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INTRODUCTION

In this paper we study the homotopy groups of the automorphism group of an AF - C* - algebra. Results on this line were previously obtained by J.Dixmier and A. Douady [8] and K.Thomsen [22]. In what concernes the computation of the homotopy groups our results contain as special cases these previous results.

Our method of computation reduces completely the computation of the groups $\pi_k(\mathrm{Aut}(A))$, $k>0$, to the computation of the homotopy groups of unitaries (A is an AF - C* - algebra, Aut(A) is the group of * - automorphisms of A endowed with the point norm topology). Using standard results concerning $\pi_k(U(n))$ we succeded a complete computation for $\pi_k(\mathrm{Aut}(A))$ for a large class of AF - C* - algebras A. If A is simple, $A\neq K$ (the algebra of compact operators on a separable Hilbert space) the results are as follows: $\pi_k(\mathrm{Aut}(A))\simeq\mathrm{Hom}(K_0(A)/Z[1], K_0(A))$, $\pi_{2k-1}(\mathrm{Aut}(A))\simeq\mathrm{Ext}(K_0(A)/Z[1], K_0(A))$ for A unital ($k\geq 1$) and $\pi_{2k}(\mathrm{Aut}(A))\simeq\mathrm{Hom}(K_0(A), K_0(A))$, $\pi_{2k-1}(\mathrm{Aut}(A))\simeq\mathrm{Ext}(K_0(A), K_0(A))$ if A is not unital ($k\geq 1$). Note the similarity with results obtained by J.Cuntz in [6] , also there exists a few points of resemblence in the techniques used there by J.Cuntz and by us. If A is not simple the results are more complicated depending in a nontrivial way on the ideal structure of A. In order to handle these situations we were led to introduce the gruoups Hom_c and Ext_c which take in account the restrictions introduced by the ideals of $K_0(A)$.

The method of proof is the following. First we study $\pi_k(\mathrm{End}(A))$, the homotopy groups of the semigroup of all *-homomorphisms $A\to A$ endowed with the pointwise convergence. It turns out then that the natural embedding $\mathrm{Aut}(A) \to \mathrm{End}(A)$ induces an isomorphism $\pi_k(\mathrm{Aut}(A)) \to \pi_k(\mathrm{End}(A))$ for any $k \geq 1$, and this is the crux point of the proof. The computation of $\pi_k(\mathrm{End}(A))$ requires the knowledge of $\pi_k(U(A_n'))$ (A_n' is the commutant of the finite dimensional C*-algebra A_n in A). This type of questions enter in what is called "nonstable K-theory" (see [17], [18]); in the same order of ideas we prove that certain C*-algebras obtained from locally trivial fields of AF-C*-algebras an spheres satisfy the cancellation propriety for finitely generated projective modules, and also we classify the positive cone of K_o of these C*-algebras.

The first section contains general results: the isomorphism $\pi_k(\mathrm{Aut}(A)) \to \pi_k(\mathrm{End}(A))$ for $k \geq 1$ and the reduction of the computation of $\pi_k(\mathrm{End}(A))$ to $\pi_k(U(n))$. In the second section we introduce the class of ordered groups with large denominators and show that $\pi_k(U(A_n')) \simeq K_o(A_n')$ if $K_o(A)$ has large denomiators. Also we introduce Hom_c and Ext_c and develop briefly their proprieties, showing that $K_o(A_n') \simeq \mathrm{Hom}_c(K_o(A_n), K_o(A))$. Next to a k-loop f in Aut(A) we associate as usual a locally trivial field of AF-C*-algebras on S^{k+1} and show that for k odd this defines an element in $\mathrm{Ext}_c(K_o(A), K_o(A))$ which is trivial if and only if f is inner. The final result is theorem 2.12.

1. In this section we shall prove same general results about the homotopy groups of the group of automorphisms of an AF-C*-algebra.

For the basic results concerning AF-algebras and for the definitions not explained, such as ordered group, ideal of an ordered group, the interested reader may consult [3] or [9].

1.1. Let us introduce first some notations and fix some conventions to be used from now on.

a) $K_i(A)$, $i=0,1$ will denote the K-theory groups of a C -algebra A ([3], [21]). If A is an AF-C*-algebra $\geq$ will denote the order on $K_o(A)$, $K_o(A)_+$ will denote the positive cone of $K_o(A)$ and $\Sigma(A)$ will denote the scale of $K_o(A)$ ([3], [9]). If $f:A \to B$ is a *-morphism of C*-algebras $K_i(f):K_i(A) \to K_i(B)$ denotes the natural group morphism.

b) If A is a C*-algebra M(A) is the multiplier C*-algebra of A ([15]).

c) Let us fix a base point $p_o \in S^k$ for $k \geq 1$. If (X,x) is a pointed topologic space a k-loop in X is a continuous base point preserving function $f:(S^k,p_o) \to (X,x)$. The class of this function in $\pi_k(X)$ will be denoted by $[f]$.

d) Let A be a C*-algebra, A^+ denotes the algebra A with adjoint unit, $\pi : A^+ \to \mathbb{C}$ is the quotient map. $\tilde{A}$ denotes A if A has unit and A^+ else. U(A) is the set of those unitaries $u \in A^+$ such that $\pi(u)=1$ $ad_n(x)=u\ x\ u^*$ is the inner automorphism of A induced by $u \in U(A)$.

e) If (X,x) is a pointed topological space X^o denotes the path component of the base point.

f) If A and B are C*-algebras Hom(A,B) will denote the set of all *-morphisms $f:A \to B$. We shall topologise this set with the topology of norm-pointwise convergence. If $i:A \to B$, Hom(A,B,i) is the pointed topological space (Hom(A,B),i). $End^o(A)$ denotes ($Hom^o(A,A)$, id). id denotes various identity morphisms.

g) If $B \subset A$ are two C*-algebras B' denotes the relative commutant of B in A.

h) Let G_n, $n \in N$ be abelian groups, $\varphi_{nm}:G_m \to G_n$, $m>n$ an inverse sistem of homomorphisms. Let $\delta : \prod_{n\in\mathbb{N}} G_n \to \prod_{n\in\mathbb{N}} G_n$ given by $\delta((x_n)_{n\in\mathbb{N}}) = (x_n - \varphi_{n,n+1}(x_{n+1}))_{n\in N}$. We shall denote by $\varprojlim^1(G_n,\varphi_{nm})$ the cokernel of this moprhism. Of course $\varprojlim(G_n,\varphi_{nm}) = \ker\delta$.
If $\psi_{nm}:X_m \to X_n$, $m \geq n$ is an inverse sistem of topological spaces $\varprojlim(X_n,\psi_{nm})$ is the subspace $\{(x_n)_{n\, N},\ x_n=\psi_{n,n+1}(x_{n+1})$ of $\prod_{n\in N} X_n$. It has the induced product topology.

i) From now an A will always denote an AF-C*-algebra, $A=\overline{\cup A_n}$ and $A_n=A_n^{(1)}\oplus \ldots \oplus A_n^{(k_n)}$, $A_n^{(j)}$ being factors of type $I_{p_{nj}}$. Also we shall denote by $\alpha_{mn}:K_o(A_n)\to K_o(A_m)$ the natural morphism induced by the inclusion $i_{mn}:A_n\to A_m$, for $m\geq n$. The inclusion $A_n\to A$ will be denoted by i_n and $Hom^o(A_n,A,i_n)$ by $Hom^o(A_n,A)$.

j) Other notations: $I=[0,1]$, $IK=[0,1]\times K$, $SA=C_o(\mathbb{R},A)$. B_n is the standard n cell, $S^{n-1}=\partial B_n$.

k) By "ideal" we shall mean "closed twosided ideal".

1.2. Let us denote by ψ_n and ψ_{nm} the mappings $\psi_n:U(A)\to Hom^o(A_n, A)$, $\psi_n(u)=ad_u|A_n$, ψ_{nm}: $Hom^o(A_m,A)\to Hom^o(A_n,A)$, $\psi_{nm}(f)=f|A_n$.

Lemma $(U(A),\psi_n, Hom^o(A_n,A))$ is a locally trivial principal $U(A_n')$ - bundle and $(Hom^o(A_m,A),\psi_{nm},Hom^o(A_n,A))$ is a fibration.

Proof. The second assertion follows from the first. Indeed since a locally trivial bundle has the homotopy lifting property ([12], [24]) we may lift a homotopy on $Hom^o(A_n,A)$ to $U(A)$ first and then descend it on $Hom^o(A_m,A)$.

Let us prove now the first part. ψ_n is obvious surjective and the function $U(A)*U(A)\ni(u,v)\to \tau(u,v)=u^{-1}v\in U(A_n')$ is continuous (we have denoted, as usual, by $U(A)*U(A)$ the set of those pairs $(u,v)\in U(A)\times U(A)$ such that $\psi_n(u)=\psi_n(v)$). Also $\tau(u,.)$ and $\tau(.,v)$ are onto for any fixed u and v. This shows that $(U(A),\psi_n,Hom^o(A_n,A)$ is a principal $U(A_n')$-bundle.

Let us show that there exist a croos section for ψ_n defined in a neighbourhood of i_n. Let V be the set of those $\varphi\in Hom^o(A_n,A)$ such that $\|\varphi-i_n\|<1$. If e and f are two selfadjoint projections such that $\|e-f\|<1$ then fe has a polar decomposition $(efe\geq(1-\|e-f\|)e)$. Denote by $\Theta(f,e)$ the partial isometry arising in this polar decomposition, thus $fe=\Theta(f,e)(efe)^{1/2}$. It follows that $\Theta(f,e)\Theta(e,f)=f$ and $\Theta(f,e)=\Theta(e,f)^*$. Let (e_{ij}^k) be a matrix unit for A_n. The required cross section is defined as follows: $u(\varphi)=\sum_{k=1}^{k_n}\sum_{i=1}^{p_{nk}}\varphi(e^k_{i1})\Theta(\varphi(e^k_{11}),e^k_{11})e^k_{1i}$ (see [4]). Since U(A) acts transitively on $\mathrm{Hom}^O(A_n,A)$ a local cross section exists in the neighbourhood of each point.

1.3. Lemma. $\mathrm{End}^O(A)$ is homeomorphic to the inverse limit $\varprojlim(\mathrm{Hom}^O(A_n,A),\psi_{mn})$.

Proof. Denote by $\varphi_n(f)=f|_{A_n}$, $\varphi_n:\mathrm{End}^O(A)\to\mathrm{Hom}^O(A_n,A)$ then $\varphi_n=\psi_{nm}\circ\varphi_m$ for any $n\leq m$. Since each of φ_n is continuous they define a continuous function $\varphi=\varprojlim\varphi_n:\mathrm{End}^O(A)\to\varprojlim(\mathrm{Hom}^O(A_n,A),\psi_{mn})$. φ is obviously one-to-one and onto. φ is a homeomorphism from the very definition of the topology on $\mathrm{End}^O(A)$.

1.4. Lemma. Let K be a finite cell complex, $f:K\to\mathrm{End}^O(A)$. Then there exists a continuous function $g:IK\to\mathrm{End}^O(A)$ such that $g|_{\{1\}\times K}=f$ and $g(t,x)\in\mathrm{Aut}(A)$ for any $0\leq t<1$ and $x\in K$.

Proof. Denote by B_n the standard n-cell, $S^{n-1}=B_n$. By induction on the number of cells we reduce the problem to the following: given $f:\{1\}\times B_n\cup I\partial B_n\to\mathrm{End}^O(A)$ a continuous function, to extend this function to a continuous function g on IB_n such that $g(x)\in\mathrm{Aut}^O(A)$ for any $x\in IB_n\setminus(\{1\}\times B_n\cup I\partial B_n)$. But since the pair $(IB_n,\{1\}\times B_n\cup I\partial B_n)$ is homeomorphic to the pair $(IB_n,\{1\}\times B_n)$ it follows that we may suppose that K itself is a cell, $K=B_n$.

Since B_n is contractible and $(U(A),\psi_n,\mathrm{Hom}^O(A_n,A))$ is a fibration there exists $\Theta_m:B_n\to U(A)$ such that $f(x)|_{A_m}=\mathrm{ad}_{\Theta_m}(x)|_{A_m}$. Let $\Theta_o(x)=1$. Using that $U(A'_m)$ is connected and B_n contractible we may choose a continuous function $\Theta_m:IB_n\to U(A'_m)$ such that $\Theta_m(t,x)=1$ for $x\in B_n$, $t\in[0,1-1/m]$ and $\Theta_m(t,x)=\Theta^*_{m-1}(x)\Theta_m(x)$ for $t\in[1-1/(m+1),1]$, $x\in B_n$, $m\geq 1$. Set as in [1] $g(t,x)=\mathrm{ad}\,\Theta_1(t,x)\Theta_2(t,x)\ldots\Theta_n(x,t)$ for $t\leq 1-\frac{1}{n+1}$ and $g(1,x)=f(x)$.

We have to prove the continuity of $g(t,x)$. It is enough to show that $(t,x) \to g(t,x)|_{A_m}$ is continuous. But $g(t,x)|_{A_m} = ad_{\Theta_1(t,x)\ldots\Theta_m(t,x)}|_{A_m}$ which is obviously continuous since Θ_j are continuous.

1.5. Theorem a) The natural inclusion $j: Aut^o(A) \to End^o(A)$ induces isomorphisms $\pi_k(j): \pi_k(Aut^o(A)) \to \pi_k(End^o(A))$, $k \geq 1$.

b) There exists a short exact sequence of groups:

$$0 \to \varprojlim{}^1 (\pi_{k+1}(Hom^o(A_n,A)), \pi_{k+1}(\psi_{nm})) \to \pi_k(Aut^o(A)) \to$$
$$\varprojlim (\pi_k(Hom^o(A_n,A)), \pi_k(\psi_{nm})) \to 0 \quad (k \geq 1).$$

c) $\pi_o(Aut(A))$ is isomorphic to the group of the automorphisms of the scaled order group $(K_o(A), \Sigma(A))$.

Proof. a) follows from lemma 1.4.

b) follows from a), lemma to 1.2 and 1.3 and [24] theorem (4.8), pag.433.

There is an obvious morphism $Aut(A) \to Aut(K_o(A), \Sigma(A))$. The kernel of this morphism is $Aut^o(A)$ ([2], theorem 3.1). This morphism is surjective by a theorem of Eliott ([10]). This proves c).

1.6. Remark

Let us note that a nontrivial element of $\varprojlim{}^1(\pi_2(Hom^o(A_n,A)), \pi_2(\psi_{nm})) \subset \pi_1(Aut(A))$, for a certain AF-C*-algebra A is implicitely contained in the construction of proposition 5.1 of [7].

1.7. Remark

Using the exact sequence of a fibration we obtain if A=K (the algebra of compact operators on a separable Hilbert space) $\pi_k(Hom^o(A_n, A)) \simeq \{0\}$ for $k \neq 2$ and $\pi_2(Hom^o(A_n,A)) \simeq \pi_2(Hom^o(A_{n+1},A)) \simeq \mathbb{Z}$, the isomorphism being induced by $\pi_2(\psi_{n,n+1})$.

It follows from theorem 1.5.b) that $\pi_2(\mathrm{Aut}(K)) \simeq \mathbb{Z}$ and $\pi_k(\mathrm{Aut}(K)) \simeq \{0\}$ for $k \neq 2$. This also follows from results from [8].

2. In this section we shall go further into the structure of the homotopy groups of a certain class of AF-C*-algebras, a class which contains, for example, all simple, non type I AF-C*-algebras.

2.1. We shall need the following results concerning the homotopy groups of the unitary group $U(n)=U(M_n(\mathbb{C}))$.

Denote by i and j the following functions $i,j:U(n) \to U(m)$ $i(u)=u \oplus I_{m-n}$, $j(u)=u \oplus \ldots \oplus u \oplus I_p$ (i is defined for $m \geq n$, j is defined for $p=m-n\ell \geq 0$, u occurs ℓ-times).

Proposition ([13]) $\pi_k(j)=\ell\pi_k(i)$ and $\pi_k(j)$ is an isomorphism for $k/2<n$. Also

$$\pi_k(U(n))=\begin{cases} Z & n \quad \text{odd} \\ 0 & n \quad \text{even} \end{cases} \qquad k/2<n$$

2.2. Definition. Let (G,G_+) be an ordered group. We shall say that G has large denominators if for any $a \geq 0$ and $n \in \mathbb{N}$ there exists $b \in G$ and $m \in \mathbb{N}$ such that $nb \leq a \leq mb$.

2.3. Proposition. Suppose that A is simple, infinite dimensional, $A \neq K$ then $K_o(A)$ has large denominators.

Proof. Let $e \neq 0$ be a projection, $a=[e]$, replacing A by $eM_n(A)e$ for some large n we may suppose that $a=[1]$. Let $k \in \mathbb{N}$. Denote by $J_n = \bigoplus_{p_{nj} \geq k} A_n^{(j)} \subset A_n$. Then $i_{mn}(J_n) \subset J_m$ and hence $J=\overline{\bigcup_n J_n}$ is an ideal of A. Since A is simple it follows that $J=A$ or $J=\{0\}$. But A/J has only finite dimensional irreducible representations, this shows that $J \neq \{0\}$ is possible only if $A=K$. It follows from the above discution that $1\in J=A$. Choose n such that $1\in J_n$. Let (e^k_{ij}) be a matrix unit for $J_n=A_n=\bigoplus_{j=1}^{k_n} A_n^{(j)}$ with $A_n^{(j)}$ finite dimensional factors. $b=\sum_{j=1}^{k_n}[e_{11}^{(j)}]$ will satisfy the requirements of definition 2.2.

2.4. <u>Proposition</u>. Suppose $K_o(A)$ has large denominators. Then:

a) $K_o(A'_m)$ has large denomiators, $m\geq 1$.

b) The natural morphisms $\pi_k(U(A))\to K_1(S^kA)$ are isomorphisms.

c) The isomorphisms of b) give a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0\to & \pi_{2k}(\mathrm{Hom}^o(A_{n+1},A)) & \to & K_o(A'_{n+1}) & \to & K_o(A) & \to & \pi_{2k-1}(\mathrm{Hom}^o(A_{n+1},A)) & \to 0\\
 & \downarrow & & \downarrow & & \| & & \downarrow & \\
0\to & \pi_{2k}(\mathrm{Hom}^o(A_n,A)) & \to & K_o(A'_n) & \to & K_o(A) & \to & \pi_{2k-1}(\mathrm{Hom}^o(A_n,A)) & \to 0
\end{array}$$

d) $\varprojlim^1(\pi_{2k-1}(\mathrm{Hom}^o(A_n,A)),\pi_{2k-1}(\psi_{n,n+1}))=0$.

<u>Proof</u>. a) Suppose that A is not unital, $1\in M(A)\setminus A$.

Let (e^k_{ij}) be a matrix unit for A_n. Denote by e_n the unit of A_n. An easy computation (see [1]) shows that A'_n is isomorphic to $(1-e_n)A(1-e_n)+\bigoplus_{k=1}^{k_m} e^k_{11}Ae^k_{11}$, the isomorphism being $\varphi(a\oplus\bigoplus_{k=1}^{k_m}a_k)=$

$=a+\sum_{k=1}^{k_m}e^k_{i1}a_ke^k_{1i}$. Let J_k be the ideal generated in A by $(1-e_n)$ for $k=0$ and by e^k_{11} for $k>0$. It follows that $K_o(A'_n)\simeq K_o(J_o)\oplus\bigoplus_{k=1}^{k_m}K_o(J_k)$ since $e^k_{11}Ae^k_{11}((1-e_n)A(1-e_n))$ is a full korner in $J_k(J_o)$, to prove this use [5]. Since $K_o(J)$ has large denominators for any ideal J of A it follows that $K_o(A'_n)$ has large denominators.

For A unital the proof is similar.

b) We shall use repetedely proposition 2.1. There exists a commutative diagram of isomorphisms

$$\begin{array}{ccc}
\varinjlim \pi_k(U(n)) & \to & K_1(S^kC)\\
 & \searrow \quad \swarrow & \\
 & \tilde{K}^o(S^{k+1}) &
\end{array}$$

(see [13])

For each A_n denote by e_n its unit. Let $\ell_o > k/2$ and f_n such that $\ell_o[f_n] \leq [e_n] \leq m[f_n]$ for some $m \in \mathbb{N}$. Replacing $(A_n)_{n \in N}$ by a subsequence and the f_n's, by same equivalent projections we may suppose that $f_n \in A_{n+1}$. Replace again A_n by $e_n A_{n+1} e_n$. It follows that $A_n \simeq M_{r_1} \oplus \dots$ $\dots \oplus M_{r_j}$ and $r_1, \dots, r_j \geq \ell_o > k/2$. Then $\pi_k(U(A_n)) \to K_1(S^k A_n)$ is an isomorphism and we have isomorphisms $\pi_k(U(A)) \simeq \varinjlim \pi_k(U(A_n)) \simeq \varinjlim K_1(S^k A_n)$ $\simeq K_1(S^k A)$

(recall the convention made for $U(A)$ in 1.1d)).

c) This follows from the exact sequences of the fibration $U(A_n') \to U(A) \to \mathrm{Hom}^o(A_n, A)$ and from the commutativity of the diagram

$$\begin{tikzcd}
U(A'_{n+1}) \arrow[r] \arrow[d] & U(A) \arrow[r] \arrow[d, equal] & \mathrm{Hom}^o(A_{n+1}, A) \arrow[d] \\
U(A'_n) \arrow[r] & U(A) \arrow[r] & \mathrm{HOm}^o(A_n, A)
\end{tikzcd}$$

(Note that $\pi_{2k+1}(U(A)) \simeq K_o(A)$ and $\pi_{2k}(U(A)) \simeq \{0\}$ by b))

d) follows form the surjectivity of $\pi_{2k-1}(\psi_{n,n+1})$ as apparent from c).

The previous lemma shows that it is important to know $K_o(A_n')$ and, in view of theorem 1.5 to compute also the morphisms $K_o(A'_{n+1}) \to K_o(A_n')$. The following definition and definition 2.9 are an atempt to give a satisfactory framework for our computations.

2.5. Definition. Let H_1, H_2 be ordered groups, $i: H_1 \to H_2$ a positive morphism, $\varphi: H_1 \to H_2$ a group morphism. We shall say that φ is compatible with i if for every $x \in H_1$, $x \geq 0$ there exists $m \in N$ such that $-mi(x) \leq \varphi(x) \leq mi(x)$.

We shall denote by $\mathrm{Hom}_c(H_1, H_2, i)$ the set of morphisms $\varphi: H_1 \to H_2$ compatible with i. In the same spirit as before $\mathrm{Hom}_c(K_o(A_n), K_o(A),$ $K_o(i_n))$ will be denoted by $\mathrm{Hom}_c(K_o(A_n), K_o(A))$ and $\mathrm{Hom}_c(G,G,id)$ by $\mathrm{End}_c(G)$.

This definition is suggested by the computation of $K_o(A'_n)$ in the proof of proposition 2.4.a).

2.6. The following proposition gives the basic proprieties of Hom_c needed in the computation of $\pi_k(Aut(A))$.

<u>Proposition</u>. a) Let H_1, H_2 and H_3 be ordered groups, $i_1: H_1 \to H_2$, $i_2: H_2 \to H_3$ be positive morphisms. Then there exists natural morphisms $i_1^*: Hom_c(H_2, H_3, i_2) \to Hom_c(H_1, H_3, i_2 \circ i_1)$ and $i_{2*}: Hom_c(H_1, H_2, i_1) \to Hom_c(H_1, H_3, i_2 \circ i_1)$ given by $i_1^*(\varphi) = \varphi \circ i_1$ and $i_{2*}(\varphi) = i_2 \circ \varphi$.

b) If H_1, H_2 and i_1 are as before and H_2 is a simple ordered group and $i_1(x) \neq 0$ for $x \geq 0$, $x \neq 0$ then $Hom_c(H_1, H_2, i_1) = Hom(H_1, H_2)$.

c) Suppose H_n, $n \in N$ and H' are ordered groups, $j_{mn}: H_n \to H_m$ are positive morphisms for $n \leq m$ and $H = \varinjlim(H_n, j_{mn})$. Also let $i: H \to H'$ be an order morphism. Denote by i_n the composition $H_n \to H \xrightarrow{i} H'$ then $Hom_c(H, H', i) \simeq \varprojlim(Hom_c(H_n, H', i_n), j_{mn}^*)$.

<u>Proof</u>. a) Let $\varphi_2 \in Hom_c(H_2, H_3, i_2)$, $\varphi_1 \in Hom_c(H_1, H_2, i_1)$ we have to prove that $\varphi_2 \circ i_1$, $i_2 \circ \varphi_1 \in Hom_c(H_1, H_3, i_2 \circ i_1)$. Let $x \in H_1$, $x \geq 0$ then $i_1(x) \geq 0$. Chose m such that $-m\, i_2(i_1(x)) \leq \varphi_2(i_1(x)) \leq m i_2(i(x))$. This proves the first part. Chose m such that $-m i_1(x) \leq \varphi_1(x) \leq m i_1(x)$, since i_2 preserves the inequalities we obtain the desired conclusion.

b) Since H_2 is simple and $i_1(x) \geq 0$, $i_1(x) \neq 0$ for $x \neq 0$, $x \geq 0$ it follows that $i_1(x)$ is an order unit for H_2, namely for any $y \in H_2$ there exists an $m \in N$ such that $-m i_1(x) \leq y \leq m i_1(x)$ (see [9]). This concludes the proof.

c) Denote by j_n the positive morphism $H_n \to H$, j_n defines a morphism $j_n^*: Hom_c(H, H', i) \to Hom_c(H_n, H', i_n)$. Since $j_m \circ j_{mn} = j_n$ it follows that $j_n^* = j_{mn}^* \circ j_m^*$ and hence j_n^* colect to define a morphism $f: Hom_c(H, H', i) \to \varprojlim(Hom_c(H_n, H', i_n), j_{mn}^*)$. Let $\varphi \in Hom_c(H, H', i)$. If $f(\varphi) = 0$ then $\varphi \circ j_n = 0$ for any n and hence $\varphi = 0$. Let $\varphi_n \in Hom_c(H_n, H', i_n)$ such that $j_{mn}^*(\varphi_m) = \varphi_n$. This means that $\varphi_m \circ j_{mn} = \varphi_n$. Define $\varphi: \varinjlim H_n \to H'$ using the universal property of the inductive (direct) limit: $\varphi \in \mathrm{Hom}(H,H')$. We need to check that φ is actually in $\mathrm{Hom}_c(H,H',i)$. Let $x \in H, x \geq 0$. Then there exists n and $x_n \in H_n$, $x_n \geq 0$ such that $j_n(x_n) = x$. By the assumption that $\varphi_n \in \mathrm{Hom}_c(H_n,H',i \circ i_n)$ it follows that there exists $m \in N$ such that $-mi(i_n(x_n)) \leq \varphi_n(x_n) \leq mi(i\ (x_n))$ and hence $-mi(x) \leq \varphi(x) \leq mi(x)$.

2.7. Lemma. Let A be a unital AF-C^*-algebra such that $K_0(A)$ has large denominators. Also suppose that $A_0 = C\mathbb{1}$.

a) There exists isomorphisms $\beta_n : K_0(A'_n) \longrightarrow \mathrm{Hom}_c(K_0(A_n),K_0(A))$ and a commutative diagram

$$\begin{tikzcd}
& K_0(A'_m) \arrow[ld] \arrow[dd, "K_0(j)"] \arrow[rr, "\beta_m"] & & \mathrm{Hom}_c(K_0(A_m),K_0(A)) \arrow[ld, "\alpha^*_{om}"'] \arrow[dd, "\alpha^*_{mn}"] \\
K_0(A) \arrow[rr, "\simeq"] & & \mathrm{Hom}(Z,K_0(A)) & \\
& K_0(A'_n) \arrow[lu] \arrow[rr, "\beta_n"] & & \mathrm{Hom}_c(K_0(A_n),K_0(A)) \arrow[lu, "\alpha^*_{on}"]
\end{tikzcd} \qquad m>n$$

($j : A'_m \longrightarrow A'_n$ is the natural inclusion and α_{pq} is as in 1.1 i)).

b) There exists morphisms $\mu : \pi_{2k}(\mathrm{Hom}^o(A_n,A)) \longrightarrow \mathrm{Hom}(K_0(A_n)/Z,K_0(A))$ and $\varepsilon : \pi_{2k-1}(\mathrm{Hom}^o(A_n,A)) \longrightarrow \mathrm{Ext}(K_0(A_n)/Z,K_0(A))$ and a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \pi_{2k}(\mathrm{Hom}^o(A_n,A)) \arrow[r] \arrow[d, "\mu"] & \pi_{2k-1}(U(A'_n)) \arrow[r] \arrow[d, "\delta"] & \pi_{2k-1}(U(A)) \arrow[r] \arrow[d, "\approx"] & \pi_{2k-1}(\mathrm{Hom}^o(A_n,A)) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(K_0(A_n)/Z,K_0(A)) \arrow[r] & \mathrm{Hom}(K_0(A_n),K_0(A)) \arrow[r] & \mathrm{Hom}(Z,K_0(A)) \arrow[r, "-1"] & \mathrm{Ext}(K_0(A_n)/Z,K_0(A)) \arrow[r] & 0
\end{tikzcd}$$

($\delta : \pi_{2k-1}(U(A'_n)) \longrightarrow \mathrm{Hom}(K_0(A_n),K_0(A))$ is the composition

$\pi_{2k-1}(U(A'_n)) \rightarrow K_0(A'_n) \rightarrow \mathrm{Hom}_c(K_0(A_n),K_0(A)) \longrightarrow \mathrm{Hom}(K_0(A_n),K_0(A))$,
$\mathbb{Z}$ is embedded as $n \to n[1]$).

Proof. It follows form the proof of proposition 2.4 a) that $K_0(A'_n)$ is a subgroup of $K_0(A) \simeq \mathrm{Hom}(\mathbb{Z}^{k_n}, K_0(A)) \simeq \mathrm{Hom}(K_0(A_n),K_0(A))$. The previous isomorphism maps $K_0(A'_n)$ onto the set of those morphisms $\varphi : K_0(A_n) \rightarrow K_0(A)$ such that $\varphi([e^k_{11}])$ belongs to the ideal generated in $K_o(A)$ by $[e^k_{11}]$, namely the set of those a $K_o(A)$ such that there exists $m\in N$ such that $-m[e^k_{11}]\leq a\leq m[e^k_{11}]$. This shows that $K_o(A'_n)$ is isomorphic to $\mathrm{Hom}_c(K_o(A_n), K_o(A))$.

We shall prove that $\alpha^*_{mn}\beta_m=\beta_n K_o(j)$, the other relations are similar. Let $\alpha_{mn}=(a_{pq})$ the matrix representation of the morphism $\alpha_{mn}: K_o(A_n)\to K_o(A_m)$ $(1\leq p\leq k_m,\ 1\leq q\leq k_n)$. Let $([e], 0,\dots,0)\in K_o(A)^{k_m}\cap$ $\mathrm{Hom}_c(K_o(A_m), K_o(A))$ i.e. $[e]\in K_o(J_1)$ (we use the notations introduced in the proof of 2.4.a)). Suppose $([e],0,\dots,0)$ is represented in A'_m by $f_1=\sum_{i=1}^{p_{m1}} e^1_{i1}fe^1_{1i}$ for a projection f equivalent to $[e]$ (e^k_{ij} a matrix unit of A_m). We want to find the class of this projection in $K_o(A'_n)$. Let (e^r_{st}) be a matrix unit of A_n. We may suppose that the matrix units (e^r_{st}) and (e^k_{ij}) are compatible in the sense that each of e^r_{st} is a sum of some of e^k_{ij}. To be more precise in such a sum for e^r_{11} appear a_{kr} projections from $e^k_{11}, e^k_{22},\dots,e^k_{p_{mk}p_{mk}}$. Let $[g]$ be the r^{th} component in $K_o(A'_n)$ of f_1, this is the r^{th} component of $\alpha^*_{mn}([e], 0,\dots,0)$ in $\mathrm{Hom}_c(K_o(A'_n), K_o(A))$.

$[g]$ is represented by $\sum_{s=1} e^r_{s1}h\,e^r_{1s}$ and h is a projection equivalent to g. It is clear now who is h: $h=e^r_{11}f_1e^r_{11}=\sum_{e^1_{ii}\leq e^r_{11}} e^1_{i1}f\,e^1_{1i}$, a_{1r} terms occur in the sum and hence $[g]=[h]=a_{1r}[f]=a_{1r}[e]$. It follows that there exist a commutative diagram

$$\begin{tikzcd}
K_o(A'_m) \arrow[r] \arrow[d, "K_o(j)"] \arrow[rrr, bend left, "\simeq"] & K_o(A_m)^*\otimes K_o(A) \arrow[r, phantom, "\simeq"] \arrow[d, "\alpha^{\tau}_{mn}\otimes 1"] & \mathrm{Hom}(K_o(A_n),K_o(A)) & \mathrm{Hom}_c(K_o(A_m),K_o(A)) \arrow[l] \arrow[d, "\alpha^*_{mn}"'] \\
K_o(A'_n) \arrow[r] \arrow[rrr, bend right, "\simeq"'] & K_o(A_n)^*\otimes K_o(A) \arrow[r, phantom, "\simeq"] & \mathrm{Hom}(K_o(A_n),K_o(A)) & \mathrm{Hom}_c(K_o(A_n),K_o(A)) \arrow[l]
\end{tikzcd}$$

This diagram gives the desired conclusion.

b) Let $f: S^{2k-1}, p_o\to U(A'_n), 1$ be a 2k-1 loop. We identify S^{2k-1}

to ∂B_{2k}. Choose $g: B_{2k} \to U(M_2(A))$ an extension of $f \oplus f^*$ to B_{2k}. $\mathrm{ad}\, g$ defines a morphism $A_n \to C(B_{2k}, M_2(A)) \to C(B_{2k}, K \otimes A)$. Since f takes values in $U(A_n')$, the range of the previous morphisms is actualy in $C(B_{2k}/\partial B_{2k}, K \otimes A) \simeq C(S^{2k}, K \otimes A)$. Denote by $\varphi_f : A_n \to C(S^{2k}, K \otimes A)$ the previous defined morphism and by $\psi : A_n \to C(S^{2k}, K \otimes A)$ the morphism $\psi(a)(x) = a \oplus 0$ (the upper left corner embedding by constant functions). Using a Künneth theorem ([3],[19]) or by direct computation $K_0(C(S^{2k}, A)) \simeq K_0(A) \oplus K_0(A)$, the first summand being $K_0(\psi)(K_0(A))$ and the second being the kernel of the morphism $K_0(e): K_0(C(S^{2k}, A)) \to K_0(A)$ induced by the evaluation at S^{2k-1}/S^{2k-1} (the point obtained by collapsing $S^{2k-1} = \partial B_{2k}$ to a point).

It follows that $K_0(\varphi_f) - K_0(\psi)$ defines a morphism $\delta([f]) = K_0(\varphi_f) - K_0(\psi): K_0(A) \to \ker(K_0(e)) \simeq K_0(A)$. This morphism depends only on the class of f in ${}_{2k-1}(U(A_n'))$. This shows that δ is a well defined function.

Let us show that δ is actually a morphism.

Denote by $f * g$ the operation of concatenation of loops and by $\sigma: S^{2k} \to S^{2k} \vee S^{2k} \simeq S^{2k}/\text{ecuator}$ the obvious morphism. Note that there exists a homotopy commutative diagram :

$$\begin{tikzcd}
 & C(S^{2k}, A \otimes K) \arrow[dr] & & \\
A_n \arrow[ur, "\varphi_f"] \arrow[dr, "\varphi_g"'] \arrow[dd, equal] & & C(S^{2k} \vee S^{2k}, A \otimes K) \arrow[r, "\sigma^*"] & C(S^{2k}, A \otimes K) \arrow[dd, equal] \\
 & C(S^{2k}, A \otimes K) \arrow[ur] & & \\
A_n \arrow[rrr, "\varphi_{f*g}"] & & & C(S^{2k}, A \otimes K)
\end{tikzcd}$$

The corresponding diagram of K_0-groups looks as follows:

$$\begin{tikzcd}
 & K_0(A) \oplus K_0(A) \arrow[dr, "i_1"] & & \\
K_0(A) \arrow[ur, "{(id, \delta([f]))}"] \arrow[dr, "{(id, \delta([g]))}"'] \arrow[dd, equal] & & K_0(A) \oplus K_0(A) \oplus K_0(A) \arrow[r, "K_0(\sigma^*)"] & K_0(A) \oplus K_0(A) \arrow[dd] \\
 & K_0(A) \oplus K_0(A) \arrow[ur, "i_2"'] & & \\
K_0(A) \arrow[rrr, "{(id, \delta([f*g]))}"] & & & K_0(A) \oplus K_0(A)
\end{tikzcd}$$

$i_1(x,y)=(x,y,0)$, $i_2(x,z)=(x,0,z)$,

$K_o(\sigma^*)(x,y,z)=(x,y+z)$.

This gives the desired conclusion.

Note that $\delta([f])([1])=\delta_o([f])=$ the index of the loop f regarded as an element of $K_1(S^{2k-1}A)$. Hence, if f is homotopic to the constant loop p_o in $U(A)$ then $\delta([f])$ factors to give a well defined morphism $K_o(A_n)/Z \to K_o(A)$. This is $\mu([f])$ under the identification $\pi_{2k}(\mathrm{Hom}^o(A_n,A)) = \ker(\pi_{2k-1}(U(A_n')) \to \pi_{2k-1}(U(A)))$.

Let us define now ε. Let $f:S^{2k-1},p_o \to U(A),1$ be a 2k-1 loop. f defines a unital morphism $A_n \to C(S^{2k-1},A)$. This morphism is the Busby invariant of an unital extension

(1) $\quad 0 \to S^{2k}A \to E \to A_n \to 0$

Denote by 1 the units of E and A_n as well. This gives an extension of groups

$$0 \to K_o(A) \to K_o(E)/Z \longrightarrow K_o(A_n)/Z \longrightarrow 0$$

(Z is embedded as $n \to n[1]$).

The class of this extension in $\mathrm{Ext}(K_o(A_n)/Z,\ K_o(A))$ will be denoted by $\varepsilon([f])$. We may show that ε is a group morphism as we did for δ or as we shall do for E in theorem 2.11 . However we shall confine ourselves to note that this will follow if we shall show that the diagram

$$\begin{tikzcd}
\pi_{2k-1}(U(A)) \arrow[r] \arrow[d, "\approx"] & \pi_{2k-1}(\mathrm{Hom}^o(A_n,A)) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
\mathrm{Hom}(Z,K_o(A)) \arrow[r, "-1"] & \mathrm{Ext}(K_o(A_n)/Z,\ K_o(A)) \arrow[r] & 0
\end{tikzcd}$$

is commutative. To show this observe that (1) becomes a trivial extension after tensoring $C_o(B_{2k},A)$ by K. A lifting for $A_n \to C(S^{2k-1},A)$

$\longrightarrow C(S^{2k-1}, M_2(A))$ to $C(B_{2k}, M_2(A))$ is given by a lifting of $f \oplus f^*$. This shows that our extension of groups is isomorphic to

$$0 \to K_o(S^{2k}A) \to K_o(S^{2k}A) \oplus K_o(A_n)/_{Z(\delta_o([f]), [1])} \to K_o(A_n)/_{Z[1]} \to 0$$

and hence it is the image of the morphism $Z \to K_o(A)$ which sends 1 to $-\delta_o([f])$ in $\mathrm{Ext}(K_o(A_n)/Z\ ,\ K_o(A))$ (this also justifies the appearence of the sign -1).

The first row is exact since it is a segment of the long exact sequence of homotopy groups of a fibration ([24]). The second row is a segment of the Ext-exact sequence of homological algebra ([14]).

The following lemma shows how we can use the preceding lemma in the nonunital case.

2.8. Lemma. Let A be a nonunital AF-C^*-algebra. There exists an exact sequence

$$0 \to \pi_{2k}(\mathrm{Hom}^o(A_n, A)) \to \mathrm{Hom}_c(K_o(A_n^+), K_o(A^+)) \to \mathrm{Hom}_c(Z, K_o(A^+)) \to \pi_{2k}(\mathrm{Hom}^o(A_n, A)) \to 0$$

Proof. There exists morphisms

$$\varphi_1 : \mathrm{Hom}_c(K_o(A_n^+), K_o(A^+)) \to \mathrm{Hom}(Z, Z) \simeq Z \quad \text{and}$$
$$\varphi_2 : \mathrm{Hom}(Z, K_o(A^+)) \to \mathrm{Hom}(Z, Z) \simeq Z$$

given by $Z \xrightarrow{\sim} Z[1] \subset K_o(A_n^+)$ and $K_o(A^+) \to K_o(\mathbb{C}) \simeq \mathbb{Z}$. It follows that $K_o(A_n') \simeq \mathrm{Ker}\,\varphi_1$ and $K_o(A) \simeq \ker \varphi_2$ as easely seen from the proof of proposition 2.4.a) (we have an argument similar to that given in the first paragraph of the proof of lemma 2.7.). The rest is an application of proposition 2.4.c).

2.9. Definition. Let H_1, H_2 be ordered groups, $i: H_1 \to H_2$ an order morphism. An exact sequence of groups

[E] $$0 \longrightarrow H_2 \longrightarrow E \xrightarrow{\pi} H_1 \longrightarrow 0$$

in which E is an ordered group, π is a positive morphism, will be called compatible with i if

(a) $\pi(E_+) = H_{1+}$

(b) If $x,y \geq 0$, $x \in H_1$, $y,z \in E$ are such that $\overbrace{\pi(y)}^{x:=} = \pi(z) \geq 0$ then $z \geq 0$ if and only if there exists $m \geq 0$ such that $-mi(x) \leq z-y \leq m \cdot i(x)$ in H_2.

As in the usual case, two compatible extensions E_1 and E_2 will be called isomorphic if there exists a positive group morphism $\varphi : E_1 \longrightarrow E_2$ and a commutative diagram

(2.9.1)
$$\begin{tikzcd}
0 \arrow[r] & H_2 \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "\varphi"] & H_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_2 \arrow[r] & E_2 \arrow[r] & H_1 \arrow[r] & 0
\end{tikzcd}$$

It follows that φ is necessarily an isomorphism of ordered groups.

The set of isomorphism classes of extensions of H_1 by H_2, compatible with i, will be denoted by $\mathrm{Ext}_c(H_1,H_2,i)$.

An extension E compatible with i is called trivial if there exists a positive lifting for π. Note that in view of our definition two liftings differ by an element of $\mathrm{Hom}_c(H_1,H_2,i)$.

It is not true that a trivial extension in $\mathrm{Ext}_c(H_1,H_2,i)$ is isomorphic to $H_1 \oplus H_2$ as ordered groups.

We will need only the following results about this Ext_c.

2.10. <u>Proposition</u> a) Let H_1,H_2,H_3,H_4 be ordered groups, $i_j : H_j \longrightarrow H_{j+1}$, $j \in \{1,2,3\}$ be order morphisms. Suppose that the ideal generated in H_{j+1} by $i_j(H_j)$ is H_{j+1} $(j \in \{1,2,3\})$.

Then there exists functions

$$i_1^*: Ext_c(H_2,H_3,i_2) \to Ext_c(H_1,H_3,i_2\circ i_1)$$

$$i_{3*}: Ext_c(H_2,H_3,i_2) \to Ext_c(H_2,H_4,i_3\circ i_2)$$

with the property that $i_1^*\circ i_{3*}=i_{3*}\circ i_1^*$ as functions from $Ext_c(H_2,H_3,i_2)$ to $Ext_c(H_1,H_4,i_3\circ i_2\circ i_1)$

b) Let E_1, $E_2\in Ext_c(H_1,H_2,i)$, $d:H_1\to H_1 \oplus H_1$ $d(a)=a\oplus a$, $\sigma: H_2 \oplus H_2 \to H_2$, $\sigma(a,b)=a+b$ then $E_1 \oplus E_2\in Ext_c(H_1\oplus H_2, H_2\oplus H_2, i\oplus i)$, and $d^*(\sigma_*(E_1 \oplus E_2))=\sigma_*(d^*(E_1 \oplus E_2))\in Ext_c(H_1,H_2,i)$ defines a group structure on $Ext_c(H_1,H_2)$ with the trivial extension as a neutral element.

c) Let $H_n, n\in N$, H',i,j_{mn},i_n and H be as in proposition 2.6.e) then there exists an exact sequence of groups

$$0\to \underleftarrow{\lim}^1(Hom_c(H_n,H',i_n),j_{mn}^*)\to Ext_c(H,H',i)\to \underleftarrow{\lim}(Ext_c(H_n,H',i_n),j_{mn}^*)\to 0$$

<u>Proof</u>. a) Let $\overbrace{0\to H_3 \xrightarrow{j} E \xrightarrow{\pi} H_2 \to 0}^{[E]=}$ be an element of $Ext_c(H_2,H_3,i_2)$ Define $i_1^*([E])$ to be the class in $Ext_c(H_1, H_3, i_2\circ i_1)$ of the extension

$$0 \to H_3 \to E \amalg_{i_1} H_1 \to H_1 \to 0$$

Here $\quad E \amalg_{i_1} H_1 = \{(x,h_1)\ |\pi(x)= i_1(h_1)\}$

and $(x,h_1)\geq 0$ if and only if $x\geq 0$ and $h_1\geq 0$.

$i_{3*}([E])$ is the class in $Ext_c(H_2,H_4,i_3\circ i_2)$ of the extension

$$0 \to H_4 \to E \oplus H_4/_{(-j)\oplus i_3(H_3)} \xrightarrow{\pi_1} H_2 \to 0$$

The order on $E_1=E \oplus H_4/_{(-j)\oplus i_3(H_3)}$ has as positive cone P_1 the set of the classes of elements (x,h_4), $x\in E$, $x\geq 0$, $h_4\in H_4$ such that there exists $m\geq 0$ for which $*$ $-mi_3\circ i_2\circ \pi(x)\leq h_4\leq m\, i_3\circ i_2\circ \pi(x)$. Denote by $\widehat{(x,h_4)}$ the class of an element $(x,h_4)\in E\oplus H_4$ in E_1. We shall show that $[E_1]\in Ext_c(H_2,H_4,i_3\circ i_2)$. Let $\widehat{(x,h)}\in E_1$. There exists positive elements $x_1,x_2\in E$ such that $x=x_2-x_1$. Also, since the ideal generated by $i_3\circ i_2(H_2)$ in H_4 is the whole of H_4, there exists $x_3\geqslant 0$ such that $-i_3\circ i_2(x_3)\leqslant h\leqslant i_3\circ i_2(x_3)$. It follows that $\widehat{(x,h)}=\widehat{(x_2+x_3,h)}-$ $-\widehat{(x_1+x_3,0)}$ is the difference of two positive elements. If $(x,h)\in$ $P_1\cap(-P_1)$ then $\pi_1(\widehat{(x,h)})=\pi(x)\in E_+\cap(-E_+)$ (E_+ is the positive cone in E). This shows that $P_1\cap(-P_1)=\{0\}$.

Let $\widehat{(x_1,h_1)}$, $\widehat{(x_2,h_2)}\in E_1$ be such that $\widehat{(x_1,h_1)}$ is positive and $h=\pi(x_1)=\pi(x_2)$. Suppose that $\widehat{(x_2,h_2)}$ is also positive. Then we may suppose that $x_1,x_2\geqslant 0$, $-mi_2(h)\leqslant x_2-x_1\leqslant mi_2(h)$ and $-m\ i_3\circ i_2(h)\leqslant h_j\leqslant$ $m\ i_3\circ i_2(h)$ for some $m\in N$ and $j\in\{1,2\}$. Then $\widehat{(x_1,h_1)}-\widehat{(x_2,h_2)}=$ $=\widehat{(0,h_1-h_2+i_3(x_1-x_2))}$ satisfies $-3m\ i_3\circ i_2(h)\leqslant h_1-h_2+i_3(x_1-x_2)\leqslant$ $3m\ i_3\circ i_2(h)$. Conversely, if $-m\ i_3\circ i_2(h)\leqslant h'\leqslant m\ i_3\circ i_2(h)$ then $\widehat{(x_1,h_1)}+$ $+\widehat{(0,h')}$ is positive from the definition. Since $\pi_1(\widehat{(x_1,h_1)})=\pi(x_1)$, $\pi_1((x_2,h_2))=\pi(x_2)$ it follows that (E_1,P_1) defines an element of $Ext_c(H_2,H_4,i_3\circ i_2)$.

Note that if in (2.9.1) we suppose only that φ is a group morphism then it follows that φ is actually a morphism of ordered groups. This shows that the natural function $Ext_c(H_1,H_2,i)\longrightarrow$ $Ext(H_1,H_2)$ is injective and hence that $Ext_c(H_1,H_2,i)$ may be identified with a subset of $Ext(H_1,H_2)$. This proves the rest of a) and b).

Let E_n and β_{mn} be such that the diagrams

$$\begin{tikzcd} 0 \arrow[r] & H' \arrow[r] \arrow[d, equal] & E_n \arrow[r] \arrow[d, "\beta_{mn}"] & H_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H' \arrow[r] & E_m \arrow[r] & H_m \arrow[r] & 0 \end{tikzcd}$$

are commutative, β_{mn} positive, then

$$0\longrightarrow H'\longrightarrow \varinjlim(E_n,\beta_{mn})\longrightarrow \varinjlim(H_n,j_{mn})\longrightarrow 0$$

represents an element $[E]\in \mathrm{Ext}_c(H,H',i)$ such that its image in $\mathrm{Ext}_c(H_n,H',j_n)$ is $[E_n]$. This gives the surjectivity of $\mathrm{Ext}_c(H_n,H',j_n) \longrightarrow \varprojlim(\mathrm{Ext}_c(H_n,H',j_n),j^*_{mn})$.

Let $0 \to H' \to E \to H \to 0$ be an extension such that if $j_n: H_n \to H$ is the limit morphism then $j_n^*([E])$ is trivial. This means that there exists positive liftings $\tau_n: H_n \to E$ such that $\pi\circ\tau_n = j_n$. Let us observe that $\tau_{n+1}\circ j_{n+1,n} - \tau_n \in \mathrm{Hom}_c(H_n,H',j_n)$. If we choose other liftings τ'_n then

$\tau'_{n+1}\circ j_{n+1,n} - \tau'_n = \tau_{n+1}\circ j_{n+1,n} - \tau_n + (\tau'_{n+1}-\tau_{n+1})\circ j_{n+1,n} - (\tau'_n - \tau_n)$. It follows that $(\tau_{n+1}\circ j_{n+1,n} - \tau_n)_{n\in N}$ and $(\tau'_{n+1}\circ j_{n+1,n} - \tau'_n)_{n\in N}$ differ in $\prod_{n\in N} \mathrm{Hom}_c(H_n,H',i_n)$ by an element in the range of δ (see 1.1.h)). This gives the rest of the statement.

Let us observe that if H_1 and H_2 are unperforated then any ordered group representing an element in $\mathrm{Ext}_c(H_1,H_2,i)$ is unperforated.

We shall denote by $\mathrm{Ext}_c(K_0(A),K_0(A))$ the group $\mathrm{Ext}_c(K_0(A),K_0(A),id)$.

2.11. <u>Lemma</u>. Let $f: S^{2k-1} \to \mathrm{Aut}(A)$ be a $2k-1$ loop and suppose that $K_0(A)$ has large denominators. Let $E_f \subset C(B_{2k},A)$ be the C^*-algebra of those functions $\varphi: B_{2k} \to A$ such that $\varphi(x)=f(x)(a)$ for some $a=\pi(\varphi)\in A$ and any $x\in S^{2k-1}$. Then the semigroup $V(A)$ of projective finitely generated modules over E_f has cancelation. If $K_0(E_f)_+$ is the positive cone of $K_0(E_f)$ then $K_0(\pi)(K_0(E_f)_+)=K_0(A)_+$ (π is the quotient map $E_f \to A$). More of that, $K_0(E_f)$ represents an element in $\mathrm{Ext}_c(K_0(A), K_0(A))$.

<u>Proof</u>. We refer the reader for the notion of topological stable rank and for the theorems used in this proof to [16].

There exists an exact sequence

$$0 \to S^{2k}A \to E_f \xrightarrow{\pi} A \to 0$$

We denote as in [16] by $tsr(B)$ the topological stable rank of a C^*-algebra B. It coincides with the Bass stable rank ([11]).

We know that $tsr(SA_n)\leqslant k+1$ and hence $tsr(\varinjlim SA_n)\leqslant k+1$. Also $tsr(A)=1$ and hence $tsr(E_f)\leqslant k+1$. Analogously $tsr(eE_f e)\leqslant k+1$ for any projection $e\in E_f$. Let e_1, e_2 be two projections in $K\otimes E_f$ such that $[e_1]=[e_2]$. Replacing A by some $M_n(A)$ we may suppose that $\pi(e_1), \pi(e_2)\in A$. Since close projections generate the same ideal and $\pi(e_1)$ and $\pi(e_2)$ are equivalent consider the ideal J generated by $\pi(e_1)$ and identify e_1 and e_2 with two functions φ_1 and φ_2 in $C(B_{2k},J)\cap E_f$. Then there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S^{2k}J \arrow[r] \arrow[d] & C(B_{2k},J) \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^{2k}A \arrow[r] & E_f \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Since $K_o(J)\to K_o(A)$ and $K_o(S^{2k}J)\to K_o(S^{2k}A)$ are injective it follows that $K_o(C(B_{2k},J)\cap E_f)\to K_o(E_f)$ is injective and hence $[e_1]$ and $[e_2]$ represent the same class in $K_o(C(B_{2k},J)\cap E_f)$. This shows that we may suppose that e_1 and e_2 are full projections in E_f.

Let $n=k+1$. Choose a full projections $e\in A$ such that $n[e]\leqslant K_o(\pi)([e_1])$ We may suppose that $e\in A_p$ for some large p. Also it follows from proposition 2.4.b) and c) and from lemma 1.2 that there exists a loop of unitaries g such that $f|_{A_p}=\operatorname{ad}g|_{A_p}$. Let h be a lifting of $g\oplus g^*$ to a unitary in $C(B_{2k},M_2(A))$. Then $\operatorname{ad}_h\begin{pmatrix} e & 0\\ 0 & 0\end{pmatrix}=e_o$ is a projection in $M_2(E_f)$ such that $\pi(e_o)=e$. Similarily we may find a projection $e'\in M_r(E_f)$, for some large r, such that $n[e]+K_o(\pi)([e'])=K_o(\pi)([e_1])$. We show now that we may replace $e_o\oplus e'$ by some other projection e'' such that (1) $(n-1)[e_o]+[e'']=[e_1]$. Indeed let $x=[e_1]-n[e_o]-[e']\in K_o(S^{2k}A)$. Since e is full $p=e\oplus\pi(e')$ is also full and this means that the map $\pi_{2k-1}(U(pM_{r+1}(A)p))\to K_o(SA^{2k})$ is surjective (use proposition 2.4.b)). Let g be $(2k-1)$ loop in $U(pM_{r+1}(A)p)$ representing x in $K_o(S^{2k}A)$. Using g we may twist $e_o\oplus e'$ such that the new projection e" satisfy (1). Indeed, let h be a lifting of $g \oplus g^*$ to a unitary in $C(B_{2k}, M_{2r}(A))$, $q = ad_h\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$. It follows that $(n-1)[e] + [q] = [e_1]$.

Since e is full in A, e_0 is also full in E_f and hence $(eE_f)^{n-1} \oplus qE_f \oplus (eE_f)^m$ and $(e_1E_f) \oplus (eE_f)^m$ are isomorphic as right E_f - modules for some $m \in N$. We may use now the Warfield cancellation theorem ([16],[23]) to conclude that $(eE_f)^{n-1} \oplus qE_f$ and e_1E_f are actually isomorphic. The same argument shows that e_2E_f and $(eE)_f^{n-1} \oplus qE_f$ are isomorphic and hence we obtain that $[e_1]$ and $[e_2]$ are equivalent projections.

We have already proved that any projections in A has a lifting in E_f. This shows that $K_0(\pi)(K_0(E_f)_+) = K_0(A)_+$. To prove that

$$0 \longrightarrow K_0(A) \longrightarrow K_0(E_f) \longrightarrow K_0(A) \longrightarrow 0$$

is an element in $Ext_c(K_0(A), K_0(A))$ we have to prove 2.9(b).

Let $e_1, e_2 \in M_r(E_f)$ such that $K_0(\pi)([e_1]) = K_0(\pi)([e_2]) = [e]$. Then, as we did before, we note that e_1 and e_2 may be identified with functions $\varphi_1, \varphi_2 : B_{2k} \to J$, J being the ideal generated in A by $\pi(e_1)$. Hence $[e_1] - [e_2]$ is an element of $K_0(S^{2k}J)$. Conversely if $x \in K_0(E_f)$ is such that $[e_1] - x \in K_0(S^{2k}J) \subset K_0(S^{2k}A)$ then we may find a 2k-1 loop g in $U(\pi(e_1)M_r(A)\pi(e_1))$ such that $\delta([g]) = x - [e_1]$. Using this g we may twist e_1 to obtain a new projection $e_2 \in M_1(E_f)$ such that $[e_2] - [e_1] = x - e_1$. This conclude the proof.

Note that E_f is a locally trivial field of AF-C^*-algebras.

2.12. <u>Theorem</u>. Let A be an AF-C^*-algebra such that $K_0(A)$ has large denominators. Then there exists a commutative diagram with exact rows:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \pi_{2k}(\mathrm{Aut}(A)) \arrow[r] \arrow[d, "M"] & \mathrm{End}_c(K_o(\tilde{A})) \arrow[r] \arrow[d] & \mathrm{Hom}(Z,K_o(\tilde{A})) \arrow[r] \arrow[d, equal] & \pi_{2k-1}(\mathrm{Aut}(A)) \arrow[r, "E"] \arrow[d, "E_1"] & \mathrm{Ext}_c(K_o(\tilde{A}),K_o(\tilde{A})) \arrow[r] \arrow[dl, bend left] & 0 \\
0 \arrow[r] & \mathrm{Hom}(K_o(\tilde{A})/_Z,K_o(\tilde{A})) \arrow[r] & \mathrm{Hom}(K_o(\tilde{A}),K_o(\tilde{A})) \arrow[r] & \mathrm{Hom}(Z,K_o(\tilde{A})) \arrow[r, "-1"] & \mathrm{Ext}(K_o(\tilde{A})/Z,K_o(\tilde{A})) \arrow[d] & & \\
 & & & & \mathrm{Ext}(K_o(\tilde{A}),K_o(\tilde{A})) \arrow[r] & 0 &
\end{tikzcd}$$

<u>Proof</u>. Let us suppose first that A is unital. $\mathrm{End}_c(K_o(\tilde{A})) \to \mathrm{Hom}(K_o(\tilde{A}),K_o(\tilde{A}))$ and $\mathrm{Ext}_c(K_o(\tilde{A}),K_o(\tilde{A})) \to \mathrm{Ext}(K_o(\tilde{A}), K_o(\tilde{A}))$ are the natural maps. M is defined analogously with μ of lemma 2.7.b). E associates to $x \in \pi_{2k-1}(\mathrm{Aut}(A))$, $x=[f]$, the class in $\mathrm{Ext}_c(K_o(A),K_o(A))$ of the extension

$$0 \to K_o(S^{2k}A) \to K_o(E_f) \to K_o(A) \to 0$$

constructed as in lemma 2.11. (We identify using Bott periodicity $K_o(A)$ with $K_o(S^{2k}A)$).

Let 1 denote the units of E_f and A as well. The extension of groups

$$0 \to K_o(S^{2k}A) \to K_o(E_f)/Z[1] \to K_o(A)/Z[1] \to 0$$

will be denoted by $E_1(x)$.

We prove now that E and E_1 are group morphisms.

Denote by $f * g$ the concatenation of loops. Also let $d: A \to A \oplus A$, $d(a) = a \oplus a$ and $\sigma: S^{2k}A \oplus S^{2k}A \to S^{2k}A$ the map induced by $C_o(R^{2k}) \oplus C_o(R^{2k}) \simeq C_o(R_+^{2k}) \oplus C_o(R_-^{2k}) \to C_o(R^{2k})$. There exists a commutative diagram of extensions:

$$\begin{tikzcd}
0 \arrow[r] & S^{2k}A \arrow[r] & E_{f*g} \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & S^{2k}A \oplus S^{2k}A \arrow[r] \arrow[u, "\sigma"] \arrow[d, equal] & E \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, "d"] & 0 \\
0 \arrow[r] & S^{2k}A \oplus S^{2k}A \arrow[r] & E_f + E_g \arrow[r] & A \oplus A \arrow[r] & 0
\end{tikzcd}$$

Since $K_0(\sigma)(a,b)=a+b$ and $K_0(d)(a)=a \oplus a$ we obtain that the extensions $E([f*g])$ and $E_1([f*g])$ are the Baer sums of the extensions $E([f])$ and $E([g])$ and, respectively, of the extensions $E_1([f])$ and $E_1([g])$.

The commutativity of the diagrams follows by the naturality of the definitions (compare with lemma 2.7 and 2.8).

We have to prove the exactness of the upper row. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_{2k}(\mathrm{Aut}(A)) \arrow[r] \arrow[d] & \varprojlim(\pi_{2k}(\mathrm{Hom}^{o}(A_n,A)),\pi_{2k}(\psi_{nm})) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Hom}_c(K_0(A),K_0(A)) \arrow[r] \arrow[d] & \varprojlim(\mathrm{Hom}_c(K_0(A_n),K_0(A)),\alpha^*_{nm}) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Hom}(Z,K_0(A)) \arrow[r] \arrow[d] & \mathrm{Hom}(Z,K_0(A)) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & \varprojlim^1(\pi_{2k}(\mathrm{Hom}^{o}(A_n,A),\pi_{2k}(\psi_{nm})) \arrow[r] \arrow[d] & \pi_{2k-1}(\mathrm{Aut}(A)) \arrow[r] \arrow[d] & \varprojlim(\pi_{2k-1}(\mathrm{Hom}^{o}(A_n,A),\pi_{2k-1}(\psi_{nm})) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & \varprojlim^1(\mathrm{Hom}_c(K_0(A_n),K_0(A)),\alpha^*_{nm}) \arrow[r] \arrow[d] & \mathrm{Ext}_c(K_0(A),K_0(A)) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

In this diagram the first row is exact due to theorem 1.5.b and to proposition 2.4.d). The second row is exact due to proposition 2.6.c). It follows also from theorem 1.5.b) that the fourth row is exact. The fifth row is exact due to proposition 2.10.c). This shows that in the previous diagram all rows are exact. We want to show that the middle coloumn is exact.

Let us first observe that the composition $\mathrm{Hom}(Z,K_0(A)) \to \pi_{2k-1}(\mathrm{Aut}(A)) \to \mathrm{Ext}_c(K_0(A),K_0(A))$ is zero. Indeed if f is a 2k-1 loop in Aut(A) then there exists a 2k-1 loop g in U(A) such that $f=\mathrm{ad}\,g$ for g a 2k-1 loop in U(A). Let e be a projection in $M_n(A)$. Choose a lifting h of $g \oplus g \oplus \ldots \oplus g \oplus g^* \oplus \ldots \oplus g^*$ to a continuous function $h: B_{2k} \to U(M_{2n}(A))$ (g appears n-times and g^* also n times), then $\tau([e]) = \mathrm{ad}_h(e)$ defines a positive lifting for $K_0(\pi): K_0(E_f) \to K_0(A)$.

Let us denote by $H_i^{(j)}$ the i-th cohomology group of the j-th coloumn, $j\in\{1,2,3\}$, $i\in\{1,\dots,5\}$. We want to show that $H_i^{(2)}=0$ for any $i\in\{1,\dots,5\}$. It is obvious that $H_1^{(1)}=H_2^{(1)}=H_3^{(1)}=\{0\}$. Also $H_1^{(3)}=H_2^{(3)}=\{0\}$ by direct computation using lemma 2.7.

The computation of other cohomology groups requires the use of the $\varprojlim^1$ exact sequence (see [20]). There exists an exact sequence (we omit to write the morphisms defining the inverse systems):

$$0\to \varprojlim \pi_{2k}(\mathrm{Hom}^0(A_n,A))\to \varprojlim \mathrm{Hom}_c(K_0(A_n),K_0(A))\to$$
$$\to \varprojlim \mathrm{Hom}_c(K_0(A_n),K_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n,A))\xrightarrow{\delta}\varprojlim^1\pi_{2k}(\mathrm{Hom}^0(A_n,A))\to$$
$$\to\varprojlim^1\mathrm{Hom}_c(K_0(A_n),K_0(A))\to\varprojlim^1\mathrm{Hom}_c(K_0(A_n),K_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n,A))\to 0.$$

We obtain so that $H_4^{(1)}\simeq \mathrm{ran}(\delta)$ and that $H_5^{(1)}\simeq \varprojlim^1\mathrm{Hom}_c(K_0(A_n),K_0(A)/\pi_{2k}(\mathrm{Hom}^0(A_n,A)$.

There exists also a $\varprojlim^1$ exact sequence obtained from the exact sequence $0\to\mathrm{Hom}_c(K_0(A_n),K_0(A))/_{\pi_{2k}(\mathrm{Hom}^0(A_n,A))}\to\mathrm{Hom}(Z,K_0(A))\to\pi_{2k-1}(\mathrm{Hom}^0(A_n,A))\to 0$:

$$0\to\varprojlim \mathrm{Hom}_c(K_0(A_n),K_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n,A))\to\mathrm{Hom}(Z,K_0(A))\to\varprojlim\pi_{2k-1}(\mathrm{Hom}^0(A_n,A))\to$$
$$\varprojlim^1\mathrm{Hom}_c(K_0(A_n),K_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n,A))\to 0.$$

This shows that $H_3^{(3)}$ is isomorphic to the cokernel of the map $\varprojlim \mathrm{Hom}_c(K_0(A_n),K_0(A))\to\varprojlim \mathrm{Hom}_c(K_0(A_n),K_0(A))/\pi_{2k}\ \mathrm{Hom}^0(A_n,A))$. From the previous $\varprojlim^1$ exact sequence this cokernel is isomorphic to $\mathrm{ran}(\delta)$ and hence $H_4^{(1)}\simeq H_3^{(3)}$. Similarily $H_4^{(3)}\simeq\varprojlim^1(\mathrm{Hom}_c(K_0(A_n),K_0(A))/\pi_{2k}(\mathrm{Hom}^0(A_n,A))\simeq H_5^{(1)}$.

We have to show that the previous isomorphisms are induced by the connecting homomorphisms in the long exact sequence of cohomology groups:

$$(1)\quad \dots\to H_j^{(1)}\to H_j^{(2)}\to H_j^{(3)}\xrightarrow{\delta_j} H_{j+1}^{(1)}\to\dots$$

Let us prove first that $H_4^3 \simeq H_5^{(1)}$ is the connecting morphism in (1). Let $x=(x_n)_{n\in N} \in \varprojlim \pi_{2k-1}(\mathrm{Hom}^o(A_n,A))$. Each x_n is represented by an $y_n \in \mathrm{Hom}(Z,K_o(A))$, namely by a $(2k-1)$-loop of unitaries f_n in $U(A)$, such that $\mathrm{ad}_{f_{n+1}}\big|_{A_n} = \mathrm{ad}_{f_n}\big|_{A_n}$. It follows that $y_{n+1}-y_n$ comes from an element $z_n \in \mathrm{Hom}_c(K_o(A_n),K_o(A))$. The class of $(z_1,z_2,\ldots\ldots,z_n,\ldots)$ in $\varprojlim^1 \mathrm{Hom}_c(K_o(A_n),K_o(A))/\pi_{2k}(\mathrm{Hom}^o(A_n,A)$ coincides with the image of x in $\varprojlim^1 \mathrm{Hom}_c(K_o(A_n),K_o(A))/\pi_{2k}(\mathrm{Hom}^o(A_n,A))$ under both compositions $\varprojlim \pi_{2k-1}(\mathrm{Hom}^o(A_n,A)) \to H_4^{(3)} \simeq \varprojlim^1 \mathrm{Hom}_c(K_o(A_n),K_o(A))/$ $/\pi_{2k}(\mathrm{Hom}^o(A_n,A))$ and $\varprojlim\ {}_{2k-1}(\mathrm{Hom}^o(A_n,A)) \to H_4^{(3)} \xrightarrow{\delta_4} H_5^{(4)} \simeq \varprojlim' \mathrm{Hom}_c(K_o(A_n),$ $K_o(A))/\pi_{2k}(\mathrm{Hom}^o(A_n,A))$. This shows that the connecting map δ_4: $H_4^{(3)} \to H_5^{(4)}$ is an isomorphism.

Let f be a 2k-1 loop in U(A) such that its class in $\varprojlim \pi_{2k-1}(\mathrm{Hom}^o(A_n,A))$ is 0. Then ad_f is in $\varprojlim^1 \pi_{2k}(\mathrm{Hom}^o(A_n,A))$ and is represented by the following element: for each n there exists a loop x_n in $U(A_n')$ such that $\mathrm{ad}\widehat{f}\big|_{A_n}$ and $\mathrm{ad}\widehat{x_n}\big|_{A_n}$ are homotopic. Also $\mathrm{ad}\widehat{f}\big|_{A_n}$ and $\mathrm{ad}\widehat{x_{n+1}}\big|_{A_n}$ are homotopic. The resulting homotopy from $\mathrm{ad}\widehat{x_{n+1}}\big|_{A_n}$ and $\mathrm{ad}\widehat{x_n}\big|_{A_n}$ defines a 2k-loop in $\mathrm{Hom}^o(A_n,A)$. Denote the class of this loop by y_n. Then the class of $\mathrm{ad}\widehat{f}$ in $\varprojlim^1 \pi_{2k}(\mathrm{Hom}^o(A_n,$ $A))$ has as a representative $([y_n])_{n\in N}$. x_n defines an element of $\pi_{2k-1}(U(A_n')) \simeq \mathrm{Hom}_c(K_o(A_n),K_o(A))$ such that x_n and x_{n+1} regarded as elements of $\mathrm{Hom}_c(K_o(A_n),K_o(A))$ have the property that $x_{n+1}([1]) = [f] = x_n([1])$ and hence $x_{n+1}-x_n$ is actually in the image of $\pi_{2k}(\mathrm{Hom}^o(A_n,A))$ in $\mathrm{Hom}_c(K_o(A_n),K_o(A))$. It follows from the definition that $x_{n+1}-x_n$ corresponds to y_n. Since $(x_{n+1}-x_n)_{n\in N}$ represents $(\mathrm{ad}\widehat{f}\big|_{A_n})_{n\in N} \in \varprojlim \mathrm{Hom}_c(K_o(A_n),K_o(A))$ in $\varprojlim^1 \pi_{2k}(\mathrm{Hom}^o(A_n,A))$ it follows that we have a commutative diagram

$$
\begin{tikzcd}
H_3^{(3)} \arrow[rr, "\delta_3"] \arrow[dr] & & H_4^{(1)} \arrow[dl] \\
& \varprojlim^1 \pi_{2k}(\mathrm{Hom}^o(A_n,A)) &
\end{tikzcd}
$$

This shows that the connecting morphism δ_3 is an isomorphism. The nonunital case is similar requing also the use of lemma 2.8 and of the isomorphism $\varprojlim^1 \mathrm{Hom}_c(K_o(A_n),K_o(A)) \simeq \varprojlim^1(\mathrm{Hom}_c(K_o(A_n^+),K_o(A^+))$.

2.13. Corollary. Let A be a simple AF-C^*-algebra, A infinite dimensional, $A \neq K$. Then, if A is unital

$$\pi_{2k-1}(\mathrm{Aut}(A)) \simeq \mathrm{Ext}(K_o(A)/Z[1],\ K_o(A))$$
$$\pi_{2k}(\mathrm{Aut}(A)) \simeq \mathrm{Hom}(K_o(A)/Z[1],\ K_o(A))$$

and, if A is not unital

$$\pi_{2k-1}(\mathrm{Aut}(A)) \simeq \mathrm{Ext}(K_o(A),K_o(A))$$
$$\pi_{2k}(\mathrm{Aut}(A)) \simeq \mathrm{Hom}(K_o(A),K_o(A)),\ k \geq 1.$$

Proof. Use theorem 2.12 and proposition 2.3.

2.14. Remark. a) Suppose that A is not unital. Let A be an AF-C^*-algebra with $K_o(A)$ with large denominators. Let e_n denote the unit of A_n. Let us suppose also that $1-e_n$ is a full projection in A^+ then it is easely seen that $\pi_k(U(A_n')) \to \pi_k(U(A))$ is surjective for any $k \geq 1$ (see lemma 2.7.a)). This shows that $\mathrm{End}_c(K_o(A)) \to \mathrm{Hom}(Z, K_o(A))$ is surjective and hence $\pi_{2k-1}(\mathrm{Aut}(A)) \simeq \mathrm{Ext}_c(K_o(A),K_o(A))$.

b) If A is unital then $\pi_{2k-1}(\mathrm{Aut}(A))$ can be identified with isomorphism classes of compatible extensions with order unit

$$(1) \quad 0 \to K_o(A) \to (E,u) \xrightarrow{\pi} (K_o(A),[1]) \to 0$$

$$0 \to K_o(A) \to E \to K_o(A) \to 0$$

is an exact sequence as in definition 2.9 and u is a positive element in E such that $\pi(u)=[1]$. Two such extensions $(E_1,u_1),(E_1,u_2)$ are isomorphic if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_o(A) \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "\varphi"] & K_o(A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_o(A) \arrow[r] & E_2 \arrow[r] & K_o(A) \arrow[r] & 0
\end{tikzcd}$$

such that φ is a positive morphism (and hence necessarily an isomorphism of ordered groups) and $\varphi(u_1)=u_2$. The extension in (1) is trivial if there exists a positive lifting τ for π such that $\tau([1])=u$.

We associate to a loop f representing $x\in\pi_{2k-1}(\mathrm{Aut}(A))$ the class of the extension $K_o(E_f)$ with $[1]\in K_o(E_f)$ as order unit . It turns out that there exists a commutative diagram :

$$\begin{tikzcd}
\mathrm{Hom}_c(K_o(A),K_o(A)) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(Z,K_o(A)) \arrow[r] \arrow[d, equal] & \pi_{2k-1}(\mathrm{Aut}(A)) \arrow[r] \arrow[d] & \mathrm{Ext}_c(K_o(A),K_o(A)) \arrow[r] \arrow[d, equal] & 0 \\
\mathrm{Hom}_c(K_o(A),K_o(A)) \arrow[r] & \mathrm{Hom}(Z,K_o(A)) \arrow[r] & \mathrm{Ext}_c^u(K_o(A),K_o(A)) \arrow[r] & \mathrm{Ext}_c(K_o(A),K_o(A)) \arrow[r] & 0
\end{tikzcd}$$

($\mathrm{Ext}_c^u(K_o(A),K_o(A))$ denotes, the group of isomorphism classes of extensions as in (1), called <u>compatible unital extensions)</u>.

The morphism $\mathrm{Hom}(Z,K_o(A)) \to \mathrm{Ext}_c^u(K_o(A),K_o(A))$ sends the morphism $n \to nu$ to the class of the trivial extension

$$0 \to K_o(A) \to E \underset{\tau}{\overset{\pi}{\rightleftarrows}} K_o(A) \to 0$$

with ordered unit $\tau([1])+u$. It follows easely that the second row is exact and hence $\pi_{2k-1}(\mathrm{Aut}(A)) \simeq \mathrm{Ext}_c^u(K_o(A),K_o(A))$ from the five lemma.

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