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Gelu POPESCU*)

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*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii nr.220,79622 Bucharest, ROMANIA

ISOMETRIC DILATIONS FOR INFINITE SEQUENCES OF NONCOMMUTING OPERATORS

by

Gelu POPESCU

This paper is a continuation of [5] and develops a dilation theory for an infinite sequence $\{T_n\}_{n=1}^{\infty}$ of noncommuting operators on a Hilbert space $\mathcal{H}$ when $\sum_{n=1}^{\infty} T_n T_n^* \leq I_{\mathcal{H}}$ ($I_{\mathcal{H}}$ is the identity on $\mathcal{H}$).

Many of the results and techniques in dilation theory for one operator [8] and also for two operators [3,4] are extended to this setting.

First we extend Wold decomposition [8,4] to the case of an infinite sequence $\{V_n\}_{n=1}^{\infty}$ of isometries with orthogonal final spaces.

In Section 2 we obtain a minimal isometric dilation for $\{T_n\}_{n=1}^{\infty}$ by extending the Schaffer construction in [6,4]. Using these results we give some theorems on the geometric structure of the space of the minimal isometric dilation. Finally, we give some sufficient conditions on a sequence $\{T_n\}_{n=1}^{\infty}$ to be simultaneously quasi-similar to a sequence $\{R_n\}_{n=1}^{\infty}$ of isometries on a Hilbert space $\mathcal{K}$ with

$$\sum_{n=1}^{\infty} R_n R_n^* = I_{\mathcal{K}}$$

In Section 3 we use the above mentioned theorems to obtain the Sz-Nagy-Foiaş lifting theorem [7,8,1,4] in our setting.

In a subsequent paper we will use the results of this paper for studying the "characteristic function" associated to a sequence $\{T_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} T_n T_n^* \leq I_{\mathcal{H}}$.

1. Throughout this paper Λ stands for the set $\{1,2,\ldots,k\}$ ($k\in N$) or the set $N=\{1,2,\ldots\}$.

For every $n\in N$ let $F(n,\Lambda)$ be the set of all functions from the set $\{1,2,\ldots,n\}$ to Λ and

$$\mathcal{F} = \bigcup_{n=0}^{\infty} F(n,\Lambda), \qquad \text{where } F(0,\Lambda)=\{0\}.$$

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ be a sequence of isometries on $\mathcal{H}$. For any $f\in F(n,\Lambda)$ we denote by V_f the product $V_{f(1)}V_{f(2)}\ldots V_{f(n)}$ and $V_0=I_{\mathcal{H}}$.

A subspace $\mathcal{L}\subset\mathcal{H}$ will be called <u>wandering</u> for the sequence $\mathcal{V}$ if for any distinct functions $f,g\in\mathcal{F}$ we have

$$V_f\mathcal{L} \perp V_g\mathcal{L} \qquad (\perp \text{ means orthogonal})$$

In this case we can form the orthogonal sum

$$M_{\mathcal{F}}(\mathcal{L}) = \bigoplus_{f\in\mathcal{F}} V_f\mathcal{L}$$

A sequence $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ of isometries on $\mathcal{H}$ is called a Λ-orthogonal shift if there exists in $\mathcal{H}$ a subspace $\mathcal{L}$, which is wandering for $\mathcal{V}$ and such that $\mathcal{H}=M_{\mathcal{F}}(\mathcal{L})$

This subspace $\mathcal{L}$ is uniquely determined by $\mathcal{V}$: indeed we have $\mathcal{L}=\mathcal{H}\ominus(\bigoplus_{\lambda\in\Lambda} V_\lambda\mathcal{H})$. The dimension of $\mathcal{L}$ is called the multiplicity of the Λ-orthogonal shift. One can show, by an argument similar to the classical unilateral shift, that a Λ-orthogonal shift is determined up to unitary equivalence by its multiplicty. It is easy to see that for $\Lambda = \{1\}$ we find again the classical unilateral shift.

Let us make some simple remarks whose proofs will be omitted.

REMARK 1.1. If $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ is a Λ-orthogonal shift on $\mathcal{H}$, with the wandering subspace $\mathcal{L}$, then for any $n \in N$, $\lambda \in \Lambda$ and $f \in F(n,\Lambda)$ we have

a) $$V_\lambda^* V_f = \begin{cases} V_{f(2)} V_{f(3)} \cdots V_{f(n)} & \text{if } f(1) = \lambda \\ 0 & \text{if } f(1) \neq \lambda \end{cases}$$

and $V_\lambda^* \ell = 0$ $(\ell \in \mathcal{L})$.

b) $\sum_{\lambda \in \Lambda} V_\lambda V_\lambda^* + P_{\mathcal{L}} = I_{\mathcal{H}}$, where $P_{\mathcal{L}}$ stands for the orthogonal projection from $\mathcal{H}$ into $\mathcal{L}$.

REMARK 1.2. If $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ is a Λ-orthogonal shift on $\mathcal{H}$ then:

a) $\lim_{n \to \infty} \sum_{f \in F(n,\Lambda)} \| V_f^* h \|^2 = 0$ for an $h \in \mathcal{H}$.

b) $V_\lambda^{*k} \to 0$ (strongly) as $k \to \infty$, for any $\lambda \in \Lambda$

c) There exists no non-zero reducing subspace $\mathcal{H}_0 \subset \mathcal{H}$ for each V_λ $(\lambda \in \Lambda)$ such that

$$(I_{\mathcal{H}} - \sum_{\lambda \in \Lambda} V_\lambda V_\lambda^*)\big|_{\mathcal{H}_0} = 0$$

Let us consider a model Λ-orthogonal shift.

Form the Hilbert space

$$l^2(\mathcal{F},\mathcal{H}) = \left\{(h_f)_{f\in\mathcal{F}} ;\ \sum_{f\in\mathcal{F}} \|h_f\|^2 < \infty,\ h_f \in \mathcal{H}\right\}$$

We embed in $l^2(\mathcal{F},\mathcal{H})$ as a subspace, by indentifying the element $h\in\mathcal{H}$ with the element $(h_f)_{f\in\mathcal{F}}$, where $h_0=h$ and $h_f=0$ for any $f\in\mathcal{F}$, $f\neq 0$.

For each $\lambda\in\Lambda$ we define the operator S_λ on $l^2(\mathcal{F},\mathcal{H})$ by $S_\lambda((h_f)_{f\in\mathcal{F}})=(h'_g)_{g\in\mathcal{F}}$, where $h'_0=0$ and for $g\in F(n,\Lambda)$ $(n\geqslant 1)$

$$h'_g = \begin{cases} h_0 ; & \text{if } g\in F(1,\Lambda) \text{ and } g(1)=\lambda \\ h_f ; & \text{if } g\in F(n,\Lambda)\ (n\geqslant 2),\ f\in F(n-1,\Lambda) \\ & \text{and } g(1)=\lambda,\ g(2)=f(1),\ g(3)=f(2),\ldots,g(n)=f(n-1) \\ 0 ; & \text{otherwise} \end{cases}$$

It is easy to see that $\{S_\lambda\}_{\lambda\in\Lambda}$ is the Λ-orthogonal shift, acting on $l^2(\mathcal{F},\mathcal{H})$, with the wandering subspace $\mathcal{H}$.

This model play an important role in this paper. The following theorem is our version of Wold decomposition for a sequence of isometries.

THEOREM 1.3. Let $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ be a sequence of isometries on a Hilbert space $\mathcal{K}$ such that

$$\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* \leqslant I_{\mathcal{K}}$$

Then $\mathcal{K}$ decomposes into an orthogonal sum $\mathcal{K}=\mathcal{K}_0\oplus\mathcal{K}_1$ such that $\mathcal{K}_0$ and $\mathcal{K}_1$ reduce each operator V_λ $(\lambda\in\Lambda)$ and we have

$$(I_{\mathcal{K}}-\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^*)|_{\mathcal{K}_1}=0 \text{ and } \{V_\lambda|_{\mathcal{K}_0}\}_{\lambda\in\Lambda} \text{ is a } \Lambda\text{-orthogonal}$$

shift acting on $\mathcal{K}_0$.

This decomposition is uniquely determined, indeed we have:

$$\mathcal{K}_1=\bigcap_{n=0}^{\infty}(\bigoplus_{f\in F(n,\Lambda)} V_f\mathcal{K});\ \mathcal{K}_0=M_{\mathcal{F}}(\mathcal{L}),\ \text{where } \mathcal{L}=\mathcal{K}\ominus(\bigoplus_{\lambda\in\Lambda} V_\lambda\mathcal{K}).$$

Proof. It is easy to see that the subspace $\mathcal{L}=\mathcal{K}\ominus(\bigoplus_{\lambda\in\Lambda} V_\lambda\mathcal{K})$ is wandering for $\mathcal{V}$.

Now let $\mathcal{K}_0=M_{\mathcal{F}}(\mathcal{L})$ and $\mathcal{K}_1'=\mathcal{K}\ominus\mathcal{K}_0$. Observe that $k\in\mathcal{K}_1'$ if and only if $k\perp\bigoplus_{f\in\mathcal{F}_n} V_f\mathcal{L}$, for every $n\in N$ where $\mathcal{F}_n$ stands for $\bigcup_{k=0}^{n} F(k,\Lambda)$.

We have

$$\mathcal{L}\oplus(\bigoplus_{f\in F(1,\Lambda)} V_f\mathcal{L})\oplus\dots\oplus(\bigoplus_{g\in F(n,\Lambda)} V_g\mathcal{L})=$$

$$=\Big[\mathcal{K}\ominus(\bigoplus_{f\in F(1,\Lambda)} V_f\mathcal{K})\Big]\oplus\Big[(\bigoplus_{f\in F(1,\Lambda)} V_f\mathcal{K})\ominus(\bigoplus_{f\in F(2,\Lambda)} V_f\mathcal{K})\Big]\oplus\dots$$

$$\dots\oplus\Big[(\bigoplus_{f\in F(n,\Lambda)} V_f\mathcal{K})\ominus(\bigoplus_{f\in F(n+1,\Lambda)} V_f\mathcal{K})\Big]=\mathcal{K}\ominus(\bigoplus_{f\in F(n+1,\Lambda)} V_f\mathcal{K})$$

Thus $k\in\mathcal{K}_1'$ if and only if $k\in\bigoplus_{f\in F(n+1,\Lambda)} V_f\mathcal{K}$ for every $n\in N$. Since $\bigoplus_{f\in F(n,\Lambda)} V_f\mathcal{K}\supset\bigoplus_{f\in F(n+1,\Lambda)} V_f\mathcal{K}$ $(n\in N)$ it follows that

$$\mathcal{K}_1'=\bigcap_{n=0}^{\infty}(\bigoplus_{f\in F(n,\Lambda)} V_f\mathcal{K})=\mathcal{K}_1\ .$$

Let us notice that

$$V_\lambda\mathcal{K}_1\subset\bigcap_{n=0}^{\infty}(\bigoplus_{f\in F(n,\Lambda)} V_\lambda V_f\mathcal{K})\subset\bigcap_{n=0}^{\infty}(\bigoplus_{g\in F(n+1,\Lambda)} V_g\mathcal{K})=\mathcal{K}_1$$

$$V_\lambda^*\mathcal{K}_1\subset\bigcap_{n=1}^{\infty}(V_\lambda^*(\bigoplus_{\substack{g\in F(n,\Lambda)\\ g(1)=\lambda}} V_g\mathcal{K}))=\bigcap_{n=1}^{\infty}(\bigoplus_{f\in F(n-1,\Lambda)} V_f\mathcal{K})=\mathcal{K}_1$$

Therefore $\mathcal{K}_1$ reduces each V_λ $(\lambda\in\Lambda)$. Hence $\mathcal{K}_o$ also reduces each V_λ $(\lambda\in\Lambda)$.

Since $\mathcal{K}_1\subset \bigoplus_{\lambda\in\Lambda} V_\lambda\mathcal{K}$ it follows that $(I_{\mathcal{K}}-\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^*)|_{\mathcal{K}_1}=0$.
The fact that $\{V_\lambda|\mathcal{K}_o\}_{\lambda\in\Lambda}$ is a Λ-orthogonal shift is obviously. The uniqueness of the decomposition follows by an argument similar to the classical Wold decomposition [8, Chap. I, Thm. 1.1]. The proof is completed.

REMARK 1.4. The subspaces $\mathcal{K}_o, \mathcal{K}_1$ from Wold decomposition can be described as follows:

$$\mathcal{K}_o=\{k\in\mathcal{K}: \lim_{n\to\infty}\sum_{f\in F(n,\Lambda)}\|V_f^*k\|^2=0\}$$

$$\mathcal{K}_1=\{k\in\mathcal{K}: \sum_{f\in F(n,\Lambda)}\|V_f^*k\|^2=\|k\|^2 \text{ for every } n\in N\}.$$

We call the sequence $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ in Theorem 1.3 <u>pure</u> if $\mathcal{K}_1=0$, that is, if $\mathcal{V}$ is a Λ-orthogonal shift on $\mathcal{K}$.

2. Let $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ a sequence of contractions on a Hilbert space $\mathcal{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^*\leq I_{\mathcal{H}}$.

We say that a sequence $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ of isometries on a Hilbert space $\mathcal{K}\supset\mathcal{H}$ is a minimal isometric dilation of $\mathcal{T}$ if the following conditions hold:

a) $\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^*\leq I_{\mathcal{K}}$

b) $\mathcal{H}$ is invariant for each V_λ^* $(\lambda\in\Lambda)$ and

$$V_\lambda^*|\mathcal{H}=T_\lambda^* \qquad (\lambda\in\Lambda)$$

c) $\mathcal{K} = \bigvee_{f\in\mathcal{F}} V_f \mathcal{H}$.

Let D_* on $\mathcal{H}$ and D on $\bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda$ ($\mathcal{H}_\lambda$ is a copy of $\mathcal{H}$) be the positive operators uniquely defined by $D_* = (I_{\mathcal{H}} - \sum_{\lambda\in\Lambda} T_\lambda T_\lambda^*)^{1/2}$ and $D = D_T$, where T stands for the matrix $[T_1, T_2, \ldots]$ and $D_T = (I - T^*T)^{1/2}$.

Let us denote $\mathcal{D}_* = \overline{D_*\mathcal{H}}$ and $\mathcal{D} = \overline{D(\bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda)}$.

THEOREM 2.1. For every sequence $\mathcal{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ of non-commuting operators on a Hilbert space $\mathcal{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq I_{\mathcal{H}}$, there exists a minimal isometric dilation $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$, which is uniquely determined up to an isomorphism.

<u>Proof</u>. Let us consider the Hilbert space

$$\mathcal{K} = \mathcal{H} \oplus l^2(\mathcal{F}, \mathcal{D})$$

We embed $\mathcal{H}$ and $\mathcal{D}$ into $\mathcal{K}$ in a natural way. For each $\lambda\in\Lambda$ we define the isometry $V_\lambda : \mathcal{K} \to \mathcal{K}$ by the relation

$$V_\lambda(h \oplus (d_f)_{f\in\mathcal{F}}) = T_\lambda h \oplus (D(\underbrace{0,\ldots,0}_{\lambda-1 \text{ times}}, h, 0, \ldots) + S_\lambda (d_f)_{f\in\mathcal{F}}) \tag{2.1}$$

where $\{S_\lambda\}_{\lambda\in\Lambda}$ is Λ-orthogonal shift on $l^2(\mathcal{F},\mathcal{D})$ (see Section 1).

Obviously, for any $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$ we have

$$\text{range } S_\lambda \perp \text{range } S_\mu$$

and

$$(T_\mu^* T_\lambda h, h') = -(D^2(\underbrace{0,\ldots,0}_{\lambda-1 \text{ times}}, h, 0, \ldots), (\underbrace{0,\ldots,0}_{\mu-1 \text{ times}}, h', 0, \ldots))$$

Hence, taking into account (2.1), it follows that

$$\text{range } V_\lambda \perp \text{range } V_\mu \qquad (\lambda, \mu\in\Lambda, \lambda \neq \mu)$$

therefore $\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* \leq I_{\mathcal{K}}$.

It is easy to show that $\mathcal{H}$ is invariant for each V_λ^* $(\lambda\in\Lambda)$ and $V_\lambda^*|_{\mathcal{H}} = T_\lambda^*$ $(\lambda\in\Lambda)$.

Finally, we verify that $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ is the minimal isometric dilation of $\mathcal{T}$.

Let $\mathcal{H}_1=\mathcal{H}\vee(\bigvee_{f\in F(1,\Lambda)} V_f\mathcal{H})$ and

$$\mathcal{H}_n=\mathcal{H}_{n-1}\vee(\bigvee_{f\in F(1,\Lambda)} V_f\mathcal{H}_{n-1}) \quad \text{if } n\geq 2$$

It is easy to see that $\mathcal{H}_1=\mathcal{H}\oplus\mathcal{D}$ and $\mathcal{H}_n=\mathcal{H}\oplus\mathcal{D}\oplus(\bigoplus_{f\in F(1,\Lambda)} S_f\mathcal{D})\oplus \ldots\oplus(\bigoplus_{f\in F(n-1,\Lambda)} S_f\mathcal{D})$ if $n\geq 2$. Clearly $\mathcal{H}_n\subset\mathcal{H}_{n+1}$ and we have:

$$\bigvee_1^\infty \mathcal{H}_n=\mathcal{H}\oplus M_{\mathcal{F}}(\mathcal{D})=\mathcal{H}\oplus l^2(\mathcal{F},\mathcal{D})=\mathcal{K}$$

Therefore $\mathcal{K}=\bigvee_{f\in\mathcal{F}} V_f\mathcal{H}$.

Following Thm. 4.1 in [8 Chap. I] it is easy to show that the minimal isometric dilation $\mathcal{V}$ of $\mathcal{T}$ is unique up to a unitary operator. To be more precise, let $\mathcal{V}'=\{V'_\lambda\}_{\lambda\in\Lambda}$ be another minimal isometric dilation of $\mathcal{T}$, on a Hilbert space $\mathcal{K}'\supset\mathcal{H}$. Then there exists a unitary operator $U:\mathcal{K}\to\mathcal{K}'$ such that $V'_\lambda U=UV_\lambda$ $(\lambda\in\Lambda)$ and $Uh=h$ for every $h\in\mathcal{H}$.

This completes the proof.

REMARK 2.2. For each $\lambda\in\Lambda$, $V_\lambda^{*n}\to 0$ (strongly) as $n\to\infty$ if and only if $T_\lambda^{*n}\to 0$ (strongly) as $n\to\infty$.

From this remark and Theorem 2.1 one can easily deduce Proposition 1.1 in [5].

The following is a generalization of [2] or Theorem 1.2 in [8, Chap. II].

PROPOSITION 2.3. Let $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ be the minimal isometric dilation of $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$. Then $\mathcal{V}$ is pure if and only if

$$\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)}\|T_f^*h\|^2=0 \tag{2.2}$$

for any $h\in\mathcal{H}$.

Proof. Assume that $\mathcal{V}$ is pure. Then, by Theorem 1.3 it follows that $\mathcal{V}$ is a Λ-orthogonal shift on the space $\mathcal{K}\supset\mathcal{H}$ of the minimal isometric dilation of $\mathcal{T}$.

Taking into account Remark 1.2 and the fact that for each $f\in\mathcal{F}$, $V_f^*|_{\mathcal{H}}=T_f^*$, we have:

$$\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)}\|T_f^*h\|^2=\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)}\|V_f^*h\|^2=0,\quad \text{for any } h\in\mathcal{H}.$$

Conversely, assume that (2.2) holds. We claim that

$$\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)}\|V_f^*k\|^2=0,\ \text{for any } k\in\mathcal{K}=\bigvee_{f\in\mathcal{F}}V_f\mathcal{H}. \tag{2.3}$$

By (2.2) we have $\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)}\|V_f^*h\|^2=0 \quad (h\in\mathcal{H})$.

For each $k\in\bigvee_{\substack{f\in\mathcal{F}\\ f\neq 0}}V_f\mathcal{H}$ and any $\varepsilon>0$, there exists $k_\varepsilon=\sum'_{\substack{g\in\mathcal{F}\\ g\neq 0}}V_gh_g$ ($h_g\in\mathcal{H}$) such that $\|k-k_\varepsilon\|<\varepsilon$. (Here $\sum'$ stands for a finite sum).

Since the isometries V_λ $(\lambda\in\Lambda)$ have orthogonal final spaces, it follows that

$$\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)}\|V_f^*k\|^2=\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)}\|V_f^*(k-k_\varepsilon)\|^2\leq\|k-k_\varepsilon\|^2<\varepsilon,$$

for any $\varepsilon>0$. Thus, (2.3) holds and by Remark 1.4 we have that $\mathcal{V}$ is pure. This completes the proof.

COROLLARY 2.4. If $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq rI_{\mathcal{H}}$, $r<1$, then the minimal isometric dilation of $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ is pure.

Now let us establish when the minimal isometric dilation $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ cannot contain a Λ-orthogonal shift. The notations being the same as above we have

PROPOSITION 2.5. $\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* = I_{\mathcal{K}}$ if and only if $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* = I_{\mathcal{H}}$.

Proof. ($\Longrightarrow$) Since $V_\lambda^*|_{\mathcal{H}} = T_\lambda^*$ ($\lambda\in\Lambda$) it follows that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* h = h$ ($h\in\mathcal{H}$).

($\Longleftarrow$) If $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* = I_{\mathcal{H}}$ then $\sum_{f\in F(n,\Lambda)} \| T_f^* h\|^2 = \|h\|^2$ for any $n\in\mathbb{N}$ and $h\in\mathcal{H}$. Taking into account Theorem 1.3 let us assume that there exists $k\in\mathcal{K}\ominus(\bigoplus_{\lambda\in\Lambda} V_\lambda\mathcal{K})$, $k\neq 0$. Using Remark 1.4 it follows that

$$\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \| V_f^* k\|^2 = 0 \tag{2.4}$$

On the other hand, since $\mathcal{K} = \mathcal{H} \vee (\bigvee_{\substack{f\in\mathcal{F}\\ f\neq 0}} V_f\mathcal{H})$ and $\bigvee_{\substack{f\in\mathcal{F}\\ f\neq 0}} V_f\mathcal{H} \subset \bigoplus_{\lambda\in\Lambda} V_\lambda\mathcal{K}$ it follows that $k\in\mathcal{H}$ and by (2.4) that $\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \| T_f^* k\|^2 = 0$, contradiction. Thus we have $\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* = I_{\mathcal{K}}$ and the proof is complete.

Dropping out the minimality condition in the definition of the isometric dilation of a sequence $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$, we can prove the following.

PROPOSITION 2.6. For any sequence $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ of operators on a Hilbert space $\mathcal{H}$ such that

$$\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq I_{\mathcal{H}}$$

there exists an isometric dilation $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ on a Hilbert space $\mathcal{K}\supset\mathcal{H}$ such that

$$\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* = I_{\mathcal{K}} .$$

Proof. Taking into account Theorem 2.1 and 1.3, we show, without loss of generality, that the Λ-orthogonal shift $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ on $\mathcal{H}_o=l^2(\mathcal{F},\mathcal{E})$ ($\mathcal{E}$ is a Hilbert space) can be extended to a sequence $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ of isometries on a Hilbert space $\mathcal{K}_o\supset\mathcal{H}_o$ such that:

(2.5) $$\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* = I_{\mathcal{K}_o} \quad\text{and}\quad V_\lambda|_{\mathcal{H}_o} = S_\lambda \qquad (\lambda\in\Lambda)$$

Consider the Hilbert space

$$\mathcal{K} = [l^2(\mathcal{F},\mathcal{E}) \ominus \mathcal{E}] \oplus l^2(\mathcal{F},\mathcal{E})$$

We embed $l^2(\mathcal{F},\mathcal{E})$ into $\mathcal{K}$ by identifying the element $\{e_f\}_{f\in\mathcal{F}} \in l^2(\mathcal{F},\mathcal{E})$ with the element $0 \oplus \{e_f\}_{f\in\mathcal{F}} \in \mathcal{K}$.

Let us define the isometries V_λ $(\lambda\in\Lambda)$ on $\mathcal{K}$. For $\lambda \geq 2$ we set $V_\lambda = S_\lambda |_{l^2(\mathcal{F},\mathcal{E})\ominus\mathcal{E}} \oplus S_\lambda$.

Consider the countable set

$$\mathcal{F}' = \{f\in\mathcal{F}\setminus F(1,\Lambda) : f(1)=1\} \cup F(1,\Lambda) \cup \{0\}$$

and a one-to-one map $\gamma : \mathcal{F}\setminus\{0\} \longrightarrow \mathcal{F}'$.

For $\{e_f^*\}_{f\in\mathcal{F}\setminus\{0\}} \oplus \{e_f\}_{f\in\mathcal{F}} \in \mathcal{K}$ the isometry V_1 is defined as follows

$$V_1(0 \oplus \{e_f\}_{f\in\mathcal{F}}) = 0 \oplus S_1(\{e_f\}_{f\in\mathcal{F}})$$

$$V_1(\{e_f^*\}_{f\in\mathcal{F}\setminus\{0\}} \oplus 0)=\{e_g'^*\}_{g\in\mathcal{F}\setminus\{0\}} \oplus \{e_g'\}_{g\in\mathcal{F}}, \text{ where}$$

$$e_g'^* = \begin{cases} e_f^* \ ; & \text{if } g=\gamma(f) \\ 0 \ ; & \text{otherwise} \end{cases}$$

and

$$e_o' = e_f^* \ ; \quad \text{if } \gamma(f)=0$$

$$e_g'=0 \ ; \quad \text{if } g\in\mathcal{F}\setminus\{0\} .$$

Now it is easy to see that the relations (2.5) hold.

Following the classification of contractions from [8] we give, in what follows, a classification of the sequences of contractions.

Let $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ on a Hilbert space $\mathcal{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq I_{\mathcal{H}}$.

Consider the following subspace of $\mathcal{H}$:

(2.6) $$\mathcal{H}_o=\Big\{h\in\mathcal{H} : \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \| T_f^* h \|^2=0\Big\}$$

(2.7) $$\mathcal{H}_1=\Big\{h\in\mathcal{H} : \sum_{f\in F(n,\Lambda)} \| T_f^* h \|^2= \|h\|^2 \quad \text{for any } n\in N\Big\}$$

REMARK 2.7. The subspaces $\mathcal{H}_o$ and $\mathcal{H}_1$ are orthogonal and invariant for each operator T_λ^* $(\lambda\in\Lambda)$.

Proof. Taking into account Theorem 2.1, 1.3 and Remark 1.4 the proof is immediately.

Thus, the Hilbert space $\mathcal{H}$ decomposes into an orthogonal sum $\mathcal{H} = \mathcal{H}_o \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$.

For each $k\in\{0,1,2\}$ we shall denote by $C^{(k)}$ (respectively $C_{(k)}$) the set of all sequences $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ on $\mathcal{H}$ for which we have $\mathcal{H}_k=\{0\}$ (respectively $\mathcal{H}=\mathcal{H}_k$).

Let us mention that $\mathcal{H}_1$ is the largest subspace in $\mathcal{H}$ on which the matrix $\begin{bmatrix} T_1^* \\ T_2^* \\ \vdots \end{bmatrix}$ acts isometrically.

Consequently, a sequence $\mathcal{T} \in C^{(1)}$ will be also called completely non-coisometric (c.n.c).

In the particular case when $\mathcal{T} = \{T\}$ ($\|T\| \leq 1$) we have that $\mathcal{T} \in C^{(1)}$ if and only if T^* is completely non-isometric, that is, if there is no non-zero invariant subspace for T^* on which T^* is an isometry.

We continue this section with the study of the geometric structure of the space of the minimal isometric dilation.

For this, let $\mathcal{T} = \{T_\lambda\}_{\lambda \in \Lambda}$ be a sequence of operators on a Hilbert space $\mathcal{H}$ such that $\sum_{\lambda \in \Lambda} T_\lambda T_\lambda^* \leq I_{\mathcal{H}}$ and $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ be the minimal isometric dilation of $\mathcal{T}$ on the Hilbert space $\mathcal{K} = \mathcal{H} \oplus l^2(\mathcal{F}, \mathcal{D})$ (see Theorem 2.1).

Considering the subspace of $\mathcal{K}$

$$\mathcal{L} = \bigvee_{\lambda \in \Lambda} (V_\lambda - T_\lambda)\mathcal{H} \quad \text{and} \quad \mathcal{L}_* = \overline{(I_{\mathcal{K}} - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^*)\mathcal{H}}$$

we can generalize some of the results from [8, Chap . II, Sec. 1,2] concerning the geometric structure of the space of the minimal isometric dilation.

THEOREM 2.8. (i) The subspaces $\mathcal{L}$ and $\mathcal{L}_*$ are wandering subspaces for $\mathcal{V}$ and

$$\dim \mathcal{L} = \dim \mathcal{D} \quad ; \quad \dim \mathcal{L}_* = \dim \mathcal{D}_*$$

(ii) The space $\mathcal{K}$ can be decomposed as follows:

$$\mathcal{K} = \mathcal{R} \oplus M_{\mathcal{F}}(\mathcal{L}_*) = \mathcal{H} \oplus M_{\mathcal{F}}(\mathcal{L})$$

and the subspace $\mathcal{R}$ reduces each operator V_λ $(\lambda \in \Lambda)$

(iii) $\mathcal{L} \cap \mathcal{L}_* = 0$

(iv) The subspace $\mathcal{R}$ reduces to $\{0\}$ if and only if $\mathcal{T} \in C_{(0)}$.

<u>Proof</u>. The Wold decomposition (see Theorem 1.3) for the minimal isometric dilation $\mathcal{V}$ on the space $\mathcal{K} = \mathcal{H} \oplus l^2(\mathcal{F}, \mathcal{D})$ gives $\mathcal{K} = \mathcal{R} \oplus M_{\mathcal{F}}(\mathcal{L}_*^!)$, where $\mathcal{R} = \bigcap_{n=0}^{\infty} \left[\bigoplus_{f \in F(n,\Lambda)} V_f \mathcal{K} \right]$ reduces each operator V_λ $(\lambda \in \Lambda)$ and $\mathcal{L}_*^! = \mathcal{K} \ominus (\bigoplus_{\lambda \in \Lambda} V_\lambda \mathcal{K})$ is an wandering subspace for $\mathcal{V}$.

It is easy to see that $\mathcal{L}_*^! = \mathcal{L}_*$ and that the operator $\Phi_* : \mathcal{L}_* \longrightarrow \mathcal{D}_*$ defined by

$$\Phi_* (I_{\mathcal{K}} - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^*) h = D_* h \qquad (h \in \mathcal{H})$$

is unitary. Hence it follows that $\dim \mathcal{L}_* = \dim \mathcal{D}_*$. Equation (2.1) yields

$$\sum_{\lambda \in \Lambda} (V_\lambda - T_\lambda) h_\lambda = 0 \oplus D((h_\lambda)_{\lambda \in \Lambda}) \quad \text{for } (h_\lambda)_{\lambda \in \Lambda} \in \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda$$

($\mathcal{H}_\lambda$ is a copy of $\mathcal{H}$).

By this relation we deduce that there exists a unitary operator $\Phi : \mathcal{L} \longrightarrow \mathcal{D}$ defined by equation

$$\Phi (\sum_{\lambda \in \Lambda} (V_\lambda - T_\lambda) h_\lambda) = D((h_\lambda)_{\lambda \in \Lambda})$$

and hence that $\dim \mathcal{L} = \dim \mathcal{D}$.

The fact that $\mathcal{L}$ is an wandering subspace for $\mathcal{V}$ and that $\mathcal{H} \perp M_{\mathcal{F}}(\mathcal{L})$ follows from the form of the isometries $V_\lambda (\lambda \in \Lambda)$ defined by (2.1).

Taking into account the minimality of $\mathcal{K}$ it follows that

$$\mathcal{K} = \mathcal{H} \oplus M_{\mathcal{F}}(\mathcal{L})$$

Let us now show that $\mathcal{L} \cap \mathcal{L}_* = 0$. First we need to prove that

(2.8) $$\mathcal{L}_* \oplus \left(\bigoplus_{\lambda\in\Lambda} V_\lambda \mathcal{H}\right) = \mathcal{H} \oplus \mathcal{L}$$

This follows from the fact that, for an element $u \in \mathcal{K}$, the possibility of a representation of the form

$$u = \left(I_{\mathcal{K}} - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right) h_0 + \sum_{\lambda\in\Lambda} V_\lambda h_\lambda \;, \quad h_0 \in \mathcal{H} \;, \quad (h_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda$$

is equivalent to the possibility of a representation of the form

$$u = h^{(0)} + \sum_{\lambda\in\Lambda} (V_\lambda - T_\lambda) h^{(\lambda)} \;, \quad h^{(0)} \in \mathcal{H} \;, \quad (h^{(\lambda)})_{\lambda\in\Lambda} \in \oplus \mathcal{H}_\lambda$$

Indeed, we have only to set

$$h_0 = h^{(0)} - \sum_{\lambda\in\Lambda} T_\lambda h^{(\lambda)}$$

$$h_\lambda = T_\lambda^* h^{(0)} + h^{(\lambda)}$$

and, conversely,

$$h^{(0)} = \sum_{\lambda\in\Lambda} T_\lambda h_\lambda + \left(I_{\mathcal{H}} - \sum_{\lambda\in\Lambda} T_\lambda T_\lambda^*\right) h_0$$

$$h^{(\lambda)} = h_\lambda - T_\lambda^* h_0$$

Thus (2.8) holds. On the other hand, since

$$\mathcal{L} \subset \left(\bigoplus_{\lambda\in\Lambda} V_\lambda \mathcal{H}\right) \vee \mathcal{H} \quad \text{and} \quad \bigoplus_{\lambda\in\Lambda} V_\lambda \mathcal{H} \subset \mathcal{L} \oplus \mathcal{H}$$

we have that $\mathcal{H} \vee \left(\bigoplus_{\lambda\in\Lambda} V_\lambda \mathcal{H}\right) = \mathcal{H} \oplus \mathcal{L}$. This relation and (2.8) show that $\mathcal{L} \cap \mathcal{L}_* = \{0\}$.

The statement (iv) is contained in Proposition 2.3. The proof is complete.

PROPOSITION 2.9. For every sequence $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ of operators on $\mathcal{H}$ and for its minimal isometric dilation $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ on $\mathcal{K}$, we have

$$M_{\mathcal{F}}(\mathcal{L})\vee M_{\mathcal{F}}(\mathcal{L}_*)=\mathcal{K}\ominus\mathcal{H}_1 \tag{2.9}$$

where $\mathcal{H}_1$ is given by (2.7).

In particular, if $\mathcal{T}$ is c.n.c., then

$$M_{\mathcal{F}}(\mathcal{L})\vee M_{\mathcal{F}}(\mathcal{L}_*)=\mathcal{K} \tag{2.10}$$

Proof. Taking into account Theorem 2.8 and that $\mathcal{H}_1\subset\mathcal{R}$ it follows that $\mathcal{H}_1\perp M_{\mathcal{F}}(\mathcal{L})\vee M_{\mathcal{F}}(\mathcal{L}_*)$.

Now let $k\in\mathcal{K}$ be such that $k\perp M_{\mathcal{F}}(\mathcal{L})$ and $k\perp M_{\mathcal{F}}(\mathcal{L}_*)$.

From the same theorem it follows that $k\in\mathcal{H}$ and $k\perp V_f\mathcal{L}_*$ for every $f\in\mathcal{F}$. Hence we have

$$0=(k,\ V_f(I_{\mathcal{K}}-\sum_{\lambda\in\Lambda}V_\lambda T_\lambda^*)h)=(T_f^*k,h)-\sum_{\lambda\in\Lambda}(T_\lambda^*T_f^*k,\ T_\lambda^*h)$$

for every $h\in\mathcal{H}$.

Choosing $h=T_f^*k$ $(f\in\mathcal{F})$ we obtain

$$\|T_fk\|^2=\sum_{\lambda\in\Lambda}\|T_\lambda^*T_f^*k\|^2$$

for any $f\in\mathcal{F}$.

Hence we deduce

$$\sum_{g\in F(n,\Lambda)}\|T_g^*k\|^2=\|k\|^2$$

for any $n\in\mathbb{N}$. We conclude that $k\in\mathcal{H}_1$.

Conversely, for every $k\in\mathcal{H}_1$ it easy to see that

$$k\perp M_{\mathcal{F}}(\mathcal{L})\vee M_{\mathcal{F}}(\mathcal{L}_*)\ .$$

The relation (2.10) follows because for $\mathcal{T}$ c.n.c. we have $\mathcal{H}_1=\{0\}$.

The last aim of this section is to generalize some of the results from [8, Chap.II, Sec.3]. Throughout $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ is the minimal isometric dilation of $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$. The space of the minimal isometric dilation is

$$(2.11)\qquad \mathcal{K}=\mathcal{R}\oplus M_{\mathcal{F}}(\mathcal{L}_*)=\mathcal{H}\oplus l^2(\mathcal{F},\mathcal{D})$$

PROPOSITION 2.10. For every $h\in\mathcal{H}$ we have

$$(2.12)\qquad P_{\mathcal{H}}h=\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)} V_f T_f^* h$$

and consequently

$$(2.13)\qquad \|P_{\mathcal{H}}h\|^2=\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)} \|T_f^* h\|^2$$

where $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{K}$ into $\mathcal{H}$.

Proof. An easy computation shows that

$$\Big\|\sum_{f\in F(n+1,\Lambda)} V_f T_f^* h-\sum_{f\in F(n,\Lambda)} V_f T_f^* h\Big\|^2=\sum_{f\in F(n+1,\Lambda)}\|T_f^* h\|^2-\sum_{f\in F(n,\Lambda)}\|T_f^* h\|^2\leq$$

for every $n\in N$. This implies the convergence of the sequence $\Big\{\sum_{f\in F(n,\Lambda)} V_f T_f^* h\Big\}_{n=1}^{\infty}$ in $\mathcal{K}$. Setting

$$k=\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)} V_f T_f^* h$$

let us show that $k=P_{\mathcal{H}}h$, i.e. $k\perp M_{\mathcal{F}}(\mathcal{L}_*)$ and $h-k\in M_{\mathcal{F}}(\mathcal{L}_*)$.

Since for every $g\in\mathcal{F}$ there exists $n_o\in N$ such that

$$\sum_{f\in F(n,\Lambda)} V_f T_f^* h\perp V_g\mathcal{L}_*$$

for any $n\geq n_o$, it follows that $k\perp M_{\mathcal{F}}(\mathcal{L}_*)$.

On the other hand we have

$$h-\sum_{f\in F(n,\Lambda)} V_f T_f^* h=(I_{\mathcal{K}}-\sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*)h+\sum_{f\in F(1,\Lambda)} V_f(I_{\mathcal{K}}-\sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*)T_f^* h+\ldots$$

$$\ldots+\sum_{f\in F(n-1,\Lambda)} V_g(I_{\mathcal{K}}-\sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*)T_g^* h\in M_{\mathcal{F}}(\mathcal{L}_*)$$ and therefore $h-k=$

$=\lim_{n\to\infty}(h-\sum_{f\in F(n,\Lambda)} V_f T_f^* h)\in M_{\mathcal{F}}(\mathcal{L}_*)$. This ends the proof.

PROPOSITION 2.11. Let $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ be a sequence of operators on $\mathcal{H}$ such that the matrix $[T_1,T_2,\ldots]$ is an injection. Then

$$\overline{P_{\mathcal{R}}\mathcal{H}}=\mathcal{R}$$

Proof. Let us suppose that there exists $k\in\mathcal{R}$, $k\neq 0$ such that $k\perp P_{\mathcal{R}}\mathcal{H}$, or equivalently, such that

$$k\perp M_{\mathcal{F}}(\mathcal{L}_*) \quad \text{and} \quad k\perp\mathcal{H}$$

By Theorem 2.8 we have $\mathcal{K}=\mathcal{H}\oplus M_{\mathcal{F}}(\mathcal{L})$. It follows that $k\in M_{\mathcal{F}}(\mathcal{L})$ and hence

$$k=\sum_{f\in\mathcal{F}} V_f l_f \quad \text{where} \quad l_f\in\mathcal{L}\ (f\in\mathcal{F}) \text{ and } \sum_{f\in\mathcal{F}}\|l_f\|^2<\infty$$

Since $k\neq 0$ there exists $f_o\in\mathcal{F}$, such that $V_{f_o} l_{f_o}\neq 0$ and

$$V_{f_o}^* k=l_{f_o}+\sum_{\substack{g\in\mathcal{F}\\ g\neq 0}} V_g l'_g \qquad (l'_g\in\mathcal{L})$$

One can easily show that for every $g\in\mathcal{F}$, $g\neq 0$ $V_g\mathcal{L}\perp\mathcal{L}_*$. Since $V_{f_o}^* k\perp\mathcal{L}_*$ it follows that $l_{f_o}\perp\mathcal{L}_*$. By the relation (2.8) we deduce that $l_{f_o}\in\bigoplus_{\lambda\in\Lambda} V_\lambda\mathcal{H}$.

Therefore, there exists a non-zero $\bigoplus_{\lambda\in\Lambda} h_\lambda\in\bigoplus_{\lambda\in\Lambda}\mathcal{H}_\lambda$ such that $l_{f_o}=\sum_{\lambda\in\Lambda} V_\lambda h_\lambda$. Since $\mathcal{L}\perp\mathcal{H}$, it follows that $\sum_{\lambda\in\Lambda} T_\lambda h_\lambda=0$ which is a contradiction with the hypothesis.

Thus $\overline{P_{\mathcal{R}}\mathcal{H}} = \mathcal{R}$ and the proof is complete.

For each $\lambda\in\Lambda$ let us denote by R_λ the operator $V_\lambda|_{\mathcal{R}}$. Taking into account the Wold decomposition (Theorem 1.3) we have $\sum_{\lambda\in\Lambda} R_\lambda R_\lambda^* = I_{\mathcal{R}}$.

The following theorem is a generalization of Proposition 3.5 in [8, Chap. II].

PROPOSITION 2.12. Let $\mathcal{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ a sequence of operators on $\mathcal{H}$ such that $\mathcal{T}\in C^{(0)}$ and the matrix $[T_1, T_2, \ldots]$ is an injective contraction.

Then $\mathcal{T}$ is quasi-similar to $\{R_\lambda\}_{\lambda\in\Lambda}$, i.e., there exists a quasi-affinity Y from $\mathcal{R}$ to $\mathcal{H}$ such that

$$T_\lambda Y = YR_\lambda$$

for every $\lambda\in\Lambda$.

<u>Proof</u>. According to Proposition 2.10 we have

$$V_\lambda^* P_{\mathcal{R}} h = \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} V_\lambda^* V_f T_f^* h = \lim_{n\to\infty} \sum_{g\in F(n-1,\Lambda)} V_g T_g^* T_\lambda^* h = P_{\mathcal{R}} T_\lambda^* h$$

for all $h\in\mathcal{H}$ and each $\lambda\in\Lambda$.

Setting $X = P_{\mathcal{R}}|_{\mathcal{H}}$ it follows that $R_\lambda^* X = XT_\lambda^*$ for every $\lambda\in\Lambda$. Let us show that X is a quasi-affinity.

Since $\mathcal{T}\in C^{(0)}$ we have that

$$\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \| T_f^* h \|^2 = 0 \qquad \text{for every non-zero } h\in\mathcal{H}$$

By Proposition 2.10 we deduce that $P_{\mathcal{R}} h \neq 0$ for every non-zero $h\in\mathcal{H}$, i.e., X is an injection.

On the other hand, Proposition 2.11 shows that $\overline{X\mathcal{H}} = \mathcal{R}$.

If we take $Y=X^*$, this finishes the proof.

3. In this section we extend the Sz.-Nagy-Foiaş lifting theorem [7, 8, 1, 4] to our setting.

Let $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ be a sequence of operators on $\mathcal{H}$ with $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq I_{\mathcal{H}}$ and $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ be the minimal isometric dilation of on the Hilbert space

$$\mathcal{K}=\mathcal{H}\oplus l^2(\mathcal{F},\mathcal{D})$$

(see Theorem 2.1).

Consider the following subspaces of $\mathcal{K}$

$$\mathcal{H}_1=\mathcal{H}\vee(\bigvee_{f\in F(1,\Lambda)} V_f\mathcal{H})$$

and

$$\mathcal{H}_n=\mathcal{H}_{n-1}\vee(\bigvee_{f\in F(1,\Lambda)} V_f\mathcal{H}_{n-1}) \qquad \text{for} \quad n\geq 2.$$

Note that $\mathcal{H}_n\subset\mathcal{H}_{n+1}$ and that all the space $\mathcal{H}_n$ $(n\geq 1)$ are invariant for each operator V_λ^* $(\lambda\in\Lambda)$.

As in [7, 8, 1, 4] the n-stepped dilation of $\mathcal{T}$ is the sequence $\mathcal{T}_n=\{(T_\lambda)_n\}_{\lambda\in\Lambda}$ of operators defined by

$$(T_\lambda)_n^*=V_\lambda^*|\mathcal{H}_n \qquad (n\geq 1,\lambda\in\Lambda)$$

One can easily show that $\mathcal{V}$ is the minimal isometric dilation of $\mathcal{T}_n$ and that $\mathcal{T}_{n+1}$ is the one-step dilation of $\mathcal{T}_n$.

Let us observe that

$$\mathcal{H}_1=\mathcal{H}\oplus\mathcal{D}$$

and

$$\mathcal{H}_n=\mathcal{H}\oplus\mathcal{D}\oplus(\bigoplus_{f\in F(1,\Lambda)} S_f\mathcal{D})\oplus\dots\oplus(\bigoplus_{f\in F(n-1,\Lambda)} S_f\mathcal{D}) \qquad (n\geq 2)$$

where $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ is the Λ-orthogonal shift acting on $l^2(\mathcal{F},\mathcal{D})$.

Now Lemma 2 and Theorem 3 in [4] can be easily extended to our setting. Thus, we omit the proofs in what follows.

LEMMA 3.1. Let P_n be the orthogonal projection from $\mathcal{K}$ into $\mathcal{H}_n$.

Then

$$\bigvee_{n\geq 1}\mathcal{H}_n=\mathcal{K}$$

and for each $\lambda\in\Lambda$ we have

$$(T_\lambda)_n^*P_n \longrightarrow V_\lambda^* \text{ (strongly) as } n\to\infty .$$

Let $\mathcal{T}'=\{T_\lambda'\}_{\lambda\in\Lambda}$ be another sequence of operators on a Hilbert space $\mathcal{H}'$ with $\sum_{\lambda\in\Lambda} T_\lambda' T_\lambda'^* \leq I_{\mathcal{H}'}$ and $\mathcal{V}'=\{V_\lambda'\}_{\lambda\in\Lambda}$ be the minimal isometric dilation of $\mathcal{T}'$ acting on the Hilbert space $\mathcal{K}'=\mathcal{H}'\oplus l^2(\mathcal{F},\mathcal{D}')$.

THEOREM 3.2. Let $A:\mathcal{H}\longrightarrow\mathcal{H}'$ be a contraction such that for each $\lambda\in\Lambda$

$$T_\lambda' A = A T_\lambda$$

Then there exists a contraction $B:\mathcal{K}\longrightarrow\mathcal{K}'$ such that for each $\lambda\in\Lambda$

$$V_\lambda' B = B V_\lambda \quad \text{and} \quad B^*|_{\mathcal{H}'} = A^* .$$

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Gelu Popescu
Department of Mathematics
INCREST
Bd. Pacii 220, 79622 Bucharest
Romania.