} \eta_i^\lambda - f(x) =$$

$$= \sum_{\xi_i=1,\ \xi_i^o=0} \eta_i^\lambda + \sum_{\xi_i=1,\ \xi_i^o=1} \eta_i^\lambda - f(x) \leq$$

$$\leq \sum_{j=1}^n |\xi_j - \xi_j^o| \max_{\xi_i=1,\ \xi_i^o=0} \eta_i^\lambda + \sum_{\xi_i=1,\ \xi_i^o=1} \eta_i^\lambda - f(x) \leq$$

$$\leq f(x) - \lambda + \sum_{\xi_i=1,\ \xi_i^o=1} \eta_i^\lambda - f(x) \leq \sum_{i=1}^n \eta_i^\lambda \xi_i^o - \lambda = w_\lambda(x_o) - \lambda.$$

If $x < x_o$, then $\{i \mid \xi_i^o = 0\} \subset \{i \mid \xi_i = 0\}$, whence, by (5.7) and (5.11),

$$w_\lambda(x) - f(x) = \sum_{\xi_i^o=1} \eta_i^\lambda - \sum_{\xi_i^o=1,\ \xi_i=0} \eta_i^\lambda - f(x) \leq$$

$$\leq \sum_{\xi_i^o=1} \eta_i^\lambda - \sum_{j=1}^n |\xi_j - \xi_j^o| \min_{\xi_i^o=1,\ \xi_i=0} \eta_i^\lambda - f(x) \leq$$

$$\leq \sum_{\xi_i^o=1} \eta_i^\lambda - (\lambda - f(x)) - f(x) = \sum_{i=1}^n \eta_i^\lambda \xi_i^o - \lambda = w_\lambda(x_o) - \lambda.$$

Finally, if $x = x_o$ or $f(x) = +\infty$ (or both), then for any $w_\lambda \in W$ there holds $w_\lambda(x) - f(x) \leq w_\lambda(x_o) - \lambda$. Hence, for each $\lambda \in R$, $\lambda < f(x_o)$, there exists $w_\lambda \in W$ such that

$$f^c(w_\lambda) = \max_{x\in\{0,\,1\}^n} (w_\lambda(x) - f(x)) \le w_\lambda(x_o) - \lambda. \qquad (5.12)$$

Consequently, for each $\lambda\in R$, $\lambda < f(x_o)$, we have

$$f(x_o) \ge f^{cc^*}(x_o) = \sup_{w\in W}(w(x_o) - f^c(w)) \ge w_\lambda(x_o) - f^c(w_\lambda) \ge \lambda, \qquad (5.13)$$

and hence $f(x_o) = f^{cc^*}(x_o)$. Thus, since $x_o\in X$ was arbitrary, we have 1°. ■

Remark 5.4. We recall (see e.g. [5], theorem 3.6 or [1], p. 332) that, if X, W are arbitrary sets and $c: \overline{R}^X \to \overline{R}^W$ is any conjugation (5.1), for a function $f\in\overline{R}^X$ we have $f = f^{cc^*}$ if and only if f is the supremum of a family of "elementary functions" $\{\varphi(., w_i) \dotplus d_i\}_{i\in I}$. The set $\{f\in\overline{R}^X \mid f = f^{cc^*}\}$ is usually denoted by $\Gamma_c(X, W)$ (see [11], [5], [1]).

Corollary 5.1. Under the assumptions of theorem 5.3, if $x_o\in X$, then the following statements are equivalent:

1°. $f(x_o) = f^{cc^*}(x_o)$.

2°. Either $f(x_o) = -\infty$, or $f\in(R\cup\{+\infty\})^X$.

Proof. 1° ⇒ 2°. If $f(x_o) > -\infty$, then, by 1°, $f^{cc^*}(x_o) > -\infty$. Hence, by theorem 5.3, applied to f^{cc^*} (which satisfies 1° of theorem 5.3, i.e., $f^{cc^*} = (f^{cc^*})^{cc^*}$), we have $f^{cc^*}\in(R\cup\{+\infty\})^X$, whence, by $f \geqslant f^{cc^*}$, we obtain $f\in(R\cup\{+\infty\})^X$.

2° ⇒ 1°. If $f(x_o) = -\infty$, then, by $f\geqslant f^{cc^*}$, we have $-\infty = f(x_o) = f^{cc^*}(x_o)$. On the other hand, if $f\in(R\cup\{+\infty\})^X$, then, by theorem 5.3, we have $f = f^{cc^*}$, whence $f(x_o) = f^{cc^*}(x_o)$.

We recall that if X and W are two sets and $c: \overline{R}^X \to \overline{R}^W$ is a conjugation (5.1), then the subdifferential $\partial_c f(x_o)$ of $f: X\to\overline{R}$ at $x_o\in X$ is the set defined by

$$\partial_c f(x_o) = \{w\in W \mid \varphi(x, w) \dotplus -\varphi(x_o, w) \le f(x) \dotplus - f(x_o)\} \qquad (x\in X). \qquad (5.14)$$

Some authors (see e.g. [1], p. 332) define (5.14) only under the additional assumption $f(x_o)\in R$; however, since we shall consider only φ of (5.7) (which is finite), this assumption will not be necessary. It is well known (see e.g. [1], p. 332) that if $\partial_c f(x_o) \ne \varnothing$ and $f(x_o)\in R$, then $f^{cc^*}(x_o) = f(x_o)$.

Remark 5.5. If c is the conjugation (5.8), then φ is the natural coupling function (5.7), and hence, by (5.14), we have

$$\partial_c f(x_0) = \partial \tilde{f}(x_0) \cap W, \tag{5.14'}$$

where $\tilde{f}: R^n \to \overline{R}$ is as in remark 5.3 and $\partial \tilde{f}(x_0)$ is the usual subdifferential of $\tilde{f}$ at x_0.

Theorem. 5.4. *Let* $X = \{0, 1\}^n$, $W \subseteq R^n$ *and* c *be as in theorem 5.3, and let* $f \in \overline{R}^X$ *and* $x_0 \in X$. *The following statements are equivalent:*

1°. $\partial_c f(x_0) \neq \emptyset$.

2°. $f(x_0) \in R$ *and* $f(x) \in R \cup \{+\infty\}$ $(x \in X \setminus \{x_0\})$.

Proof. 1° ⇒ 2°. By (5.14) and (5.7), we have $w_0 \in \partial_c f(x_0)$ if and only if

$$w_0(x) - w_0(x_0) \leq f(x) \dot{+} - f(x_0) \qquad (x \in X), \tag{5.15}$$

with $w_0(x) - w_0(x_0) \in R$ $(x \in X)$. Now, if 2° does not hold, then we have either $f(x_0) \notin R$, in contradiction with (5.15) for $x = x_0$, or $f(x') = -\infty$ for some $x' \in X \setminus \{x_0\}$, which, again, contradicts (5.15).

2° ⇒ 1°. If 2° holds, then, since $f(x_0) \in R$, the argument of the above proof of theorem 5.3, implication 2° ⇒ (5.12), works also for $\lambda = f(x_0)$, whence $f^c(w_{f(x_0)}) \leq w_{f(x_0)}(x_0) - f(x_0)$, and hence, by (5.8), we have

$$w_{f(x_0)}(x) - f(x) \leq w_{f(x_0)}(x_0) - f(x_0) \qquad (x \in X). \tag{5.16}$$

Consequently (considering the cases $f(x) \in R$ and $f(x) = +\infty$), we obtain

$$w_{f(x_0)}(x) - w_{f(x_0)}(x_0) \leq f(x) \dot{+} - f(x_0) \qquad (x \in X), \tag{5.17}$$

i.e., $w_{f(x_0)} \in \partial_c f(x_0)$. ∎

Remark 5.6. Recently, Fujishige has shown ([7], theorem 3.1) that if X is a distributive sublattice of $\{0, 1\}^n$, $W = (R^n)^*$, and $c: \overline{R}^X \to \overline{R}^W$ is the conjugation (5.8), with $\{0, 1\}^n$ replaced by X, then for every submodular function $f: X \to R$ we have

$$f(x) = \max_{w \in (R^n)^*} (w(x) - f^c(w)) \qquad (x \in X). \tag{5.18}$$

From the above it follows that in this result *the assumption of submodularity of* f *is superflous*, even for $(R^n)^*$ replaced by $W \subseteq (R^n)^*$ of theorem 5.3. Indeed, by theorem 5.3 (extended, with the same proof, to any distributive sublattice X of $\{0, 1\}^n$), for every function $f: X \to R$ we have

$$f(x_0) = f^{cc^*}(x_0) = \sup_{w \in W} (w(x_0) - f^c(w)) \qquad (x_0 \in X), \tag{5.19}$$

so it remains to observe that, by $f^c(w_{f(x_0)}) \leq w_{f(x_0)}(x_0) - f(x_0)$ and (5.8) we have $f^c(w_{f(x_0)}) = w_{f(x_0)}(x_0) - f(x_0)$, whence the sup in (5.19) is attained for $w_{f(x_0)}$.

From theorems 5.3 and 5.4 and corollary 5.1, we obtain

Corollary 5.2. a) *Under the assumptions of theorem 5.4, the following statements are equivalent:*

1°. $\partial_c f(x_0) \neq \emptyset$.

2°. $f = f^{cc^*}$, $f \not\equiv -\infty$ *and* $f(x_0) < +\infty$.

3°. $f = f^{cc^*}$ and $f(x_0) \in R$.

4°. $f \in (R \cup \{+\infty\})^X$ and $f(x_0) \in R$.

5°. $f^{cc^*}(x_0) = f(x_0) \in R$.

b) *For* $f \in \overline{R}^X$, *the following statements are equivalent:*

1°. *There exists* $x_0 \in X$ *such that* $\partial_c f(x_0) \neq \emptyset$.

2°. $f = f^{cc^*}$ ______ *and* $f \not\equiv +\infty$.

c) *For* $f \in \overline{R}^X$, *the following statements are equivalent:*

1°. $\partial_c f(x_0) \neq \emptyset$ $(x_0 \in X)$.

2°. ______ $f \in R^X$.

Remark 5.7. By remark 5.5, the statements of corollary 5.2 a) are equivalent to

6°. $\partial \tilde{f}(x_0) \cap W \neq \emptyset$.

Let us prove now the following result, corresponding to corollary 3.1:

Corollary 5.3. a) *Every function* $f: X = \{0, 1\}^n \to R \cup \{+\infty\}$ *can be extended to a proper lower semi-continuous convex function* $\hat{f}$ *on* R^n.

b) *Every function* $f: X = \{0, 1\}^n \to R$ *can be extended to a polyhedral convex function* $\hat{f}$ *on* R^n.

Proof. a) If $f \equiv +\infty$, one can take, for example,

$$\hat{f} = \chi_{\{x = \{\xi_j\}_1^n \in R^n \mid \sum_{j=1}^n \xi_j \leq -1\}} \tag{5.20}$$

If $f \not\equiv +\infty$, define $\hat{f}: R^n \to \overline{R}$ by

$$\hat{f}(x) = \sup_{w \in (R^n)^*} (w(x) - f^c(w)) \qquad (x \in R^n). \tag{5.21}$$

Then $\hat{f}$ is a lower semi-continuous convex function on R^n and, by theorem 5.3 and formula (5.19), we have $\hat{f}|_X = f$, whence $\hat{f} \not\equiv +\infty$ (since $f \not\equiv +\infty$). Finally, if we had $\hat{f}(x_0) = -\infty$ for some $x_0 \in R^n$, then, by (5.21), similarly to the proof of theorem 5.3, implication $1° \Rightarrow 2°$, it would follow that $\hat{f} \equiv -\infty$, whence $f \equiv -\infty$, in contradiction with our assumption on f. Thus, $\hat{f}$ is proper.

b) Assume now that $f:\{0, 1\}^n \to R$ and, for each $x_0 \in X = \{0, 1\}^n$, choose $w_{f(x_0)} \in W$ as in the above proof of theorem 5.4 (in fact, any $w_{x_0} \in \partial_c f(x_0)$ works, with a similar argument). Let

$$\hat{f}(x) = \max_{x_0 \in \{0, 1\}^n} \left(w_{f(x_0)}(x - x_0) + f(x_0)\right) \qquad (x \in R^n). \qquad (5.22)$$

Then $\hat{f}$ is a polyhedral convex function on R^n and, taking $x_0 = x \in \{0, 1\}^n$ in (5.22), we obtain

$$\hat{f}(x) \geq f(x) \qquad (x \in X = \{0, 1\}^n). \qquad (5.23)$$

On the other hand, by $f^c(w_{f(x_0)}) = w_{f(x_0)}(x_0) - f(x_0)$ (see remark 5.6) and (5.19), we have

$$\hat{f}(x) = \max_{x_0 \in \{0, 1\}^n} \left(w_{f(x_0)}(x) - f^c(w_{f(x_0)})\right) \leq \sup_{w \in W} \left(w(x) - f^c(w)\right) = f(x) \qquad (x \in X),$$

whence, by (5.23), we obtain $\hat{f}|_X = f$. ∎

Remark 5.8. Corollary 5.2 b) has been obtained, with a different method, in [19], theorem 4.1 (see also [19], remark 4.2b)).

Now we shall interchange the roles of X and W, considering the conjugation $c^*: \overline{R}^W \to \overline{R}^X$ defined by (5.9), i.e., the conjugation with respect to the coupling function $\varphi^*: W \times X \to R$ of (5.6), with φ of (5.7). Then, for any subset X of R^n and any subset W of $(R^n)^*$, the subdifferential $\partial_{c^*} g(w_0)$ of $g: W \to \overline{R}$ at $w_0 \in W$ becomes

$$\partial_{c^*} g(w_0) = \{x \in X \mid w(x) - w_0(x) \leq g(w) \dotplus -g(w_0) \quad (w \in W)\}. \qquad (5.24)$$

Remark 5.9. We have

$$\partial_{c^*} g(w_0) = \partial \tilde{g}(w_0) \cap X, \qquad (5.24')$$

where $\tilde{g}: R^n \to \overline{R}$ is as in remark 5.3 and $\partial \tilde{g}(w_0)$ is the usual subdifferential of $\tilde{g}$ at w_0.

We shall prove the following result, corresponding to corollary 5.2 a):

Theorem 5.5. *Let X be a finite subset of* R^n, W *a subset of* $(R^n)^*$, *and let* $c^*: \overline{R}^W \to \overline{R}^X$ *be as above. For* $w_o \in W$ and $g \in \overline{R}^W$, *the following statements are equivalent:*

1°. $\partial_{c^*} g(w_o) \neq \emptyset$.

2.° $g^{c^*c}(w_o) = g(w_o) \in R$.

Proof. Observe that, by (5.8), we have

$$g(w_o) = g^{c^*c}(w_o) = \max_{x \in X} (w_o(x) - g^{c^*}(x)) \tag{5.25}$$

if and only if there exists $x_{w_o} \in X$ such that

$$g(w_o) = w_o(x_{w_o}) - g^{c^*}(x_{w_o}). \tag{5.26}$$

1° ⇒ 2°. If 1° holds, then $g(w_o) \in R$ (since $w_o \in W$, $g(w_o) \notin R$ would imply, taking $w = w_o$ in (5.24), that $\partial_{c^*} g(w_o) = \emptyset$).

Now let $w_o \in W$ and $x_o \in \partial_{c^*} g(w_o)$. Then, by (5.24), we have $g(w) \in R \cup \{+\infty\}$ $(w \in W)$. Thus, by (5.24) and $g(w_o) \in R$,

$$w(x_o) - g(w) \leq w_o(x_o) - g(w_o) \qquad (w \in W),$$

whence, by (5.9), we obtain (5.26) with $x_{w_o} = x_o$.

2° ⇒ 1°. If 2° holds, let $w_o \in W$ and let $x_{w_o} \in X$ be as in (5.26). Then, by $g(w_o) \in R$ and (5.26) we have $g^{c^*}(x_{w_o}) \in R$, whence, by (5.9), we get $g(w) \in R \cup \{+\infty\}$ $(w \in W)$. Hence, by $g(w_o) \in R$, (5.26), (5.9) and (5.24), we obtain $x_{w_o} \in \partial_{c^*} g(w_o)$. ∎

Remark 5.10. By remark 5.9, the statements of theorem 5.5 are equivalent to

3°. $\partial \tilde{g}(w_o) \cap X \neq \emptyset$.

Corollary 5.4. Let X be a finite subset of R^n, W a subset of $(R^n)^*$, and let $c^*: \overline{R}^W \to \overline{R}^X$ be as above. For a function $g \in \overline{R}^W$, the following statements are equivalent:

1°. $\partial_{c^*} g(w_o) \neq \emptyset$ $(w_o \in W)$.

2°. $g = g^{c^*c}$ and $g \in R^W$.

Remark 5.11. By remark 5.9 (or, remark 5.10), the statements of corollary 5.4 are equivalent to

3°. $\partial \tilde{g}(w_o) \cap X \neq \emptyset$ $(w_o \in W)$.

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