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by

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# ALMOST LINEAR SPACES WITH INNER PRODUCTS

by

G. GODINI

1. The normed almost linear spaces were first introduced in [1], where we showed that they constitute a natural framework for the theory of best simultaneous approximation in a normed linear space. An example of a normed almost linear space is the set of all nonempty, bounded and convex subsets of a real normed linear space (see Example 2.7 below). Weakening the axioms of a linear space but increasing the number of the axioms of a norm on a linear space in such a way that these spaces to generalize the normed linear spaces, in [1]-[5] we began to develop a theory for them, similar with that of normed linear spaces. Thus, we defined the dual space of a normed almost linear space (where the functionals are no longer linear but almost linear), the bounded, linear and almost linear operators between two such spaces and we obtained in this more general framework basic results from the theory of normed linear spaces. The main tool for the theory of normed almost linear spaces was given in [4] (see Theorem 2.2 below) where we proved that any normed almost linear space can be "embedded" in a normed linear space (here the embedding mapping is not one-to-one in general). This result permits us to use the techniques of normed linear spaces to prove certain results in a normed almost linear space. For example, we showed ([4], Corollary 3.3) that on any normed almost linear space there exists a semi-metric $\rho$ with good properties.

In the present paper we study the almost linear spaces X with an inner product $(\cdot,\cdot)$, a concept which generalizes the inner product on a linear space. The inner product generates a norm on the almost linear space X but, in contrast to the linear case, such a space may be not strictly convex or smooth. The main result (Theorem 3.5) shows that any almost linear space with an inner product can be "embedded" in an inner product linear space and as a consequence we can define a semi-metric $\rho_1$ on X such that $\rho_1(x,y) \leq \rho(x,y)$, $x,y \in X$. Now we can use the techniques of inner product linear spaces to solve certain problems in our more general spaces. Unfortunately, some other differences between the "linear" and "almost linear" case appear. For an element $x \in X \setminus \{0\}$ it is possible that for no $y \in X \setminus \{0\}$ to have $x \perp y$ (i.e., $(x,y) = 0$). However, when $X_1$ is a complete linear subspace of X then $X = X_1 \oplus X_2$, where $X_2$ is an almost linear subspace of X such that $X_1 \perp X_2$ (Proposition 3.9). Another unpleasant fact occurs for the dual space $X^*$ since it is possible that no inner product on $X^*$ to exist such that $(f,f) = \|f\|^2$, for each $f \in X^*$. That is why we study a certain almost linear subspace of $X^*$ which has some properties similar with the linear case and when X is a Hilbert space this subspace equals $X^*$. Examples are scattered throughout this paper to clarify the discussed problems.

2. Besides notation, in this section we recall some definitions and results from our previous papers. As in these papers, we assume that <u>all spaces are over the real field</u> R. We denote by $R_+$ the set $\{\lambda \in R : \lambda \geq 0\}$.

An <u>almost linear space</u> is a set X together with two mappings $s: X \times X \longrightarrow X$ and $m: R \times X \longrightarrow X$ satisfying $(L_1)$-$(L_8)$

below. We denote $s(x,y)$ by $x+y$ (or $x\dot{+}y$) and $m(\lambda,x)$ by $\lambda\circ x$ (or $\lambda x$). Let $x,y,z\in X$ and $\lambda,\mu\in R$. $(L_1)$ $x+(y+z)=(x+y)+z$ ; $(L_2)$ $x+y=y+x$ ; $(L_3)$ There exists an element $0\in X$ such that $x+0=x$ for each $x\in X$ ; $(L_4)$ $1\circ x=x$ ; $(L_5)$ $0\circ x=0$ ; $(L_7)$ $\lambda\circ(\mu\circ x)=$ $=(\lambda\mu)\circ x$ ; $(L_8)$ $(\lambda+\mu)\,x=\lambda x+\mu x$ for $\lambda,\mu\in R_+$ .

Let $V_X=\{x\in X : x+(-1\circ x)=0\}$ and $W_X=\{x\in X : x=-1\circ x\}$. These are almost linear subspaces of X (i.e., closed under addition and multiplication by reals), and $V_X$ is a linear space. Clearly, an almost linear space X is a linear space iff $X=V_X$, iff $W_X=\{0\}$.

In an almost linear space X we shall always use the notation $\lambda\circ x$ (in particular $-1\circ x$) for $m(\lambda,x)$ (for $m(-1,x)$), the notation $\lambda x$ (in particular $-x$) being used only in a linear space.

A normed almost linear space is an almost linear space X together with a norm $|||\cdot||| : X\longrightarrow R$ satisfying $(N_1)-(N_4)$ below. Let $x,y\in X$, $w\in W_X$ and $\lambda\in R$. $(N_1)$ $|||x+y|||\leq|||x|||+|||y|||$ ; $(N_2)$ $|||x|||=0$ iff $x=0$ ; $(N_3)$ $|||\lambda\circ x|||=|\lambda|\,|||x|||$ ; $(N_4)$ $|||x|||\leq|||x+w|||$. Note that $|||x|||\geq 0$ for each $x\in X$. We denote by $S_X$ the set $\{x\in X : |||x|||=1\}$.

Let X,Y be two almost linear spaces. A mapping $T:X\longrightarrow Y$ is called a linear operator if $T(\lambda_1\circ x_1+\lambda_2\circ x_2)=\lambda_1\circ T(x_1)+$ $+\lambda_2\circ T(x_2)$ , $x_i\in X$, $\lambda_i\in R$, $i=1,2$. When X and Y are normed almost linear spaces, a linear operator $T:X\longrightarrow Y$ is called a linear isometry if $|||T(x)|||=|||x|||$ for each $x\in X$. Here we note that a linear isometry is not always one-to-one (see examples in the next section).

2.1. REMARK. ( [4] , Remark 3.1). If T is a linear isometry of X onto Y then $T(V_X) = V_Y$, $T(W_X) = W_Y$ and the restriction $T|V_X$ is one-to-one.

2.2. THEOREM. ( [4] , Theorem 3.2). For any normed almost linear space $(X, |||\cdot|||)$ there exist a normed linear space $(E, \|\cdot\|)$ and a mapping $\omega : X \longrightarrow E$ with the following properties:

(i) $E = \omega(X) - \omega(X)$ and $\omega(X)$ can be organized as an almost linear space where the addition and the multiplication by non-negative reals are the same as in E.

(ii) For $z \in E$ we have

$$\| z \| = \inf \{ |||x||| + |||y||| \; ; \; x,y \in X, \; z = \omega(x) - \omega(y) \}$$

and $(\omega(X), \|\cdot\|)$ is a normed almost linear space.

(iii) $\omega$ is a linear isometry of $(X, |||\cdot|||)$ onto $(\omega(X), \|\cdot\|)$.

2.3. COROLLARY. ( [4] , Corollary 3.3). For $(X, |||\cdot|||)$ the function $\rho(x,y) = \| \omega(x) - \omega(y) \|$, $x, y \in X$, is a semi-metric on X.

The proof of the following lemma is contained in the proof of ( [4] , Theorem 3.2, (iv), fact I).

2.4. LEMMA. Let $(X, |||\cdot|||)$ be a normed almost linear space and $x, y \in X$. If $\omega(x) = \omega(y)$, then for each $\varepsilon > 0$ there exist $x_\varepsilon, y_\varepsilon, u_\varepsilon \in X$ such that $|||x_\varepsilon||| + |||y_\varepsilon||| < \varepsilon$ and $x + y_\varepsilon + u_\varepsilon = y + x_\varepsilon + u_\varepsilon$ .

We define now the dual of a normed almost linear space X. A functional $f : X \longrightarrow R$ is called an almost linear functional if f is additive, positively homogeneous and the restriction

$f|W_X \geq 0$. Let $X^{\#}$ be the set of all almost linear functionals on X. Define the addition in $X^{\#}$ by $(f_1+f_2)(x)=f_1(x)+f_2(x)$, $x \in X$ and the multiplication by reals $\lambda \in R$ by $(\lambda \circ f)(x)=f(\lambda \circ x)$, $x \in X$. The element $0 \in X^{\#}$ is the functional which is 0 at each $x \in X$. Then $X^{\#}$ is an almost linear space. For $f \in X^{\#}$ define $|||f||| = \sup \{|f(x)| : |||x||| \leq 1\}$ and let $X^* = \{f \in X^{\#} : |||f||| < \infty\}$. Then $X^*$ is a normed almost linear space ( [1] ) called the <u>dual</u> space of X. The dual space $X^*$ is $\neq \{0\}$ if $X \neq \{0\}$ ( [4] ).

Let E be a normed linear space. For a subset $A \subset E$ and $f \in E^*$ we denote by Int A (cl A, resp.) the interior (closure, resp.) of A in the norm topology and $\sup f(A) = \sup \{f(a) : a \in A\}$, $\inf f(A) = \inf \{f(a) : a \in A\}$.

We conclude this section with some examples from [1], [2] which will be used in the next section.

2.5. EXAMPLE. Let $X = \{(\alpha,\beta) \in R^2 : \alpha,\beta \in R_+\}$. Define the addition and the multiplication by non-negative reals as in $R^2$ and define also $-1 \circ x = x$ for each $x \in X$. Then X is an almost linear space such that $X = W_X$.

2.6. EXAMPLE. Let $X = \{(\alpha,\beta) \in R^2 : \beta > 0\} \cup \{(0,0)\}$. Define the addition and the multiplication by non-negative reals as in $R^2$ and define $-1 \circ x = x$, $x \in X$. Then X is an almost linear space such that $X = W_X$.

2.7. EXAMPLE. Let $(E, \|\cdot\|)$ be a normed linear space and let X be the collection of all nonempty, bounded, convex (and closed) subsets A of E. For $A_1, A_2 \in X$ and $\lambda \in R$ define $A_1+A_2 = \{a_1+a_2 : a_i \in A_i,\ i=1,2\}$ $(A_1 \dot{+} A_2 = \mathrm{cl}(A_1+A_2))$, $\lambda \circ A_1 = \{\lambda a_1 : a_1 \in A_1\}$

and 0 in X is the set $\{0\}$. Then X is an almost linear space. For $A \in X$ define $|||A||| = \sup_{a \in A} \|a\|$. Then $(X, |||\cdot|||)$ is a normed almost linear space.

3. Let X be an almost linear space. An <u>inner product</u> on X is a function $(\cdot,\cdot): X \times X \longrightarrow R$ satisfying $(I_1)$-$(I_7)$ below.

$(I_1)$ $(x,x) > 0$ $\qquad (x \in X \setminus \{0\})$

$(I_2)$ $(x,y) = (y,x)$ $\qquad (x,y \in X)$

$(I_3)$ $(x+y,z) = (x,z) + (y,z)$ $\qquad (x,y,z \in X)$

$(I_4)$ $(\lambda \circ x,y) = \lambda(x,y)$ $\qquad (x,y \in X,\ \lambda \in R_+)$

$(I_5)$ $(x,w) \geqslant 0$ $\qquad (x \in X,\ w \in W_X)$

$(I_6)$ $(-1\circ x,-1\circ y) = (x,y)$ $\qquad (x,y \in X)$

$(I_7)$ $(x,y)^2 \leqq (x,x)(y,y)$ $\qquad (x,y \in X \setminus V_X)$

As the following simple example shows, condition $(I_7)$ is not a consequence of $(I_1)$-$(I_6)$.

3.1. EXAMPLE. Let X be the almost linear space described in Example 2.5. For $x=(\lambda_1,\lambda_2) \in X$, $y=(\mu_1,\mu_2) \in X$ let $(x,y) = \lambda_1\mu_1+2\lambda_1\mu_2+\lambda_2\mu_2 + 2\lambda_2\mu_1$. Then $(\cdot,\cdot): X \times X \longrightarrow R$ satisfies $(I_1)$-$(I_6)$, but $(I_7)$ does not hold e.g., for $x=(1,0)$ and $y=(0,1)$. Note that in this example there exists a norm $|||\cdot|||$ on X such that $|(x,y)| \leq |||x|||\,|||y|||$, $x,y \in X$ (e.g., for $x=(\lambda_1,\lambda_2) \in X$, define $|||x||| = 2(\lambda_1+\lambda_2)$).

3.2. REMARK. Let X be an almost linear space with an inner product. Let $x,y \in X$, $v \in V_X$, $w \in W_X$ and $\lambda \in R$. We have:

(i) $(x,0) = 0$

(ii) $(x,\lambda\circ y) = (\lambda\circ x,y)$

(iii) $(x,\lambda\circ v) = \lambda(x,v)$

(iv) $(x,v)^2 \leq (x,x)(y,y)$

(v) $(w,v) = 0$

As in the case of an inner product linear space, when X is an almost linear space with an inner product, we define

$$(3.1) \qquad |||x|||^2 = (x,x) \qquad (x \in X)$$

It is easy to show that X together with this norm is a normed almost linear space.

3.3. REMARK. Let $X \neq \{0\}$ be the normed almost linear space described in Example 2.7.

(i) There exists no inner product on X such that $(A,A)^{1/2} = \sup_{a \in A} \|a\|$ for each $A \in X$. Indeed, suppose such an inner product exists and let $a \in S_E$. For $A = \{\lambda a: -1 \leq \lambda \leq 1\} \in W_X$ and $B = \{a\} \in V_X$ we have by our assumption $(A,A) = (B,B) = 1$ and by Remark 3.2 (v), $(A,B) = 0$. Then $(A+B,A+B)^{1/2} = 2^{1/2}$ and $\sup_{c \in A+B} \|c\| = 2$, a contradiction.

(ii) Let $X_1$ be the almost linear subspace of X defined by $X_1 = \{A \in X: \mathrm{Int}\, A \neq \emptyset\} \cup \{0\}$ and let $f \in S_{E^*}$. For $A,B \in X_1$ define:

$$(3.2) \qquad (A,B)_f = \sup f(A) \sup f(B) + \inf f(A) \inf f(B)$$

Then $(A,B)_f$ is an inner product on $X_1$. Here we note that if we define $(A,B)_f$ as in (3.2) for $A,B \in X$ and we set $(A,B) = \sup\{(A,B)_f : f \in S_{E^*}\}$ then $(\cdot,\cdot): X \times X \longrightarrow R$ satisfies $(I_1)$, $(I_2)$, $(I_4)$-$(I_7)$. When $\dim E = 1$ then $(\cdot,\cdot)$ is an inner product on X. Even when E is a Hilbert space, dim $E > 1$, simple examples show that $(I_3)$ is not satisfied and we were not able to determine an inner product on the almost linear space X.

In the sequel, if otherwise not stated, we shall always consider an almost linear space X with an inner product equipped with the norm given by (3.1).

3.3. LEMMA. Let X be an almost linear space with an inner product and let $x_1, x_2 \in X$. If $\omega(x_1) = \omega(x_2)$ then $(x_1,u) = (x_2,u)$ for each $u \in X$.

Proof. Since $\omega(x_1) = \omega(x_2)$, by Lemma 2.4, for each $\varepsilon > 0$ there exist $x'_\varepsilon, x''_\varepsilon, u_\varepsilon \in X$ such that $x_1+x''_\varepsilon+u_\varepsilon = x_2+x'_\varepsilon+u_\varepsilon$ and $|||x'_\varepsilon||| + |||x''_\varepsilon||| < \varepsilon$. Then for $u \in X$ we have $(x_1+x''_\varepsilon+u_\varepsilon, u) = (x_2+x'_\varepsilon+u_\varepsilon, u)$ and so $|(x_1,u)-(x_2,u)| = |(x'_\varepsilon,u)-(x''_\varepsilon,u)| \leq \leq |||u|||(|||x'_\varepsilon||| + |||x''_\varepsilon|||) < \varepsilon |||u|||$, whence the result follows.

The main result of this paper is the following:

3.5. THEOREM. Let X be an almost linear space with an inner product $(\cdot,\cdot)$. There exist a linear space H with an inner product $\langle\cdot,\cdot\rangle$ and a mapping $T:X \longrightarrow H$ satisfying the following properties:

(i) T(X) is a convex cone such that $H = T(X)-T(X)$ and T(X) can be organized as an almost linear space such that the addition and the multiplication by non-negative reals are the same as in H.

(ii) $\langle\cdot,\cdot\rangle$ is an inner product on the almost linear space T(X) and $\langle T(x),T(y)\rangle = (x,y)$ for $x,y \in X$

(iii) T is a linear isometry of X onto T(X).

Proof. Let E and $\omega$ be given by Theorem 2.2. We first note that if $x_i, y_i \in X$, $i=1,2$ are such that $\omega(x_1) = \omega(x_2)$ and $\omega(y_1) = \omega(y_2)$ then $(x_1,y_1) = (x_2,y_2)$. Indeed, by our assumptions and Lemma 3.4 we get $(x_1,y_1) = (x_2,y_1)$ and $(y_1,x_2) = (y_2,x_2)$, whence $(x_1,y_1) = (x_2,y_2)$. Let us define for $z_i = \omega(x_i) - \omega(y_i) \in E$, $x_i, y_i \in X$, $i=1,2$

$$(3.3) \qquad \langle z_1, z_2 \rangle = (x_1,x_2)-(x_1,y_2)-(x_2,y_1)+(y_1,y_2)$$

We show that $\langle\cdot,\cdot\rangle : E \times E \longrightarrow R$ is well-defined by (3.3). Suppose $x_i', y_i' \in X$, $i=1,2$ are such that

$$(3.4) \qquad z_i = \omega(x_i)-\omega(y_i) = \omega(x_i')-\omega(y_i') \qquad (i=1,2)$$

and we prove that

$$(3.5) \qquad (x_1,x_2)-(x_1,y_2)-(x_2,y_1)+(y_1,y_2)=(x_1',x_2')-(x_1',y_2')-(x_2',y_1')+ \\ +(y_1',y_2')$$

By (3.4) we get

$$(3.6) \qquad \omega(x_i+y_i') = \omega(x_i'+y_i) \qquad (i=1,2)$$

By (3.6) for i=1,2 and Lemma 3.4 we get

$$(3.7) \qquad (x_1+y_1', x_2+y_2') = (x_1'+y_1, x_2'+y_2)$$
$$(3.8) \qquad (x_1'+y_1, y_2+y_2') = (x_1+y_1', y_2+y_2')$$
$$(3.9) \qquad (x_2'+y_2, y_1+y_1') = (x_2+y_2', y_1+y_1')$$

If we add side by side the above three equalities (3.7)-(3.9), simple computations show that we get (3.5). Consequently, $\langle\cdot,\cdot\rangle : E \times E \longrightarrow R$ is well-defined by (3.3) and it is easy to prove that $\langle\cdot,\cdot\rangle$ is a Hermitian form on $E \times E$ such that $\langle z,z\rangle \geqslant 0$, $z \in E$. Let $M = \{ z \in E : \langle z,z\rangle = 0\}$, $H = E/M$ and $\omega_1$ be the canonical mapping of E onto H. It is well-known that for the linear space H the following function on $H \times H$ is an inner product:

$$(3.10) \qquad \langle \omega_1(z_1),\omega_1(z_2)\rangle = \langle z_1,z_2\rangle \qquad (z_i \in E,\ i=1,2)$$

We show that H together with the inner product defined by (3.10) and the mapping $T = \omega_1\omega$ satisfy all the required conditions. Clearly T(X) is a convex cone such that $H = T(X)-T(X)$. We organize T(X) as an almost linear space where the addition and the multiplication by non-negative reals are the same as in H, while for $T(x) \in T(X)$ we define:

$$(3.11) \qquad -1\circ T(x) = T(-1\circ x)$$

In order that this be well-defined, we must show that for $x,y \in X$ such that $T(x) = T(y)$ we have $T(-1\circ x) = T(-1\circ y)$. Since $\omega_1(\omega(x)) = \omega_1(\omega(y))$, there exists $m \in M$, $m = \omega(x_1)-\omega(y_1)$ for some $x_1,y_1 \in X$, such that $\omega(x) = \omega(y)+m$. Let $m_1 = \omega(-1\circ x_1)-\omega(-1\circ y_1) \in E$. By (3.3) we get $\langle m_1,m_1\rangle = \langle m,m\rangle = 0$, i.e., $m_1 \in M$. Since $\omega(x+y_1) = \omega(x_1+y)$, by the properties of $\omega$ we get $\omega(-1\circ x) = \omega(-1\circ y)+m_1$, i.e., $T(-1\circ x) = T(-1\circ y)$. Clearly $T: X \longrightarrow T(X)$ is a linear operator and by (3.3) and (3.10) we get $\langle T(x_1),T(x_2)\rangle = (x_1,x_2)$. Hence T is a linear isometry of X onto T(X). It is immediate that $\langle\cdot,\cdot\rangle: T(X)\times T(X)\longrightarrow R$ satisfies $(I_1)$-$(I_7)$, which completes the proof.

The following example shows that the linear subspace M of E defined in the proof of Theorem 3.5 can be $\neq\{0\}$.

3.6. EXAMPLE. Let X be the almost linear space given by Example 2.5. For $x=(\lambda_1,\lambda_2)\in X$, $y=(\mu_1,\mu_2)\in X$, define $(x,y)=(\lambda_1+\lambda_2)(\mu_1+\mu_2)$. Then $(\cdot,\cdot): X\times X\longrightarrow R$ satisfies $(I_1)$-$(I_7)$. We have $E=R^2$ and for $z=(\lambda_1,\lambda_2)\in E$, $\|z\|=|\lambda_1|+|\lambda_2|$. Here $\omega$ is the identity mapping on X. The linear subspace M of E is $\{(\lambda_1,\lambda_2)\in R^2:\lambda_1+\lambda_2=0\}$ and we have dim H = 1.

As in the case of normed linear spaces, a normed almost linear space X is called <u>strictly convex</u> if the relations $x,y\in X$, $|||x|||=|||y|||$, $x\neq y$ imply that $|||x+y|||<|||x|||+|||y|||$, and X is called <u>smooth</u> if for each $x\in S_X$ there exists a <u>unique</u> $f\in S_{X^*}$ such that $f(x)=|||x|||$ (the existence of at least such an f is guaranteed by [4], Corollary 3.4). The above example shows that, in contrast to the linear case, when X is an almost linear space with an inner product then X is not always strictly convex or smooth. Clearly, if X is strictly convex, then the same is $\omega(X)$ (the converse is not always true, as simple example shows).

3.7. REMARK. Let X be an almost linear space with an inner product. If $\omega(X)$ (in particular X) is strictly convex, then in Theorem 3.5 the linear space H equals E and $T=\omega$. Indeed, we show that the linear space M given in the proof of Theorem 3.5 is $\{0\}$. Let $m\in M$, say, $m=\omega(x_1)-\omega(x_2)$, $x_1,x_2\in X$. Since

$\langle m,m \rangle = 0$, by (3.3) and $(I_7)$ we get $|||x_1|||^2 + |||x_2|||^2 = 2(x_1,x_2) \leq$ $\leq 2|||x_1|||\, |||x_2|||$ and so $|||x_1||| = |||x_2||| = \alpha$. Then $(x_1+x_2,x_1+x_2)=4\alpha^2$ and so $\| \omega(x_1)+\omega(x_2)\| = \| \omega(x_1+x_2)\| = ||| x_1+x_2||| = 2\alpha$. Since $\| \omega(x_1)\| = \| \omega(x_2)\| = \alpha$ and $\omega(X)$ is strictly convex, it follows $\omega(x_1) = \omega(x_2)$, i.e., m=0, which completes the proof. Let us note that if we replace "strictly convex" by "smooth" in the above remark, the conclusion is no longer true. Indeed, let $X = \{(\alpha,\beta)\in X : \alpha > 0,\ \beta > 0\} \cup \{(0,0)\}$. We organize X as an almost linear space with an inner product as in Example 3.6. Then X $(= \omega(X))$ is smooth and $M \neq \{0\}$.

An immediate consequence of Theorem 3.5, using also the definition of T given in the proof of this theorem and $(I_6)$ is the following:

3.8. COROLLARY. Let X be an almost linear space with an inner product. The function $\rho_1 : X \times X \longrightarrow R$ defined by

$$\rho_1(x,y) = \| T(x)-T(y)\| \qquad (x,y\in X) \tag{3.12}$$

is a semi-metric on X with the following properties:

$$| \, |||x||| - |||y||| \, | \leq \rho_1(x,y) \qquad (x,y\in X)$$

$$\rho_1(x,v) = ||| x-v ||| \qquad (x\in X,\ v\in V_X)$$

$$\rho_1(x+z,y+z) = \rho_1(x,y) \qquad (x,y,z\in X)$$

$$\rho_1(\lambda\circ x,\lambda\circ y) = |\lambda|\, \rho_1(x,y) \qquad (x,y\in X,\ \lambda\in R)$$

$$\lim_{\lambda_n\to\lambda_o} \rho_1(\lambda_n\circ x,y) = \rho_1(\lambda_o\circ x,y) \qquad (x,y\in X,\ \lambda_o > 0)$$

$$\rho_1(x,y) \leq \rho(x,y) \qquad (x,y\in X)$$

where $\rho$ is defined in Corollary 2.3.

On an almost linear space X with an inner product there exist two semi-metrics $\rho$ and $\rho_1$, which are not equal in general (use Example 3.6). In the sequel we say that X is complete if it is complete with respect to the semi-metric $\rho_1$. Clearly $\rho_1$ is a metric on X iff T is one-to-one. Hence by Remark 2.1, $\rho_1$ is a metric on $V_X$. Note that even when $\rho_1$ is not a metric on X we can use sequences instead of nets.

Maintaining the same definition and notation from the linear case for two elements (or two subsets) of X to be orthogonal, when $X \neq V_X$ it is possible that for some $x \in X$ no element $y \in X \setminus \{0\}$ to satisfy $x \perp y$ (use Example 3.6 where X is also complete ; this may happen when in addition $\rho_1$ is a metric on X as one can see in Example 3.11 (ii) below) Consequently, when X is complete and $X_1$ is an almost linear subspace of X which is $\rho_1$-closed (i.e., closed in the topology generated by the semi-metric $\rho_1$), we can not hope to find an orthogonal complement $X_2$ such that $X = X_1 \oplus X_2$ (i.e., for each $x \in X$ to exist unique $x_i \in X_i$, $i=1,2$ such that $x=x_1+x_2$). This is however true when $X_1$ is a closed subspace of $V_X$ as the next result shows. Note that $W_X \perp V_X$ by Remark 3.2 (v).

3.9. PROPOSITION. Let X be an almost linear space with an inner product and let $X_1$ be a complete linear subspace of $V_X$. There exists an almost linear subspace $X_2$ of X which is $\rho_1$-closed such that $W_X \subset X_2$, $X_1 \perp X_2$ and $X = X_1 \oplus X_2$.

Proof. Let H and T be given by Theorem 3.5. By Remark 2.1, $T(X_1) \subset V_{T(X)}$ and $T(X_1)$ is a complete linear subspace of H. Consequently, there exists a closed linear subspace $H_2$ of H such that $T(X_1) \perp H_2$ and $H = T(X_1) \oplus H_2$. Clearly, $T(W_X) =$

$= W_{T(X)} \subset H_2$ . Let $X_2 = \{x \in X : T(x) \in H_2\}$. By the properties of T given in Theorem 3.5 and Remark 2.1, it follows that $X_2$ is an almost linear subspace of X such that $W_X \subset X_2$ , $X_1 \perp X_2$ and $X = X_1 \oplus X_2$ . Moreover, $X_2$ is $\rho_1$-closed.

The remainder of this section is devoted to the study of the dual space of an almost linear space with an inner product.

3.10. PROPOSITION. Let X be an almost linear space with an inner product and let $\Psi_X = \Psi : X \to X^*$ be defined for $x \in X$ by $(\Psi(x))(y) = (x,y)$, $y \in X$. Then $\Psi$ is a linear isometry and

$$(3.13) \qquad (\Psi(x), \Psi(y)) = (x,y) \qquad (x,y \in X)$$

is an inner product on the almost linear space $\Psi(X)$. Consequently, the norm on $\Psi(X)$ is the same with that generated by the inner product (3.13).

Proof. Clearly, for each $x \in X$, $\Psi(x)$ is an almost linear functional on X such that $|||\Psi(x)||| = |||x|||$ . Using the properties of the inner product on X, it is easy to show that $\Psi : X \to X^*$ is a linear operator. Hence $\Psi(X)$ is an almost linear subspace of $X^*$ . We show now that (3.13) is well defined. Let $x_i, y_i \in X$, i=1,2 be such that $\Psi(x_1) = \Psi(x_2)$ and $\Psi(y_1) = \Psi(y_2)$. Then for each $y \in X$ we have $(x_1,y) = (x_2,y)$ and $(y_1,y) = (y_2,y)$. Hence $(x_1,y_1) = (x_2,y_1) = (y_2,x_2)$ which shows that (3.13) is well defined. The fact that $(\cdot,\cdot) : \Psi(X) \times \Psi(X) \to R$ defined by (3.13) is an inner product follows by the properties of $\Psi$, Remark 2.1 and $(I_1)$-$(I_7)$ for $(\cdot,\cdot) : X \times X \to R$ . The last assertion of the proposition is obvious, which completes the proof.

When $X = V_X$ it is known that $\Psi : X \longrightarrow X^*$ is one-to-one and if in addition X is complete then $\Psi(X) = X^*$ . These assertions are no longer true when $X \neq V_X$ . In this case it is possible to exist no inner product on $X^*$ such that for each $f \in X^*$ to have $(f,f) = |||f|||^2$ . This may happen even when T is one-to-one and X is complete. The condition $\Psi(X) = X^*$ does not imply that T is one-to-one. These will be seen in the following simple examples;

3.11. EXAMPLES. (i) Let X and $(\cdot,\cdot)$ be given as in Example 3.6. We have H = R with the usual inner product and $T((\alpha,\beta)) = \alpha + \beta$, $(\alpha,\beta) \in X$. Let $f \in S_{X^*}$ be defined by $f((\alpha,\beta)) = \alpha + \beta$, $(\alpha,\beta) \in X$. Then for each $x \in S_X$ we have $\Psi(x) = f$, i.e., $\Psi$ is not one-to-one.

(ii) Let $X = \{(\alpha,\beta) \in R^2 : 0 \leq \beta \leq \alpha\}$. We organize X as an almost linear space similarly with the space described in Example 2.5. If $x_i = (\alpha_i, \beta_i) \in X$, i=1,2, define $(x_1,x_2) = \alpha_1\alpha_2 + \beta_1\beta_2$ . Then $(\cdot,\cdot)$ is an inner product on X. The space H given by Theorem 3.5 is $R^2$ endowed with the Euclidean norm and T is the identity mapping on X. Clearly T is one-to-one and X is complete. Suppose there exists an inner product on $X^*$ such that $(f,f) = |||f|||^2$ for each $f \in X^*$ . Since $X = W_X$ it follows that $X^* = W_{X^*}$ and so $(f,g) \geq 0$ for $f,g \in X^*$ . Let $f_i \in S_{X^*}$ , i=1,2 be defined for $x = (\alpha,\beta) \in X$ by $f_1(x) = \alpha - \beta$ and $f_2(x) = 2^{1/2}\beta$ . Note that $f_i \notin \Psi(X)$, i=1,2. By our assumptions we have $(f_i,f_i) = 1$, i=1,2 and $(f_1,f_2) = \mu \geq 0$. Let

$f=(2^{1/2}/2)f_1 + f_2$ . Clearly $f \in S_{X^*}$ and so $(f,f) = 1$. Then $1 = ((2^{1/2}/2)f_1 + f_2, (2^{1/2}/2)f_1 + f_2) = (3/2)+2^{1/2}\mu$ , which is impossible since $\mu \geq 0$.

(iii) Let X be the almost linear space given in Example 2.6. Define for $x_i = (\alpha_i, \beta_i) \in X$, i=1,2, $(x_1,x_2) = \beta_1 \beta_2$ . Then $(\cdot,\cdot)$ is an inner product on X. We have $X^* = \{\lambda \circ f_o : \lambda \in R\}$, where $f_o$ is defined by $f_o((\alpha,\beta)) = \beta$ , $(\alpha,\beta) \in X$. Clearly, $X^* = \Psi(X)$. Here H given by Theorem 3.5 is R with the usual inner product, $T:X \longrightarrow H$ is defined by $T((\alpha,\beta)) = \beta$ , $(\alpha,\beta) \in X$ and T is not one-to-one.

3.12. PROPOSITION. Let X be an almost linear space with an inner product. The mapping T (given by Theorem 3.5) is one-to-one iff the mapping $\Psi$ (given by Proposition 3.10) is one-to-one.

Proof. Suppose T one-to-one and let $x_1,x_2 \in X$ such that $\Psi(x_1) = \Psi(x_2)$. Let $f_i \in H^*$ , i=1,2, be defined for $h \in H$ by $f_i(h) = \langle T(x_i),h \rangle$. For $h \in H$, $h=T(x)-T(y)$, $x,y \in X$ we have

$$f_1(h)= \langle T(x_1),T(x)\rangle - \langle T(x_1),T(y)\rangle =(x_1,x)-(x_1,y) =$$

$$= (\Psi(x_1))(x)-(\Psi(x_1))(y) = (\Psi(x_2))(x)-(\Psi(x_2))(y) = f_2(h),$$

i.e., $f_1=f_2$ and so $T(x_1) = T(x_2)$. By our assumption it follows that $x_1=x_2$ and so $\Psi$ is one-to-one.

Conversely, suppose $\Psi$ one-to-one and let $x_1,x_2 \in X$ such that $T(x_1) = T(x_2)$. Let $f \in H^*$ be defined by $f(h)= \langle T(x_1),h \rangle$ , $h \in H$. Then for each $x \in X$ we have $(\Psi(x_1))(x) = (x_1,x) =$

$$= \langle T(x_1),T(x)\rangle = \langle T(x_2),T(x)\rangle = (\Psi(x_2))(x)$$

and so $\Psi(x_1) = \Psi(x_2)$. Hence $x_1=x_2$ which proves that T is one-to-one.

3.13. REMARK. The above proof shows that for $x_1,x_2 \in X$

we have $T(x_1) = T(x_2)$ iff $\Psi(x_1) = \Psi(x_2)$.

3.14. REMARK. For the almost linear space $\Psi(X)$ with the inner product given by Proposition 3.10, the mapping $\Psi_{\Psi(X)}$ is one-to-one. Indeed, let $f_1, f_2 \in \Psi(X)$, say, $f_i = \Psi(x_i)$, $x_i \in X$, i=1,2, such that $\Psi_{\Psi(X)}(f_1) = \Psi_{\Psi(X)}(f_2)$. For each $x \in X$ we have $(\Psi_{\Psi(X)}(f_i))(\Psi(x)) = (f_i, \Psi(x)) = (\Psi(x_i), \Psi(x)) =$ $= (x_i, x) = f_i(x)$, i=1,2. By our assumption it follows that $f_1(x) = f_2(x)$ for each $x \in X$, i.e., $f_1 = f_2$.

Let us define for each $x \in X$ the following functional $Q_x$ on the almost linear space $\Psi(X)$ with the inner product given by (3.13):

$$Q_x(f) = f(x) \qquad (f \in \Psi(X)) \tag{3.14}$$

It is easy to show that $Q_x$ is an almost linear functional on $\Psi(X)$ and

$$|||Q_x||| = |||x||| \tag{3.15}$$

3.15. PROPOSITION. Let X be an almost linear space with an inner product and let $Q: X \longrightarrow \Psi(X)^*$ be defined by (3.14). We have:

$$Q = \Psi_{\Psi(X)} \Psi \tag{3.16}$$

and Q is a linear isometry which is one-to-one iff T is one-to-one. $(Q_x, Q_y) = (x, y)$, $x, y \in X$ is an inner product on the almost linear subspace $Q(X)$ of $\Psi(X)^*$ and the norm on $Q(X)$ is the same with that generated by this inner product.

Proof. Let $x_0 \in X$ and $f \in \Psi(X)$, say, $f=\Psi(x)$ for some $x \in X$. Then $Q_{x_0}(f)=f(x_0)=(x,x_0)$ . Let $f_0=\Psi(x_0)$. Then $f_0(x) = =(x_0,x)=Q_{x_0}(f)$ and $(\Psi_{\Psi(X)}(f_0))(f)=(f_0,f)=(\Psi(x_0),\Psi(x))= =(x_0,x)=Q_{x_0}(f)$ which proves (3.16). The last assertions follow now by Proposition 3.10, formulae (3.15), Proposition 3.12 and Remark 3.14.

3.16. PROPOSITION. Let $X$ be an almost linear space with an inner product. If $\Psi(X) = X^*$ then $Q$ is a linear isometry of $X$ onto $X^{**}$ and $(Q_x,Q_y) = (x,y)$, $x,y \in X$ is an inner product on $X^{**}$ . Moreover, the norm of $X^{**}$ is the same with that generated by this inner product.

Proof. By Proposition 3.15 it is enough to show that if $\Psi(X) = X^*$ then $\Psi_{X^*}(X^*) = X^{**}$. Let $\Phi \in X^{**}$ and let $f_0= \Phi\Psi$ . Since for each $w \in W_X$ we have $\Psi(w) \in W_{X^*}$ , it is easy to show that $f_0 \in X^*$ and $|||f_0||| = |||\Phi|||$. By our assumption, there exists $x_0 \in X$ such that $f_0=\Psi(x_0)$. Let now $f \in X^*$ . Then $f=\Psi(x)$ for some $x \in X$ and we have $\Phi(f)=\Phi(\Psi(x))=f_0(x)=(\Psi(x_0))(x)= =(x_0,x)=(\Psi(x_0),\Psi(x))=(f_0,f)$, i.e., $\Psi_{X^*}(f_0)=\Phi$.

We proved in [4] that for any normed almost linear space $X$ the following formulae holds:

$$S_{X^*} = \{ f\omega : f \in S_{E^*} ,\ f|W_{\omega(X)} \geq 0 \}$$

where $E$ and $\omega$ are given by Theorem 2.2. If $X$ is an almost linear space with an inner product and if we replace $E$ and $\omega$ (and $S_{X^*}$)

by H and T given in Theorem 3.5 (and by $S_{\Psi(X)}$), a similar formulae is no longer true (use Example 3.8 (ii)). The next result gives a formulae for $S_{\Psi(X)}$.

3.17. PROPOSITION. Let X be an almost linear space with an inner product. We have:

$$(3.17) \quad S_{\Psi(X)} = \{fT : f \in S_{H^*},\ f \text{ attains its norm at } S_{T(X)}\}$$

Proof. Let $f_o \in S_{\Psi(X)}$ and let $x_o \in S_X$ such that $f_o(x)=(x_o,x)$ for each $x \in X$. Define $f \in S_{H^*}$ by $f(h)= \langle T(x_o),h \rangle$, $h \in H$. Then f attains its norm at $T(x_o) \in S_{T(X)}$ and $f(T(x)) = \langle T(x_o),T(x) \rangle = (x_o,x) = f_o(x)$, $x \in X$, i.e., $fT=f_o$, which proves the inclusion $\subset$ in (3.17). Let now $f \in S_{H^*}$ and suppose that $f(T(x_o)) = 1$ for some $x_o \in S_X$. Then $f(h) = \langle T(x_o),h \rangle$ for each $h \in H$. Let $f_o= \Psi(x_o)$ ($\in S_{\Psi(X)}$). We have $f_o(x) = (x_o,x) = \langle T(x_o),T(x) \rangle = f(T(x))$, $x \in X$, i.e., $f_o=fT$, which completes the proof.

A consequence of this result is the following:

3.18. COROLLARY. Let X be an almost linear space with an inner product. If $\Psi(X) = X^*$ then X is complete.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X which does not converge to any $x \in X$. Let H and T be given by Theorem 3.5 and let $\tilde{H}$ be the completion of H. Since $\{T(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in H, it converges to some $\tilde{h} \in \tilde{H} \setminus T(X)$. Let $f \in S_{H^*}$ be defined by $f(h)= \langle \tilde{h}/\|\tilde{h}\|,h \rangle$, $h \in H$. We have $fT \in S_{X^*}$ and by Proposition 3.17 (since f does not attain its norm at $S_{T(X)}$)

we get $fT \in S_{\psi(X)}$ , contradicting the hypothesis that $\psi(X)=X^*$ .

REFERENCES

1. G. GODINI : A framework for best simultaneous approximation : normed almost linear spaces, J. Approximation Theory, 43 no. 4 (1985), 338-358

2. G. GODINI : An approach to generalizing Banach spaces : normed almost linear spaces, Proceedings of the 12th Winter School on Abstract Analysis (Srni 1984). Suppl. Rend. Circ. Mat. Palermo Serie II, no. 5 (1984), 33-50

3. G. GODINI : Best approximation in normed almost linear spaces, in "Constructive Theory of Functions". Proceedings of the International Conference on Constructive Theory of Functions (Varna 1984). Publishing House of the Bulgarian Academy of Sciences, Sofia, 1984, 356-363

4. G. GODINI : On normed almost linear spaces, Math. Ann.,(to appear)

5. G. GODINI : Operators in normed almost linear spaces, Preprint Series Math. INCREST, Bucureşti, No. 22/1986, ISSN 0250 3638

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