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H-CONES OF FUNCTIONS

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INTRODUCTION

In this paper we show that giving a standard H-cone of functions $\mathcal{S}$ on a set X such-that X is semi-saturated with respect to $\mathcal{S}$, for any subset A of X, any point $x_o \in X$ and any ultrafilter $\mathcal{F}$ on X converging to x_o the limit λ of the measures $(\varepsilon_x^A)_{x,\mathcal{F}}$ along the filter $\mathcal{F}$ is a measure on X of the form

$$\lambda = \alpha \varepsilon_{x_o} + (1-\alpha)\varepsilon_{x_o}^{A \setminus \{x_o\}}$$

We generalize in the framework of standard H-cone the result of O. Forstman [7] on the behaviour of the solutions of Dirichlet problem at the irregular points. Our result extends also a similar one given by N. Boboc and A. Cornea [5] for harmonic spaces. Other related topics may be find in [8], [9], [10].

Let $\mathcal{S}$ be a standard H-cone of functions on a semisaturated set X. We denote by K_1 the convex compact subset of $\mathcal{S}^*$ (with respect to the natural topology on $\mathcal{S}^*$) given by

$$K_1 = \{\mu \in \mathcal{S}^* \mid \mu(1) \leq 1\}$$

If X_1 is the set of all non-zero extreme points of K_1 we have $X \subset X_1$ and the closure of X (in K_1) is a compact subset of K_1. For any element μ of $\mathcal{S}^*$ such that $\mu(1) < \infty$ there exists a unique finite measure m_μ on X such that

$$\mu(s) = \int s\, dm_\mu$$

Always we identify m_μ with μ. If $(\mu_i)_i$ is a generalized sequence of measures on X which are dominated by a finite measure μ_o on X and if $(\mu_i)_i$ converges weakly to a measure $\bar{\mu}$ on $\bar{X}$ then $(\mu_i)_i$ converges naturally (with respect to the natural topology on $\mathcal{S}^*$) to a measure μ on X for which we have

$$\mu(p) = \int p\, d\bar{\mu} \qquad (\forall)\ p \in \mathcal{S}_o \, .$$

Generally $\bar{\mu}$ is not carried by X and therefore we don't expect the measures μ and $\bar{\mu}$ be equal.

For any subset A of X and any measure μ on X there exists a unique measure μ^A on X such that for any $p \in \mathcal{S}$ we have

$$\mu^A(p) = \mu(B^A p) \, .$$

We remember [6] the following results:

$(B\text{-}H)_1$. If A_1, A_2 are two arbitrary subsets of X and μ is a finite measure on X we have

$$\mu^{A_1 \cup A_2} \precsim \mu^{A_1} + \mu^{A_2}$$

where the symbol $\precsim$ stands for the specific order in the dual H-cone $\mathcal{S}^*$ or equivalently for the usual inequality between the Borel measures $\mu^{A_1 \cup A_2}$ and $\mu^{A_1} + \mu^{A_2}$ on X.

$(B\text{-}H)_2$. If A_1, A_2 are two Borel and fine closed subsets of X, $A_1 \subset A_2$ and μ is a measure which does not charge the set $(A_1 \setminus b(A_1)) \cap b(A_2)$ we have

$$\mu^{A_1} = \mu^{A_2}/A_1 + (\mu^{A_2}/CA_1)^{A_1}$$

Definition. A family $(\mu_i)_i$ of finite measures on X is termed convergent (resp. strongly convergent) to a measure μ on X if for any universally continuous element p of $\mathcal{S}$ (resp. any bounded continuous function f on X) we have

$$\lim_i \mu_i(p) = \mu(p) \quad (\text{resp. } \lim_i \mu_i(f) = \mu(f)) .$$

Obviously if $(\mu_i)_i$ is strongly convergent to μ then $(\mu_i)_i$ converges to μ.

Lemma 1. *A family $(\mu_i)_i$ of finite measures on X is strongly convergent to a finite measure μ on X iff the family $(\mu_i)_i$ of measures (considered as measures on compact space $\overline{X}$) is weakly convergent to the measure μ.*

Proof. Obviously if the family $(\mu_i)_i$ is strongly convergent to μ then this family is weakly convergent to μ on $\overline{X}$. Let us suppose that $(\mu_i)_i$ is weakly convergent to μ on $\overline{X}$ and let f be a bounded continuous function on X. If we denote by $\hat{f}$ (resp. $\check{f}$) the lower (resp. upper) semicontinuous regularization of f on $\overline{X}$ we have

$$\mu(f) = \mu(\hat{f}) \leq \lim_i \inf \mu_i(\hat{f}) = \lim_i \inf \mu_i(f)$$

$$\mu(f) = \mu(\check{f}) \geq \lim_i \sup \mu_i(\check{f}) = \lim_i \sup \mu_i(f)$$

and therefore

$$\mu(f)=\lim_i \mu_i(f)$$

Remark. If $(\mu_i)_i$ is a family of uniformly bounded measures on X then this family is strongly convergent to a measure μ on X iff we have $\lim_i \mu_i(p)=\mu(p)$ for any $p\in\mathcal{D}$ where $\mathcal{D}$ is a uniformly dense subset of the min-stable cone generated by $\mathcal{S}_o$ and by positive constant functions on X.

Theorem 2. Let $\mathcal{F}$ be a base of ultrafilter on X which converges to a point $x_o\in X$ and let A be a subset of X. Then the family $(\varepsilon_x^A)_x$ of measures on X is strongly convergent along $\mathcal{F}$ to a measure λ on X of the form

$$\lambda=\alpha\varepsilon_{x_o}+(1-\alpha)\varepsilon_{x_o}^{A\setminus\{x_o\}}$$

where $\alpha:=\lambda(\{x_o\})$. In particular (taking as $\mathcal{F}$ the filter of all subsets of X containing x_o) we have

$$\varepsilon_{x_o}^A=\beta\varepsilon_{x_o}+(1-\beta)\varepsilon_{x_o}^{A\setminus\{x_o\}}\ ,\ \beta:=\varepsilon_{x_o}^A(\{x_o\})\ .$$

Proof. Let U be an open neighbourhood in K_1 of x_o. From the property $(B-H)_1$ we have,

$$\varepsilon_x^A\preceq\varepsilon_x^{A\setminus U}+\varepsilon_x^{A\cap U}$$

for any $x\in X$. Since for any bounded element p of X the function

$$X\ni x\longrightarrow\varepsilon_x^{A\setminus U}(p)$$

is continuous on $X \cap U$ we deduce that for any bounded and Borel function f on K_1 we have

$$\lim_{\mathcal{F}} \varepsilon_x^{A\setminus U}(f) = \varepsilon_{x_0}^{A\setminus U}(f)$$

If we denote by λ and λ_U the weak limit in K_1 of the measures $(\varepsilon_x^A)_x$ respectively $(\varepsilon_x^{A\cap U})_x$ along the base of ultrafilter $\mathcal{F}$ we deduce that λ_U is a measure on K_1 carried by the set $\overline{U}$ and we have

$$\lambda \precsim \varepsilon_{x_0}^{A\setminus U} + \lambda_U$$

The neighbourhood U of x_0 being arbitrary we deduce that λ is a measure on X, the family $(\varepsilon_x^A)_x$ is strongly convergent to λ along the base of ultrafilter $\mathcal{F}$ and moreover we have

$$\lambda \precsim \varepsilon_{x_0}^{A\setminus\{x_0\}} + \varepsilon_{x_0}$$

For any $p \in \mathcal{S}_0$ we have

$$\varepsilon_{x_0}^A(p) = B_p^A(x_0) \leq \liminf_{x \to x_0} \varepsilon_x^A(p) \leq \lim_{\mathcal{F}} \varepsilon_x^A(p) = \lambda(p) \quad ,$$

$$\lambda(p) = \lim_{\mathcal{F}} \varepsilon_x^A(p) \leq \lim_{\mathcal{F}} p(x) = p(x_0)$$

Hence, we deduce

$$\varepsilon_{x_0}^A \leq \lambda \leq \varepsilon_{x_0} \ , \ \lambda \precsim \varepsilon_{x_0}^{A\setminus\{x_0\}} + \varepsilon_{x_0}$$

and therefore if $\lambda \neq \varepsilon_{x_0}$ we have $\alpha := \lambda(\{x_0\}) \neq 1$. In this case, if we put

$$\lambda = \alpha \varepsilon_{x_0} + (1-\alpha)\mu$$

then we have $\mu \leq \varepsilon_{x_0}$. Without loss of generality we may suppose that A is a Borel subset of X and A is also fine closed. From the above considerations it follows that μ is carried by the set $A \setminus \{x_0\}$. We show now that $\mu = \varepsilon_{x_0}^{A \setminus \{x_0\}}$. Let p by a universally continuous element of $\mathcal{S}$.

Using [4], Proposition 5.3.1 we deduce that for any $\varepsilon > 0$ we may choose an element $s \in \mathcal{S}$ such that s=p on $A \setminus \{x_0\}$ and $s(x_0) < B_{p(x_0)}^{A \setminus \{x_0\}} + \varepsilon$. We have

$$\mu(p) = \mu(s) \leq s(x_0) < B_{p(x_0)}^{A \setminus \{x_0\}} + \varepsilon \ , \ \mu(p) < \varepsilon + \varepsilon_{x_0}^{A \setminus \{x_0\}}(p)$$

The number $\varepsilon > 0$ and the element p of $\mathcal{S}_0$ being arbitrary we obtain $\mu \leq \varepsilon_{x_0}^{A \setminus \{x_0\}}$.

For the inequality $\varepsilon_{x_0}^{A \setminus \{x_0\}} \leq \mu$ we suppose first that X is a Souslinean space. Let $p \in \mathcal{S}_0$ and let U be an open neighbourhood of x_0. Using the assertion $(B\text{-}H)_2$ for any $x \in U$ we have

$$\varepsilon_x^{A \setminus U}(p) = \varepsilon_x^A(R^{A \setminus U}p), \quad \varepsilon_x^k(p) \leq \varepsilon_x^A(R^K p)$$

for any compact subset K of $A \setminus U$.

Using [4], Proposition 5.2.4 we deduce that

$$R^k p = \inf \{ q \in \mathcal{S}_0 \mid q = p \text{ on } k \}$$

for any compact subset k of X and therefore the function $R^K p$ is bounded and upper semicontinuous on X. Since the measure

ε_x^A is strongly convergent along the ultrafilter $\mathcal{F}$ to λ we have

$$\alpha R^K p(x_o)+(1-\alpha)\mu(R^K p)\geq \limsup_{\mathcal{F}} \varepsilon_x^A(R^K p)\geq \lim_{\mathcal{F}} \varepsilon_x^K(p) \quad ,$$

$$\alpha R^K p(x_o)+(1-\alpha)\mu(R^K p)\geq \varepsilon_{x_o}^K(p)$$

for any compact subset of $A\setminus U$. Hence we get

$$\alpha B^{A\setminus U}p(x_o)+(1-\alpha)\mu(p)\geq \varepsilon_{x_o}^{A\setminus U}(p), \quad \mu(p)\geq \varepsilon_{x_o}^{A\setminus U}(p) .$$

The open neighbourhood U of x_o being arbitrary we deduce

$$\mu(p)\geq \varepsilon_{x_o}^{A\setminus\{x_o\}}(p) , \qquad \mu=\varepsilon_{x_o}^{A\setminus\{x_o\}} .$$

If X is an arbitrary semisaturated set then for an arbitrary subset A of X there exists a Borel subset A_1 of X_1 such that

$$A\subset A_1 , \quad B^A=B^{A_1} , \quad B^{A\setminus\{x_o\}}=B^{A_1\setminus\{x_o\}} .$$

From the above considerations we get that the family of measures $(\varepsilon_x^A)_x=(\varepsilon_x^{A_1})_x$ is strongly convergent along the base of ultrafilter $\mathcal{F}$ to a measure λ on X_1 such that

$$\lambda=\alpha\,\varepsilon_{x_o}+(1-\alpha)\,{}_{x_o}^{A_1\setminus\{x_o\}}=\alpha\,\varepsilon_{x_o}+(1-\alpha)\,{}_{x_o}^{A\setminus\{x_o\}}$$

The proof is finished if we use Lemma 1.

<u>Theorem 3</u>. If the base $\mathcal{F}$ of ultrafilter on X converges to x_o then the family $(\varepsilon_x^A)_x$ of measures on X is strongly

convergent along $\mathcal{F}$ to a measure λ of the form

$$\lambda = \beta \varepsilon_{x_o} + (1-\beta)\varepsilon_{x_o}^{A} .$$

Proof. From the relation $\varepsilon_{x_o}^{A} \leq \lambda$ and using Theorem 2 we get

$$\lambda = \alpha \varepsilon_{x_o} + (1-\alpha)\varepsilon_{x_o}^{A\setminus\{x_o\}} , \quad \varepsilon_{x_o}^{A} = \alpha' \varepsilon_{x_o} + (1-\alpha')\varepsilon_{x_o}^{A\setminus\{x_o\}} ,$$

$$(\alpha-\alpha')\varepsilon_{x_o}^{A\setminus\{x_o\}} \leq (\alpha-\alpha')\varepsilon_{x_o}$$

and therefore if $\varepsilon_{x_o}^{A\setminus\{x_o\}} \neq \varepsilon_{x_o}$ we have $\alpha' < \alpha$.

Hence if we suppose that $x_o \notin b(A)$ then $\alpha' < 1$, $\alpha' \leq \alpha$ and therefore

$$\lambda = \alpha\varepsilon_{x_o} + \frac{1-\alpha}{1-\alpha'}(\varepsilon_{x_o}^{A} - \alpha'\varepsilon_{x_o}) = \frac{\alpha-\alpha'}{1-\alpha'}\varepsilon_{x_o} + \frac{1-\alpha}{1-\alpha'}\varepsilon_{x_o}^{A}$$

If $x_o \in b(A)$ obviously we have $\lambda = \varepsilon_{x_o}$.

Let now $\mathcal{U}_{x_o}$ be the set of all bases of ultrafilter on X which converges to x_o. For any subset A of X and any base of ultrafilter $\mathcal{F}$ from $\mathcal{U}_{x_o}$ we denote by $\alpha_{\mathcal{F}}^{A}$ the positive number equal 1 if $x_o \in b(A)$ and equal with the unique coeficient $\alpha \in [0,1]$ from the decomposition

$$\lambda = \alpha\varepsilon_{x_o} + (1-\alpha)\varepsilon_{x_o}^{A}$$

where λ is the strong limit, along the ultrafilter $\mathcal{F}$, of the family $(\varepsilon_{x}^{A})_x$ if $x_o \notin b(A)$.

Theorem 4. For any subset A of X and any neighbourhood U of x_o we have $\alpha_{\mathcal{F}}^{A}=\alpha_{\mathcal{F}}^{A\cap U}$ for any $\mathcal{F}\in\mathcal{U}_{x_o}$.

Proof. Obviously we may suppose that A is Borel and finely closed. The relation $\alpha_{\mathcal{F}}^{A}=\alpha_{\mathcal{F}}^{A\cap U}$ is obvious if $x_o\in b(A)$ and therefore we may suppose that $x_o\notin b(A)$. Let $\mathcal{F}\in\mathcal{U}_{x_o}$ and let U be an arbitrary neighbourhood of x_o.

Considering the measure $\varepsilon_{x_o}^{A}+\varepsilon_{x_o}^{A\setminus U}$ we may choose an open neighbourhood V of x_o, $\overline{V}\subset U$, such that $\varepsilon_{x_o}^{A}+\varepsilon_{x_o}^{A\cap U}(\overline{V}\setminus V)=0$. It is obvious now that for any family $(\mu_i)_i$ of uniformely bounded measures on X which is strongly convergent to a measure μ on X and for any subset M of X for which $\mu(\partial M)=0$ we have

$$\lim_{i}\mu_i(f.1_M)=\mu(f.1_M)$$

for any bounded and continuous function f on M. Using the property $(B\text{-}H)_2$, for any $x\in V$ we have

$$\varepsilon_x^{A\cap\overline{V}}=\varepsilon_x^{A}/_{\overline{V}}+(\varepsilon_x^{A}/_{C\overline{V}})^{A\cap\overline{V}}$$

and therefore for any element $p\in\mathcal{S}_o$ we get

$$\alpha_{\mathcal{F}}^{A\cap\overline{V}}p(x_o)+(1-\alpha_{\mathcal{F}}^{A\cap\overline{V}})B^{A\cap\overline{V}}p(x_o)=\alpha_{\mathcal{F}}^{A}.p(x_o)+(1-\alpha_{\mathcal{F}}^{A}).\int_{\overline{V}}p\,d\varepsilon_{x_o}^{A}+$$

$$+\lim_{\mathcal{F}}\int_{C\overline{V}}B^{A\cap\overline{V}}p\,d\varepsilon_x^{A}=\alpha_{\mathcal{F}}^{A}p(x_o)+(1-\alpha_{\mathcal{F}}^{A})\left[\varepsilon_{x_o}^{A}|_{\overline{V}}+(\varepsilon_{x_o}^{A}|_{C\overline{V}})^{A\cap\overline{V}}\right](p)=$$

$$=\alpha_{\mathcal{F}}^{A}p(x_o)+(1-\alpha_{\mathcal{F}}^{A})B^{A\cap\overline{V}}p(x_o)\quad.$$

From the previous relation we get $\alpha_{\mathcal{F}}^{A\cap\overline{V}}=\alpha_{\mathcal{F}}^{A}$. In a similar manner taking $A\cap U$ instead of A we obtain $\alpha_{\mathcal{F}}^{A\cap\overline{V}}=\alpha_{\mathcal{F}}^{A\cap U}$ and therefore $\alpha_{\mathcal{F}}^{A\cap U}=\alpha_{\mathcal{F}}^{A}$.

Remark. For the particular cases of harmonic spaces or balayage spaces the above theorem was proved by W. Hansen [8] in a rather different way.

Proposition 5. Suppose that X is semisaturated and the pair $(X,\mathcal{S})$ satisfies axiom D_o i.e. for any open sets G_1, G_2 such that $G_1\cup G_2=X$ we have $B^{G_1}B^{G_2}=B^{G_2}B^{G_1}$. If A is an arbitrary subset of X and if $x_o\in\overline{C}_A$ then the following assertions are equivalent:

1. $\varepsilon_{x_o}\neq\varepsilon_{x_o}^{A}$ and for any ultrafilter $\mathcal{F}$ on C_A converging to x_o the family of measures $(\varepsilon_x^A)_x$ is strongly convergent along $\mathcal{F}$ to the measure $\varepsilon_{x_o}^{A}$;

2. there is no ultrafilter $\mathcal{F}$ on C_A converging to x_o and such that the family of measures $(\varepsilon_x^A)_x$ is strongly convergent along $\mathcal{F}$ to ε_{x_o}.

Proof. Obviously $1)\Rightarrow 2)$. Let now $p\in\mathcal{S}$ be a bounded continuous function on X such that the linear space generated by the set of its specific minorants from $\mathcal{S}$ is uniformely dense in the set of all bounded and uniformly continuous functions on X.

If the assertion 2) is fullfiled then we have

$\limsup_{C_A\ni x\to x_o} B^A p(x)<p(x_o)$. Indeed, in the contrary case there exists

an ultrafilter $\mathcal{F}$ on C_A converging to x_o such that

$$\lim_{\mathcal{F}} \varepsilon_x^A(p) = \varepsilon_{x_o}(p)$$

Since for any specific minorant $p' \in \mathcal{S}$ of the element p we have

$$\lim_{\mathcal{F}} \varepsilon_x^A(p') \leq \lim_{\mathcal{F}} \varepsilon_x(p') = p'(x_o)$$

we get

$$\lim_{\mathcal{F}} \varepsilon_x^A(p') = \varepsilon_{x_o}(p')$$

and therefore, using the above considerations and Lemma 1, we deduce that the family $(\varepsilon_x^A)_x$ is strongly convergent, along $\mathcal{F}$, to the measure ε_{x_o}.

Let now V be an open neighbourhood of x_o and let δ be a strictly positive number such that $B^A p(x) < p(x) - \delta$ on $V \setminus A$.

We consider also, for a fixed point $x^* \in C_A$, a decreasing sequence of open neighbourhoods $(V_n)_n$ of A such that $x^* \notin V_n$ and $\lim_{n \to \infty} B^{V_n} p(x^*) = B^A p(x^*)$.

Since $B^{V_n} q \geq B^A q$ for any $n \in N$ and any $q \in \mathcal{S}$ we deduce that we have

$$\lim_{n \to \infty} B^{V_n} q(x^*) = B^A q(x^*)$$

for any specific minorant $q \in \mathcal{S}$ of p. It follows, using again Lemma 1, that the sequence $(\varepsilon_{x^*}^{V_n})_n$ is strongly convergent to the measure $\varepsilon_{x^*}^A$. Hence we deduce

$$\liminf_{n \to \infty} \varepsilon_{x^*}^{V_n}(V) \geq \varepsilon_{x^*}^A(V) \quad .$$

On the other hand, from the property $(B\text{-}H)_2$ we have, for any $n\in N$,

$$\varepsilon_{x^*}^{A}(p)=\int R^A p\, d\varepsilon_{x^*}^{V_n}\leq \int p\, d\,\varepsilon_{x^*}^{V_n}-\delta\,\varepsilon_{x^*}^{V_n}(V\setminus A)$$

Since the measure $\varepsilon_{x^*}^{V_n}$ is carried by ∂V_n (see [1],[2]) we get

$$\varepsilon_{x^*}^{A}(p)\leq \varepsilon_{x^*}^{V_n}(p)-\delta\,\varepsilon_{x^*}^{V_n}(V)\ ,$$

$$\varepsilon_{x^*}^{A}(p)\leq \varepsilon_{x^*}^{A}(p)-\delta\,\varepsilon_{x^*}^{A}(V)\ ,\qquad \varepsilon_{x^*}^{A}(V)=0\ .$$

Let now $\mathcal{F}$ be an ultrafilter on $V\cap(X\setminus A)$ which converges to x_o. Using the property $(B\text{-}H)_1$ for any $y\in V\cap(X\setminus A)$ we have

$$\varepsilon_y^{A}\precsim\varepsilon_y^{A\cap V}+\varepsilon_y^{A\cap(X\setminus V)}\ .$$

From the above considerations we deduce

$$\varepsilon_y^{A}\precsim\varepsilon_y^{A\cap(X\setminus V)}\ .$$

From Theorem 3 the family of measures $(\varepsilon_y^A)_y$ is strongly convergent, along $\mathcal{F}$, to the measure $\lambda=\alpha_{\mathcal{F}}\varepsilon_{x_o}+(1-\alpha_{\mathcal{F}})\varepsilon_{x_o}^{A}$.

On the other hand

$$(\alpha_{\mathcal{F}}\varepsilon_{x_o}+(1-\alpha_{\mathcal{F}})\varepsilon_{x_o}^{A})(V)\leq \liminf_{\mathcal{F}}\varepsilon_y^{A}(V)\leq \liminf_{y\to x_o}\varepsilon_y^{A\cap(X\setminus V)}(V)=0$$

Hence $\alpha_{\mathcal{F}}=0$, $\varepsilon_{x_o}\neq\varepsilon_{x_o}^{A}$, $\lambda=\varepsilon_{x_o}^{A}$.

<u>Theorem 6</u>. If X is semisaturated and the pair $(X,\mathcal{S})$ satisfies axiom D_o then for any subset A of X and any $x_o\in X$ we have

$$\{\varepsilon_{\mathcal{F}}^{A} \mid \mathcal{F}\in\mathcal{U}_{x_0}\}\subset\{0,1\} \quad \text{or} \quad \{\varepsilon_{\mathcal{F}}^{A} \mid \mathcal{F}\in\mathcal{U}_{x_0}\}=[0,1]$$

where $\mathcal{U}_{x_0}$ is the set of all ultrafilter on X converging to x_0 and for any $\mathcal{F}\in\mathcal{U}_{x_0}$, $\alpha_{\mathcal{F}}^{A}\in[0,1]$ is such that the family $(\varepsilon_x^A)_x$ of measures on X is strongly convergent, along $\mathcal{F}$, to the measure $\alpha_{\mathcal{F}}^{A}\varepsilon_{x_0}+(1-\alpha_{\mathcal{F}}^{A})\varepsilon_{x_0}^{A}$.

<u>Proof</u>. Without loss of generality we may suppose A Borel and fine closed. The assertion is obvious if $x_0\in b(A)$; more precisely in this case we have

$$\{\alpha_{\mathcal{F}}^{A} \mid \mathcal{F}\in\mathcal{U}_{x_0}\}=\{1\}.$$

Also is $x_0\in X\setminus\bar{A}$ we have

$$\{\alpha_{\mathcal{F}}^{A} \mid \mathcal{F}\in\mathcal{U}_{x_0}\}=\{0\}$$

So let $x_0\in\partial A$ and suppose that there exists $\alpha_0\in(0,1)$ such that for any $\mathcal{F}\in\mathcal{U}_{x_0}$ the family $(\varepsilon_x^A)_x$ of measures on X is strongly convergent along $\mathcal{F}$ to the measure $\alpha\varepsilon_{x_0}+(1-\alpha)\varepsilon_{x_0}^{A}$ where $\alpha\neq\alpha_0$. In this case for any bounded continuous generator p of $\mathcal{S}$ we have

$$\lim_{\mathcal{F}}\varepsilon_x^A(p)=\alpha p(x_0)+(1-\alpha)\varepsilon_{x_0}^{A}(p)\neq\alpha_0 p(x_0)+(1-\alpha_0)\varepsilon_{x_0}^{A}(p)$$

Hence there exists an open neighbourhood V of x_0 such that for any $x\in V$ we have

$$\varepsilon_x^A p\neq\alpha_0 p(x_0)+(1-\alpha_0)\varepsilon_{x_0}^{A}(p)=:\beta$$

and such that for any $x\in V$ we have $p(x)>\beta$.

Let us denote

$$V_+=\{x\in V\,|\,\varepsilon_x^A(p)>\beta\}\,,\quad V_-=\{x\in V\,|\,\varepsilon_x^A(p)<\beta\}$$

Obviously we have $V=V_+\cup V_-$, $V_+\cap V_-=\emptyset$, V_+ open, V_- finely open and $x_o\in V_-$. We remark that the fine closure of the set $V_-\setminus A$ contains V_-. Indeed in the contrary case the fine interior of the set $V_-\cap A$ is non empty and therefore in any fine-interior point x of the set $V_-\cap A$ we have the contradictory relations

$$\beta < p(x)=B^A p(x) <\beta .$$

First we show that for any $\mathcal{F}\in\mathcal{U}_{x_o}$ such that $V_-\in\mathcal{F}$ we have $\lim\limits_{\mathcal{F}} B_p^A(x)=\varepsilon_{x_o}^A(p)$ i.e. $\alpha_{\mathcal{F}}^A=0$. Using the preceding remark it will be sufficient to consider that $V_-\setminus A\in\mathcal{F}$. Also for any $x\in V_-$ we have

$$\varepsilon_x^{V_+}(V_+)=0\,,\quad \varepsilon_x^{A\cup V_+\cup C_V}\precsim \varepsilon_x^{A\cup C_V}+\varepsilon_x^{V_+}\quad ,$$

and on the other hand the measure $\varepsilon_x^{V_+}$ is carried by $\overline{V}_+^f\setminus V_+$ and therefore by the fine boundary of V_+ which is contained in ∂V. But using [6], Proposition 1.4 we deduce that $\varepsilon_x^{A\cup V_+\cup C_V}(M)\leq\varepsilon_x^{A\cup C_V}(M)$ for any Borel subset M of ∂V and any $x\in V_-$. Hence

$$\varepsilon_x^{A\cup V_+\cup C_V}\precsim\varepsilon_x^{A\cup C_V}\,,\quad \varepsilon_x^{A\cup V_+\cup C_V}=\varepsilon_x^{A\cup C_V}$$

for any $x\in V_-$.

Obviously, since $\varepsilon_x^A(p) < \beta = \alpha_o p(x_o) + (1-\alpha_o)\varepsilon_{x_o}^A(p)$ for any $x \in V_- \setminus A$ we deduce that $\alpha_{\mathcal{F}}^A \le \alpha_o$ and therefore, using Theorem 4, we have

$$\alpha_{\mathcal{F}}^{A\cup V_+ \cup C_V} = \alpha_{\mathcal{F}}^{A\cup C_V} = \alpha_{\mathcal{F}}^A \le \alpha_o < 1 .$$

for any $\mathcal{F} \in \mathcal{U}_{x_o}$ such that $V_- \setminus A \in \mathcal{F}$. Since $V_- \setminus A = X \setminus (A \cup V_+ \cup C_V)$ we conclude, using Proposition 5, that the family $(\varepsilon_x^{A\cap C_V})_x$ of measures on X is strongly convergent, along $\mathcal{F}$, to the measure $\varepsilon_{x_o}^{A\cup C_V}$. Using again Theorem 4 we deduce that for any $\mathcal{F} \in \mathcal{U}_{x_o}$ with $V_- \setminus A \in \mathcal{F}$ we have $\lim\limits_{\mathcal{F}} B^A p(x) = \varepsilon_{x_o}^A(p)$, $\alpha_{\mathcal{F}}^A = 0$.

Now, we show that for any $\mathcal{F} \in \mathcal{U}_{x_o}$ such that $V_+ \setminus A \in \mathcal{F}$ the family $(\varepsilon_x^A)_x$ of measures on X is strongly convergent along $\mathcal{F}$ to the measure ε_{x_o}. Since $x_o \in V_-$ obviously we have

$$\lim_{\substack{x \to x_o \\ x \in V_+ \setminus A}} \varepsilon_x^{A\cup V_-}(p) = p(x_o) \qquad (\forall)\quad p \in \mathcal{S}$$

On the other hand, for any $x \in V_+$, we have

$$\varepsilon_x^{A\cup V_-} \succcurlyeq \varepsilon_x^A + \varepsilon_x^{V_- \setminus A}$$

and therefore, for our purpose, it will be sufficient to prove that

$$\lim_{\substack{x \to x_o \\ x \in V_+ \setminus A}} \varepsilon_x^{A\cup V_-}(V_- \setminus A) = 0 \quad .$$

Let p be a continuous bounded generator of $\mathcal{S}$ such that the linear space generated by the set of its specific minorants from $\mathcal{S}$ is uniformly dense in the set of all bounded and uniformly continuous functions on X .

Let $\delta > 0$ be an arbitrary positive number. Since

$\lim_{V_- \ni x \to x_o} \varepsilon_x^A(p) = B_p^A(x_o) =: b$ and $B_p^A(x) > a > b$ on V_+, changing eventualy V, we may suppose that

$$\varepsilon_x^A(p) > ap(x) \quad \text{on} \quad V_+$$

$$\varepsilon_x^A(p) < (b+\delta)p(x) \quad \text{on} \quad V_-$$

$$|p(x)-p(x_o)| < \delta \quad \text{on} \quad V$$

Without loss of generality we may suppose $p(x_o)=1$. Since $x_o \in V_-$ then we have

$$\lim_{x \to x_o} \varepsilon_x^{A \cup V_-} = p(x_o)$$

and therefore there exists a neighbourhood U_δ of x_o for which we have

$$\int_{\partial V} p d\varepsilon_x^{A \cup V_-} < \delta \qquad (\forall) \quad x \in U_\delta$$

We remark now that for any $x \in V_+ \setminus A$ and any closed subset F of X, $F \subset V_-$ there exists an open subset $D_\delta =: D$ such that

$$F \subset D \subset \bar{D} \subset V \setminus \{x\}, \int_A p d\varepsilon_x^{A \cup V_-} \leq \int_A p d\varepsilon_x^{A \cup V_- \cup D} + \delta .$$

Indeed, let us consider a closed subset H of A such that $\int_{A\setminus H} p d\varepsilon_x^{A\cup V_-} < \delta/2$, let G be an open subset of X such that $F\subset G\subset\overline{G}\subset V\setminus(H\cup\{x\})$ and let (D_n') be a decreasing sequence of open subsets of X such that

$$A\cup V_-\subset D_n' \quad \text{for any } n\in N, \quad \lim_n \int p d\varepsilon_x^{D_n'} = \int p d\varepsilon_x^{A\cup V_-}$$

Noting $D_n = A\cup V_-\cup(G\cap D_n')$ we have

$$\lim_n \int p d\varepsilon_x^{D_n} = \int p d\varepsilon_x^{A\cup V_-}$$

and therefore the sequence $(\varepsilon_x^{D_n})_n$ of measures on X is strongly convergent to the measure $\varepsilon_x^{A\cup V_-}$. Taking $f:X\to[0,1]$ a continuous function on X such that f=1 on H, f=0 on $\overline{G}$ we have

$$\lim_{n\to\infty} \int f.p d\varepsilon_x^{D_n} = \int f p d \,_x^{A\cup V_-}$$

Using [6], Proposition 1.4 we have

$$\varepsilon_x^{D_n}(M) \le \varepsilon_x^{A\cup V_-}(M)$$

for any Borel subset M of X, $M\subset X\setminus G\subset X\setminus D_n$ and therefore

$$\lim_{n\to\infty} \int_H p d\varepsilon_x^{D_n} = \lim_{n\to\infty} \int_H f p d\varepsilon_x^{D_n} = \int_H p d\varepsilon_x^{A\cup V_-} \quad ,$$

$$\int_A p d\varepsilon_x^{A\cup V_-} - \int_A p d\varepsilon_x^{D_n} \le \left(\int_{A\setminus H} p d\varepsilon_x^{A\cup V_-} - \int_{A\setminus H} p d\varepsilon_x^{D_n}\right) + \left(\int_H p d\varepsilon_x^{A\cup V_-} - \int_H p.d\varepsilon_x^{D_n}\right) <$$

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$$< \frac{\delta}{2} + \left(\int_H p\,d\varepsilon_x^{A\cup V_-} - \int_H p\,d\varepsilon_x^{D_n}\right)$$

We choose $D = D_\varepsilon := G\cup D'_n$ for a sufficient large n such that

$$\int_H p\,d\varepsilon_x^{A\cup V_-} - \int_H p\,d\varepsilon_x^{D_n} < \frac{\delta}{2}$$

Let now $x\in(V_+\setminus A)\cap U_\delta$, let F be an arbitrary closed subset of X, $F\subset V_-\setminus A$ and let D be an open subset of X such that

$$F\subset D\subset\bar{D}\subset V\setminus\{x\},\ \int_A p\,d\varepsilon_x^{A\cup V_-} - \int_A p\,d\varepsilon_x^{A\cup V_-\cup D} < \delta .$$

If we denote

$$\mu = \varepsilon_x^{A\cup V_-},\quad \nu = \varepsilon_x^{A\cup V_-\cup D}$$

then using the properties $(B\text{-}H)_1$ and $(B\text{-}H)_2$ we have

$$\mu = \nu|_{A\cup V_-} + (\nu|_{X\setminus(A\cup V_-)})^{A\setminus V_-} ,$$

$$\int_F p\,d\varepsilon_x^{A\cup V_-} = \int_{X\setminus(A\cup V_-)} \varepsilon_y^{A\cup V_-}(p.1_F)\,d\nu =$$

$$\int_{V_+\setminus A} \varepsilon_y^{A\cup V_-}(p.1_F)\,d\nu + \int_{\partial V} \varepsilon_y^{A\cup V_-}(p.1_F)\,d\nu \le \int_{V_+\setminus A} p\,d\nu + \int_{\partial V} p\,d\mu ,$$

$$(1)\quad \int_F p\,d\mu \le \int_{V_+\setminus A} p\,d\nu + \delta .$$

From the relations

$$\varepsilon_x^A = \mu/A + (\mu/_{X\setminus A})^A,\quad \varepsilon_x^A = \nu/_A + (\nu/_{X\setminus A})^A$$

we have

$$B_p^A(x)=\int_A p\,d\mu+\int_{X\setminus A} B^A p\,d\mu=\int_A p\,d\mu+\int_{V_-\setminus A} B^A p\,d\mu+\int_{\partial V} B^A p\,d\mu \quad ,$$

(2) $$B^A p(x)\leq \int_A p\,d\mu+(b+\delta)\int_{V_-\setminus A} p\,d\mu+\delta$$

(3) $$B^A p(x)\geq \int_A p\,d\nu+\int_{V_+\setminus A} B^A p\,d\nu\geq\int_A p\,d\nu+a\int_{V_+\setminus A} p\,d\nu \quad .$$

The above relations (1), (2), (3) give us

$$\int_F p\,d\mu\leq\delta+\frac{1}{a}\left(B^A p(x)-\int_A p\,d\nu\right)\leq$$

$$\leq\delta+\frac{1}{a}\left[\int_A p\,d\mu+(b+\delta)\int_{V_-\setminus A} p\,d\mu+\delta-\int_A p\,d\nu\right]\leq\delta+\frac{1}{a}\left[\delta+(b+\delta)\int_{V_-\setminus A} p\,d\mu\right],$$

$$\int_F p\,d\mu-\frac{b+\delta}{a}\int_{V_-\setminus A} p\,d\mu\leq\delta\left(1+\frac{1}{a}\right) \quad .$$

If F was chosen such that $\int_{V_-\setminus A} p\,d\mu<\delta+\int_F p\,d\mu$ then

$$\left(1-\frac{b+\delta}{a}\right)\int_{V_-\setminus A} p\,d\mu\leq\delta\left(2+\frac{1}{a}\right)$$

The number δ being arbitrary we have

$$\lim_{V_+\setminus A\ni x\to x_0}\int_{V_-\setminus A} p\,d\varepsilon_x^{A\cup V_-}=0, \qquad \lim_{V_+\setminus A\ni x\to x_0}\varepsilon_x^{A\cup V_-}(p)=p(x_0)$$

Obviously we have

$$\lim_{b(A)\ni x\to x_0}\varepsilon_x^{A\cup V_-}(p)=p(x_0)$$

and therefore

$$\lim_{V_+\cup b(A)\setminus A\ni x\to x_o} \varepsilon_x^{A\cup V_-}(p)=p(x_o)$$

The proof is finished since the fine closure of the set $V_+\cup b(A)\setminus A$ contains V_+.

In the last part of this paper we show that if the axiom D_o drop then the assertion from theorem 6 drop also.

Indeed let us consider, for instance, the unit disk U from R^2 and the subset A of U of the forms $A=\bigcup_{k=1}^{\infty}[a_k,a_{k-1}]$ where $(a_k)_k$ is a strictly decreasing sequence of real numbers from the interval $(0,1]$ of the real line such that the series of functions $\sum_{k=1}^{\infty} B_1^{[a_k,a_{k-1}]}$ is convergent at the point z=0.

We remark that the set A is thin at z=0 in the same time with the set $\bigcup_{k\geq n}[a_k, a_{k-1}]$ for any $n\in N$. On the other hand we have

$$1>\sum_{k\geq n} B_1^{[a_k,a_{k-1}]}(0)>B_1^{\bigcup_{k\geq n}[a_k,a_{k-1}]}(0)$$

for n sufficiently large and therefore the set $\bigcup_{k\geq n}[a_k,a_{k-1}]$ is thin at z=0 and consequently the set A is also thin at z=0.

We denote by s the positive superharmonic function on U given by $s:=B_1^A$. Since A is thin at z=0 we have $s(0)=B_1^A(0)<1$.

We choose two real numbers r_1,r_2 such that $s(0)<r_1<r_2<1$ and we denote by M the set

$$M=\{x\in U \mid r_1\leq s(x)\leq r_2\}\ .$$

We consider now the H-cone of functions S' on the set $U\setminus M$ given by

$$S'=\{B^{U\setminus M}t \mid t\in \mathcal{S}_+(U)\}$$

where $\mathcal{S}_+(U)$ is the H-cone of all positive superharmonic functions on U. Obviously $1=B_1^{U\setminus M}$ on $U\setminus M$ and

$$B_1^A=B^A(B_1^{U\setminus M})='B_1^A \quad \text{on} \quad U\setminus M$$

where $'B_t^A$ means the balayage on the set A in the new H-cone of functions S'. From the preceding considerations it follows that if $\mathfrak{F}$ is an ultrafilter on $U\setminus M$ which converges at z=0 we have

$$\lim_{\mathfrak{F}}\ 'B_1^A=\lim_{\mathfrak{F}}\ B_1^A\in[s(0),r_1]\cup[r_1,1]$$

and on the other hand there exists $\mathfrak{F}'$, $\mathfrak{F}''$ ultrafilter on $U\setminus M$ for which $\lim_{\mathfrak{F}'}\ 'B_1^A=s(0)$, $\lim_{\mathfrak{F}''}\ 'B_1^A=1$.

So the assertion from Theorem 6 is not true for the standard H-cone of function S'.

REFERENCES

1 N. Boboc, Gh. Bucur : Natural localization and natural sheaf property in standard H-cones of functions: Preprint Series in Math. no.32, 1982, INCREST, Bucharest.

2 N. Boboc, Gh. Bucur: Natural localization and natural sheaf property in standard H-cones of functions I; II. Rev. Roum. Math. Pures et Appl. 30 (1985), 1, 1-21; 30 (1985), 3, 193-213.

3 N. Boboc, Gh. Bucur. Potentials and supermedian functions on fine open sets in standard H-cones. Rev. Roumaine Math. Pures et Appl. 31 (1986), 745-775.

4 N. Boboc, Gh. Bucur, A. Cornea: Order and convexity in Potential Theory: H-cones, Lecture Notes in Mathematics, 853, Springer Verlag, Berlin-Heiseller-New York, 1981.

5 N. Boboc, A.Cornea: Comportement des balayees des mesures ponctuelles. Comportement des solution du probleme de Dirichlet aux points irreguliers. C.R. Acad. Sci. Paris, Ser. A 254 (1967), 995-997.

6 Gh. Bucur, W. Hansen: Balayage, quasi-balayage and fine decomposition properties in standard H-cones of functions, Rev. Roumaine Math. Pures Appl. 29 (1984), 19-41.

7 O. Frosman: Les points irreguliers dans la theorie du potentiel et le critere de Wiener. Medd Lunds. Univ. Mat. Sem. 4, 1-10 (1939).

8 W. Hansen: Convergence of Balayage Measures. Math. Ann. 264 (1983), 437-446.

9 J. Köhn, M. Sieveking: Reguläre und extremale Randpunkte in der Potentialtheorie, Rev. Roum. Math. Pures Appl. 12 (1967), 1489-1502.

10 J. Lukes, J. Maly: On the behaviour of the Perrin generalized solution. Math. Ann. 257 (1981), 355-366.