INSTITUTUL
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MATEMATICA

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PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

TWO PAPERS ON OPTIMIZATION

by

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PREPRINT SERIES IN MATHEMATICS

No. 24/1987

BUCURESTI

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June 1987

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A GENERAL THEORY OF DUAL OPTIMIZATION PROBLEMS. II: ON THE PERTURBATIONAL DUAL PROBLEM CORRESPONDING TO AN UNPERTURBATIONAL DUAL PROBLEM

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Distinguishing between a problem and its instances, we redefine the perturbational dual problem corresponding to an unperturbational dual problem, by means of explicit formulas, instead of "the scheme" of "formal replacements" of [5]. We show the relations between some main perturbational dual problems and the perturbational dual problems corresponding to some main unperturbational dual problems.

§0. INTRODUCTION

In the paper [5] we have constructed a unified theory of dual optimization problems, which encompasses, as particular cases, the known dual problems. In this theory, we have defined a general concept of a dual problem and then, for any "unperturbational" dual problem (i.e., defined without assuming a perturbation of the primal problem), we have defined, via a certain "scheme" of "formal replacements", a corresponding "perturbational" dual problem, mentioning (see [5], p.96) that this scheme is given "similarly to the idea of universally defined multifunctions [4]". This has permitted to obtain, in [5], new connections between the unperturbational and perturbational versions of various known classes of dual problems (e.g., Lagrangian duals, surrogate duals, etc.).

The aim of the present paper is to give some improvements and complements to these parts of [5]. Instead of using the above mentioned "scheme" of "formal replacements", we shall define here the perturbational dual problem corresponding to an unperturbational dual problem, by means of explicit formulas. This new definition will not only shed an additional light on the "scheme" of [5], but will also reveal the necessity of a certain condition (formula (2.14) below), which was only implicitly assumed in [5]. Finally, we shall show the relations between some main perturbational dual problems (namely, Lagrangian, surrogate and Tind-Wolsey-type) and the perturbational dual problems corresponding to some main unperturbational dual problems (namely, Lagrangian, surrogate, and Tind-Wolsey-type, respectively).

While in our previous papers, in which we have studied the dependence of optimization problems on their parameters (see e.g. [5], and the references therein), we have used the same term "problem", both for a problem and for an instance of a problem, in the present paper we shall find it important to distinguish between these concepts (following e.g. [1], [2]), since this will lead to a deeper understanding of the parameters of dual optimization problems. Also, an essential feature of the dual problems considered in [5] is that their constraint set and objective function (and hence their value, etc.) are "universally defined" in the sense of [4], i.e., they are defined, "by the same formula", for a whole collection of instances, where each instance is obtained by specifying particular values for all the problem parameters. One of the basic methods of [5] has been to discover connections between different instances of the same problem, or, of two different problems. Roughly speaking, an instance of the perturbational dual problem (Q_p) corresponding to an unperturbational problem (Q) and to a perturbation p of the primal problem, is defined by means of an instance (involving p) of the unperturbational dual problem (Q).

We shall denote by Ens the family of all sets. For any $G \in$ Ens, we shall denote by $\overline{R}^G$ the set of all functions $h: G \to \overline{R}$, where $\overline{R} = [-\infty, +\infty]$ is endowed with the usual "upper addition" $\dot{+}$ and "lower addition" $\underset{\cdot}{+}$ of Moreau (see e.g. [5], p. 80). For any $F, X \in$ Ens, and $x_o \in X$, we shall denote the set $F \times \{x_o\} \subset F \times X$ by (F, x_o), and we shall use the canonical embedding $R^F \times \overline{R}^X \subset \overline{R}^{F \times X}$, given by

$$(v,w)(x,y) = v(y) + w(x) \qquad (v \in R^F,\ w \in \overline{R}^X,\ y \in F,\ x \in X). \qquad (0.1)$$

Finally, we mention that for $G \in Ens$ and $h \in \overline{R}^G$, we shall use the notations $\inf h(G) = \inf_{g \in G} h(g)$, $\sup h(G) = \sup_{g \in G} h(g)$.

§1. PRIMAL AND DUAL PROBLEMS. SOME UNPERTURBATIONAL DUAL PROBLEMS

For our purposes, an instance of an infimization problem will be the "task" $(P_{G,h})$ of computing, for a given ordered pair (G,h), where $\emptyset \neq G \in Ens$ and $h \in \overline{R}^G$, the number

$$\alpha_{G,h} = \inf h(G), \tag{1.1}$$

called the "value" of the instance (we shall not be concerned here with the related questions usually subsumed under the same generic name, e.g., that of finding the "optimal solutions" $g_o \in G$, etc.). An infimization problem is a set $\mathcal{J}$ of instances of an infimization problem; the "constraint set" G and the "objective function" $h: G \to \overline{R}$ are called the parameters of the infimization problem, and each instance is obtained by specifying particular values of these parameters G and h. Thus,

$$\mathcal{J} = \left\{ (P_{G,h}) \mid (G,h) \in \mathcal{O} \right\}, \tag{1.2}$$

where $\mathcal{O}$ denotes a set of ordered pairs (G,h), with $\emptyset \neq G \in Ens$ and $h \in \overline{R}^G$.

An instance of a supremization problem and a supremization problem $\mathcal{J}$ are defined similarly, replacing inf by sup in (1.1).

The (instances of) infimization and supremization problems are called (instances of) optimization problems.

In the sequel, instead of "problem (1.2), with $(P_{G,h})$ of the form (1.1)", we shall also use the term "problem of the form (1.1)", or, briefly (when this will lead to no confusion), "problem (1.1)", and we shall proceed similarly for supremization problems, too.

Remark 1.1. a) The above definition of a primal infimization problem is slightly different from that of an "optimization problem" given in [2], p.4, definition 1.1; indeed, in the latter, the parameters are a set G and a function $h \in R^G$ (not $\overline{R}^G$), and the problem is to find a globally optimal solution $g_o \in G$ of (1.1) (which, however, need not exist).

b) The parameters G and h of problem (1.2) are not independent, since h must be defined on G, but we shall not write h_G

instead of h, in order to simplify the notations. Similarly, in the sequel, when this will lead to no confusion, we shall omit some of the parameters, leaving only those on which we want to concentrate.

c) Often the parameters G, h (or, one of them) are defined with the aid of other objects, which, in turn, may be regarded as parameters of problem (1.2). Note also that one may fix some of the parameters of a given optimization problem and consider only the instances obtained by letting the other parameters vary.

Let us give now a definition of dual optimization problems.

Definition 1.1. a) By a dual problem to a (primal) infimization problem (1.2), we shall mean any supremization problem of the form

$$\beta_{G,h,\psi,\varphi}=\sup \mu(W), \tag{1.3}$$

where $(G,h)\in\mathcal{O}$ (with $\mathcal{O}$ being the same as in (1.2), which we shall understand throughout the sequel, without special mention), $\psi\in(\text{Ens}\setminus\emptyset)^{\mathcal{O}}$, $\emptyset\neq W=\psi(G,h)\in\text{Ens}$, $\varphi:(G,h)\to\bar{R}^W=\bar{R}^{\psi(G,h)}$ $((G,h)\in\mathcal{O})$ and $\mu=\varphi_{G,h}\in\bar{R}^W$ (we use the notation $\varphi_{G,h}$ instead of $\varphi(G,h)$, since in the sequel we shall consider the values $\varphi_{G,h}(w)$, $w\in W$). An instance of a dual problem (1.3) is obtained by specifying particular values for all the problem parameters G, h, ψ and φ.

b) Given a (primal) infimization problem (1.2), for each (fixed) parameter φ as above, we shall call (1.3) a φ-dual problem to problem (1.2). An instance of a φ-dual problem (1.3) is obtained by specifying the values of the parameters G, h and ψ.

c) Given a (primal) infimization problem (1.2), for each (fixed) ordered pair of parameters (ψ,φ) as above, we shall call (1.3) the (ψ,φ)-dual problem to problem (1.2). An instance of the (ψ,φ)-dual problem (1.3) is obtained by specifying particular values for the problem parameters G and h.

Remark 1.2. a) The parameters G and h of problem (1.3) are the same as for the primal problem, while the parameter ψ is a multi-function which, to each $(G,h)\in\mathcal{O}$ assigns a "dual constraint set" $W=\psi(G,h)$, and the parameter φ is a "vector field", which, to each $(G,h)\in\mathcal{O}$, assigns a "dual objective function", $\mu=\varphi_{G,h}:W=$ $=\psi(G,h)\to\bar{R}$ (where, by remark 1.1 b), we write $\mu=\varphi_{G,h}$ instead of

$\mu_W=\varphi_{G,h,W}$). For each instance $(P_{G',h'})$ of (1.2), there exists a unique instance of (1.3), having the same $(G',h')\in\mathcal{O}$, called sometimes (see e.g. [2], p.78) "a dual problem to the given instance $(P_{G',h'})$", namely,

$$\beta_{G',h',\psi,\varphi}=\sup \mu'(W') , \qquad (1.4)$$

where $W'=\psi(G',h')$, and $\mu'=\varphi_{G',h'}$. Some general relations between various properties of the parameters ψ,φ and corresponding properties of instances of (1.3), will be given elsewhere.

b) In the sequel, we shall usually consider dual problems involving more parameters (hence, they will be subclasses of (1.3)), e.g., dual problems

$$\beta_{G,F,\bar{h},\psi,\varphi}=\sup \mu(W), \qquad (1.5)$$

where $F\in Ens$, $F\supseteq G$, $\bar{h}\in\bar{R}^F$ is an extension of h (i.e., $\bar{h}|_G=h$), and $W=\psi(G,F,\bar{h})$, $\mu=\varphi(G,F,\bar{h})$; according to remark 1.1. b), we simplify the notations for these problems, writing $(G,F,\bar{h})$ instead of $(G,h,F,\bar{h})$. Note that, for "constrained" optimization, a set $F\supseteq G$ and a function $\bar{h}\in\bar{R}^F$ with $\bar{h}|_G=h$, appear already in the formulation of the primal problem (see e.g. remark 2.1 d), with $F=R^n$), denoted, in this case, by $(P_{G,F,\bar{h}})$, although in (1.1) only G and $h=\bar{h}|_G$ are used; then, of course, it is natural to define a dual problem, to such a primal problem, by (1.5), with the same F and $\bar{h}$ as in the corresponding instance of the primal problem. We shall proceed in this way in §2, where we shall consider only constrained primal infimization problems $(P_{G,F,\bar{h}})$. However, in this section we shall adopt the more general point of view, that a set $F\supseteq G$ and a function $\bar{h}\in\bar{R}^F$ need not appear in any instance of a primal problem $(P_{G,h})$, even if we consider a dual problem of the form (1.5); then, in some cases, it may be useful to select from (1.5) the class of instances with $F=G$ and $\bar{h}=h$, reducing thereby (1.5) to the dual problem (1.3) (we have proceeded in this way in [6], in order to express some min-max results of combinatorial optimization as unperturbational Lagrangian duality theorems).

The above general concept of a dual problem encompasses both the "unperturbational" dual problems (i.e., the dual problems defined directly, without assuming any perturbation of the primal problem) and the "perturbational" ones (i.e., those defined with the aid of a perturbation of the primal problem). Now we shall redefine three main concepts of unperturbational dual problems (considered in [5]), in a form showing that they are indeed dual problems in the sense of definition 1.1.

Definition 1.2. a) By an unperturbational Lagrangian dual problem to a (primal) infimization problem (1.2), we shall mean any supremization problem of the form

$$\beta^{\lambda}_{G,F,\overline{h},\psi^{\lambda},\varphi^{\lambda}}=\sup \mu^{\lambda}(W^{\lambda}), \tag{1.6}$$

where $F\in Ens$, $F\supseteq G$, $\overline{h}\in\overline{R}^F$ is an extension of h (i.e., $\overline{h}|_G=h$), $W^{\lambda}=\psi^{\lambda}(G,F,\overline{h})$ is a non-empty subset of $\overline{R}^F$, satisfying

$$\inf w(G)<+\infty \qquad (w\in W^{\lambda}), \tag{1.7}$$

and $\mu^{\lambda}=\varphi^{\lambda}_{G,F,\overline{h}}\in\overline{R}^{W^{\lambda}}$ is defined by

$$\mu^{\lambda}(w)=\varphi^{\lambda}_{G,F,\overline{h}}(w)=\inf_{y\in F}\{\overline{h}(y)\dot{+}-w(y)\}\dot{+}\inf w(G) \qquad (w\in W^{\lambda}). \tag{1.8}$$

An instance of problem (1.6)-(1.8) is obtained by specifying the values of the parameters G, F, $\overline{h}$ and φ^{λ} (whence also W^{λ}, φ^{λ} and μ^{λ}).

b) Given a (primal) infimization problem (1.2), for each (fixed) function ψ^{λ} as above, we shall call (1.6)-(1.8) the unperturbational ψ^{λ}-Lagrangian dual problem to (1.2). An instance of the unperturbational ψ^{λ}-Lagrangian dual problem (1.6)-(1.8) is obtained by specifying particular values for the parameters G, F and $\overline{h}$.

Remark 1.3. a) In the terminology of definition 1.1, an unperturbational Lagrangian (respectively, ψ^{λ}-Lagrangian) dual problem is a φ^{λ}-dual (respectively, $(\psi^{\lambda},\varphi^{\lambda})$-dual) problem, with φ^{λ} of (1.8).

b) When F is a linear (respectively, a topological linear) space, if one takes $\psi^{\lambda}(G,F,\overline{h})=W^{\lambda}=F^{\#}$ (respectively, F^{*}), the algebraic (respectively, topological linear) dual of F, then (1.7) is satisfied. For examples and applications of some other multi-functions ψ^{λ}, see e.g. [5] and the references therein.

c) The letter λ stands for: Lagrangian.

Definition 1.3. By an unperturbational surrogate dual problem to a (primal) infimization problem (1.2), we shall mean any supremization problem of the form

$$\beta^{\sigma}_{\Delta}=\beta^{\sigma}_{G,F,\bar{h},\psi^{\sigma},\Delta,\varphi^{\sigma}}=\sup \mu^{\sigma}_{\Delta}(W^{\sigma}), \tag{1.9}$$

where $F\in Ens$, $F\supseteq G$, $\bar{h}\in\bar{R}^F$, $\bar{h}|_G=h$, $\emptyset\neq W^{\sigma}=\psi^{\sigma}(G,F,\bar{h})\subset\bar{R}^F$, Δ is a multi--function which assigns to each triple (G,F,w) $(w\in W^{\sigma})$, with G, F and W^{σ} as above, a set $\Delta_{G,F,w}\subseteq F$, not depending on $\bar{h}$, and $\mu^{\sigma}_{\Delta}=\varphi^{\sigma}_{G,F,\bar{h},\Delta}\in\bar{R}^{W^{\sigma}}$ is defined by

$$\mu^{\sigma}_{\Delta}(w)=\varphi^{\sigma}_{G,F,\bar{h},\Delta}(w)=\inf \bar{h}(\Delta_{G,F,w}) \qquad (w\in W^{\sigma}). \tag{1.10}$$

An <u>instance</u> of problem (1.9), (1.10) is obtained by specifying the values of the parameters G, F, $\bar{h}$, ψ^{σ} and Δ (whence also W^{σ}, φ^{σ} and μ^{σ}).

<u>Remark 1.4</u>. One can make, for defintion 1.3, similar observations to those of remark 1.3 a), and the letter σ stands now for "surrogate". Fixing one or both of the parameters ψ^{σ}, Δ, one obtains a "<u>ψ^{σ}-surrogate dual</u>", a "<u>Δ-surrogate dual</u>" , and "<u>the (ψ^{σ},Δ)-surrogate dual problem</u>", respectively. As an example of a Δ-surrogate dual problem, take $\Delta_{G,F,w}=\{y\in F\,|\,w(y)\geqslant\inf w(G)\}$ $(w\in W^{\sigma})$; for other examples, see e.g. [5].

<u>Definition 1.4</u>. By the <u>unperturbational τ-dual problem</u> to a (primal) infimization problem (1.2), we shall mean the supremization problem

$$\beta^{\tau}_{G,F,\bar{h},\psi^{\tau},\varphi^{\tau}}=\sup \mu^{\tau}(W^{\tau}), \tag{1.11}$$

where $F\in Ens$, $F\supseteq G$, $\bar{h}\in\bar{R}^F$, $\bar{h}|_G=h$, and where $W^{\tau}=\psi^{\tau}(G,F,\bar{h})\subset\bar{R}^F$ and $\mu^{\tau}=\varphi^{\tau}_{G,F,\bar{h}}\in\bar{R}^{W^{\tau}}$ are defined by

$$W^{\tau}=\psi^{\tau}(G,F,\bar{h})=\{w\in\bar{R}^F\,|\,w\leqslant\bar{h}\}, \tag{1.12}$$

$$\mu^{\tau}(w)=\varphi^{\tau}_{G,F,\bar{h}}(w)=\inf w(G) \qquad (w\in W^{\tau}). \tag{1.13}$$

An <u>instance</u> of problem (1.11)-(1.13) is obtained by specifying the values of the parameters G, F and $\bar{h}$ (whence also ψ^{τ}, W^{τ}, φ^{τ} and μ^{τ}).

<u>Remark 1.5</u>. One can also define a more general concept of unperturbational τ-dual problem, replacing $\bar{R}^F$, in (1.12), by a set $W'=\rho(G,F,\bar{h})\subseteq\bar{R}^F$, such that $W^{\tau}=\psi^{\tau}(G,F,\bar{h},\rho)=\{w\in W'\,|\,w\leqslant\bar{h}\}\neq\emptyset$ (then,

$\mu^{\tau}=\varphi^{\tau}_{G,F,\overline{h},\rho}$ in (1.13)); in fact, the particular case when F is a linear space containing G, and $W'=F^{\#}$ (the algebraic dual of F), has been considered in [5], remark 2.9 f). The letter τ is used here to indicate the connections with the dual problems studied by Tind and Wolsey [7] (see [5], remark 3.6 c)).

§2. THE PERTURBATIONAL DUAL PROBLEM CORRESPONDING TO AN UNPERTURBATIONAL DUAL PROBLEM

It is well-known that each embedding of (an instance ______ of) a primal infimization problem, ______ into a family of (instances of) "perturbed" infimization problems, permits to define (an instance of) a dual problem to the given (instance of) primal problem, with respect to the given embedding. Such a dual problem has been called, in [5], a "perturbational" dual problem, since it is defined with respect to a perturbation of the primal problem. Let us first redefine three main concepts of perturbational dual problems, showing that they are indeed dual problems in the sense of definition 1.1.

<u>Definition 2.1</u>. Let each instance $(P_{G,F,\overline{h}})$ of a (constrained) primal infimization problem (see remark 1.2 b)) be "embedded" into a family __ of instances ______ of "perturbed" infimization problems

$$\{(P^{x}_{G,F,\overline{h}})\mid x\in X\}\ , \tag{2.1}$$

with $(P^{x}_{G,F,\overline{h}})$ being the task of computing

$$\alpha^{x}_{G,F,\overline{h}}=\inf_{y\in F} p(y,x)=f(x)=f_{G,F,\overline{h}}(x) \qquad (x\in X), \tag{2.2}$$

where $X=X_{G,F,\overline{h}}$ is a set (of "perturbations" $x\in X$), $x_o=x_o^{G,F,\overline{h}}\in X$ is a "marked element" of X, and $p=p_{G,F,\overline{h}}:F\times X\to\overline{R}$ is a function (called "perturbation function"), such that

$$p(y,x_o)=\begin{cases} h(y) & \text{if } y\in G \\ +\infty & \text{if } y\in F\setminus G. \end{cases} \tag{2.3}$$

By a <u>perturbational Lagrangian dual problem to problem $(P_{G,F,\overline{h}})$, with respect to the "perturbation triple"</u> (X,x_o,p), we shall mean any supremization problem of the form

$$\beta^{\pi\lambda}_{G,F,\bar{h},\psi^{\pi\lambda},\varphi^{\pi\lambda},X,x_o,p} = \sup \mu^{\pi\lambda}(W^{\pi\lambda}), \qquad (2.4)$$

where ______________ $W^{\pi\lambda}=\psi^{\pi\lambda}(G,F,\bar{h},X,x_o,p)$ is a non-empty subset of $\bar{R}^X$, satisfying

$$w(x_o)<+\infty \qquad (w\in W^{\pi\lambda}), \qquad (2.5)$$

and $\mu^{\pi\lambda}=\varphi^{\pi\lambda}_{G,F,\bar{h},X,x_o,p}\in\bar{R}^{W^{\pi\lambda}}$ is defined by

$$\mu^{\pi\lambda}(w)= \inf_{(y,x)\in F\times X} \{p(y,x)\dot{+}-w(x)\}\dot{+}w(x_o) \qquad (w\in W^{\pi\lambda}). \qquad (2.6)$$

An instance of problem (2.4)-(2.6) is obtained by specifying the values of the parameters.

Remark 2.1. a) When X is a linear (respectively, a topological linear) space, if one takes $\psi^{\pi\lambda}(G,F,\bar{h},X,x_o,p)=W^{\pi\lambda}=X^{\#}$ (respectively, X^{*}), then (2.5) is satisfied, for any $x_o\in X$; for other $\psi^{\pi\lambda}$'s, see e.g. [5] and the references therein.

b) Often X is called the set of "parameters" (instead of "perturbations") $x\in X$, but we shall not use here this term, since it might lead to confusion e.g. with the same term for G and h of problem (1.2)).

c) The letters $\pi\lambda$ stand for: perturbational Lagrangian.

d) An instance of linear programming can be defined (see e.g. [2], p.5 and [3], p.301) by taking

$$G=\{y\in R^n | y\geq 0,\ u(y)=0\},\ h=\bar{h}|_G, \qquad (2.7)$$

where $u:R^n\to R^m$ is an affine operator and $\bar{h}:R^n\to R$ is a linear function. Then, for $F=R^n$, $X=X_{G,F}=R^m$ and

$$p(y,x)=\begin{cases}\bar{h}(y), & \text{for } y\in R^n,\ x\in R^m,\ y\geq 0,\ u(y)=x\\ +\infty & \text{otherwise,}\end{cases} \qquad (2.8)$$

we have (2.3), with $x_o=0$. Let us mention that, often one wants to find a "best representation" (2.7) of (the fixed set) G, in some sense, e.g., via an $X'=X'_{G,F}=R^{m'}$ (and $u':R^n\to R^{m'}$) with the smallest possible m', in order to obtain a dual problem for which $W^{\pi\lambda}=(R^{m'})^{*}$

has the smallest dimension. For various optimization problems, there are also other criteria for choosing X, x_o and p (see e.g. [5] and the references therein).

e) Fixing the parameter $\psi^{\pi\lambda}$, one obtains "the perturbational $\psi^{\pi\lambda}$-Lagrangian dual problem to problem (1.2), with respect to (X, x_o, p)".

Definition 2.2. By a perturbational $\tilde{\Delta}$-surrogate dual problem to an infimization problem $(P_{G,F,\bar{h}})$, with respect to a perturbation triple (X, x_o, p) (as in definition 2.1), we shall mean any supremization problem of the form

$$\beta^{\pi\sigma}_{\tilde{\Delta}} = \beta^{\pi\sigma}_{G,F,\bar{h},\psi^{\pi\sigma},\tilde{\Delta},\varphi^{\pi\sigma},X,x_o,p} = \sup \mu^{\pi\sigma}_{\tilde{\Delta}}(W^{\pi\sigma}), \tag{2.9}$$

where ______________ $\emptyset \neq W^{\pi\sigma} = \psi^{\pi\sigma}(G,F,\bar{h},X,x_o,p) \subseteq \bar{R}^X$, $\tilde{\Delta}$ is a multi-function which assigns, to each triple $((F,x_o), F \times X, w)$ $(w \in W^{\pi\sigma})$, with F, x_o, X and $W^{\pi\sigma}$ as above, a set $\tilde{\Delta}_{(F,x_o),F\times X,w} \subseteq F \times X$, not depending on p, and $\mu^{\pi\sigma}_{\tilde{\Delta}} = \varphi^{\pi\sigma}_{G,F,\bar{h},\tilde{\Delta},X,x_o,p} \in \bar{R}^{W^{\pi\sigma}}$ is defined by

$$\mu^{\pi\sigma}_{\tilde{\Delta}}(w) = \inf p(\tilde{\Delta}_{(F,x_o),F\times X,w}) \qquad (w \in W^{\pi\sigma}). \tag{2.10}$$

An instance of problem (2.9), (2.10) is obtained by specifying the values of the parameters.

Remark 2.2. One can make, again, some remarks corresponding to those of remark 2.1, and the letters $\pi\sigma$ stand for: perturbational surrogate.

Definition 2.3. By the perturbational τ-dual problem to an infimization problem $(P_{G,F,\bar{h}})$, with respect to a perturbation triple (X, x_o, p) (as in definition 2.1), we shall mean the supremization problem

$$\beta^{\pi\tau}_{G,F,\bar{h},\psi^{\pi\tau},\varphi^{\pi\tau},X,x_o,p} = \sup \mu^{\pi\tau}(W^{\pi\tau}), \tag{2.11}$$

where ______________ $W^{\pi\tau} = \psi^{\pi\tau}(G,F,\bar{h},X,x_o,p) \subseteq \bar{R}^X$

and $\mu^{\pi\tau} = \varphi^{\pi\tau}_{G,F,\bar{h},X,x_o,p} \in \bar{R}^{W^{\pi\tau}}$ are defined by

$$W^{\pi\tau}=\{w\in\overline{\overline{R}}^X \mid w\leq p(y,\cdot)\quad (y\in F)\}, \tag{2.12}$$

$$\mu^{\pi\tau}(w)=w(x_o) \qquad (w\in W^{\pi\tau}). \tag{2.13}$$

An instance of problem (2.11)-(2.13) is obtained by specifying the values of the parameters.

Remark 2.3. Similarly to remark 1.5, one can also define a more general concept of perturbational τ-dual problem. The case when $\overline{\overline{R}}^X$ is replaced, in (2.12), by $X^{\#}$, has been considered in [5], remark 3.6 c). The letters $\pi\tau$ stand for: perturbational τ(-dual).

Let us redefine now the concept of "the perturbational dual problem corresponding to an unperturbational dual problem" [5], using explicit formulas instead of the "scheme" of "formal replacements" of [5].

Definition 2.4. Assume that the (primal) infimization problem $(P_{G,F,\overline{h}})$ is embedded into a family (2.1) of perturbed optimization problems, as in definition 2.1, and let (1.5) be an unperturbational dual problem to $(P_{G,F,\overline{h}})$, such that $\psi=\psi(G,F,\overline{h})$ and the perturbation triple (X,x_o,p) satisfy (for all G, F and $\overline{h}$)

$$(0,\psi(\{x_o\},X,f))\subseteq\psi((F,x_o),F\times X,p)\ (\subseteq\overline{\overline{R}}^{F\times X}), \tag{2.14}$$

where $f=f_{G,F,\overline{h}}:X\to\overline{R}$ is the "value function" (2.1) and $\psi(\{x_o\},X,f)\subseteq\overline{\overline{R}}^X$. Then, by the perturbational dual problem to $(P_{G,F,\overline{h}})$, with respect to the perturbation triple (X,x_o,p), corresponding to problem (1.5), we shall mean the supremization problem

$$\beta^{\pi}_{G,F,\overline{h},\psi^{\pi},\varphi^{\pi},X,x_o,p}=\sup\mu^{\pi}(W^{\pi}), \tag{2.15}$$

where $W^{\pi}=\psi^{\pi}(G,F,\overline{h},X,x_o,p)\subseteq\overline{\overline{R}}^X$ and $\mu^{\pi}=\varphi^{\pi}_{G,F,\overline{h},X,x_o,p}:W^{\pi}\to\overline{R}$ are defined by

$$W^{\pi}=\psi(\{x_o\},X,f), \tag{2.16}$$

$$\mu^{\pi}(w)=\varphi_{(F,x_o),\,F\times X,p}(0,w) \qquad (w\in W^{\pi}). \tag{2.17}$$

An instance of problem (2.15)-(2.17) is obtained by specifying the values of the parameters.

Remark 2.4. Definition 2.4 should be understood as follows. Consider the primal infimization problem $(P_{(F,x_o),p|_{(F,x_o)}})$, that is,

$$\inf p(F,x_o). \tag{2.18}$$

Then, since (1.5) is "universally defined", in the sense mentioned in §0 (i.e., well defined, for all values of the parameters), for any $Z\in Ens$, $Z\supseteq(F,x_o)$, $\overline{p}\in\overline{\overline{R}}^Z$, $\overline{p}|_{(F,x_o)}=p|_{(F,x_o)}$, we may consider the instance

$$\beta_{(F,x_o),Z,\overline{\overline{p}},\psi,\mu}=\sup \mu'(W') \tag{2.19}$$

of (1.5), where $W'=\psi((F,x_o),Z,\overline{p})\subseteq\overline{\overline{R}}^Z$, $\mu'=\varphi_{(F,x_o),Z,\overline{\overline{p}}}:W'\to\overline{\overline{R}}$. Applying this, in particular, to $Z=F\times X$, $\overline{\overline{p}}=p$ and $(0,W^\pi)\subseteq W'$ (see (2.16), (2.14)), $\mu^\pi=\mu'|_{(0,W^\pi)}$, we get the right hand sides of (2.17), (2.15).

We shall give now some relations between the perturbational dual problems defined in the first part of this section and the perturbational dual problems corresponding to the unperturbational dual problems of §1.

Theorem 2.1. Assume that the primal infimization problem $(P_{G,F,\overline{h}})$ is embedded into a family (2.1) of perturbed infimization problems, as in definition 2.1.

a) If ψ^λ is a multi-function which assigns, to each triple $(G,F,\overline{h})$ as in (2.1), a non-empty set $W^\lambda=\psi^\lambda(G,F,\overline{h})\subseteq\overline{R}^F$, satisfying (1.7) and (2.14), then, choosing

$$\psi^{\pi\lambda}=(\psi^\lambda)^\pi \tag{2.20}$$

and taking $W^{\pi\lambda}$, $\mu^{\pi\lambda}$ and $\beta^{\pi\lambda}$ as in definition 2.2, we have (2.5) and

$$(W^\lambda)^\pi=W^{\pi\lambda}, \quad (\mu^\lambda)^\pi=\mu^{\pi\lambda}, \quad (\beta^\lambda)^\pi=\beta^{\pi\lambda}. \tag{2.21}$$

Thus, the perturbational dual problem to $(P_{G,F,\overline{h}})$, with respect to (X,x_o,p), corresponding to an unperturbational Lagrangian dual problem to (1.2) (satisfying (2.14)), is a perturbational Lagrangian dual problem to (1.2), with respect to (X,x_o,p).

b) In particular, if F and X are linear spaces and $x_o=0$, and if $W^\lambda=\psi^\lambda(G,F,\overline{h})=F^\#$, $W^{\pi\lambda}=\psi^{\pi\lambda}(G,F,\overline{h},X,0,p)=X^\#$ (as in remarks 1.3 b) and

2.1 a)), then we have (1.7), (2.14) and (2.20), whence also (2.21).

Proof. By definition 2.4 (with $\psi=\psi^{\lambda}$, $W=W^{\lambda}$), (2.20) and definition 2.1, we have

$$(W^{\lambda})^{\pi}=(\psi^{\lambda})^{\pi}(G,F,\bar{h},X,x_o,p)=\psi^{\pi\lambda}(G,F,\bar{h},X,x_o,p)=W^{\pi\lambda}, \qquad (2.22)$$

whence, by (2.16) and (2.14) (with $\psi=\psi^{\lambda}$, $W=W^{\lambda}$), we obtain

$$(0,W^{\pi\lambda})=(0,(W^{\lambda})^{\pi})=(0,\psi^{\lambda}(\{x_o\},X,f))\subseteq\psi^{\lambda}((F,x_o),\ F\times X,p). \qquad (2.23)$$

Hence, since (1.7) is assumed to hold for all $G,F,\bar{h}$ and $W^{\lambda}=\psi^{\lambda}(G,F,\bar{h})$, applying it to $G,F,\bar{h}$ and W^{λ} replaced, respectively, by (F,x_o), $F\times X,p$ and $\psi^{\lambda}((F,x_o),\ F\times X,p)$, and taking into account that (by (0.1))

$$\inf\ (0,w)(F,x_o)=w(x_o) \qquad (w\in\bar{\bar{R}}^{X}), \qquad (2.24)$$

we obtain (2.5).

Furthermore, for each $w\in(W^{\lambda})^{\pi}=W^{\pi\lambda}$ we have, by (2.17), (1.8), (0.1) and (2.6),

$$(\mu^{\lambda})^{\pi}(w)=\varphi^{\lambda}_{(F,x_o),F\times X,p}(0,w)=$$

$$=\inf_{(y,x)\in F\times X}\{p(y,x)\dot{+}-(0,w)(y,x)\}\dot{+}\inf(0,w)(F,x_o)=$$

$$=\inf_{(y,x)\in F\times X}\{p(y,x)\dot{+}-w(x)\}\dot{+}w(x_o)=\mu^{\pi\lambda}(w). \qquad (2.25)$$

Finally, from (2.22) and (2.25), we obtain

$$(\beta^{\lambda})^{\pi}=\sup\ (\mu^{\lambda})^{\pi}((W^{\lambda})^{\pi})=\sup\ \mu^{\pi\lambda}(W^{\pi\lambda})=\beta^{\pi\lambda}.$$

b) Under the assumptions of b), there holds, clearly (1.7). Also, by (0.1), we have

$$(0,\psi^{\lambda}(\{0\},X,f))=(0,X^{\#})\subseteq(F\times X)^{\#}=\psi^{\lambda}((F,0),F\times X,p), \qquad (2.26)$$

that is, (2.14). Finally, from definition 2.1 and (2.16), we obtain

$$\psi^{\pi\lambda}(G,F,\bar{h},X,0,p)=W^{\pi\lambda}=X^{\#}=\psi^{\lambda}(\{0\},X,f)=$$

$$=(\psi^{\lambda})^{\pi}(G,F,\bar{h},X,0,p),$$

i.e., (2.20), whence also (2.21) (by part a) above).

Remark 2.5. In the situation of theorem 2.1 b), if $X=F$ and $x_o=0$, then for $p=p_{G,F,\bar{h}}: F\times F\to\bar{R}$ defined by

$$p(y,x)=\begin{cases}\bar{h}(y) & \text{if } y\in G+x\\ +\infty & \text{if } y\in F\setminus(G+x),\end{cases} \tag{2.27}$$

we have (see e.g. [5], corollary 3.8)

$$\psi^{\pi\lambda}=\psi^{\lambda},\ W^{\pi\lambda}=W^{\lambda},\ \mu^{\pi\lambda}=\mu^{\lambda},\ \beta^{\pi\lambda}=\beta^{\lambda}, \tag{2.28}$$

so the perturbational Lagrangian dual problem (with respect to the above (X,x_o,p)) coincides with the unperturbational Lagrangian dual problem of definition 1.2. More generally, often one can regard a (not necessarily Lagrangian) perturbational dual problem, with respect to some concrete perturbation triple (X,x_o,p) (such that after the computations, p does not occur explicitly), as an unperturbational dual problem. However, it is also important to interpret (2.28) in the opposite direction, i.e., as the observation that, for linear spaces, an unperturbational Lagrangian dual problem is a particular perturbational Lagrangian dual problem (i.e., with respect to a suitable perturbation triple (X,x_o,p)). Such a remark remains also valid for other unperturbational dual problems (e.g., for surrogate duality [5]) and for the dual problems (mentioned above) defined initially without the aid of any perturbation and thus appearing as unperturbational ones, but being essentially perturbational for suitable p, which does not appear explicitly (e.g. as in the case of linear programming duality); actually, some natural dual problems of this latter type have been ____ the main models for introducing perturbational dual problems (see e.g. [3]).

Theorem 2.2. Assume that the primal infimization problem $(P_{G,F,\bar{h}})$ is embedded into a family (2.1) of perturbed infimization problems, as in defintion 2.1.

a) If ψ^{σ} is a multi-function which assigns, to each triple $(G,F,\bar{h})$ as in (2.1), a non-empty set $W^{\sigma}=\psi^{\sigma}(G,F,\bar{h})\subseteq\bar{R}^{F}$, satisfying (2.14), and Δ is a multi-function which assigns to each triple (G,F,w) $(w\in W^{\sigma})$, a set $\Delta_{G,F,w}\subseteq F$, not depending on $\bar{h}$, then, choosing

$$\psi^{\pi\sigma}=(\psi^{\sigma})^{\pi}, \tag{2.29}$$

$$\tilde{\Delta}_{(F,x_o),F\times X,w}=\Delta_{(F,x_o),F\times X,(0,w)}\ (\subseteq F\times X)\qquad (w\in(W^{\sigma})^{\pi}), \tag{2.30}$$

and taking $W^{\pi\sigma}$, $\mu^{\pi\sigma}_{\tilde{\Delta}}$ and $\beta^{\pi\sigma}_{\tilde{\Delta}}$ as in definition 2.2, we have

$$(W^{\sigma})^{\pi}=W^{\pi\sigma}, \quad (\mu^{\sigma}_{\Delta})^{\pi}=\mu^{\pi\sigma}_{\tilde{\Delta}}, \quad (\beta^{\sigma}_{\Delta})^{\pi}=\beta^{\pi\sigma}_{\tilde{\Delta}} \quad . \tag{2.31}$$

Thus, the perturbational dual problem to $(P_{G,F,\bar{h}})$, with respect to (X,x_o,p), corresponding to an unperturbational surrogate dual problem to (1.2) (satisfying (2.14)), is a perturbational surrogate dual problem to (1.2), with respect to (X,x_o,p).

b) In particular, if F and X are linear spaces and $x_o=0$, and if $W^{\sigma}=\psi^{\sigma}(G,F,\bar{h})=F^{\#}$, $W^{\pi\sigma}=\psi^{\pi\sigma}(G,F,\bar{h},X,0,p)=X^{\#}$, then we have (2.14) and (2.29), whence also (2.31).

Proof. a) By definition 2.4 (with $\psi=\psi^{\sigma}$, $W=W^{\sigma}$), (2.29) and definition 2.2, we have

$$(W^{\sigma})^{\pi}=(\psi^{\sigma})^{\pi}(G,F,\bar{h},X,x_o,p)=\psi^{\pi\sigma}(G,F,\bar{h},X,x_o,p)=W^{\pi\sigma}. \tag{2.32}$$

Furthermore, for each $w\in(W^{\sigma})^{\pi}=W^{\pi\sigma}$ we have, by (2.17), (1.10), (2.30) and (2.10),

$$\begin{aligned}(\mu^{\sigma}_{\Delta})^{\pi}(w)&=\varphi^{\sigma}_{(F,x_o),F\times X,p,\Delta}(0,w)=\inf p(\Delta_{(F,x_o),F\times X,(0,w)})=\\ &=\inf p(\tilde{\Delta}_{(F,x_o),F\times X,w})=\mu^{\pi\sigma}_{\tilde{\Delta}}(w) \quad .\end{aligned} \tag{2.33}$$

Finally, from (2.32) and (2.33) we obtain

$$(\beta^{\sigma}_{\Delta})^{\pi}=\sup(\mu^{\sigma}_{\Delta})^{\pi}((W^{\sigma})^{\pi})=\sup\ \mu^{\pi\sigma}_{\tilde{\Delta}}(W^{\pi\sigma})=\beta^{\pi\sigma}_{\tilde{\Delta}} \quad .$$

b) The proof of b) is similar to that of theorem 2.1 b) (replacing λ by σ).

Theorem 2.3. Assume that the primal infimization problem $(P_{G,F,\bar{h}})$ is embedded into a family (2.1) of perturbed infimization problems, as in definition 2.1. Then, the multi-function ψ^{τ} defined by (1.12) satisfies (2.14), and for $\psi^{\pi\tau}$, $W^{\pi\tau}$, $\mu^{\pi\tau}$ and $\beta^{\pi\tau}$ as in definition 2.3, we have

$$(\psi^{\tau})^{\pi}=\psi^{\pi\tau}, \quad (W^{\tau})^{\pi}=W^{\pi\tau}, \quad (\mu^{\tau})^{\pi}=\mu^{\pi\tau}, \quad (\beta^{\tau})^{\pi}=\beta^{\pi\tau} \quad . \tag{2.34}$$

Thus, each perturbational τ-dual problem to $(P_{G,F,\bar{h}})$, with respect to (X,x_o,p), is nothing else than the perturbational dual problem to $(P_{G,F,\bar{h}})$, with respect to (X,x_o,p), corresponding to the unperturbational τ-dual problem to $(P_{G,F,\bar{h}})$.

Proof. For any $w\in\overline{\overline{R}}^X$ we have, by (1.12), (2.2) and (0.1), the equivalences

$$w\in\psi^{\tau}(\{x_o\},X,f) \Longleftrightarrow w\leq f \Longleftrightarrow w(x)\leq \inf_{y\in F} p(y,x) \quad (x\in X)$$

$$\Longleftrightarrow (0,w)(y,x)\leq p(y,x) \quad (y\in F, x\in X) \Longleftrightarrow (0,w)\leq p$$

$$\Longleftrightarrow (0,w)\in\psi^{\tau}((F,x_o),F\times X,p),$$

so ψ^{τ} satisfies (2.14). Also, by definition 2.4 (with $\psi=\psi^{\tau}$, $W=W^{\tau}$), (2.16), (1.12) and (2.12), we obtain

$$(W^{\tau})^{\pi}=(\psi^{\tau})^{\pi}(G,F,\overline{h},X,x_o,p)=\psi^{\tau}(\{x_o\},X,f)=\{w\in\overline{\overline{R}}^X \mid w\leq f\}=$$

$$=\{w\in\overline{\overline{R}}^X \mid w\leq p(y,\cdot) \quad (y\in F)\}=\psi^{\pi\tau}(G,F,\overline{h},X,x_o,p)=W^{\pi\tau} . \qquad (2.35)$$

Furthermore, for each $w\in(W^{\tau})^{\pi}=W^{\pi\tau}$ we have, by (2.17), (1.13), (0.1) and (2.13),

$$(\mu^{\tau})^{\pi}(w)=\varphi^{\tau}_{(F,x_o),F\times X,p}(0,w)=$$

$$=\inf\ (0,w)(F,x_o)=w(x_o)=\mu^{\pi\tau}(w) . \qquad (2.36)$$

Finally, from (2.35) and (2.36), we obtain

$$(\beta^{\tau})^{\pi}=\sup\ (\mu^{\tau})^{\pi}((W^{\tau})^{\pi})=\sup\ \mu^{\pi\tau}(W^{\pi\tau})=\beta^{\pi\tau} .$$

Remark 2.6. Other dual problems, e.g., of [5], can be treated similarly.

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A GENERAL DUALITY APPROACH TO MIN-MAX RESULTS IN COMBINATORIAL OPTIMIZATION, VIA COUPLING FUNCTIONS. II: FURTHER RESULTS

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We apply the duality approach of [4], via coupling functions, to some combinatorial min-max equalities which are not of the ("all-cardinality covering-packing") type studied in [4], namely, to min-max equalities for "B-colourings", "A-cover packings", "weighted B-packings" and "weighted A-covers" in incidence triples (A,B,ρ), non-bipartite matchings, flows in networks and matroid intersections.

§0. Introduction

In [4] we have proposed to study min-max equalities in combinatorial optimization, i.e., equalities of the form

$$\min h(G)=\max \mu(W), \tag{0.1}$$

where G and W are (non-empty) finite sets and $h:G\to R=(-\infty,+\infty)$, $\mu:W\to R$, in conjunction with the "Lagrangian duality equalities"

$$\min h(G)=\max_{w\in W}\min_{g\in G} \varphi(g,w), \tag{0.2}$$

where $\varphi:G\times W\to R$ is a function, called "coupling function", satisfying the "bounding" and "dual bounding" inequalities

$$h(g) \geq \varphi(g,w) \geq \mu(w) \qquad (g \in G,\ w \in W), \qquad (0.3)$$

and we have given some applications to "all-cardinality" min-max equalities (0.1) "of covering-packing type" (i.e., in which both h and μ are the cardinality function and the elements g of G and w of W are "coverings", and, respectively, "packings", in some sense). In order to give a unified framework for applications, we have introduced in [4], for any "incidence triple" (A,B,ϱ) (i.e., any triple consisting of two non-empty finite sets A, B and a binary "incidence" relation $\varrho \subseteq A \times B$), the concepts of an "A-cover" (for B) and a "B-packing" (for A); namely, a subset g of A is called an A-cover (for B), if for each $b \in B$ there exists $a \in g$ such that $a\varrho b$, and a subset w of B is called a B-packing (for A), if for each $a \in A$ there exists at most one $b \in w$ such that $a\varrho b$. In these terms, the min-max equalities "of covering-packing type" are the equalities (0.1) with

$$G = \text{the collection of all A-covers } g, \qquad (0.4)$$

$$W = \text{the collection of all B-packings } w. \qquad (0.5)$$

Also, in [4] we have introduced, for any incidence triple (A,B,ϱ), four coupling functions $\varphi_i : 2^A \times 2^B \rightarrow Z_+$ $(i=1,2,3,4)$, of which the most important one has been φ_3 (which we shall use also here), defined (on $2^A \times 2^B$, which is larger than $G \times W$, for reasons explained in [4]) by

$$\varphi_3(y,w) = |\{(a,b) \in y \times w \mid a\varrho b\}| \qquad (y \in 2^A,\ w \in 2^B), \qquad (0.6)$$

where, for any (finite) set M, 2^M denotes the collection of all subsets of M, and $|M|$ denotes the cardinality of M, and where Z_+ denotes the set of all non-negative integers.

In the present paper (announced also in [4], §0), we shall apply the duality approach of [4] to other min-max equalities (0.1), which are not of "all cardinality covering-packing type", introducing other coupling functions.

In §1 we shall introduce "B-colourings", for an arbitrary incidence triple (A,B,ϱ), and we shall study an all-cardinality min-max equality (0.1) for minimum cardinality B-colourings, which is not of covering-packing type (i.e., in which G and W are not those of (0.4), (0.5)). To this end, we shall construct a new incidence triple (A',B',ϱ') and sets G',W' with $G\subseteq G'$, $W\subseteq W'$, such that the initial equalities (0.1), (0.2), will become "equivalent" to an all-cardinality covering-packing type min-max equality and, respectively, to the Lagrangian duality equality for this new system, with the coupling function φ_3 of (0.6) (for A',B',ϱ')), to which we shall then apply the results of [4]. We shall also give some applications to König's edge colouring theorem in bipartite graphs.

In §2 we shall introduce "A-cover packings", for an arbitrary incidence triple (A,B,ϱ), and we shall study an all-cardinality min-max equality (0.1) for maximum cardinality A-cover packings, which is not of covering-packing type. To this end, we shall use a similar method to that of §1, which will be somewhat simpler, since we shall not need to change the set W. We shall also give some applications to a theorem of Gupta on edge-cover packings in bipartite graphs.

In §3 we shall consider "weighted B-packings", for an arbitrary incidence triple (A,B,ϱ), and we shall study a min-max equality (0.1) for maximum weight B-packings, jointly with the Lagrangian duality equality (0.2) with a suitable $\varphi=\varphi_5:(Z_+)^A\times 2^B\rightarrow Z_+$, where $(Z_+)^A$ denotes the collection of all functions $g:A\rightarrow Z_+$. We shall also give some applications to Egerváry's weighted matching theorem in bipartite graphs.

In §4 we shall consider "weighted A-covers", for an arbitrary incidence triple (A,B,ϱ), and we shall study a min-max equality (0.1) for minimum weight A-covers, jointly with the equality (0.2) with a suitable $\varphi=\varphi_6: 2^A \times (Z_+)^B \rightarrow Z_+$. We shall also give some applications to Egerváry's weighted covering theorem in bipartite graphs. Note that one can apply the results of §§1-4 to other incidence triples (A,B,ϱ) as well.

In §5 we shall study Edmonds' (non-bipartite) matching theorem (0.1), with suitable (G,h,W,μ), jointly with the Lagrangian duality equality (0.2) with $\varphi=\varphi_3$ of (0.6), where the incidence triple (A,B,ϱ) will be different from the one used in [4] for matchings in bipartite graphs.

In §6 we shall study the max flow-min cut theorem (0.1) in networks, with suitable (G,h,W,μ), jointly with the Lagrangian duality equality (0.2) with a coupling function $\varphi=\varphi_7: 2^U \times (R_+)^U \rightarrow R_+=[0,+\infty)$, where U denotes the collection of all arcs of the network, and where no incidence triple is used in the definition of φ_7.

Finally, in §7 we shall give similar results for Edmonds' matroid intersection theorem, with a coupling function $\varphi=\varphi_8: (2^S \times 2^S) \times 2^S \rightarrow Z_+$, ______ where S denotes the common ground set of the two matroids and where, again, no incidence triple is used in the definition of φ_8.

Let us mention now some complements to the notations and results of [4], which we shall need in the sequel.

For an incidence triple (A,B,ϱ), denoting, as in [4],

$$\varrho(a)=\{b\in B \mid a\varrho b\} \qquad (a\in A), \qquad (0.7)$$

$$\varrho^{-1}(b)=\{a\in A \mid a\varrho b\} \qquad (b\in B), \qquad (0.8)$$

let us note that $g\in 2^A$ is an A-cover (for B) if and only if

$$g\cap\varrho^{-1}(b)\neq\emptyset \qquad (b\in B); \qquad (0.9$$

in order to ensure the existence of A-covers, we shall assume, as [4], that

$$\varrho^{-1}(b)\neq\emptyset \qquad (b\in B). \qquad (0.10$$

Furthermore, note that $w\in 2^B$ is a B-packing (for A) if and only if

$$\varrho^{-1}(b_1)\cap\varrho^{-1}(b_2)=\emptyset \qquad (b_1,b_2\in w,\ b_1\neq b_2). \qquad (0.11)$$

Definition 0.1. Let (A,B,ϱ) be an incidence triple. For any $a\in$ we shall call the number $|\varrho(a)|$ the degree of a. Similarly, for any $b\in B$ we shall call $|\varrho^{-1}(b)|$ the degree of b.

For any subset M of a set S, we shall denote by χ_M the characteristic function (called also "incidence function", "incidence vector") of M, defined by

$$\chi_M(s)=\begin{cases}1 & \text{if } s\in M\\ 0 & \text{if } s\in S\setminus M.\end{cases} \qquad (0.12)$$

For any function $f:S\to R$, we shall denote by supp f the "support" of f, i.e. the set

$$\operatorname{supp} f=\{s\in S \mid f(s)\neq 0\}. \qquad (0.13)$$

For two functions $f_1,f_2:S\to R$, we shall write $f_1\leq f_2$ if $f_1(s)\leq f_2(s)$ for all $s\in S$.

The following theorem collects some results of [4] (we use here the term "optimal solution" instead of the term "optimal element" of [4]) in a slightly improved form:

Theorem 0.1. Let (A,B,ϱ) be an incidence triple and let G,W be as in (0.4), (0.5). Furthermore, assume that the min-max equality (0.1) holds, with $h(g)=|g|$ $(g\in G)$, $\mu(w)=|w|$ $(w\in W)$, i.e., that

$$\min_{g\in G}|g|=\max_{w\in W}|w|, \qquad (0.14)$$

and let $\varphi_3: G \times W \to R$ be the coupling function (0.6). Then

a) We have the Lagrangian duality equality

$$\min_{g \in G} |g| = \max_{w \in W} \min_{g \in G} \varphi_3(g,w). \tag{0.15}$$

b) For any $(g_o, w_o) \in G \times W$, the following statements are equivalent:

1°. $g_o \in G$ is an optimal solution of the (primal) minimization problem

$$\text{(P)} \qquad \min_{g \in G} |g|, \tag{0.16}$$

and $w_o \in W$ is an optimal solution of the (dual) maximization problem

$$\text{(Q)} \qquad \max_{w \in W} |w|. \tag{0.17}$$

2°. Each $a \in g_o$ is incident with exactly one $b \in w_o$ and for each $b \in w_o$ there exists exactly one $a \in w_o$ such that a and b are incident.

3°. The number of incidences between g_o and w_o is

$$\varphi_3(g_o, w_o) = |g_o| = |w_o|. \tag{0.18}$$

Proof. We have a) by [4], theorem 1.2 a).

b) The implication $1° \Rightarrow 2°$ holds by [4], theorem 2.2, and the implication $2° \Rightarrow 3°$ is obvious (see [4], remark 2.4 b)). Finally, the implication $3° \Rightarrow 1°$ (even $|g_o| = |w_o| \Rightarrow 1°$) holds by [4], theorem 1.2 d) (note also that the implication $1° \Rightarrow 3°$ has been given in [4], theorem 2.1).

Finally, for the simplicity of some applications, by a "graph" we shall mean, as in [4], a finite simple graph, without isolated vertices.

§1. B-colourings

Definition 1.1. Let (A, B, ρ) be an incidence triple. We shall call B-colouring an assignment of a colour to each $b \in B$, such that any $b_1, b_2 \in B$ with $b_1 \neq b_2$, $\rho^{-1}(b_1) \cap \rho^{-1}(b_2) \neq \emptyset$, have different colours.

Remark 1.1. The B-colourings can be identified with the partitions of B into B-packings. Indeed, by the characterization (0.11) of B-packings, for each colour c of a B-coloring, the set $\{b\in B \mid b$ has colour $c\}$ is a B-packing, so each B-colouring is a partition of B into B-packings; and, conversely, if for each B-packing σ belonging to a partition of B into B-packings, we assign the same colour c_σ to all $b\in\sigma$, with $c_{\sigma'}\neq c_{\sigma''}$ $(\sigma'\neq\sigma'')$, then we obtain a B-colouring.

We shall denote

$$G=\text{the collection of all partitions } g \text{ of } B \text{ into B-packings}, \quad (1.1)$$

$$W=\{\rho(a) \mid a\in A\}. \quad (1.2)$$

Remark 1.2. It may happen that $a_1\neq a_2$ and $\rho(a_1)=\rho(a_2)$, but this will cause no confusion in (1.2), since for each $w\in W$ we can fix (arbitrarily) an element $a\in A$ such that $w=\rho(a)$.

We shall consider the min-max equality

$$\min_{g\in G} |g| = \max_{\rho(a)\in W} |\rho(a)| . \quad (1.3)$$

This is an "all-cardinality" min-max equality (i.e., in which both the primal and the dual objective functions are the cardinality functions), but, in contrast with the situation of [4], the elements of the constraint sets G and W of (1.3) are not A-covers and B-packings. Therefore, in order to arrive at the situation of [4], we shall construct now a new incidence triple (A',B',ρ') and sets G' and W', preserving the extrema of the cardinality functions and the optimal solutions.

Let

$$A'=\text{the collection of all non-empty B-packings } a' \text{ (for A)}, \quad (1.4)$$

$$B'=B, \quad (1.5)$$

$$a'\rho' b \iff b\in a'. \quad (1.6)$$

Remark 1.3. a) We have $A'\subset 2^B$ and $a'\neq\emptyset$ $(a'\in A')$, $\bigcup_{a'\in A'} a'=B$, so

(B,A') is a hypergraph and, moreover, the incidence ϱ' of (1.6) is that of [4], example 2.3.

b) $g'\in 2^{A'}$ <u>is an</u> A'-<u>cover if and only if it is a covering of B by B-packings</u>. Hence, every $g\in G$ (of (1.1)) is an A'-cover and, conversely, every A'-cover g' generates, in the usual way, an element $g\in G$ such that $|g|\leq|g'|$ (if $g'=\{M_1,M_2,M_3,\ldots\}$, define g by eliminating from $\{M_1,M_2\setminus M_1,M_3\setminus(M_1\cup M_2),\ldots\}$ the empty sets). Thus, if

$$G'=\text{the collection of all } A'\text{-covers } g', \tag{1.7}$$

then $G\subseteq G'$ and

$$\min_{g\in G}|g| = \min_{g'\in G'}|g'|. \tag{1.8}$$

c) $w'\in 2^B$ <u>is a</u> B-<u>packing for</u> A' (<u>i.e., the relations</u> $a'\in A'$, $b_1,b_2\in w'$, $a'\varrho' b_1$, $a'\varrho' b_2$ <u>imply</u> $b_1=b_2$) <u>if and only if there exists</u> $a_o\in A$ <u>such that</u> $w'\subseteq\varrho(a_o)$. Hence, every $\varrho(a)\in W$ is a B-packing for A', and, conversely, every B-packing w' for A' is contained in some $\varrho(a_o)\in W$ (whence $|w'|\leq|\varrho(a_o)|$). Thus, if

$$W'=\text{the collection of all B-packings } w' \text{ for } A'= =\{w'\in 2^B \mid \exists a_o\in A,\ w'\subseteq\varrho(a_o)\}, \tag{1.9}$$

then $W\subseteq W'$ and

$$\max_{\varrho(a)\in W}|\varrho(a)| = \max_{w'\in W'}|w'|. \tag{1.10}$$

Let us also consider, as in [4], example 2.3 (using now ϱ' of (1.6)) the coupling function $\varphi_3: 2^{A'}\times 2^B\to R$ defined by

$$\varphi_3(g,w)=|\{(a',b)\in g\times w \mid b\in a'\}| \qquad (g\in 2^{A'},\ w\in 2^B). \tag{1.11}$$

<u>Proposition 1.1</u>. <u>For</u> G,W,G',W' <u>and</u> φ_3 <u>as above, we have</u>

$$\min_{g\in G}\varphi_3(g,w) = \min_{g'\in G'}\varphi_3(g',w) \qquad (w\in 2^B), \tag{1.12}$$

$$\max_{\rho(a)\in W}\ \min_{g\in G} \varphi_3(g,\rho(a)) = \max_{w'\in W'}\ \min_{g'\in G'} \varphi_3(g',w'). \tag{1.13}$$

Proof. By $G\subseteq G'$, we have

$$\min_{g\in G} \varphi_3(g,w) \geq \min_{g'\in G'} \varphi_3(g',w) \qquad (w\in 2^B). \tag{1.14}$$

On the other hand, for each $g'=\{a'_1, a'_2, a'_3, \ldots\}\in G'$, the partition $g\in G$ of remark 1.3 b) satisfies $a'_2\setminus a'_1 \subseteq a'_2$, $a'_3\setminus(a'_1\cup a'_2)\subseteq a'_3, \ldots$, whence

$$\varphi_3(g,w) = |\{(a',b)\in g\times w \mid b\in a'\}| \leq$$
$$\leq |\{(a',b)\in g'\times w \mid b\in a'\}| = \varphi_3(g',w) \qquad (w\in 2^B). \tag{1.15}$$

whence we obtain the opposite inequality to (1.14), and thus the equality (1.12).

Furthermore, by $W\subseteq W'$, we have

$$\max_{\rho(a)\in W}\ \min_{g'\in G'} \varphi_3(g',\rho(a)) \leq \max_{w'\in W'}\ \min_{g'\in G'} \varphi_3(g',w'). \tag{1.16}$$

On the other hand, by (1.9), each $w'\in W'$ is a subset of some $\rho(a_o)\in W$ (where $a_o\in A$), whence

$$\varphi_3(g',w') = |\{(a',b)\in g'\times w' \mid b\in a'\}| \leq$$
$$\leq |\{(a',b)\in g'\times \rho(a_o) \mid b\in a'\}| = \varphi_3(g',\rho(a_o)) \qquad (g'\in G'). \tag{1.17}$$

Thus, for each $w'\in W'$ there exists $\rho(a_o)\in W$ such that

$$\min_{g'\in G'} \varphi_3(g',w') \leq \min_{g'\in G'} \varphi_3(g',\rho(a_o)), \tag{1.18}$$

whence we obtain the opposite inequality to (1.16), and hence the equality

$$\max_{\rho(a)\in W}\ \min_{g'\in G'} \varphi_3(g',\rho(a)) = \max_{w'\in W'}\ \min_{g'\in G'} \varphi_3(g',w'), \tag{1.19}$$

which, together with (1.12) (for $w=\rho(a)\in W$), yields (1.13).

By the above, the min-max equality and the Lagrangian duality equality for G', W' of (1.7), (1.9), i.e., the equalities

$$\min_{g'\in G'} |g'| = \max_{w'\in W'} |w'|, \tag{1.20}$$

$$\min_{g'\in G'} |g'| = \max_{w'\in W'} \min_{g'\in G'} \varphi_3(g',w'), \tag{1.21}$$

become now, respectively, (1.3) and

$$\min_{g\in G} |g| = \max_{\rho(a)\in W} \min_{g\in G} \varphi_3(g, \rho(a)) . \tag{1.22}$$

Hence, since the bounding and dual bounding inequalities (0.3) hold for G', W', φ_3, and since $G\subseteq G'$, $W\subseteq W'$, from theorem 0.1 we obtain

Theorem 1.1. Assume that the min-max equality (1.3) holds and let φ_3 be the coupling function (1.11). Then

a) The Lagrangian duality equality (1.22) holds.

b) For any $(g_o, \rho(a_o))\in G\times W$, the following statements are equivalent:

1°. g_o is a minimum cardinality partition of B into B-packings and $a_o\in A$ is an element of maximum degree.

2°. Each B-packing $a'\in g_o$ contains exactly one element $b\in\rho(a_o)$ and each $b\in\rho(a_o)$ belongs to exactly one B-packing $a'\in g_o$.

3°. The number of incidences between g_o and $\rho(a_o)$ is

$$\varphi_3(g_o, \rho(a_o)) = |g_o| = |\rho(a_o)| . \tag{1.23}$$

Corollary 1.1. Assume that the min-max equality (1.3) holds. Then

a) Given any minimum cardinality partition $g_o=\{a'_1,\dots,a'_q\}$ of B into B-packings, from each B-packing $a'_i\in g_o$ one can select an element $b_i\in a'_i$, in such a way that $\{b_1,\dots,b_q\}=\rho(a_o)$, for some $a_o\in A$ of maximum degree.

b) Given any $a_o\in A$ of maximum degree, for each $b_i\in\rho(a_o)=\{b_1,\dots,b_q\}$ one can select a B-packing $a'_i\in A'$, containing b_i, in such a way that $g_o=\{a'_1,\dots,a'_q\}$ is a minimum cardinality partition of B into B-packings.

Proof. a) By [4], corollary 2.2 a), from each $a_i' \in g_o = \{a_1', \ldots a_q'\}$ $(\in G \subseteq G')$ one can select an element $b_i \in a_i'$, in such a way that $w' = \{b_1, \ldots, b_q\}$ is a maximum cardinality element of W'. But, by (1.9), every maximum cardinality element w' of W' is of the form $\varrho(a_o)$, for some $a_o \in A$ of maximum degree.

b) By [4], corollary 2.2 b), for each $b_i \in \varrho(a_o) = \{b_1, \ldots, b_q\}$ $(\in W \subseteq W')$ one can select a B-packing $c_i' \in A'$, containing b_i, in such a way that $g_o' = \{c_1', \ldots, c_q'\}$ is a minimum cardinality element of G'. Then, since $c_i' \in A'$ (i.e., c_i' is a B-packing for A), we have $|c_i' \cap \varrho(a_o)| \leq 1$, whence, since $b_i \in c_i' \cap \varrho(a_o)$ $(i=1, \ldots, q)$, we obtain

$$b_j \notin c_i' \qquad (j \neq i;\ j,\ i \in \{1, \ldots, q\}). \tag{1.24}$$

Hence, for $g_o = \{a_1', \ldots, a_q'\} \in G$, where $a_1' = c_1'$, $a_2' = c_2' \setminus c_1'$, $a_3' = c_3' \setminus (c_1' \cup c_2'), \ldots$ (see remark 1.3 b)), we have

$$b_i \in a_i' \neq \emptyset \qquad (i=1, \ldots, q). \tag{1.25}$$

Proposition 1.1 and theorem 1.1 a) can be strengthened as follows.

Theorem 1.2. For G, G' and φ_3 of (1.1), (1.7) and (1.11) (with A' of (1.4)), we have

$$\varphi_3(g,w) = |w| = \min_{g' \in G'} \varphi_3(g',w) \qquad (g \in G,\ w \in 2^B). \tag{1.26}$$

Proof. Let $g = \{a_1', \ldots, a_q'\}$ be any partition of B into subsets (not necessarily B-packings) and let $w \in 2^B$. Then, by (1.11), we have

$$\varphi_3(g,w) = |w \cap a_1'| + \ldots + |w \cap a_q'| = |w|.$$

Finally, since for each $b \in B$ we have $\{b\} \in A'$ (see [4], remark 2.2 b)), from [4], theorem 2.4, we obtain the second equality in (1.26).

Remark 1.4. The first equality of (1.26) shows that $\varphi_3|_{G \times W}$ is the "trivial" coupling function of [4], formula (1.37).

One can apply the above results to various incidence triples

(A,B,ρ). In order to give an example, let us recall (see e.g. [3], theorem 3)

"König's edge-colouring theorem". In a bipartite graph $\mathcal{G}=(V=V'\cup V'',E)$, the minimum number of colours needed to colour the edges of $\mathcal{G}$ so that no two intersecting edges have the same colour (or, equivalently, the minimum cardinality of a partition of E into matchings) is equal to the maximum degree of $\mathcal{G}$, i.e., to $\max_{v\in V} |\delta(v)|$, where $\delta(v)=\{e\in E | v\in e\}$ $(v\in V)$.

As has been observed in [4], §3, for the incidence triple (A,B,ρ) defined by

$$A=V,\quad B=E\subset 2^V\setminus\emptyset,\quad a\rho b \Leftrightarrow a\in b, \tag{1.27}$$

the B-packings coincide with the matchings of $\mathcal{G}$; also, clearly, $\rho(v)=\delta(v)$ $(v\in V)$. Hence, for G,W of (1.1), (1.2), the min-max equality (1.3) is now König's edge-colouring theorem, and thus, from theorem 1.1 and corollary 1.1 we obtain

Theorem 1.3. a) We have the Lagrangian duality equality

$$\min_{g\in G} |g| = \max_{\delta(v)\in W} \min_{g\in G} \varphi_3(g,\delta(v)) \;. \tag{1.28}$$

b) For any $(g_o,\delta(v_o))\in G\times W$, the following statements are equivalent:

1°. $g_o\in G$ is a minimum cardinality partition of E into matchings, and $v_o\in V$ is a vertex of maximum degree.

2°. Each matching $a'\in g_o$ contains exactly one edge $e\in\delta(v_o)$ and each edge $e\in\delta(v_o)$ belongs to exactly one matching $a'\in g_o$.

3°. The number of incidences between g_o and $\delta(v_o)$ is

$$\varphi_3(g_o,\delta(v_o)) = |g_o| = |\delta(v_o)| \;. \tag{1.29}$$

Corollary 1.2. a) Given any minimum cardinality partition

$g_o=\{a'_1,\ldots,a'_q\}$ of E into matchings, from each matching $a'_i \in g_o$ one can select an edge $e_i \in a'_i$, in such a way that $\{e_1,\ldots,e_q\}=\delta(v_o)$, for some $v_o \in V$ of maximum degree.

b) Given any $v_o \in V$ of maximum degree, for each $e_i \in \delta(v_o)=\{e_1,\ldots,e_q\}$ one can select a matching $a'_i \in A'$, containing e_i, in such a way that $g_o=\{a'_1,\ldots,a'_q\}$ is a minimum cardinality partition of E into matchings.

§2. A-cover packings

Definition 2.1. Let (A,B,ρ) be an incidence triple. We shall call A-cover packing a packing of A-covers (for B) into 2^A, i.e., a collection of pairwise disjoint A-covers.

We shall denote

$$G=\{\rho^{-1}(b)\mid b\in B\}, \tag{2.1}$$

$$W=\text{the collection of all A-cover packings } w. \tag{2.2}$$

Remark 2.1. Similarly to remark 1.2, the fact that the mapping $b\to\rho^{-1}(b)$ of B onto G is not one-to-one, will cause no confusion in (2.1), since for each $g\in G$ we can fix (arbitrarily) an element $b\in B$ such that $g=\rho^{-1}(b)$.

We shall consider the min-max equality

$$\min_{\rho^{-1}(b)\in G} |\rho^{-1}(b)| = \max_{w\in W} |w|. \tag{2.3}$$

This is an "all cardinality" min-max equality, but the elements of the constraint sets G and W are not A-covers and B-packings. Therefore, we shall construct now a new incidence triple (A',B',ρ') and a new set G', preserving the minimum of the cardinality function and the optimal solutions (we shall not need to change W).

Let

$$A'=A, \tag{2.4}$$

B'=the collection of all A-covers b' (for B), (2.5)

$a\varrho' b' \Longleftrightarrow a\in b'$. (2.6)

Remark 2.2. a) We have $B'\subset 2^A$ and $b'\neq\emptyset$ $(b'\in B')$, $\bigcup_{b'\in B'} b'=A$, so (A,B') is a hypergraph and, moreover, the incidence ϱ of (2.6) is that of [4], example 2.2.

b) $g'\in 2^A$ is an A-cover for B' (i.e., for each $b'\in B'$ there exists $a\in g'$ such that $a\in b'$) if and only if there exists $b_o\in B$ such that $\varrho^{-1}(b_o)\subseteq g'$ (indeed, if such an element b_o exists and if $b'\in B'$, then, by (0.9), $b'\cap\varrho^{-1}(b_o)\neq\emptyset$, and any $a\in b'\cap\varrho^{-1}(b_o)$ satisfies $a\in g'$, so g' is an A-cover for B'; on the other hand, if no such b_o exists, i.e. if for each $b\in B$ there exists an element $a_b\in\varrho^{-1}(b)\setminus g'$, then for $b'=\{a_b | b\in B\}\in 2^A$ we have $b'\in B'$ and $a_b\notin g'$ for all $a_b\in b'$, so g' is not an A-cover for B'). Hence, every $\varrho^{-1}(b)\in G$ is an A-cover for B', and, conversely, every A-cover g' for B' contains some $\varrho^{-1}(b_o)\in G$ (whence $|\varrho^{-1}(b_o)|\leq|g'|$). Thus, if

G'=the collection of all A-covers g' for B'=

$$=\{g'\in 2^A | \exists b_o\in B,\ \varrho^{-1}(b_o)\subseteq g'\}, \tag{2.7}$$

then $G\subseteq G'$ and

$$\min_{\varrho^{-1}(b)\in G} |\varrho^{-1}(b)| = \min_{g'\in G'} |g'|. \tag{2.8}$$

c) $w'\in 2^{B'}$ is a B'-packing (i.e., $b_1'\cap b_2'=\emptyset$ for all $b_1',b_2'\in w'$, $b_1'\neq b_2'$) if and only if it is an A-cover packing. Thus, by (2.2),

W=the collection of all B'-packings. (2.9)

Let us also consider, as in [4], example 2.2 (using now ϱ' of (2.6)), the coupling function $\varphi_3: 2^A\times 2^{B'}\to R$ defined by

$$\varphi_3(g,w)=|\{(a,b')\in g\times w \mid a\in b'\}| \qquad (g\in 2^A,\ w\in 2^{B'}). \tag{2.10}$$

Proposition 2.1. For G, G', B' and φ_3 as above, we have

$$\min_{\rho^{-1}(b)\in G} \varphi_3(\rho^{-1}(b),w)=\min_{g'\in G'} \varphi_3(g',w) \qquad (w\in 2^{B'}). \tag{2.11}$$

Proof. By $G\subseteq G'$, we have

$$\min_{\rho^{-1}(b)\in G} \varphi_3(\rho^{-1}(b),w)\geq \min_{g'\in G'} \varphi_3(g',w) \qquad (w\in 2^{B'}). \tag{2.12}$$

On the other hand, by (2.7), each $g'\in G'$ contains some $\rho^{-1}(b_o)\in G$ (where $b_o\in B$), whence

$$\varphi_3(g',w)=|\{(a,b')\in g'\times w \mid a\in b'\}|\geq$$
$$\geq |\{(a,b')\in \rho^{-1}(b_o)\times w \mid a\in b'\}|=\varphi_3(\rho^{-1}(b_o),w) \qquad (w\in 2^{B'}), \tag{2.13}$$

whence we obtain the opposite inequality to (2.12), and thus the equality (2.11).

By the above, the min-max equality and the Lagrangian duality equality for G', W of (2.7), (2.2), i.e., the equalities

$$\min_{g'\in G'} |g'|=\max_{w\in W} |w|, \tag{2.14}$$

$$\min_{g'\in G'} |g'|=\max_{w\in W} \min_{g'\in G'} \varphi_3(g',w), \tag{2.15}$$

become now, respectively, (2.3) and

$$\min_{g\in G} |g|=\max_{w\in W} \min_{\rho^{-1}(b)\in G} \varphi_3(\rho^{-1}(b),w). \tag{2.16}$$

Hence, since the bounding and dual bounding inequalities (0.3) hold for G', W, φ_3, and since $G\subseteq G'$, from theorem 0.1 we obtain

Theorem 2.1. Assume that the min-max equality (2.3) holds and let φ_3 be the coupling function (2.10). Then

a) The Lagrangian duality equality (2.16) holds.

b) For any $(\rho^{-1}(b_o),w_o)\in G\times W$, the following statements are equivalent:

1°. $b_o \in B$ *is an element of minimum degree, and* w_o *is a maximum cardinality A-cover packing.*

2°. *Each* $a \in \varrho^{-1}(b_o)$ *belongs to exactly one A-cover* $b' \in w_o$ *and each A-cover* $b' \in w_o$ *contains exactly one element* $a \in \varrho^{-1}(b_o)$.

3°. *The number of incidences between* $\varrho^{-1}(b_o)$ *and* w_o *is*

$$\varphi_3(\varrho^{-1}(b_o), w_o) = |\varrho^{-1}(b_o)| = |w_o|. \tag{2.17}$$

Corollary 2.1. Assume that the min-max equality (2.3) *holds. Then*

a) *Given any* $b_o \in B$ *of minimum degree, for each* $a_i \in \varrho^{-1}(b_o) = \{a_1, \ldots, a_q\}$ *one can select an A-cover* $b'_i \in B'$, *containing* a_i, *in such a way that* $w_o = \{b'_1, \ldots, b'_q\}$ *is a maximum cardinality A-cover packing.*

b) *Given any maximum cardinality A-cover packing* $w_o = \{b'_1, \ldots, b'_q\}$, *from each A-cover* $b'_i \in w_o$ *one can select an element* $a_i \in b'_i$, *in such a way that* $\{a_1, \ldots, a_q\} = \varrho^{-1}(b_o)$, *for some* $b_o \in B$ *of minimum degree.*

Proof. Part a) follows from $G \subseteq G'$ and [4], corollary 2.2 a).

b) By [4], corollary 2.2 b), from each $b'_i \in w_o = \{b'_1, \ldots, b'_q\}$ $(\subseteq W)$ one can select an element $a_i \in b'_i$, in such a way that $g' = \{a_1, \ldots, a_q\}$ is a minimum cardinality element of G'. But, by (2.7), every minimum cardinality element g' of G' is of the form $\varrho^{-1}(b_o)$, for some $b_o \in B$ of minimum degree.

Since by (0.10) we have $A \in B'$ (see [4], remark 2.2 a)), from [4], theorem 2.3, we obtain

Theorem 2.2. For W *and* φ_3 *of* (2.2), (2.10), *we have*

$$|y| = \max_{w \in W} \varphi_3(y, w) \qquad (y \in 2^A). \tag{2.18}$$

Remark 2.3. Corresponding to the first equality in (1.26), we have now

$$\varphi_3(y, w) = |y| \qquad (w = \{b'_1, \ldots, b'_q\} \in W,\ y \subseteq \bigcup_{i=1}^{q} b'_i). \tag{2.19}$$

Indeed, if $w=\{b'_1,\ldots,b'_q\}$ is any collection of pairwise disjoint subsets of A (not necessarily A-covers) and $y\subseteq\bigcup_{i=1}^{q} b'_i$, then, by (2.10),

$$\varphi_3(y,w)=|y\cap b'_1|+\ldots+|y\cap b'_q|=|y|.$$

One can apply the above results to various incidence triples (A,B,ϱ). In order to give an example, let us recall (see e.g. [3], corollary 3 a))

"Gupta's theorem". In a bipartite graph $\mathcal{G}=(V=V'\cup V'',E)$, the maximum cardinality of a collection of pairwise disjoint edge-covers is equal to the minimum degree of $\mathcal{G}$, i.e., to $\min_{v\in V}|\delta(v)|$.

As has been observed in [4], §4, for the incidence triple (A,B,ϱ) defined by

$$A=E\subset 2^V\setminus\emptyset,\quad B=V,\quad e\varrho v \Longleftrightarrow v\in e, \tag{2.20}$$

condition (0.10) is satisfied and the A-covers coincide with the edge-covers of $\mathcal{G}$; also, clearly, $\varrho^{-1}(v)=\delta(v)$ $(v\in V)$. Hence, for G, W of (2.1), (2.2), the min-max equality (2.3) is now Gupta's theorem, and thus, from theorem 2.1 and corollary 2.1 we obtain

Theorem 2.3. a) We have the Lagrangian duality equality

$$\min_{\delta(v)\in G}|\delta(v)|=\max_{w\in W}\ \min_{\delta(v)\in G}\ \varphi_3(\delta(v),w). \tag{2.21}$$

b) For any $(\delta(v_o),w_o)\in G\times W$, the following statements are equivalent:

1°. $v_o\in V$ is a vertex of minimum degree, and $w_o\in W$ is a maximum cardinality edge cover packing.

2°. Each edge $e\in\delta(v_o)$ belongs to exactly one edge cover $b'\in w_o$, and each edge cover $b'\in w_o$ contains exactly one edge $e\in\delta(v_o)$.

3°. The number of incidences between $\delta(v_o)$ and w_o is

$$\varphi_3(\delta(v_o),w_o)=|\delta(v_o)|=|w_o| . \tag{2.22}$$

Corollary 2.2. a) Given any $v_o \in V$ of minimum degree, for each $e_i \in \delta(v_o)=\{e_1,\ldots,e_q\}$ one can select an edge cover $b'_i \in B'$, containing e_i, in such a way that $w_o=\{b'_1,\ldots,b'_q\}$ is a maximum cardinality edge cover packing.

b) Given any maximum cardinality edge cover packing $w_o=\{b'_1,\ldots,b'_q\}$, from each edge cover $b'_i \in w_o$ one can select an edge $e_i \in b'_i$, in such a way that $\{e_1,\ldots,e_q\}=\delta(v_o)$, for some $v_o \in V$ of minimum degree.

§3. Weighted B-packings

Definition 3.1. Let (A,B,ϱ) be an incidence triple and $\nu \in Z^B$ a "weight function" on B. We shall say that $g \in (Z_+)^A$ is a generalized A-cover (for B) with respect to ν, or, briefly, a ν-A-cover (for B), if

$$\sum_{a \in \varrho^{-1}(b)} g(a) \geqslant \nu(b) \qquad (b \in B). \tag{3.1}$$

We shall denote

$$G_\nu = \text{the collection of all } \nu\text{-A-covers } g. \tag{3.2}$$

Remark 3.1. a) The set G_ν is infinite, since

$$g' \in G_\nu \Rightarrow g \in G_\nu \qquad (g \in (Z_+)^A,\ g \geqslant g'); \tag{3.3}$$

in particular, by (3.3) and $G_\nu \subset (Z_+)^A$, we have

$$ng_1+g_2 \in G \qquad (g_1 \in G_\nu, g_2 \in (Z_+)^A,\ n \in Z_+ \setminus \{0\}). \tag{3.4}$$

b) For the particular weight function

$$\nu_o(b)=1 \qquad (b \in B), \tag{3.5}$$

the characteristic function $g=\chi_M \in \{0,1\}^A$ of a subset M of A satisfies $\chi_M \in G_{\nu_o}$ if and only if M is an A-cover; indeed, by (0.12) and (0.8), condition (3.1) (for $\nu=\nu_o$) becomes

$$\sum_{a \in \varrho^{-1}(b) \cap M} 1 = |\{a \in M | a\varrho b\}| \geqslant 1 \qquad (b \in B).$$

c) For ν_0 of (3.5) and any $g\in(Z_+)^A$, we have

$$g\in G_{\nu_0} \iff \chi_{\text{supp } g}\in G_{\nu_0} \iff \text{supp } g \text{ is an } A\text{-cover}, \tag{3.6}$$

$$\chi_{\text{supp } g}\leq g. \tag{3.7}$$

Indeed, for any $g\in(Z_+)^A$ we have

$$\chi_{\text{supp } g}(a)=\begin{cases} 1 & \text{if } g(a)\geq 1 \\ 0 & \text{if } g(a)=0, \end{cases} \tag{3.8}$$

which implies (3.7) and which also shows that g satisfies (3.1) with $\nu=\nu_0$ if and only if so does $\chi_{\text{supp } g}$; hence, by b) above, we obtain (3.6).

d) For any weight function $\nu\in(Z_+\setminus\{0\})^B$, we have the implication

$$g\in G_\nu \implies \text{supp } g \text{ is an A-cover}. \tag{3.9}$$

Indeed, for any $\nu\in(Z_+\setminus\{0\})^B$ we have $\nu_0\leq\nu$, whence $G\subseteq G_{\nu_0}$, which, by (3.6), yields (3.9).

Now, for an arbitrary weight function $\nu\in Z^B$, we shall consider the primal optimization problem

$$\min h(G_\nu), \tag{3.10}$$

where G_ν is the set (3.2) and $h:G_\nu\to Z_+$ is defined by

$$h(g)=\sum_{a\in A} g(a) \qquad (g\in G_\nu), \tag{3.11}$$

the dual optimization problem

$$\max \nu(W), \tag{3.12}$$

where W is the set (0.5) and $\nu:W\to Z$ is defined by

$$\nu(w)=\sum_{b\in w}\nu(b) \qquad (w\in W), \tag{3.13}$$

the min-max equality

$$\min h(G_\nu)=\max \nu(W), \tag{3.14}$$

and the Lagrangian duality equality

$$\min h(G_\nu)=\max_{w\in W}\min_{g\in G_\nu} \varphi_5(g,w), \tag{3.15}$$

where $\varphi_5 : (Z_+)^A \times 2^B \to Z_+$ is the coupling function defined by

$$\varphi_5(g,w) = \sum_{b\in w} \sum_{a\in \varrho^{-1}(b)} g(a) \qquad (g\in(Z_+)^A,\ w\in 2^B). \tag{3.16}$$

Remark 3.2. a) By (3.11), (0.11), (3.16), (3.1) and (3.13), we have the bounding and dual bounding inequalities

$$h(g) = \sum_{a\in A} g(a) \geq \sum_{b\in w} \sum_{a\in \varrho^{-1}(b)} g(a) = \varphi_5(g,w) \geq$$

$$\geq \sum_{b\in w} \nu(b) = \nu(w) \qquad (g\in G_\nu,\ w\in W). \tag{3.17}$$

b) By (3.16), (0.12) and (0.6) we have

$$\varphi_5(\chi_{\mathrm{supp}\ g}, w) = \sum_{b\in w} \sum_{a\in \varrho^{-1}(b)} \chi_{\mathrm{supp}\ g}(a) =$$

$$= \left| \{(a,b)\in (\mathrm{supp}\ g)\times w \mid a\varrho b\} \right| = \varphi_3(\mathrm{supp}\ g, w) \qquad (g\in(Z_+)^A,\ w\in 2^B). \tag{3.18}$$

c) For $\nu = \nu_o$ (of (3.5)), ν of (3.13) becomes the cardinality function $\nu(w) = |w|$ $(w\in W)$, considered in [4].

Proposition 3.1. a) $g\in(Z_+)^A$ is a ν-A-cover (i.e., $g\in G_\nu$) if and only if

$$\varphi_5(g, \{b\}) \geq \nu(b) \qquad (b\in B). \tag{3.19}$$

b) $w\in 2^B$ is a B-packing (i.e., $w\in W$) if and only if

$$\varphi_5(\chi_{\{a\}}, w) \leq 1 \qquad (a\in A). \tag{3.20}$$

Proof. a) By (3.16), we have $\varphi_5(g,\{b\}) = \sum_{a\in \varrho^{-1}(b)} g(a)$, so (3.19) coincides with (3.1).

b) This follows from (3.18) (applied to $g = \chi_{\{a\}}$).

Remark 3.3. If $B = \{b_1,\ldots,b_m\}$, then, defining $u : (Z_+)^A \to (Z_+)^m$ by

$$u(g) = \{\varphi_5(g,\{b_1\}),\ldots,\varphi_5(g,\{b_m\})\} \qquad (g\in(Z_+)^A), \tag{3.21}$$

we can also write (3.19) in the form

$$u(g) \geq \{\nu(b_1),\ldots,\nu(b_m)\} \qquad (g\in(Z_+)^A); \tag{3.22}$$

in connection with (3.4), note also that

$$u(g_1+g_2)=u(g_1)+u(g_2) \qquad (g_1,g_2\in(Z_+)^A). \tag{3.23}$$

<u>Theorem 3.1. Assume that the min-max equality</u> (3.14) <u>holds and let</u> φ_5 <u>be the coupling function</u> (3.16). <u>Then</u>

a) <u>The Lagrangian duality equality</u> (3.15) <u>holds.</u>

b) <u>For any</u> $(g_o,w_o)\in G_\nu\times W$, <u>the following statements are equivalent:</u>

1°. g_o <u>and</u> w_o <u>are optimal solutions of</u> (3.10) <u>and</u> (3.12), <u>respectively.</u>

2°. <u>We have</u>

$$\operatorname{supp} g_o \subseteq \bigcup_{b\in w_o} \rho^{-1}(b), \tag{3.24}$$

$$\sum_{a\in\rho^{-1}(b)} g_o(a)=\nu(b) \qquad (b\in w_o). \tag{3.25}$$

<u>Proof.</u> a) This follows from [4], theorem 1.2 a).

b) By [4], theorem 1.2 d), $(g_o,w_o)\in G_\nu\times W$ satisfies 1° if and only if

$$h(g_o)=\varphi_5(g_o,w_o)=\nu(w_o), \tag{3.26}$$

that is, if and only if

$$\sum_{a\in A} g_o(a)=\sum_{b\in w_o}\ \sum_{a\in\rho^{-1}(b)} g_o(a)=\sum_{b\in w_o}\nu(b) \ . \tag{3.27}$$

Now, by (0.11) (for $w=w_o\in W$), the first equality in (3.27) holds if and only if we have (3.24). Finally, by (3.1) (for $g=g_o\in G_\nu$), the second equality in (3.27) holds if and only if we have (3.25).

<u>Theorem 3.2. Assume that the min-max equality</u> (3.14) <u>holds for</u> $\nu=\nu_o$ <u>of</u> (3.5). <u>Then, for any</u> $g_o\in G_{\nu_o}$, <u>the following statements are equivalent:</u>

1°. g_o <u>is an optimal solution of</u> (3.10) (<u>with</u> $\nu=\nu_o$).

2°. $g_o\in\{0,1\}^A$ <u>and</u> supp g_o <u>is a minimum cardinality</u> A-<u>cover</u>.

Proof. $1° \Rightarrow 2°$. If $g_o \in G_{\nu_o} \setminus \{0,1\}^A$, then, since $G_{\nu_o} \subset (Z_+)^A$, there exists $a_o \in A$ such that $g_o(a_o) \geq 2$. Then, for $g'_o \in (Z_+)^A$ defined by

$$g'_o(a) = \begin{cases} 1 & \text{if } a = a_o \\ g_o(a) & \text{if } a \in A \setminus \{a_o\}, \end{cases} \tag{3.28}$$

and for any $b \in B$ we have, by (3.1) (with $g = g_o$, $\nu = \nu_o$),

$$\sum_{a \in \varrho^{-1}(b)} g'_o(a) = \begin{cases} \sum_{a \in \varrho^{-1}(b) \setminus \{a_o\}} g_o(a) + g'_o(a_o) \geq 1 & \text{if } a_o \in \varrho^{-1}(b) \\ \sum_{a \in \varrho^{-1}(b)} g_o(a) \geq 1 & \text{if } a_o \notin \varrho^{-1}(b), \end{cases}$$

so $g'_o \in G_{\nu_o}$. Also, by $g_o(a_o) \geq 2$ and (3.28), we have $h(g'_o) = \sum_{a \in A} g'_o(a) <$

$< \sum_{a \in A} g_o(a) = h(g_o)$, so g_o is not an optimal solution of (3.10). Thus

$1° \Rightarrow g_o \in \{0,1\}^A$, whence $g_o = \chi_{\text{supp } g_o}$. Now, let $w_o \in W$ be any optimal solution of (3.12). Then, by $1°$ and (3.27), (3.5),

$$|\text{supp } g_o| = \sum_{a \in A} \chi_{\text{supp } g_o}(a) = \sum_{a \in A} g_o(a) = \sum_{b \in w_o} 1 = |w_o|,$$

whence, by (3.6) and [4], theorem 1.2 d), supp g_o is a minimum cardinality A-cover.

$2° \Rightarrow 1°$. If $2°$ holds, then, by $g_o = \chi_{\text{supp } g_o}$, (3.6) and (3.7),

$$\sum_{a \in A} g_o(a) = \sum_{a \in A} \chi_{\text{supp } g_o}(a) = |\text{supp } g_o| \leq |\text{supp } g| =$$
$$= \sum_{a \in A} \chi_{\text{supp } g}(a) \leq \sum_{a \in A} g(a) \qquad (g \in G_{\nu_o}).$$

One can apply the above results to various incidence triples (A,B,ϱ). In order to give an example, let us recall (see e.g. [3], p. 450)

"Egerváry's weighted matching theorem". Given a bipartite graph $\mathcal{G}=(V=V'\cup V'',E)$ and a weight function $\nu\in Z^E$, the maximum weight of a matching is equal to the minimum of $\sum_{v\in V} g(v)$, taken over all $g\in(Z_+)^V$ satisfying

$$g(v')+g(v'')\geqslant\nu(e) \qquad (e=(v',v'')\in E). \tag{3.29}$$

We can write this theorem in the form (3.14), by choosing

$$A=V,\ B=E\subset 2^V\setminus\emptyset,\ a\rho b \Longleftrightarrow a\in b, \tag{3.30}$$

G_ν as in (3.2), and W to be the collection of all matchings w; indeed, since $\rho^{-1}(e)=\{v',v''\}$ for all $e=(v',v'')\in E$, condition (3.1) becomes (3.29). Thus, since the min-max equality (3.14) is now Egerváry's weighted matching theorem, from theorem 3.1 we obtain

Theorem 3.3. a) We have the Lagrangian duality equality (3.15).

b) For any $(g_o,w_o)\in G_\nu\times W$, the following statements are equivalent:

1°. g_o and w_o are optimal solutions of (3.10) and (3.12), respectively.

2°. We have

$$\operatorname{supp} g_o \subseteq \bigcup_{(v',v'')=e\in w_o} \{v',v''\}, \tag{3.31}$$

$$g_o(v')+g_o(v'')=\nu(e) \qquad (e=(v',v'')\in w_o) . \tag{3.32}$$

The inclusion in (3.31) (and hence in (3.24)) may be strict, as shown by

Example 3.1. Let $V'=\{v_1'\}$, $V''=\{v_1''\}$, $E=\{e_1=(v_1',v_1'')\}$ (singletons), $\mathcal{G}=(V'\cup V'',E)$, $\nu(e_1)=\nu_o(e_1)=1$, $g_o(v_1')=1$, $g_o(v_1'')=0$, $w_o=\{e_1\}$. Then we have 1° and 2° of theorem 3.3 b), with strict inclusion in (3.31) (since $v_1''\notin\operatorname{supp} g_o$).

Corresponding to theorem 1.2, there holds now only the following result:

Theorem 3.4. *For* G_ν *and* φ_5 *of* (3.2) *and* (3.16), *with* (A,B,ϱ) *of* (3.30), *we have*

$$\nu(w)=\min_{g\in G_\nu} \varphi_5(g,w) \qquad (w\in W). \tag{3.33}$$

Proof. Let $w=\{e_1=(v_1',v_1''),\dots,e_p=(v_p',v_p'')\}\in W$. Then, since $\mathcal{G}=(V'\cup V'',E)$ is bipartite and since w is a matching, the vertices $v_1',v_1'',\dots,v_p',v_p''$ are all distinct. Hence, the system of p linear equations

$$g_o(v_i')+g_o(v_i'')=\nu(e_i) \qquad (i=1,\dots,p), \tag{3.34}$$

with 2p unknowns $g_o(v_1')$, $g_o(v_1''),\dots,g_o(v_p')$, $g_o(v_p'')$, is compatible, so there exists $g_o\in G_\nu$ satisfying (3.34). Then, by (3.2) and (3.29),

$$\nu(w)=\sum_{i=1}^{p}\nu(e_i)=\sum_{i=1}^{p}(g_o(v_i')+g_o(v_i''))=\varphi_5(g_o,w)\leq$$

$$\leq\sum_{i=1}^{p}(g(v_i')+g(v_i''))=\varphi_5(g,w) \qquad (g\in G_\nu), \tag{3.35}$$

which proves (3.33) (with the min attained for $g=g_o$).

In particular, from (3.35) for $\nu=\nu_o$ of (3.5) we obtain, by (3.18) and (3.6),

$$|w|=\varphi_3(\mathrm{supp}\, g_o,w)=\min_{g\in G_{\nu_o}}\varphi_3(\mathrm{supp}\, g,w) \qquad (w\in W). \tag{3.36}$$

Since by (3.6) and [4], theorem 3.2, we have (3.36) even for all $w\in 2^E$, it is natural to ask whether (3.33) above remains valid for all $w\in 2^E$. The answer is negative, as shown by

Example 3.2. Let $V'=\{v_1',v_2'\}$, $V''=\{v_1'',v_2''\}$, $E=\{e_1=(v_1',v_1''),\ e_2=(v_1',v_2''),\ e_3=(v_2',v_1''),\ e_4=(v_2',v_2'')\}$, $\mathcal{G}=(V'\cup V'',E)$ (=the complete bipartite graph $K_{2,2}$), $w=E\in 2^E\setminus W$, and let

$$\nu(e_1)=\nu(e_2)=\nu(e_3)=1, \qquad \nu(e_4)=2. \tag{3.37}$$

Then, the equations (3.34) become

$$g_o(v_1')+g_o(v_1'')=1, \tag{3.38}$$

$$g_o(v_1')+g_o(v_2'')=1, \tag{3.39}$$

$$g_o(v_2')+g_o(v_1'')=1, \tag{3.40}$$

$$g_o(v_2')+g_o(v_2'')=2, \tag{3.41}$$

i.e., a linear system of four equations with only four unknowns. But, by (3.38) and (3.39), we have $g_o(v_1'')-g_o(v_2'')=0$, while (3.40) and (3.41) yield $g_o(v_1'')-g_o(v_2'')=-1$, so the system of equations is incompatible. Hence, by (3.29) and (3.16),

$$\nu(w)=\sum_{i=1}^{4}\nu(e_i)=5<2\sum_{i=1}^{2}(g(v_i')+g(v_i''))=\varphi_5(g,w) \qquad (g\in G_\nu), \tag{3.42}$$

whence (although G_ν is infinite), since $\varphi_5(g,\ w)\in Z_+$ $(g\in G_\nu)$, we obtain the strict inequality

$$\nu(w)<\min_{g\in G}\ \varphi_5(g,w). \tag{3.43}$$

§4. Weighted A-covers

Definition 4.1. Let (A,B,ρ) be an incidence triple and $\nu\in Z^A$ a "weight function" on A. We shall say that $w\in(Z_+)^B$ is a generalized B-packing (for A) with respect to ν, or, briefly, a ν-B-packing (for A), if

$$(0\leqslant)\sum_{b\in\rho(a)}w(b)\leqslant\nu(a) \qquad (a\in A). \tag{4.1}$$

We shall denote

$$W_\nu=\text{the collection of all } \nu\text{-B-packings } w. \tag{4.2}$$

Remark 4.1. a) We have

$$w'\in W_\nu \Longrightarrow w\in W_\nu \qquad (w\in(Z_+)^A,\ w\leqslant w'); \tag{4.3}$$

however, in contrast with the situation of remark 3.1 a), the set W is finite (by (4.1) and since A and B are finite).

b) If a ν-B-packing exists (i.e., if $W_\nu\neq\emptyset$), then, by $W_\nu\subset(Z_+)^B$

and (4.1), we must have $\nu \geqslant 0$, whence

$$\nu \in (Z_+)^A ; \tag{4.4}$$

also, by (3.7) (applied to $w \in (Z_+)^B$) and (4.3), we have

$$w \in W_\nu \Leftrightarrow \chi_{\mathrm{supp}\, w} \in W_\nu . \tag{4.5}$$

c) For the particular weight function

$$\nu_1(a) = 1 \qquad (a \in A), \tag{4.6}$$

the characteristic function $w = \chi_N \in \{0,1\}^B$ of a subset N of B satisfies $\chi_N \in W_{\nu_1}$ <u>if and only if</u> N <u>is a</u> B-<u>packing</u>; indeed, by (0.12) and (0.7), condition (4.1) (for $\nu = \nu_1$) becomes

$$\sum_{b \in \rho(a) \cap N} 1 = |\{b \in N \mid a \rho b\}| \leqslant 1 \qquad (a \in A).$$

d) By (4.6) and (4.1), for each $w \in W_{\nu_1}$ we have $0 \leqslant w \leqslant 1$, whence, by $w \in (Z_+)^B$, we obtain $w \in \{0,1\}^B$. Thus,

$$W_{\nu_1} \subset \{0,1\}^B , \tag{4.7}$$

$$\sum_{b \in B} w(b) = \sum_{b \in B} \chi_{\mathrm{supp}\, w}(b) = |\mathrm{supp}\, w| \qquad (w \in W_{\nu_1}), \tag{4.8}$$

and, by c) above,

$$w \in W_{\nu_1} \Leftrightarrow \chi_{\mathrm{supp}\, w} \in W_{\nu_1} \Leftrightarrow \mathrm{supp}\, w \text{ is a B-packing.} \tag{4.9}$$

Note also that for any weight function $\nu \in (Z_+ \setminus \{0\})^A$ we have $\nu_1 \leqslant \nu$, whence $W_{\nu_1} \subset W_\nu$.

Now, for an arbitrary weight function $\nu \in (Z_+)^A$, we shall consider the primal optimization problem

$$\min \nu(G), \tag{4.10}$$

where G is the set (0.4) and $\nu : G \rightarrow Z_+$ is defined by

$$\nu(g) = \sum_{a \in g} \nu(a) \qquad (g \in G), \tag{4.11}$$

the dual optimization problem

$$\max \mu(W_\nu), \tag{4.12}$$

where W_ν is the set (4.2) and $\mu: W \to Z_+$ is defined by

$$\mu(w) = \sum_{b\in B} w(b) \qquad (w\in W_\nu), \tag{4.13}$$

the min-max equality

$$\min \nu(G) = \max \mu(W_\nu), \tag{4.14}$$

and the Lagrangian duality equality

$$\min \nu(G) = \max_{w\in W_\nu} \min_{g\in G} \varphi_6(g,w), \tag{4.15}$$

where $\varphi_6 : 2^A \times (Z_+)^B \to Z_+$ is the coupling function defined by

$$\varphi_6(g,w) = \sum_{a\in g} \sum_{b\in\varrho(a)} w(b) \qquad (g\in 2^A,\ w\in (Z_+)^B). \tag{4.16}$$

Remark 4.2. a) By (4.11), (4.1), (4.16), (0.9) and (4.13), we have the bounding and dual bounding inequalities

$$\nu(g) = \sum_{a\in g} \nu(a) \geqslant \sum_{a\in g} \sum_{b\in\varrho(a)} w(b) = \varphi_6(g,w) \geqslant$$
$$\geqslant \sum_{b\in B} w(b) = \mu(w) \qquad (g\in G,\ w\in W_\nu). \tag{4.17}$$

b) By (4.16), (0.12) and (0.6), we have

$$\varphi_6(g, \chi_{\mathrm{supp}\, w}) = \sum_{a\in g} \sum_{b\in\varrho(a)} \chi_{\mathrm{supp}\, w}(b) =$$

$$= \left|\{(a,b)\in g\times \mathrm{supp}\, w \mid a\varrho b\}\right| = \varphi_3(g, \mathrm{supp}\, w) \qquad (g\in 2^A,\ w\in(Z_+)^B). \tag{4.18}$$

c) For $\nu = \nu_1$ (of (4.6)), ν of (4.11) becomes the cardinality function $\nu(g) = |g|$ $(g\in G)$, considered in [4].

Proposition 4.1. a) $g\in 2^A$ is an A-cover (i.e., $g\in G$) if and only if

$$\varphi_6(g, \chi_{\{b\}}) \geqslant 1 \qquad (b\in B). \tag{4.19}$$

b) $w\in (Z_+)^B$ is a ν-B-packing (i.e., $w\in W_\nu$) if and only if

$$\varphi_6(\{a\}, w) \leqslant \nu(a) \qquad (a\in A). \tag{4.20}$$

Proof. a) This follows from (4.18) (applied to $w=\chi_{\{b\}}$).

b) By (4.16), we have $\varphi_6(\{a\},w)=\sum_{b\in\rho(a)} w(b)$, so (4.20) coincides with (4.1).

Remark 4.3. If $A=\{a_1,\ldots,a_n\}$, then, defining $u':(Z_+)^B\to(Z_+)^n$ by

$$u'(w)=\{\varphi_6(\{a_1\},w),\ldots,\varphi_6(\{a_n\},w)\} \qquad (w\in(Z_+)^B), \tag{4.21}$$

we can also write (4.20) in the form

$$u'(w)\leq\{\nu(a_1),\ldots,\nu(a_n)\} \qquad (w\in(Z_+)^B). \tag{4.22}$$

Theorem 4.1. Assume that the min-max equality (4.14) holds and let φ_6 be the coupling function (4.16). Then

a) The Lagrangian duality equality (4.15) holds.

b) For any $(g_o,w_o)\in G\times W_\nu$, the following statements are equivalent:

1°. g_o and w_o are optimal solutions of (4.10) and (4.12), respectively.

2°. We have

$$\operatorname{supp} w_o\subseteq\{b\in B \mid |g_o\cap\rho^{-1}(b)|=1\}, \tag{4.23}$$

$$\sum_{b\in\rho(a)} w_o(b)=\nu(a) \qquad (a\in g_o). \tag{4.24}$$

Proof. a) This follows from [4], theorem 1.2 a).

b) By [4], theorem 1.2 d), $g_o\in G$ and $w_o\in W_\nu$ are optimal solutions of (4.10) and (4.12) respectively, if and only if

$$\nu(g_o)=\varphi_6(g_o,w_o)=\mu(w_o), \tag{4.25}$$

that is, if and only if

$$\sum_{a\in g_o}\nu(a)=\sum_{a\in g_o}\sum_{b\in\rho(a)} w_o(b)=\sum_{b\in B} w_o(b) . \tag{4.26}$$

Now, by (4.1) (for $w=w_o$), the first equality in (4.26) holds if and only if we have (4.24).

Furthermore, since $g_o\in G$, we have

$$B=\bigcup_{a\in g_o}\rho(a), \tag{4.27}$$

and hence the second equality in (4.26) holds if and only if

$$\sum_{b\in B} |g_o \cap \varrho^{-1}(b)|\, w_o(b) = \sum_{b\in B} w_o(b), \qquad (4.28)$$

which, by (0.9) (for $g=g_o$) and $w_o \in (Z_+)^B$, is equivalent to (4.23).

Remark 4.4. Corresponding to theorem 3.2, we have now that for $\nu=\nu_1$ of (4.6) and for any $w_o \in W_{\nu_1}$, the following statements are equivalent:

1°. w_o is an optimal solution of (4.12) (with $\nu=\nu_1$);

2°. supp w_o is a maximum cardinality B-packing.

Indeed, this is an immediate consequence of (4.13) and (4.7)-(4.9).

One can apply the above results to various incidence triples (A,B,ϱ). In order to give an example, let us recall (see e.g. [3], p. 450)

"Egerváry's weighted covering theorem". Given a bipartite graph $\mathcal{G}=(V=V'\cup V'',E)$ and a weight function $\nu\in Z^E$, the minimum weight of an edge cover is equal to the maximum of $\sum_{e\in E} w(e)$, taken over all $w\in (Z_+)^V$ satisfying

$$(0\leq)\, w(v')+w(v'')\leq \nu(e) \qquad (e=(v',v'')\in E). \qquad (4.29)$$

We can write this theorem in the form (4.14), by choosing

$$A=E\subset 2^V\setminus\emptyset,\quad B=V,\quad a\varrho b \Leftrightarrow b\in a, \qquad (4.30)$$

W_ν as in (4.2) and G to be the collection of all edge covers g; indeed, since $\varrho(e)=\{v',v''\}$ for all $e=(v',v'')\in E$, condition (4.1) becomes (4.29). Thus, since the min-max equality (4.14) is now Egerváry's weighted covering theorem, from theorem 4.1 we obtain

Theorem 4.2. a) We have the Lagrangian duality equality (4.15).

b) For any $(g_o,w_o)\in G\times W_\nu$, the following statements are equivalent:

1°. g_o and w_o are optimal solutions of (4.10) and (4.12) respectively.

2°. We have

$$\text{supp } w_o \subseteq \{v\in V \mid \exists e\in g_o \text{ unique, such that } v\in e\}, \qquad (4.31)$$

$$w_o(v')+w_o(v'')=\nu(e) \qquad (e=(v',v'')\in g_o). \qquad (4.32)$$

The inclusion in (4.31) (and hence in (4.23)) may be strict, as shown by

Example 4.1. Let $\mathcal{G}=(V'\cup V'',E)$ and ν be as in example 3.1, $g_o=\{e_1\}$,

$w_o(v_1')=1$, $w_o(v_1'')=0$. Then we have 1° and 2° of theorem 4.2, with strict inclusion in (4.31) (since $v_1'' \notin \text{supp}\, w_o$).

Theorem 3.4 above and [4], theorem 4.2, suggest the question, whether for W_ν and φ_6 of (4.2) and (4.16), with (A,B,ϱ) of (4.30), we have

$$\nu(g)=\max_{w\in W_\nu} \varphi_6(g,w) \qquad (g\in G). \tag{4.33}$$

The answer is negative, as shown by

Example 4.2. Let $\mathcal{G}=(V'\cup V'',E)$ and ν be as in example 3.2, and let $g_o=E\in G$, so $\nu(g_o)=\sum_{e\in g_o}\nu(e)=5$. Then the system of four linear equations

$$w_o(v_1')+w_o(v_1'')=1 \ , \tag{4.34}$$

$$w_o(v_1')+w_o(v_2'')=1, \tag{4.35}$$

$$w_o(v_2')+w_o(v_1'')=1, \tag{4.36}$$

$$w_o(v_2')+w_o(v_2'')=2, \tag{4.37}$$

is incompatible (see example 3.2), whence, by (4.16) (for $g=g_o$) and (4.29),

$$\max_{w\in W_\nu} \varphi_6(g_o,w)=\max_{w\in W_\nu} \sum_{i=1}^{2}\sum_{j=1}^{2}\{w(v_i')+w(v_j'')\} < 5=\nu(g_o);$$

actually, a direct inspection shows that $\max_{w\in W_\nu}\varphi_6(g_o,w)=2$. Note also that, in this example, $\nu(g_o)=\min \nu(G)=\min_{g\in G}\sum_{e\in g}\nu(e)=2$ (attained for $g'=\{e_2, e_3\}\in G$).

§5. Non-bipartite matchings

Let us first recall some definitions (see e.g. [1], p. 241), for an arbitrary (not necessarily bipartite) graph $\mathcal{G}=(V,E)$. An odd-set $N\subseteq V$ (i.e., a set of vertices containing an odd number of elements) is said to cover an edge $e\in E$, if

$$\begin{cases} e \text{ has one vertex in } N, \text{ when } |N|=1, \\ e \text{ has both vertices in } N, \text{ when } |N|\geq 3. \end{cases} \tag{5.1}$$

A family $g=\{N_1,\dots,N_q\}$ of odd-sets of vertices is said to be an odd-set cover, if each $e\in E$ is covered by at least one $N_i\in g$.

The capacity of an odd-set $N\subseteq V$ is, by definition, the number

$$c(N)=\max(\frac{|N|-1}{2},1)=\begin{cases}1 & \text{if} \quad |N|=1 \\ \frac{|N|-1}{2} & \text{if} \quad |N|\geqslant 3,\end{cases} \tag{5.2}$$

and the capacity of a family $g=\{N_1,\ldots,N_q\}$ of odd-sets of vertices is defined by

$$c(g)=c(\{N_1,\ldots,N_q\})=\sum_{i=1}^{q} c(N_i). \tag{5.3}$$

Now we can recall (see e.g. [1], theorem 7.1)

"Edmonds' matching theorem". For any graph $\mathcal{G}=(V,E)$, the maximum cardinality of a matching is equal to the minimum capactiy of an odd set cover.

We can write this theorem in the form of a min-max equality

$$\min c(G)=\max_{w\in W} |w|, \tag{5.4}$$

where

$$G=\text{the collection of all odd-set covers } g, \tag{5.5}$$

$$W=\text{the collection of all matchings } w. \tag{5.6}$$

Let

$$A=\text{the collection of all odd-sets of vertices}, \tag{5.7}$$

$$B=E, \tag{5.8}$$

$$N\varrho e \Leftrightarrow N \text{ covers } e \qquad (N\in A,\ e\in E). \tag{5.9}$$

Remark 5.1. a) The incidence triple (A,B,ϱ) defined by (5.7)-(5.9) is an "extension" of the incidence triple (V,E,ϵ) considered in [4], §3, since $V\subset A$ (if we identify each $v\in V$ with the singleton $\{v\}\in A$), $B=E$ and $v\varrho e \Longleftrightarrow v\in e$ (where we identify each edge with the set of its two endpoints) Moreover, we have $B\subset 2^V\subset 2^A$ and $b\neq\emptyset$ $(b\in B)$, but $\bigcup_{b\in B} b\neq A$, so (A,B) is a set system (with ground set A), which is not a hypergraph; also, the incidence ϱ of (5.9) does not coincide with the incidence ϵ used (for a hypergraph (A,B)) in [4], example 2.2.

b) Clearly, $g\in 2^A$ is an A-cover if and only if it is an odd-set cover, so the set G of (5.5) coincides with the set of all A-covers.

c) $w\in 2^B$ is a B-packing if and only if it is either the empty set $\emptyset$, or a singleton $\{b\}$, where $b\in B$. Indeed, the "if" part has been observed in [4], remark 2.2 b), and the "only if" part is obvious when $|V|\leqslant 2$. Finally, if $|V|\geqslant 3$, then no edge set $w\in 2^B$ with $|w|\geqslant 2$ is a B-packing (for A), since for the odd-set $N=V$ (when $|V|$ is odd), respectively, $N=V\setminus\{v_o\}$, where $v_o\in V$ is arbitrary (when $|V|$ is even), and for any $e_1,e_2\in w$, $e_1\neq e_2$, we have $N\varrho e_i$ $(i=1,2)$. Thus, although every

B-packing is a matching, the set W of (5.6) is considerably larger than the set of all B-packings.

Let us also consider the Lagrangian duality equality

$$\min c(G)=\max_{w\in W}\min_{g\in G}\varphi_3(g,w), \tag{5.10}$$

where, by (0.6) (for the incidence triple (A,B,ρ) of (5.7)-(5.9)),

$$\varphi_3(g,w)=|\{(N,e)\in g\times w \mid N\rho e\}| \qquad (g\in 2^A, w\in 2^E). \tag{5.11}$$

Remark 5.2. We have the bounding and dual bounding inequalities

$$c(g)\geq\varphi_3(g,w)\geq|w| \qquad (g\in G,\ w\in W). \tag{5.12}$$

Indeed, by (5.11), (5.9), (5.1) and (5.2), for any odd-set $N\subseteq V$ and any matching $w\in W$ we have

$$\varphi_3(N,w)=|\{e\in w \mid N \text{ covers } e\}|\leq c(N), \tag{5.13}$$

whence, by (5.11) and (5.3),

$$\varphi_3(g,w)=\sum_{N_i\in g}\varphi_3(N_i,w)\leq\sum_{N_i\in g}c(N_i)=c(g) \quad (g=\{N_1,\dots,N_q\}\in 2^A,\ w\in W), \tag{5.14}$$

and, on the other hand, by the definition of odd-set covers,

$$|w|\leq\varphi_3(g,\ w) \qquad (g\in G,\ w\in 2^E). \tag{5.15}$$

Theorem 5.1. a) We have the Lagrangian duality equality (5.10).

b) For any $(g_o,w_o)\in G\times W$, the following statements are equivalent:

1°. $g_o=\{N_1,\dots,N_q\}\in G$ is an optimal solution of

$$\min c(G), \tag{5.16}$$

i.e., a minimum capacity odd-set cover, and $w_o\in W$ is a maximum cardinality matching.

2°. For each $i\in\{1,\dots,q\}$ we have

$$|\{e\in w_o \mid N_i\in e\}|=1, \quad \text{if} \quad |N_i|=1, \tag{5.17}$$

$$|\{e\in w_o \mid N_i\ni e\}|=\frac{|N_i|-1}{2}, \text{ if } |N_i|\geq 3, \tag{5.18}$$

and there holds

$$|w_o|=|\{i \mid N_i\in g_o,\ |N_i|=1\}|+\sum_{N_i\in g_o,\ |N_i|\geq 3}\frac{|N_i|-1}{2}. \tag{5.19}$$

Proof. a) This follows from [4], theorem 1.2 a), since the min-max equality (5.4) is now Edmonds' matching theorem.

b) By [4], theorem 1.2 d), $(g_o,w_o)\in G\times W$ satisfies 1° if and only if

$$c(g_o)=\varphi_3(g_o,w_o)=|w_o|. \tag{5.20}$$

Now, by (5.13) and (5.14), the first equality in (5.20) holds for $g_o=\{N_1,\dots,N_q\}\in G$ if and only if

$$c(N_i)=\varphi_3(N_i,w_o) \qquad (i=1,\dots,q), \qquad (5.21)$$

which, by (5.2), (5.13) and (5.1), is equivalent to (5.17), (5.18). Finally, by (5.3), (5.2), the equality $|w_o|=c(g_o)$ in (5.20) holds if and only if we have (5.19).

Theorem 5.2. For G and φ_3 of (5.5), (5.11), we have

$$|w|=\min_{g\in G}\varphi_3(g,w) \qquad (w\in 2^E). \qquad (5.22)$$

Proof. If $|V|=2n-1$ for some $n\geq 1$, then for $g_o=\{V\}$ we have $g_o\in G$ and

$$\varphi_3(g_o,w)=|w| \qquad (w\in 2^E), \qquad (5.23)$$

which, together with (5.15), yields (5.22). On the other hand, if $|V|=2n$ for some $n\geq 1$, say $V=\{v_1,\dots,v_{2n}\}$, let

$$N_1=\{v_1,\dots,v_{2n-1}\},\quad N_2=\{v_{2n}\}, \qquad (5.24)$$

and, for any $w\in 2^E\setminus\emptyset$, let

$$w_1=\{e\in w|N_1\varrho e\}, \qquad w_2=\{e\in w|N_2\varrho e\}, \qquad (5.25)$$

so $w=w_1\cup w_2$. If $n=1$, then $N_1=\{v_1\}$, $N_2=\{v_2\}$, and, choosing $g_o=\{N_1\}$, we have $g_o\in G$ and (5.23) (since E consists of the single edge $e=(v_1,v_2)$), which, together with (5.15), yields (5.22). Finally, if $n\geq 2$, so $|N_1|\geq 3$, then for any $w\in 2^E\setminus\emptyset$ we have, by (5.25), (5.9) and (5.1),

$$w_1=\{e\in w|\text{e has both vertices in } N_1\}, \qquad (5.26)$$

$$w_2=\{e\in w|\text{e has one vertex in } N_1 \text{ and one in } N_2\}, \qquad (5.27)$$

whence $w_1\cap w_2=\emptyset$, $|w|=|w_1|+|w_2|$. Hence, choosing

$$g_o=\{N_1,N_2\}, \qquad (5.28)$$

we obtain $g_o\in G$ and

$$\varphi_3(g_o,w)=\varphi_3(g_o,w_1)+\varphi_3(g_o,w_2)=|w_1|+|w_2|=|w|,$$

which, together with (5.15), yields (5.22).

Remark 5.3. The "universal" (i.e., independent on w) element $g_o\in G$ of (5.28) need not be an optimal solution of problem (5.16), as shown e.g. by $\mathcal{G}$=the union of two disjoint triangles, since then $c(g_o)=3>2=$ $=\min c(G)$.

§6. Flows in networks

Let us first recall some definitions (see e.g. [1], [5], [2]). A network is a directed graph $\mathcal{D}=(V,U)$, together with a function $c:U\to R_+$

(c(u) is called "the capacity of u", for each arc $u \in U$), a "source" $r \in V$ and a "sink" $s \in V$. A <u>flow</u> in a network $\mathcal{N}=(\mathcal{D}=(V,U),c,r,s)$ is a function $w:U \to R_+$ (w(u) is called "the flow in u", for each arc $u \in U$), such that

$$(0 \leq) w(u) \leq c(u) \qquad (u \in U), \tag{6.1}$$

$$w(u)=0 \qquad (u \in \delta^-(r) \cup \delta^+(s)), \tag{6.2}$$

$$\sum_{u \in \delta^+(v)} w(u) = \sum_{u \in \delta^-(v)} w(u) \qquad (v \in V \setminus \{r,s\}), \tag{6.3}$$

where $\delta^+(v)$ (respectively, $\delta^-(v)$) denotes the set of all arcs leaving (respectively, having as "positive" endpoint) the vertex v.

The <u>value</u> of a function $w \in (R_+)^U$ is the number

$$\mu(w) = \sum_{u \in \delta^+(r)} w(u) \ . \tag{6.4}$$

A <u>cut in a network</u> $\mathcal{N}=(\mathcal{D}=(V,U), c,r,s)$ is an "r-s-cut" of the directed graph $\mathcal{D}=(V,U)$, in the sense used in [4], §7, i.e., a set of arcs $g=\overrightarrow{P,V\setminus P}$ of the form

$$g=\overrightarrow{P,V\setminus P}=\{u=\overrightarrow{v_i v_j} \in U \mid v_i \in P,\ v_j \in V\setminus P\}, \tag{6.5}$$

where

$$r \in P \subset V, \quad s \in V \setminus P. \tag{6.6}$$

The <u>capacity of a set of arcs</u> $g \in 2^U$ is the number

$$c(g) = \sum_{u \in g} c(u) \ . \tag{6.7}$$

Now we can recall (see e.g. [1], p. 113, theorem 2.3) the

<u>"Max flow-min cut theorem"</u>. <u>In any network $\mathcal{N}=(\mathcal{D}=(V,U),c,r,s)$, the maximum value of a flow is equal to the minimum capacity of a cut.</u>

We can write this theorem in the form of a min-max equality

$$\min c(G) = \max \mu(W), \tag{6.8}$$

where

$$G = \text{the collection of all cuts } g=\overrightarrow{P,V\setminus P}, \tag{6.9}$$

$$W = \text{the collection of all flows } w. \tag{6.10}$$

Let us also consider the Lagrangian duality equality

$$\min c(G) = \max_{w \in W} \min_{g \in G} \varphi_7(g,w), \tag{6.11}$$

where $\varphi_7 : 2^U \times (R_+)^U \to R_+$ is the coupling function defined by

$$\varphi_7(g,w) = \sum_{u \in g} w(u) \qquad (g \in 2^U,\ w \in (R_+)^U). \tag{6.12}$$

<u>Remark 6.1</u>. a) One can also define an incidence relation ϱ between arcs $u \in U$ and functions $w \in (R_+)^U$, by

$$u \varrho w \Longleftrightarrow u \in \text{supp } w, \tag{6.13}$$

and (using [4], definition 2.1) one can extend it to an incidence relation between sets of arcs $g \in 2^U$ and functions $w \in (R_+)^U$, by

$$g \varrho w \Longleftrightarrow g \cap \text{supp } w \neq \emptyset. \tag{6.14}$$

Then, since for any $g \in 2^U$ and $w \in (R_+)^U$ we have

$$\varphi_7(g, \chi_{\text{supp } w}) = \sum_{u \in g} \chi_{\text{supp } w}(u) = |g \cap \text{supp } w|, \tag{6.15}$$

we obtain

$$g \varrho w \Longleftrightarrow \varphi_7(g, \chi_{\text{supp } w}) \neq 0 . \tag{6.16}$$

However, here we shall use only [4], §1 (not involving incidence).

b) c, φ_7 and μ satisfy the bounding and dual bounding inequalities. Indeed, by (6.7), (6.1) and (6.12), we have

$$c(g) = \sum_{u \in g} c(u) \geq \sum_{u \in g} w(u) = \varphi_7(g, w) \qquad (g \in 2^U, w \in W), \tag{6.17}$$

and, on the other hand, from $W \subset (R_+)^U$ and the well-known formula (see e.g. [5], p.326)

$$\sum_{u \in g = \overrightarrow{P, V \setminus P}} w(u) = \mu(w) + \sum_{u \in \overrightarrow{V \setminus P, P}} w(u) \qquad (g = \overrightarrow{P, V \setminus P} \in G, \ w \in W), \tag{6.18}$$

it follows that

$$\varphi_7(g, w) = \sum_{u \in g} w(u) \geq \mu(w) \qquad (g \in G, w \in W). \tag{6.19}$$

<u>Theorem 6.1</u>. a) <u>We have the Lagrangian duality equality</u> (6.11).

b) <u>For any</u> $(g_o, w_o) \in G \times W$, <u>the following statements are equivalent</u>:

1°. $g_o = \overrightarrow{P_o, V \setminus P_o} \in G$ <u>is an optimal solution of</u>

$$\min c(G), \tag{6.20}$$

<u>i.e. a minimum capactiy cut, and</u> $w_o \in W$ <u>is an optimal solution of</u>

$$\max \mu(W), \tag{6.21}$$

<u>i.e., a maximum value flow.</u>

2°. <u>We have</u>

$$w_o(u) = c(u) \qquad (u \in g_o), \tag{6.22}$$

$$w_o(u) = 0 \qquad (u \in \overrightarrow{V \setminus P_o, P_o}). \tag{6.23}$$

<u>Proof</u>. a) This follows from [4], theorem 1.2 a), since the min-max equality (6.8) is now the max flow-min cut theorem.

b) By [4], theorem 1.2 d), $(g_o, w_o) \in G \times W$ satisfies 1° if and only if

$$c(g_o) = \varphi_7(g_o, w_o) = \mu(w_o). \tag{6.24}$$

Now, by (6.12), (6.1) and (6.7), the first equality in (6.24) holds if and only if we have (6.22). Finally, by (6.12), (6.18) and $W\subset(R_+)^U$, the second equality in (6.24) holds if and only if we have (6.23).

Remark 6.2. a) The main part of theorem 6.1 b) is essentially known (see e.g. [5], p.326, corollary 2 a).

b) As in the corresponding results of the preceding section, involving functions, conditions (6.22), (6.23) can be also expressed in terms of supports, namely,

$$\text{supp } (c-w_o)\subseteq U\backslash g_o \ , \qquad (6.25)$$

$$\text{supp } w_o\subseteq U\backslash \overrightarrow{P_o, V\backslash P_o} \ . \qquad (6.26)$$

Theorem 6.2. For G, W and φ_7 of (6.9), (6.10) and (6.12), we have

$$|w| = \min_{g\in G} \varphi_7(g,w) \qquad (w\in W). \qquad (6.27)$$

Proof. Let

$$g_o = \overrightarrow{\{r\}, V\backslash\{r\}} = \delta^+(r) \ . \qquad (6.28)$$

Then $g_o\in G$ (since $r\in\{r\}$, $s\in V\backslash\{r\}$) and, by (6.12) and (6.4), we have

$$\varphi_7(g_o,w) = \sum_{u\in\delta^+(r)} w(u) = \mu(w) \qquad (w\in(R_+)^U), \qquad (6.29)$$

which, together with (6.19), yields (6.27).

Remark 6.3. a) The "universal" (i.e., independent on w) cut $g_o\in G$ of (6.28) need not be a minimum capacity cut.

b) Concerning the problem of finding other possible coupling functions $\varphi: G\times W\to R$, note that for the "naturally defined" coupling function

$$\varphi_7'(g,w) = \sum_{u\in g} w(u) - \sum_{u\in\overrightarrow{V\backslash P, P}} w(u) \qquad (g=\overrightarrow{P, V\backslash P}\in G,\ w\in(R_+)^U) \qquad (6.30)$$

we have, by (6.18),

$$\varphi_7'(g,w) = \mu(w) \qquad (g\in G,\ w\in W), \qquad (6.31)$$

so $\varphi_7'|_{G\times W}$ coincides with the "trivial" coupling function of [4], formula (1.37).

§7. Matroid intersections

Let us first recall (see e.g. [3], theorem 24)

"Edmonds' matroid intersection theorem". Let $\mathcal{M}_1=(S, \mathcal{J}_1)$ and $\mathcal{M}_2=(S, \mathcal{J}_2)$ be two matroids (on the same ground set S, with collections $\mathcal{J}_1$ and $\mathcal{J}_2$ of independent sets, respectively), with rank functions r_1

and r_2 respectively. Then the maximum cardinality of a set in $\mathcal{J}_1 \cap \mathcal{J}_2$ is equal to

$$\min_{S' \subseteq S} \{r_1(S') + r_2(S \setminus S')\} . \tag{7.1}$$

We can write this theorem in the form of a min-max equality

$$\min h(G) = \max_{w \in W} |w| , \tag{7.2}$$

where

$$G = \text{the collection of all pairs } g = (S', S \setminus S'), \text{ where } S' \subseteq S, \tag{7.3}$$

$$h(g) = r_1(S') + r_2(S'') \qquad (g = (S', S'') \in 2^S \times 2^S), \tag{7.4}$$

$$W = \mathcal{J}_1 \cap \mathcal{J}_2 . \tag{7.5}$$

Let us also consider the Lagrangian duality equality

$$\min h(G) = \max_{w \in W} \min_{g \in G} \varphi_8(g, w), \tag{7.6}$$

where $\varphi_8 : (2^S \times 2^S) \times 2^S \to Z_+$ is the coupling function defined by

$$\varphi_8(g, w) = |w \cap S'| + r_2(S'') \qquad (g = (S', S'') \in 2^S \times 2^S,\ w \in 2^S). \tag{7.7}$$

Remark 7.1. We have the bounding and dual bounding inequalities. Indeed, if $w \in \mathcal{J}_1$, then $w \cap S' \in \mathcal{J}_1$ $(S' \in 2^S)$, whence

$$h(g) = r_1(S') + r_2(S'') \geq r_1(w \cap S') + r_2(S'') = |w \cap S'| + r_2(S'') = \varphi_8(g, w)$$
$$(g = (S', S'') \in 2^S \times 2^S,\ w \in \mathcal{J}_1), \tag{7.8}$$

and, on the other hand,

$$\varphi_8(g, w) \geq |w \cap S'| + r_2(w \cap (S \setminus S')) = |w \cap S'| +$$
$$+ |w \cap (S \setminus S')| = |w| \qquad (g = (S', S \setminus S') \in G,\ w \in \mathcal{J}_2). \tag{7.9}$$

Theorem 7.1. a) We have the Lagrangian duality equality (7.6).

b) For any $(g_o, w_o) \in G \times W$, the following statements are equivalent:

1°. $g_o = (S_o, S \setminus S_o) \in G$ is an optimal solution of

$$\min h(G), \tag{7.10}$$

and $w_o \in W$ is a maximum cardinality set in $\mathcal{J}_1 \cap \mathcal{J}_2$.

2°. We have

$$r_1(S_o) = r_1(w_o \cap S_o), \tag{7.11}$$

$$r_2(S \setminus S_o) = r_2(w_o \cap (S \setminus S_o)). \tag{7.12}$$

Proof. a) This follows from [4], theorem 1.2 a), since the min-max equality (7.2) is now the matroid intersection theorem.

b) By [4], theorem 1.2 d), $(g_o, w_o) \in G \times W$ satisfies 1° if and only if

$$h(g_o)=\varphi_8(g_o,w_o)=|w_o|. \tag{7.13}$$

Now, by (7.5) and (7.8), the first equality in (7.13) holds if and only if we have (7.11). Finally, by (7.7), (7.5) and (7.9), the second equality in (7.13) holds if and only if we have (7.12).

Theorem 7.2. *For G and φ_8 of (7.3) and (7.7), we have*

$$|w|=\min_{g\in G}\varphi_8(g,w) \qquad (w\in\mathcal{J}_2). \tag{7.14}$$

Proof. Let

$$g_o=(S,\emptyset)\in 2^S\times 2^S. \tag{7.15}$$

Then $g_o\in G$ (since $\emptyset=S\setminus S$) and, by (7.7), we have

$$\varphi_8(g_o,w)=|w\cap S|+r_2(\emptyset)=|w| \qquad (w\in\mathcal{J}_2), \tag{7.16}$$

which, together with (7.9), yields (7.14).

Remark 7.2. a) The universal pair g_o of (7.15) need not be an optimal solution of problem (7.10).

b) Concerning the problem of finding other possible coupling functions $\varphi:G\times W\to R$, note that for the "naturally defined" coupling function

$$\varphi_8'(g,w)=r_1(w\cap S')+r_2(w\cap(S\setminus S')) \qquad (g=(S',S\setminus S')\in G,\ w\in 2^S), \tag{7.17}$$

we have, by (7.5), the equalities

$$\varphi_8'(g,w)=|w\cap S'|+|w\cap(S\setminus S')|=|w| \qquad (g=(S',S\setminus S')\in G,\ w\in W), \tag{7.18}$$

so φ_8' coincides with the "trivial" coupling function of [4], formula (1.37). Furthermore, for the ____________ coupling function

$$\varphi_8''(g,w)=r_1(w\cap S') \qquad (g=(S',S\setminus S')\in G,\ w\in 2^S), \tag{7.19}$$

we have $\varphi_8''(g,w)\le|w|$ $(g\in G,\ w\in 2^S)$, and φ_8'' does not satisfy the "dual bounding inequalities".

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