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**F.-H. VASILESCU**

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*\*) Department of Mathematics, The National Institute for  
Scientifical and Technical Creation, Bd. Pacii 220, 79622  
Bucharest, Romania.*



# HOMOGENEOUS OPERATORS AND ESSENTIAL COMPLEXES

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F.-H. Vasilescu

## 1. INTRODUCTION

The aim of this work is to present a new approach to the concept of essential Fredholm complex of Banach spaces ([10],[2]; see also [11],[4],[6],[7] etc. for further connections), by using non-linear homogeneous mappings. We obtain some generalized homotopic properties of the class of essential Fredholm complexes, in our sense, which are then applied to establish its relationship with similar concepts. We also prove the stability of this class under small perturbations with respect to the gap topology.

Throughout this paper we shall work with linear spaces over the field  $\mathbb{K}$ , which is either the real field  $\mathbb{R}$  or the complex one  $\mathbb{C}$ . If  $X, Y$  are Banach spaces over  $\mathbb{K}$ , we denote by  $\mathcal{L}(X, Y)$  the space of all linear and continuous operators from  $X$  into  $Y$ . The subspace of  $\mathcal{L}(X, Y)$  consisting of all compact operators will be designated by  $\mathcal{K}(X, Y)$ . Let  $\mathcal{H}(X, Y)$  be the space of all  $\mathbb{K}$ -homogeneous and continuous operators from  $X$  into  $Y$ . Endowed with the usual operations and with the norm defined as in the case of linear operators, the space  $\mathcal{H}(X, Y)$  becomes a Banach space which contains  $\mathcal{L}(X, Y)$  as a closed subspace. Moreover, if  $\mathcal{KH}(X, Y)$  is the subspace of all compact operators from  $\mathcal{H}(X, Y)$  (the definition of a compact  $\mathbb{K}$ -homogeneous operator is the same as for a linear operator), then  $\mathcal{KH}(X, Y)$  is a closed subspace of  $\mathcal{H}(X, Y)$  and it obviously contains  $\mathcal{K}(X, Y)$  (see [12], [13] or [15] for details).

Let  $\text{Ban}_{\mathbb{K}}$  be the category whose objects are Banach spaces over  $\mathbb{K}$  and whose morphisms are bounded  $\mathbb{K}$ -linear operators. In the present <sup>work</sup>

a complex in the category  $\text{Ban}_K$  is a sequence  $A = (A_p)_{p \geq 0}$ , where  $A_p \in \mathcal{L}(X_p, X_{p-1})$ ,  $A_p A_{p+1} = 0$  and  $X_p$  is a Banach space over  $K$  for every integer  $p \geq 0$ , with  $X_{-1} = \{0\}$ , and therefore  $A_0 = 0$ . In addition, we assume that there exists an integer  $n \geq 0$  (depending upon  $A$ ) such that  $X_p = \{0\}$  for all  $p \geq n+1$  (and therefore  $A_p = \{0\}$  if  $p \geq n+1$ ). In other words, we work only with complexes of finite length. A complex  $A = (A_p)_{p \geq 0}$  can be represented in the usual way, that is, as a sequence of the form

$$0 \longrightarrow X_n \xrightarrow{A_n} X_{n-1} \xrightarrow{A_{n-1}} \dots \xrightarrow{A_1} X_0 \longrightarrow 0.$$

Nevertheless, since the space  $X_p$  is determined by the operator  $A_p$  (and it will be called in the sequel the domain of definition of  $A_p$ ), we prefer the concentrated notation  $A = (A_p)_{p \geq 0}$ .

For a given complex  $A = (A_p)_{p \geq 0}$ , we shall denote by  $H_p(A)$  the quotient  $N(A_p)/R(A_{p+1})$ , where  $N(A_p)$  is the null-space of  $A_p$  and  $R(A_{p+1})$  is the range of  $A_{p+1}$  (as a rule, we shall use the notation from [9]), that is, the homology of the complex  $A$ . When  $H_p(A) = \{0\}$  for all integers  $p \geq 0$ , the complex  $A$  is said to be exact. More generally, if  $\dim_K H_p(A) < \infty$  for all  $p \geq 0$ , then the complex  $A$  is said to be Fredholm. In this case, one can define the index of  $A$  by the formula

$$\text{ind}_K(A) = \sum_{p \geq 0} (-1)^p \dim_K H_p(A),$$

which has some stability properties (see [4], [14], [1], [2] etc.).

An essential complex [10], [2] in the category  $\text{Ban}_K$  is a sequence  $A = (A_p)_{p \geq 0}$ , where  $A_p \in \mathcal{L}(X_p, X_{p-1})$ ,  $A_p A_{p+1} \in \mathcal{K}(X_{p+1}, X_{p-1})$  and  $X_p$  is a Banach space over  $K$  for each integer  $p \geq 0$ , with  $X_{-1} = \{0\}$ . In addition, there exists an integer  $n \geq 0$  (depending upon  $A$ ) such that  $A_p = 0$  for all  $p \geq n+1$ .



Obviously, every complex is an essential complex.

A standard procedure to study essential complexes is to transform them into ordinary ones by means of certain functors. We shall describe in the following three functors of this type.

1° The functor  $\lambda_Z$ . Let  $Z$  be a fixed Banach space. For every Banach space  $X$ , we denote by  $\lambda_Z(X)$  the quotient  $\mathcal{L}(Z, X) / \mathcal{K}(Z, X)$ . Note that every linear operator  $S \in \mathcal{L}(X, Y)$  induces, by left multiplication, a linear and continuous operator  $\lambda_Z(S)$  from  $\lambda_Z(X)$  into  $\lambda_Z(Y)$ . It is easily seen that the assignment  $\lambda_Z$  defines a covariant functor from the category  $\text{Ban}_{\mathbb{K}}$  into itself. In addition, one has  $\lambda_Z(S) = 0$  for every Banach space  $Z$  iff  $S \in \mathcal{K}(X, Y)$ . This shows that if  $A = (A_p)_{p \geq 0}$  is an essential complex, then  $\lambda_Z(A) = (\lambda_Z(A_p))_{p \geq 0}$  is a complex.

The idea of studying classes of linear operators that are equivalent modulo compact operators goes back to Calkin. In connection with essential complexes, the functor  $\lambda_Z$  has been used in [10].

2° The functor  $\chi_Z$ . Let  $Z$  be again a fixed Banach space. For every Banach space  $X$  we denote by  $\chi_Z(X)$  the quotient  $\mathcal{H}(Z, X) / \mathcal{KH}(Z, X)$ . If  $S \in \mathcal{L}(X, Y)$ , then the left multiplication by  $S$  induces a linear and continuous operator from  $\chi_Z(X)$  into  $\chi_Z(Y)$ . It is straightforward to see that  $\chi_Z$  defines a covariant functor in the category  $\text{Ban}_{\mathbb{K}}$ . Moreover, we have  $\chi_Z(S) = 0$  for all Banach spaces  $Z$  iff  $S$  is compact. Therefore the functor  $\chi_Z$  also maps the class of essential complexes into the class of ordinary ones. This functor is seemingly new.

3° The functor  $\mathcal{L}$ . For every Banach space  $X$  let  $\ell^\infty(X)$  (resp.  $\mathcal{T}(X)$ ) be the Banach space of all bounded (resp. totally bounded) sequences consisting of elements of  $X$ , endowed with the

natural linear structure and topology. Then we consider the quotient  $\mathfrak{K}(X) = \ell^\infty(X)/\tau(X)$ . Every operator  $S \in \mathcal{L}(X, Y)$  induces a linear and continuous operator  $\mathfrak{K}(S)$  from  $\mathfrak{K}(X)$  into  $\mathfrak{K}(Y)$  by its action on coordinates. We obtain again a covariant functor in the category  $\text{Ban}_{\mathbb{K}}$ , such that  $\mathfrak{K}(S) = 0$  iff  $S$  is compact.

The functor  $\mathfrak{K}$ , which had been known for some time (see, for instance, [3]), was used to study essential complexes in [6], [7], [2] etc.

The above functors are needed to define various concepts of essential Fredholm complexes.

1.1. DEFINITION. Let  $A = (A_p)_{p \geq 0}$  be an essential complex in the category  $\text{Ban}_{\mathbb{K}}$ . We say that  $A$  is

(1)  $\lambda$ -Fredholm if the complex  $\lambda_Z(A) = (\lambda_Z(A_p))_{p \geq 0}$  is exact for every Banach space  $Z$ ;

(2)  $\chi$ -Fredholm if the complex  $\chi_Z(A) = (\chi_Z(A_p))_{p \geq 0}$  is exact for every Banach space  $Z$ ;

(3)  $\mathfrak{K}$ -Fredholm if the complex  $\mathfrak{K}(A) = (\mathfrak{K}(A_p))_{p \geq 0}$  is exact.

Let us mention that an essential  $\lambda$ -Fredholm complex is called in [10] simply Fredholm, and an essential  $\mathfrak{K}$ -Fredholm complex is called in [2] essentially Fredholm.

The purpose of this paper is to introduce the class of essential  $\chi$ -Fredholm complexes, to establish its relationship with the other classes of essential Fredholm complexes via some generalized homotopic properties, and to prove its stability under small perturbations with respect to the gap topology. Unlike the class of essential  $\lambda$ -Fredholm complexes, the class of essential  $\chi$ -Fredholm complexes contains the family of all Fredholm complexes (see Theorem 2.7). Nevertheless, some of the properties of essential  $\lambda$ -Fredholm complexes can be restated in the context of essential  $\chi$ -Fredholm complexes, as we shall see in



the next section.

## 2. GENERALIZED HOMOTOPIC PROPERTIES

Let  $A = (A_p)_{p \geq 0}$  and  $B = (B_p)_{p \geq 0}$  be two essential complexes in the category  $\text{Ban}_K$ . Let also  $X_p$  (resp.  $Y_p$ ) be the domain of definition of  $A_p$  (resp.  $B_p$ ) for every  $p \geq 0$ .

An essential morphism of  $A$  into  $B$  is a family of linear operators  $F = (F_p)_{p \geq 0}$  such that  $F_p \in \mathcal{L}(X_p, Y_p)$  and  $B_{p+1}F_{p+1} - F_p A_{p+1} \in \mathcal{K}(X_{p+1}, Y_p)$  for all  $p \geq 0$ .

The families  $1 = (1_p)_{p \geq 0}$  and  $0 = (0_p)_{p \geq 0}$ , where  $1_p$  is the identity on  $X_p$  and  $0_p$  is the zero map on  $X_p$ , are obviously essential morphism of the essential complex  $A = (A_p)_{p \geq 0}$  into itself.

2.1. REMARK. The essential morphism  $F = (F_p)_{p \geq 0}$  of  $A = (A_p)_{p \geq 0}$  into  $B = (B_p)_{p \geq 0}$  induces a linear map from  $H_p(\mathcal{X}_Z(A))$  into  $H_p(\mathcal{X}_Z(B))$  for each  $p \geq 0$  and every Banach space  $Z$ . Indeed, if  $\sigma \in N(\mathcal{X}_Z(A_p))$  and  $\sigma_0 \in \mathcal{H}(Z, X_p)$  is in the equivalence class  $\sigma$ , then  $A_p \sigma_0 \in \mathcal{K}\mathcal{H}(Z, X_{p-1})$ . Therefore  $B_p F_p \sigma_0 \in \mathcal{K}\mathcal{H}(Z, Y_{p-1})$ , so that the coset of  $F_p \sigma_0$  is in the null-space of  $\mathcal{X}_Z(B_p)$ . If, in addition,  $\sigma \in R(\mathcal{X}_Z(A_{p+1}))$ , a similar argument shows that the coset of  $F_p \sigma_0$  is in  $R(\mathcal{X}_Z(B_{p+1}))$ .

2.2. DEFINITION. Let  $A = (A_p)_{p \geq 0}$  and  $B = (B_p)_{p \geq 0}$  be two essential complexes and let  $X_p$  (resp.  $Y_p$ ) be the domain of definition of  $A_p$  (resp.  $B_p$ ). Let also  $F = (F_p)_{p \geq 0}$  and  $G = (G_p)_{p \geq 0}$  be two essential morphisms of  $A$  into  $B$ . We say that  $F$  and  $G$  are  $\mathcal{X}$ -homotopic if there exists a family of homogeneous operators  $\theta = (\theta_p)_{p \geq 0}$ , where  $\theta_p \in \mathcal{H}(X_p, Y_{p+1})$ , such that

$$(2.1) \quad F_p - G_p - B_{p+1} \theta_p - \theta_{p-1} A_p \in \mathcal{K}\mathcal{H}(X_p, Y_p), \quad p \geq 0,$$

with  $\theta_{-1} = 0$ .

Note that if  $\theta' = (\theta'_p)_{p \geq 0}$  and  $\theta'' = (\theta''_p)_{p \geq 0}$  satisfy (2.1), then for every  $t \in [0, 1]$  the family  $\theta(t) = (t \theta'_p + (1-t) \theta''_p)_{p \geq 0}$  also satisfies (2.1).

If in Definition 2.2 we can choose the operators  $(\theta^p)_{p \geq 0}$  to be  $\mathbb{K}$ -linear, then we say that  $E$  and  $G$  are  $\lambda$ -homotopic.

2.3. PROPOSITION. Let  $A, B, E, G$  be as in Definition 2.2. If  $E$  and  $G$  are  $\mathcal{X}$ -homotopic, then they induce (by Remark 2.1) the same map from  $H_p(\mathcal{X}_Z(A))$  into  $H_p(\mathcal{X}_Z(B))$  for each  $p \geq 0$  and every Banach space  $Z$ .

Proof. Since  $E$  and  $G$  are  $\mathcal{X}$ -homotopic iff  $F - G = (F_p - G_p)_{p \geq 0}$  and zero are  $\mathcal{X}$ -homotopic, with no loss of generality we may assume that  $F$  and  $G = 0$  are  $\mathcal{X}$ -homotopic. Then we shall show that the map induced by  $F$  from  $H_p(\mathcal{X}_Z(A))$  into  $H_p(\mathcal{X}_Z(B))$  is equal to zero. Indeed, if  $\sigma \in N(\mathcal{X}_Z(A_p))$  and  $\sigma_0 \in \sigma$  (as in Remark 2.1), then, from (2.1), we infer that  $F_p \sigma_0 - B_{p+1} \theta_p \sigma_0 \in \mathcal{KH}(Z, Y_p)$  (since  $\theta_{p-1} A_p \sigma_0$  is compact). This shows that the coset of  $F_p \sigma_0$  equals the coset of  $B_{p+1} \theta_p \sigma_0$ , that is, it is in  $R(\mathcal{X}_Z(B_{p+1}))$ . Hence the action induced by  $E$  from  $H_p(\mathcal{X}_Z(A))$  into  $H_p(\mathcal{X}_Z(B))$  must be null.

If  $A = (A_p)_{p \geq 0}$  is an essential  $\mathcal{X}$ -Fredholm complex and  $X_p$  is the domain of definition of  $A_p$ , then there are linear operators  $B_p \in \mathcal{L}(X_p, X_{p+1})$  and  $C_p \in \mathcal{K}(X_p, X_p)$  such that  $A_{p+1} B_p + B_{p-1} A_p = 1_p - C_p$  for all  $p \geq 0$ . The converse is also true (see [10], Proposition 2.3). In other words, an essential complex is  $\lambda$ -Fredholm iff the identity and the zero morphism on it are  $\lambda$ -homotopic. A similar result holds for essential  $\mathcal{X}$ -Fredholm complexes.

2.4. THEOREM. An essential complex  $A = (A_p)_{p \geq 0}$  is  $\mathcal{X}$ -Fredholm if and only if the identity on  $A$  is  $\mathcal{X}$ -homotopic to zero.



Proof. If the identity of  $A$  is  $\mathcal{K}$ -homotopic to zero, it follows from Proposition 2.3 that the identity on  $H_p(\mathcal{K}_Z(A))$  is zero for each integer  $p \geq 0$  and every Banach space  $Z$ . This forces the spaces  $H_p(\mathcal{K}_Z(A))$  to be zero, that is, the essential complex  $A$  is  $\mathcal{K}$ -Fredholm.

Conversely, we shall use an inductive argument inspired by the proof of Proposition 2.3 from [10]. We assume that the complex  $\mathcal{K}_Z(A)$  is exact for every Banach space  $Z$ . We shall show that there are operators  $\theta_p \in \mathcal{H}(X_p, X_{p+1})$  and  $\nu_p \in \mathcal{KH}(X_p, X_p)$  such that

$$(2.2) \quad A_{p+1} \theta_p + \theta_{p-1} A_p = 1_p - \nu_p, \quad p \geq 0,$$

with  $\theta_{-1} = 0$ , where  $X_p$  is the domain of definition of  $A_p$ . We find the operators  $\theta_p$  and  $\nu_p$  by induction with respect to  $p$ .

If  $p=0$ , then the exactness of the complex  $\mathcal{K}_Z(A)$  for  $Z=X_0$  shows that we can choose an operator  $\theta_0 \in \mathcal{H}(X_0, X_1)$  such that  $A_1 \theta_0 - 1_0 \in \mathcal{KH}(X_0, X_0)$ . Hence we may define  $\nu_0 = 1_0 - A_1 \theta_0$ .

Assume now that we have found the mappings  $\theta_q$  and  $\nu_q$  for all  $q \leq p$ . Note that

$$\begin{aligned} A_{p+1}(1_{p+1} - \theta_p A_{p+1}) &= A_{p+1} - (1_p - \nu_p - \theta_{p-1} A_p) A_{p+1} = \\ &= \nu_p A_{p+1} + \theta_{p-1} A_p A_{p+1} \in \mathcal{KH}(X_{p+1}, X_p), \end{aligned}$$

by (2.2). From the exactness of the complex  $\mathcal{K}_Z(A)$  with  $Z=X_{p+1}$ , we deduce the existence of an operator  $\theta_{p+1} \in \mathcal{H}(X_{p+1}, X_{p+2})$  such that  $A_{p+2} \theta_{p+1} = 1_{p+1} - \theta_p A_{p+1} - \nu_{p+1}$ , where  $\nu_{p+1} \in \mathcal{KH}(X_{p+1}, X_{p+1})$ .

This shows that equations (2.2) have solutions for all  $p \geq 0$ , that is, the identity on  $A$  and zero are  $\mathcal{K}$ -homotopic.

2.5. COROLLARY. Let  $A = (A_p)_{p \geq 0}$  be an essential complex and

let  $X_p$  be the domain of definition of  $A_p$  .Then  $A$  is  $\mathcal{X}$ -Fredholm if and only if the complex  $\mathcal{X}_Z(A)$  is exact for each  $Z$  in the family  $(X_p)_{p \geq 0}$  .

Proof. It follows from the proof of Theorem 2.4 that equations (2.2) have solutions when  $\mathcal{X}_Z(A)$  is exact only for those  $Z$  in the family  $(X_p)_{p \geq 0}$  .

2.6. COROLLARY. Let  $A = (A_p)_{p \geq 0}$  be an essential complex.

(1) If  $A$  is  $\lambda$ -Fredholm, then  $A$  is  $\mathcal{X}$ -Fredholm.

(2) If  $A$  is  $\mathcal{X}$ -Fredholm, then  $A$  is  $\mathcal{X}$ -Fredholm.

Proof. The assertion (1) follows from Proposition 2.3 from [10], via Theorem 2.4.

Let us prove the assertion (2). Let us fix an integer  $p \geq 0$ , let  $X_p$  be the domain of definition of  $A_p$  and let  $\{x_k\}_k$  be a sequence in  $\ell^\infty(X_p)$  such that  $\{A_p x_k\}_k \in \tau(X_{p-1})$ . If we choose  $\theta_p$  as in (2.2), we can see that  $\{A_{p+1} y_k - x_k\}_k \in \tau(X_p)$ , where  $y_k = \theta_p(x_k)$ , and hence  $\{y_k\}_k \in \ell^\infty(X_{p+1})$ . In other words, the complex  $\mathcal{X}(A)$  is exact, that is,  $A$  is  $\mathcal{X}$ -Fredholm.

The class of essential  $\mathcal{X}$ -Fredholm complexes contains the class of Fredholm complexes, via Theorem 2.4.

2.7. THEOREM. Let  $A = (A_p)_{p \geq 0}$  be a complex in the category  $\text{Ban}_{\mathbb{K}}$ . Then  $A$  is Fredholm iff the identity of  $A$  and zero are  $\mathcal{X}$ -homotopic.

Proof. Assume first that  $A$  is a Fredholm complex, that is,  $\dim_{\mathbb{K}} H_p(A) < \infty$  for every  $p \geq 0$ . In particular, the space  $R(A_{p+1})$  is closed and of finite codimension in  $N(A_p)$ . Let  $\pi_{1,p}$  be a linear projection of  $N(A_p)$  onto  $R(A_{p+1})$ . Let also  $\pi_{2,p}$  be a homogeneous projection of  $X_p$  onto  $N(A_p)$  (see, for instance, Corollary 1.2 from [15]). Then  $\pi_p$  is a homogeneous projection of  $X_p$  onto  $R(A_{p+1})$ ,



where  $\pi_p = \pi'_{1,p} \pi_{2,p}$ . Let  $C_p : X_p \longrightarrow X_p/N(A_p)$  be the canonical mapping and let  $\int_p : X_p/N(A_p) \longrightarrow X_p$  be the homogeneous lifting associated with  $\pi_{2,p}$  [15], that is,  $\int_p C_p = 1_p - \pi_{2,p}$ . We define a mapping  $\theta_p \in \mathcal{H}(X_p, X_{p+1})$  in the following way. Let

$$B_{p+1} : X_{p+1}/N(A_{p+1}) \longrightarrow R(A_{p+1})$$

be the bijective operator induced by  $A_{p+1}$ . Then  $(B_{p+1})^{-1}$  is a bounded operator, by the closed graph theorem. Set  $\theta_p = \int_{p+1} (B_{p+1})^{-1} \pi_p$ , which is a continuous and homogeneous mapping from  $X_p$  into  $X_{p+1}$  for all  $p \geq 0$ . We shall show that the mappings  $\theta_p$  satisfy (2.2).

Indeed, let  $x \in X_p$ , where  $p \geq 0$  is fixed. Note that

$$\begin{aligned} \theta_{p-1}(A_p x) &= (\int_p (B_p)^{-1} \pi_{p-1} A_p)(x) = (\int_p (B_p)^{-1})(A_p x) = \\ &= \int_p (C_p x) = x - \pi_{2,p}(x). \end{aligned}$$

On the other hand, since  $\pi_p(x) \in R(A_{p+1})$ , we can write  $\pi_p(x) = A_{p+1} v$  for some  $v \in X_{p+1}$ . Hence

$$\begin{aligned} A_{p+1} \theta_p(x) &= (A_{p+1} \int_{p+1} (B_{p+1})^{-1} \pi_p)(x) = \\ &= (A_{p+1} \int_{p+1} (B_{p+1})^{-1})(A_{p+1} v) = (A_{p+1} \int_{p+1} C_{p+1})(v) = \\ &= A_{p+1}(v - \pi_{2,p+1}(v)) = A_{p+1} v = \pi_p(x). \end{aligned}$$

Consequently,

$$A_{p+1} \theta_p(x) + \theta_{p-1}(A_p x) = x - \gamma_p(x), \quad x \in X_p,$$

and the homogeneous mapping

$$\gamma_p = \pi_{2,p} - \pi_p = (1_p|N(A_p) - \pi'_{1,p}) \pi_{2,p}$$

is compact, since the linear operator  $1_p|N(A_p) - \pi'_{1,p}$  is a finite rank projection. Therefore the identity on the essential complex  $A$  is  $\gamma$ -homotopic to zero.

Conversely, let  $A$  be a complex such that its identity and zero

are  $\mathcal{X}$ -homotopic. Then we can find operators  $\theta_p \in \mathcal{H}(X_p, X_{p+1})$  and  $\gamma_p \in \mathcal{K} \mathcal{H}(X_p, X_p)$  such that (2.2) is fulfilled. By using (2.2), we shall derive that the complex  $A$  is Fredholm.

We first prove that  $R(A_p)$  is a closed subspace of  $X_{p-1}$  for every  $p \geq 0$ . If this were not true for some  $p$ , then there would exist a sequence  $\{y_k\}_k \subset R(A_p)$ ,  $y_k = A_p(x_k)$ , such that  $y_k \rightarrow 0$  ( $k \rightarrow \infty$ ) but  $\text{dist}(x_k, N(A_p)) = 1$  for all  $k$  (where "dist" stands for distance). The sequence  $\{x_k\}_k$  can obviously be assumed bounded. From the equality

$$A_{p+1} \theta_p(x_k) + \theta_{p-1}(y_k) = x_k - \gamma_p(x_k),$$

which follows from (2.2), we infer that the sequence  $\{x_k - A_{p+1} \theta_p(x_k)\}_k$  contains a convergent subsequence. Therefore, with no loss of generality, we may assume that

$$x_k - A_{p+1} \theta_p(x_k) \rightarrow v \in X_p \quad (k \rightarrow \infty).$$

Then

$$A_p(x_k - A_{p+1} \theta_p(x_k)) = A_p x_k = y_k \rightarrow 0 \quad (k \rightarrow \infty),$$

so that  $v \in N(A_p)$ . Hence

$$\text{dist}(x_k, N(A_p)) \leq \|x_k - A_{p+1} \theta_p(x_k) - v\| \rightarrow 0 \quad (k \rightarrow \infty),$$

which contradicts the choice of the sequence  $\{y_k\}_k$ . Therefore  $R(A_p)$  must be closed for all  $p \geq 0$ .

We now prove that  $\dim_{\mathbb{K}} H_p(A) < \infty$  for each  $p \geq 0$ . If this were not true for a certain  $p$ , then in the Banach space  $N(A_p)/R(A_{p+1})$  there would exist a bounded sequence  $\{\xi_k\}_k$  which wouldn't contain any convergent subsequence. Let  $x_k \in \xi_k$  be chosen such that the sequence  $\{x_k\}_k$  be bounded in  $N(A_p)$ . Then we have

$$A_{p+1} \theta_p(x_k) - \theta_{p-1}(A_p x_k) = A_{p+1} \theta_p(x_k) = x_k - \gamma_p(x_k),$$

as a consequence of (2.2). Since  $\gamma_p$  is compact, the sequence  $\{\gamma_p(x_k)\}_k$



contains a convergent subsequence. Therefore

$$\xi_k = x_k + R(A_{p+1}) = \mathcal{V}_p(x_k) + R(A_{p+1})$$

contains a convergent subsequence, which contradicts the choice of the sequence  $\{\xi_k\}_k$ .

Consequently the complex  $A$  is Fredholm.

2.8. COROLLARY. Let  $A = (A_p)_{p \geq 0}$  be a complex in the category  $\text{Ban}_K$ . The following assertions are equivalent :

- (1)  $A$  is Fredholm ;
- (2)  $A$  is  $\mathcal{V}$ -Fredholm;
- (3)  $A$  is  $\mathcal{K}$ -Fredholm.

Proof. The equivalence  $(1) \iff (2)$  follows from Theorems 2.4 and 2.7 whereas the equivalence  $(1) \iff (3)$  follows from the results of [6] or [7] (from which one can also derive the "sufficiency" in Theorem 2.7).

2.9. REMARK. The class of Fredholm complexes is strictly larger than the class of those complexes that are  $\lambda$ -Fredholm. Indeed, let  $X_1$  be a Banach space which contains a closed linear subspace  $X_2$  that is not complemented in  $X_1$ . Let also  $X_0 = X_1/X_2$ . If  $A_2 : X_2 \rightarrow X_1$  is the inclusion and  $A_1 : X_1 \rightarrow X_0$  is the canonical mapping, then the complex  $A = (0, A_2, A_1, 0)$  is Fredholm but not  $\lambda$ -Fredholm, in virtue of Theorem 2.7 from [6].

It is known that for every essential  $\lambda$ -Fredholm complex one can define an index that is stable under compact and small perturbations, and coincides with the natural one in the case of Fredholm complexes (see [10] for details). Therefore, a natural question arises in our context: Is it possible to assign an index to every essential  $\mathcal{K}$ -Fredholm complex? Of course, such an index should be stable under small and compact perturbations and coincide with the usual one in the case of Fredholm complexes. This problem, whose answer is not known

to the author of this text, is intimately connected with the Problem 5.1 from [1] as well as with the problem raised in [2], Remark 4.8(3), which have received so far only partial answers (see, for instance, [8]).

Finally, let us remark that Theorem 2.7 leads to a non-linear version of a well-known characterization of Fredholm operators.

2.10. COROLLARY. Let  $X_1, X_0$  be Banach spaces. A linear operator  $A_1 \in \mathcal{L}(X_1, X_0)$  is Fredholm if and only if there exists a homogeneous operator  $\theta_1 \in \mathcal{H}(X_0, X_1)$  such that  $A_1 \theta_1 - 1_0 \in \mathcal{KH}(X_0, X_0)$  and  $\theta_1 A_1 - 1_1 \in \mathcal{KH}(X_1, X_1)$ .

### 3. STABILITY UNDER SMALL PERTURBATIONS

In this section we shall prove that the class of essential  $\mathcal{K}$ -Fredholm complexes is stable under perturbations that are small with respect to the gap topology (see [6], [7], [10], [2] etc. for similar results concerning the other types of essential Fredholm complexes). We shall rely heavily upon some perturbation results from [2].

Let  $\mathcal{X}$  be a fixed Banach space over the field  $\mathbb{K}$ . We denote by  $\mathcal{G}(\mathcal{X})$  the family of all closed linear subspaces of  $\mathcal{X}$ . The space  $\mathcal{G}(\mathcal{X})$  is endowed with the gap topology, which is defined by means of the functions  $\delta(X, Y)$  and  $\hat{\delta}(X, Y)$ , where  $X, Y \in \mathcal{G}(\mathcal{X})$  (see [9], IV.2.1).

3.1. LEMMA. Let  $X, Y \in \mathcal{G}(\mathcal{X})$  and let  $\delta > \delta(X, Y)$ . Then there exists an operator  $\tau \in \mathcal{H}(X, Y)$  such that  $\|x - \tau(x)\| \leq \delta \|x\|$  for each  $x \in X$ .

Proof. Let  $S(X)$  be the unit sphere of  $X$  and let  $x \in S(X)$ . Then there exists a vector  $y_x \in Y$  such that  $\|x - y_x\| < \delta$ . Let  $V_x$  be an open neighbourhood of  $x$  in  $S(X)$  such that  $\|v - y_x\| < \delta$  for all  $v \in V_x$ . The family  $\{V_x\}_{x \in S(X)}$  is an open cover of the metric (and therefore paracompact) space  $S(X)$ , so that we can choose a



partition of unity subordinated to this cover. In other words, there exists a family of continuous and non-negative functions  $\{f_\alpha\}_{\alpha \in A}$  on  $S(X)$  such that  $\text{supp}(f_\alpha) \subset V_{x_\alpha}$  for every  $\alpha \in A$ , the family of closed sets  $\{\text{supp}(f_\alpha)\}_{\alpha \in A}$  is locally finite and  $\sum_{\alpha \in A} f_\alpha(x) = 1$  for all  $x \in S(X)$ . We define a continuous  $Y$ -valued function by the formula

$$\tau_0(x) = \sum_{\alpha \in A} f_\alpha(x) y_{x_\alpha}, \quad x \in S(X),$$

where  $y_{x_\alpha}$  corresponds to the vector  $x_\alpha \in V_{x_\alpha}$ , as above. Obviously,  $\|x - \tau_0(x)\| \leq \delta$  for all  $x \in S(X)$ . Since  $\|y_{x_\alpha}\| \leq \delta + \|x_\alpha\| \leq \delta + 1$ , the function  $\tau_0$  is bounded.

We shall obtain from  $\tau_0$  a homogeneous operator by using a procedure inspired from [16], Thm. 1.1. We first extend the function  $\tau_0$  to the whole space  $X$  by setting  $\tau_1(x) = \|x\| \tau_0(x/\|x\|)$  if  $x \neq 0$  and  $\tau_1(0) = 0$ . Since  $\tau_0$  is bounded, it is easily seen that  $\tau_1$  is continuous on  $X$ . Moreover,  $\tau_1$  is positive-homogeneous. In addition, if  $x \neq 0$ ,

$$\|x - \tau_1(x)\| = \|x\| \|x/\|x\| - \tau_0(x/\|x\|)\| \leq \delta \|x\|,$$

and the final estimate also holds for  $x = 0$ .

We shall discuss two cases. If  $K = \mathbb{R}$ , then we define

$$\tau(x) = 2^{-1}(\tau_1(x) - \tau_1(-x)).$$

The function  $\tau$  is  $\mathbb{R}$ -linear,  $Y$ -valued and continuous on  $X$ . Moreover,

$$\|x - \tau(x)\| \leq 2^{-1} \|x - \tau_1(x)\| + 2^{-1} \|x + \tau_1(-x)\| \leq \delta \|x\|$$

for all  $x \in X$ .

If  $K = \mathbb{C}$ , then we define

$$\tau(x) = (2\pi)^{-1} \int_0^{2\pi} e^{-it} \tau_1(e^{it} x) dt, \quad x \in X.$$

It is easy to check that  $\tau$  is continuous,  $Y$ -valued and  $\mathbb{C}$ -homoge-

neous. Furthermore,

$$\begin{aligned} \|x - \tau(x)\| &= (2\pi)^{-1} \left\| \int_0^{2\pi} e^{-it} (e^{it}x - \tau_1(e^{it}x)) dt \right\| \leq \\ &\leq (2\pi)^{-1} \int_0^{2\pi} \|e^{it}x - \tau_1(e^{it}x)\| dt \leq \delta \|x\|, \end{aligned}$$

and the proof of the lemma is complete.

3.2. LEMMA: Let  $X, Y \in \mathcal{G}(\mathcal{X})$  . Then for every Banach space  $Z$  we have the estimates

$$\delta(\mathcal{H}(Z, X), \mathcal{H}(Z, Y)) \leq \delta(X, Y),$$

$$\delta(\mathcal{KH}(Z, X), \mathcal{KH}(Z, Y)) \leq \delta(X, Y).$$

Proof. Let  $\delta > \delta(X, Y)$  . By the previous lemma, we can choose  $\tau \in \mathcal{H}(X, Y)$  such that  $\|x - \tau(x)\| \leq \delta \|x\|$  for all  $x \in X$  . Let  $\sigma \in \mathcal{H}(Z, X)$  be arbitrary and let  $\theta = \tau\sigma \in \mathcal{H}(Z, Y)$  . Then

$$\|\sigma - \theta\| = \sup_{\|x\| \leq 1} \|\sigma(x) - \tau(\sigma(x))\| \leq \delta \sup_{\|x\| \leq 1} \|\sigma(x)\| \leq \delta \|\sigma\|.$$

Therefore  $\delta(\mathcal{H}(Z, X), \mathcal{H}(Z, Y)) \leq \delta$  . Since  $\delta > \delta(X, Y)$  is arbitrary, we infer easily the first estimate from the statement.

To obtain the second estimate from the statement, we only note that if  $\sigma \in \mathcal{KH}(Z, X)$  , then  $\theta = \tau\sigma \in \mathcal{KH}(Z, Y)$  . A similar argument then leads to the desired conclusion.

3.3. LEMMA. Let  $X_1, Y_1, X_2, Y_2 \in \mathcal{G}(\mathcal{X})$  , let  $S_1 \in \mathcal{L}(X_1, Y_1)$  and let  $S_2 \in \mathcal{L}(X_2, Y_2)$  . Then

$$\delta(X_1, X_2) \leq (1 + \|S_1\|) \delta(S_1, S_2).$$

In addition, if  $(1 + \|S_1\|) \delta(S_2, S_1) < 1$  , then

$$\|S_2\| \leq (1 - (1 + \|S_1\|)\delta(S_2, S_1))^{-1} ((1 + \|S_1\|)\delta(S_2, S_1) + \|S_1\|).$$

This result is contained in Lemma 4.6 from [2].

3.4. LEMMA. Let  $X_1, Y_1, X_2, Y_2 \in \mathcal{G}(\mathcal{X})$  , and let  $S_j \in \mathcal{L}(X_j, Y_j)$  ( $j=1,2$ ) . We define the operators  $M_j : \mathcal{H}(Z, X_j) \rightarrow \mathcal{H}(Z, Y_j)$  by the



equality  $M_j \sigma = S_j \cdot \sigma (\sigma \in \mathcal{H}(Z, X_j))$ , where  $Z$  is an arbitrary Banach space. Then we have the estimate  $\delta(M_1, M_2) \leq 2^{1/2} \delta(S_1, S_2)$ .

Proof. We have, by definition [9],

$$\delta(S_1, S_2) = \delta(G(S_1), G(S_2)),$$

where  $G(S_j) \in \mathcal{G}(\mathcal{X} \times \mathcal{X})$  is the graph of the operator  $S_j$  ( $j=1,2$ ).

Let  $\delta > \delta(S_1, S_2)$ . By virtue of Lemma 3.1, there exists an operator  $\tau \in \mathcal{H}(G(S_1), G(S_2))$  such that

$$(3.1) \quad \|(x, S_1 x) - \tau(x, S_1 x)\| \leq \delta \|(x, S_1 x)\|, \quad x \in X_1.$$

We fix an element  $\sigma_1 \in \mathcal{H}(Z, X_1)$ . Then the map

$$Z \ni z \longrightarrow \tau(\sigma_1(z), S_1 \sigma_1(z)) \in G(S_2)$$

is continuous and  $\mathbb{K}$ -homogeneous. Therefore it must be of the form

$$\tau(\sigma_1(z), S_1 \sigma_1(z)) = (\sigma_2(z), S_2 \sigma_2(z)), \quad z \in Z,$$

where  $\sigma_2 \in \mathcal{H}(Z, X_2)$ . If we assume that the norm of  $\mathcal{X} \times \mathcal{X}$  is given by  $\|(x_1, x_2)\|^2 = \|x_1\|^2 + \|x_2\|^2$ , from the estimate (3.1) we infer that

$$\begin{aligned} & \|\sigma_1(z) - \sigma_2(z)\|^2 + \|S_1 \sigma_1(z) - S_2 \sigma_2(z)\|^2 \leq \\ & \leq \delta^2 (\|\sigma_1(z)\|^2 + \|S_1 \sigma_1(z)\|^2) \leq \delta^2 (\|\sigma_1\|^2 + \|S_1 \sigma_1\|^2) \|z\|^2, \end{aligned}$$

so that

$$\begin{aligned} \|\sigma_1 - \sigma_2\|^2 & \leq \delta^2 (\|\sigma_1\|^2 + \|S_1 \sigma_1\|^2), \\ \|S_1 \sigma_1 - S_2 \sigma_2\|^2 & \leq \delta^2 (\|\sigma_1\|^2 + \|S_1 \sigma_1\|^2). \end{aligned}$$

Hence

$$\|(\sigma_1, M_1 \sigma_1) - (\sigma_2, M_2 \sigma_2)\|^2 \leq 2 \delta^2 (\|\sigma_1\|^2 + \|M_1 \sigma_1\|^2),$$

which leads to the desired estimate.

To state the next lemma, we need a special notation from [2].

Let  $X, X_0, Y, Y_0 \in \mathcal{G}(\mathcal{X})$  be such that  $X_0 \subset X$  and  $Y_0 \subset Y$ , and let  $S \in \mathcal{L}(X/X_0, Y/Y_0)$ . Then we set

$$G_0(S) = \{(x, y) \in \mathcal{X} \times \mathcal{X} : S(x + X_0) = y + Y_0\}.$$

If  $\tilde{X}, \tilde{X}_0, \tilde{Y}, \tilde{Y}_0 \in \mathcal{G}(\mathcal{X})$  have similar properties and  $\tilde{S} \in \mathcal{L}(\tilde{X}/\tilde{X}_0, \tilde{Y}/\tilde{Y}_0)$ , then we define

$$(3.2) \quad \begin{aligned} \delta_0(S, \tilde{S}) &= \delta(G_0(S), G_0(\tilde{S})) , \\ \hat{\delta}_0(S, \tilde{S}) &= \max \{ \delta_0(S, \tilde{S}), \delta_0(\tilde{S}, S) \} , \end{aligned}$$

which are computed in  $\mathcal{G}(\mathcal{X} \times \mathcal{X})$ .

3.5. LEMMA. Let  $X_1, Y_1, X_2, Y_2 \in \mathcal{G}(\mathcal{X})$  and let  $S_j \in \mathcal{L}(X_j, Y_j)$  ( $j=1,2$ ). Then for every Banach space  $Z$  we have the estimate

$$\delta_0(\chi_Z(S_1), \chi_Z(S_2)) \leq 3(1 + \|S_1\|) \max \{ \delta(Y_1, Y_2), \delta(S_1, S_2) \}.$$

Proof. Let  $\delta_0 > \delta(Y_1, Y_2)$ , let  $\delta > \delta(S_1, S_2)$  and let  $(\sigma_1, \theta_1) \in G_0(\chi_Z(S_1))$ . Then  $\theta_1 - S_1\sigma_1 \in \mathcal{KH}(Z, Y_1)$ . By Lemma 3.2, we can choose  $\nu_2 \in \mathcal{KH}(Z, Y_2)$  such that  $\|\theta_1 - S_1\sigma_1 - \nu_2\| < \delta_0 \|\theta_1 - S_1\sigma_1\|$ .

With the notation of Lemma 3.4,  $(\sigma_1, S_1\sigma_1) \in G(M_1)$ . Then, by this lemma, we can find an element  $(\sigma_2, S_2\sigma_2) \in G(M_2)$  such that

$$\|\sigma_1 - \sigma_2\|^2 + \|S_1\sigma_1 - S_2\sigma_2\|^2 \leq 2\delta^2(\|\sigma_1\|^2 + \|S_1\sigma_1\|^2).$$

If we set  $\theta_2 = S_2\sigma_2 + \nu_2 \in \mathcal{H}(Z, Y_2)$ , then  $(\sigma_2, \theta_2) \in G_0(\chi_Z(S_2))$ .

In addition,

$$\begin{aligned} \|\sigma_1 - \sigma_2\|^2 + \|\theta_1 - \theta_2\|^2 &\leq \|\sigma_1 - \sigma_2\|^2 + (\|\sigma_1 - S_1\sigma_1 - \nu_2\| + \\ &+ \|S_1\sigma_1 - S_2\sigma_2\|)^2 \leq 2\delta^2(\|\sigma_1\|^2 + \|S_1\sigma_1\|^2) + \\ &+ 2\delta_0^2\|\theta_1 - S_1\sigma_1\|^2 + 4\delta^2(\|\sigma_1\|^2 + \|S_1\sigma_1\|^2) < \\ &< 9(1 + \|S_1\|^2) \max \{ \delta_0^2, \delta^2 \} (\|\sigma_1\|^2 + \|\theta_1\|^2), \end{aligned}$$

whence we obtain the claimed estimate.

Let  $C = (C_p)_{p \geq 0}$  be a complex such that the domain of definition of  $C_p$  is of the form  $X_p/Y_p$ , where  $X_p, Y_p \in \mathcal{G}(\mathcal{X})$ ,  $Y_p \subset X_p$  for all  $p \geq 0$ . Let also  $\tilde{C} = (\tilde{C}_p)_{p \geq 0}$  be a complex of the same type, that



is,  $\tilde{C}_p$  is defined on  $\tilde{X}_p/\tilde{Y}_p$ , with  $\tilde{X}_p, \tilde{Y}_p \in \mathcal{G}(\mathcal{X})$ ,  $\tilde{Y}_p \subset \tilde{X}_p$ . Then one can define the quantity [2]

$$(3.3) \quad \hat{\delta}_0(C, \tilde{C}) = \sup_{p \geq 0} \hat{\delta}_0(C_p, \tilde{C}_p),$$

with  $\hat{\delta}_0(C_p, \tilde{C}_p)$  given by (3.2). We also set [2]

$$(3.4) \quad \gamma(C) = \inf_{p \geq 0} \gamma(C_p),$$

where  $\gamma(C_p)$  is the reduced minimum modulus of  $C_p$  [9]. If  $C$  is Fredholm, in particular if  $C$  is exact, then obviously  $\gamma(C) > 0$ .

3.6. PROPOSITION. Let  $C$  and  $\tilde{C}$  be as above. If  $C$  is exact and  $3 \hat{\delta}_0(C, \tilde{C})(1 + \gamma(C)^{-2})^{1/2} < 1$ , then  $\tilde{C}$  is also exact.

Proof. The assertion is a consequence of Corollary 2.12 from [2].

Indeed, let us define the numbers

$$\begin{aligned} r &= \gamma(C)^{-1} \geq \max \{ \gamma(C_p)^{-1}, \gamma(C_{p+1})^{-1} \}, \\ \delta_0 &= \hat{\delta}_0(C, \tilde{C}) \geq \max \{ \delta_0(\tilde{C}_p, C_p), \delta_0(C_{p+1}, \tilde{C}_{p+1}) \}, \\ \delta &= (1 + \gamma(C)^{-2})^{1/2} \hat{\delta}_0(C, \tilde{C}) \geq \max \{ (1 + \gamma(C_p)^{-2})^{1/2} \delta_0(\tilde{C}_p, C_p), \\ &\quad (1 + \gamma(C_{p+1})^{-2})^{1/2} \delta_0(C_{p+1}, \tilde{C}_{p+1}) \}. \end{aligned}$$

Then we have

$$\begin{aligned} \delta + \delta_0(1 + \delta)(1 + r^2)^{1/2} &\leq \delta + 2\delta_0(1 + r^2)^{1/2} = \\ &= 3 \hat{\delta}_0(C, \tilde{C})(1 + \gamma(C)^{-2})^{1/2} < 1, \end{aligned}$$

which insures, by Corollary 2.12 from [2], that  $R(\tilde{C}_{p+1}) = N(\tilde{C}_p)$  for all  $p \geq 0$ , that is, the exactness of the complex  $\tilde{C}$ .

We can now establish the exactness of the class of  $\mathcal{H}$ -Fredholm complexes under small perturbations. We shall denote by  $\partial_e(\mathcal{X})$  the family of those essential complexes  $A = (A_p)_{p \geq 0}$  such that  $X_p \in \mathcal{G}(\mathcal{X})$  for all  $p \geq 0$ , where  $X_p$  is the domain of definition of  $A_p$ .

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If  $A = (A_p)_{p \geq 0}$ ,  $B = (B_p)_{p \geq 0}$  are members of  $\mathcal{D}_e(\mathcal{X})$ , we set

$$(3.5) \quad \widehat{\delta}(A, B) = \sup_{p \geq 0} \widehat{\delta}(A_p, B_p),$$

where  $\widehat{\delta}(A_p, B_p) = \max \{ \delta(A_p, B_p), \delta(B_p, A_p) \}$  ([9]; see also Lemma 3.4).

Let also

$$(3.6) \quad \|A\| = \sup_{p \geq 0} \|A_p\|, \quad A = (A_p)_{p \geq 0} \in \mathcal{D}_e(\mathcal{X}).$$

2.7. THEOREM. Let  $A = (A_p)_{p \geq 0} \in \mathcal{D}_e(\mathcal{X})$  be  $\mathcal{X}$ -Fredholm. Then there exists a positive number  $\delta_A$  such that if  $B = (B_p)_{p \geq 0} \in \mathcal{D}_e(\mathcal{X})$  and  $\widehat{\delta}(A, B) < \delta_A$ , then  $B$  is also  $\mathcal{X}$ -Fredholm.

Proof. Let  $X_p$  be the domain of definition of  $A_p$  and let  $(\theta_p)_{p \geq 0}$  be a fixed family of homogeneous operators such that  $\theta_p \in \mathcal{H}(X_p, X_{p+1})$  and  $A_{p+1}\theta_p + \theta_{p-1}A_p - 1_p \in \mathcal{K}\mathcal{H}(X_p, X_p)$  for all  $p \geq 0$  (which exists by Theorem 2.4). With no loss of generality we may assume that  $\widehat{\delta}(A, B)$  is small enough so that

$$(3.7) \quad (1 + \|A\|) \widehat{\delta}(A, B) \leq 2^{-1}.$$

In particular,  $(1 + \|A_p\|) \delta(B_p, A_p) \leq 2^{-1}$ , and hence

$$\begin{aligned} \|B_p\| &\leq (1 - (1 + \|A_p\|) \delta(B_p, A_p))^{-1} ((1 + \|A_p\|) \delta(B_p, A_p) + \|A_p\|) \leq \\ &\leq 2(2^{-1} + \|A_p\|) = 1 + 2\|A_p\|, \end{aligned}$$

by Lemma 3.3. Therefore

$$(3.8) \quad \|B\| = \sup_{p \geq 0} \|B_p\| \leq 1 + 2\|A\|.$$

Let  $Z$  be a fixed Banach space. With  $(\theta_p)_{p \geq 0}$  as above (note that  $\theta_p = 0$  for all but a finite family of indices), we have

$$(3.9) \quad \gamma(\chi_Z(A))^{-1} \leq r < \infty,$$

where  $r = \max \{ \|\theta_p\| : p \geq 0 \}$ . Indeed, let  $\varepsilon > 0$  be given, let

$\sigma \in N(\chi_Z(A_p))$  for some  $p$  and let  $\sigma_0 \in \sigma$  be such that

$$\|\sigma_0\| \leq (1 + \varepsilon) \|\sigma\|. \text{ Then, from (2.2),}$$



$$\chi_Z(A_{p+1})(\theta_p \sigma_0 + \mathcal{K} \mathcal{H}(z, x_{p+1})) = \sigma .$$

Moreover,

$$\|\theta_p \sigma_0 + \mathcal{K} \mathcal{H}(z, x_{p+1})\| \leq \|\theta_p \sigma_0\| \leq (1+\varepsilon) \|\theta_p\| \|\sigma\| .$$

Using the definition of  $\gamma(\chi_Z(A_{p+1}))$ , we deduce that

$$\gamma(\chi_Z(A_{p+1}))^{-1} \leq (1+\varepsilon) \|\theta_p\| .$$

Since  $\varepsilon > 0$  is arbitrary, from the last estimate we derive easily (3.9).

Next we prove that

$$(3.10) \quad \hat{\delta}_0(\chi_Z(A), \chi_Z(B)) \leq 12(1+\|A\|)^2 \hat{\delta}(A, B) .$$

Indeed, by Lemma 3.3,

$$\begin{aligned} \hat{\delta}(X_p, Y_p) &\leq (1 + \max\{\|A_p\|, \|B_p\|\}) \hat{\delta}(A_p, B_p) \leq \\ &\leq (1 + \max\{\|A\|, \|B\|\}) \hat{\delta}(A, B) \leq 2(1 + \|A\|) \hat{\delta}(A, B) , \end{aligned}$$

where we have used (3.8). Therefore, Lemma 3.5 and the above calculation show that

$$\begin{aligned} \hat{\delta}_0(\chi_Z(A_p), \chi_Z(B_p)) &\leq 3(1 + \max\{\|A_p\|, \|B_p\|\}) \times \\ &\times \max\{\hat{\delta}(X_{p-1}, Y_{p-1}), \hat{\delta}(A_p, B_p)\} \leq \\ &\leq 12(1 + \|A\|)^2 \hat{\delta}(A, B) , \end{aligned}$$

where  $Y_p$  is the domain of definition of  $B_p$ , which obviously leads to (3.10).

To obtain the final conclusion, we intend to apply Proposition 3.6.

Note that

$$\begin{aligned} (3.11) \quad &3 \hat{\delta}_0(\chi_Z(A), \chi_Z(B)) (1 + \gamma(\chi_Z(A))^{-2})^{1/2} \leq \\ &\leq 36(1 + \|A\|)^2 (1 + r^2)^{1/2} \hat{\delta}(A, B) , \end{aligned}$$

which is given by (3.9) and (3.10). Hence, if

$$(3.12) \quad 36(1 + \|A\|)^2 (1 + \max_p \|\theta_p\|^2)^{1/2} \hat{\delta}(A, B) < 1 ,$$

then, by Proposition 3.6, the complex  $\chi_Z(B)$  is also exact. Since the

coefficient of  $\hat{\mathcal{S}}(A,B)$  in (3.12) does not depend on  $Z$ , the fulfillment of (3.12) (provided (3.4) is also fulfilled), implies the exactness of  $\mathcal{K}_Z(B)$  for every Banach space  $Z$ , that is, the essential complex  $B$  is  $\mathcal{K}$ -Fredholm.

2.8. COROLLARY. The family of  $\mathcal{K}$ -Fredholm complexes is open in  $\partial_e(\mathcal{X})$ .



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