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On Quasi-Similarity of Contractions with Finite Defect Indices

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Necessary and sufficient conditions are obtained, under which two C_{10} contractions with finite defect indices and the same Fredholm index are quasi-similar. Some invariants under quasi-similarity and some examples are given.

Finally we point out some consequences of the main result of this paper.

0. INTRODUCTION

For those C_{10} contractions T on a Hilbert space H for which the defect indices are finite ($d_T < \infty$, $d_{T^*} < \infty$) and $d_T - d_{T^*} = -1$, a complete invariant under quasi-similarity has been obtained by V. I. Vasyunin and N. G. Makarov [1]. (For the terminology see Section 1.)

In this paper we find necessary and sufficient conditions under which two C_{10} contractions T_1, T_2 with finite defect indices and $d_{T_1} - d_{T_1^*} = d_{T_2} - d_{T_2^*} = -k$ ($k = 1,2,\ldots$) are quasi-similar. We point out some invariants under quasi-similarity of such kind of contractions and we give some examples proving the existence of C_{-k} operators which are not quasi-similar (for k = 1 see [8, 1]).

In Section 3, as a consequence of the main result of Section 2, we find some new necessary and sufficient conditions on a contraction with finite defect indices in order to be quasi-similar to a unilateral shift.

Finally, our results give the possibility to add some new equivalent conditions to [11, Theorem 3] and [12, Theorem 2] in terms of the multiplicity of the operators.

1. PRELIMINARIES

In the following all the operators are acting on complex, separable Hilbert spaces. The main reference is the book of Sz.–Nagy and Foias [6].

Recall that for operators T_1 and T_2 on H_1 and H_2, respectively, $T_1 \overset{i}{\prec} T_2$ (resp. $T_1 \overset{d}{\prec} T_2$) denotes that there exists an operator $X : H_1 \to H_2$ which is injective (resp. has dense range) such that $XT_1 = T_2X$. If X is both injective and with dense range (called quasi-affinity), then we denote this by $T_1 \prec T_2$. T_1, T_2 are quasi-similar ($T_1 \sim T_2$) if $T_1 \prec T_2$ and $T_2 \prec T_1$. For an operator T on H, let μ_T denote the multiplicity of T, that is the least cardinal number of a subset M of elements in H for which $H = \bigvee_{n=0}^{\infty} T^n M$. Note that if $T_1 \prec T_2$ then $\mu_{T_1} \geq \mu_{T_2}$.

Let T be a contraction on H. The defect indices of T are, by definition, $d_T = \operatorname{rank}(I - T^*T)^{\frac{1}{2}}$ and $d_{T^*} = \operatorname{rank}(I - TT^*)^{\frac{1}{2}}$. If $d_T < \infty$ and $d_{T^*} < \infty$ then T is a Fredholm operator and the Fredholm index $\operatorname{ind} T$ is equal to $d_T - d_{T^*}$.

Recall that $T \in C_{1.}$ (resp. $C_{.1}$) if $T^n h \not\to 0$ (resp. $T^{*n} h \not\to 0$) for all $h \neq 0$; $C_{11} = C_{1.} \cap C_{.1}$. For every $T \in C_{1.}$ we have $d_T \leq d_{T^*}$. $T \in C_{0.}$ (resp. $C_{.0}$) if $T^n h \to 0$ (resp. $T^{*n} h \to 0$) for all h; $C_{10} = C_{1.} \cap C_{.0}$.

Let $\mathbb{C}$ be the complex plane. For a positive integer n, let L^2_n and H^2_n denote the standard Lebesgue and Hardy spaces of $\mathbb{C}$-valued functions defined on the unit circle $\partial\mathbb{D}$. We will use "e^{it}" to denote the argument of a function defined on $\partial\mathbb{D}$ and for an analytic function, we will freely identify $h(e^{it})$ on the circle with its extension to the unit disk $h(\lambda)$ (see [2]).

2. THE QUASI-SIMILARITY OF C_{-k} CONTRACTIONS

Let us define for $k = 1,2,\ldots$

$$C_{-k} = \{T \in C_{10};\ d_T < \infty,\ d_{T^*} < \infty,\ \operatorname{ind} T = -k\}.$$

Note that $S_k \in C_{-k}$, where S_k denote the unilateral shift on H^2_k.

For $T \in C_{-k}$ and $d_{T^*} = n$ we have $d_T = n - k$ and we can consider its characteristic function Θ acting from $\mathbb{C}^{n-k}$ to $\mathbb{C}^n$. Therefore Θ is an $n \times (n-k)$ matrix over H^∞. Since $T \in C_{10}$, Θ is both inner and $*$-outer function (cf. Prop. 3.5 in [6, Chap. VI]). The functional model of T is defined on $H(\Theta) = H^2_n \ominus \Theta H^2_{n-k}$ by

$$T_\Theta f = P_{H(\Theta)}(e^{it} f), \quad f \in H(\Theta) \tag{2.1}$$

where $P_{H(\Theta)}$ denotes the orthogonal projection from H^2_n into $H(\Theta)$.

Let $\{e_i\}_{i=1}^n$ be the standard base of $\mathbb{C}^n$. Since Θ is inner its values $\Theta(e^{it})$ on $\partial\mathbb{D}$ are isometries a.e. (cf. Prop. 2.2 in [6, Chap. V.]) we have (see [8]):

$$\sum_{\sigma} |\Theta_\sigma(e^{it})|^2 = 1 \quad \text{a.e. on } \partial\mathbb{D}, \tag{2.2}$$

where σ runs over the set Σ_k^n of subsets $\sigma = \{i_1, \ldots, i_k\}$ $(i_1 < i_2 < \cdots < i_k)$ of the set $\{1,2,\ldots,n\}$, and

$$\Theta_\sigma = \det(e_{i_1}, \ldots, e_{i_k}, \Theta). \tag{2.3}$$

On the other hand Θ is a $*$-outer function and from Theorem on Outer Functions [3, p.21] it follows that the greatest common inner divisor of $\{\Theta_\sigma\}_{\sigma\in\Sigma_k^n}$ is the constant inner function, that is,

$$\bigwedge_{\sigma} \Theta_\sigma^i = 1\,, \tag{2.4}$$

where Θ_σ^i stands for the inner part of Θ_σ.

We start by proving the following two preliminary lemmas.

LEMMA 2. 1. *For $f_1 \in H^2_n$ the equality*

$$\det(f_1, f_2, \ldots, f_k, \Theta) \equiv 0 \tag{2.5}$$

holds for any $f_2, \ldots, f_k \in H^2_n$ if and only if $f_1 \in \Theta H^2_{n-k}$.

PROOF. ($\Leftarrow$) is obviously.

($\Rightarrow$) Taking $f_2, \ldots, f_k \in \{e_i\}_{i=1}^n$ we infer that $\operatorname{rank}(f_1(e^{it})), \Theta(e^{it})) = n - k$ a.e. on $\partial\mathbb{D}$. Now since $\operatorname{rank}\Theta(e^{it}) = n - k$ a.e. on $\partial\mathbb{D}$ it follows that $f_1(e^{it})$ is a linear combination of the columns of the matrix $\Theta(e^{it})$ a.e. on $\partial\mathbb{D}$, in other words $f_1 = \Theta g$ with $g \in L^2_{n-k}$.

From (2.2) we deduce that there exists at least one $\sigma \in \Sigma_k^n$ for which $\Theta_\sigma(e^{it})$ is non-zero on a set of positive measure and therefore a.e. on $\partial\mathbb{D}$.

For any such $\sigma = \{i_1, \ldots, i_k\}$ we apply the matrix $(e_{i_1}, \ldots, e_{i_k}, \Theta)$ to the equality

$$f_1 = (e_{i_1}, \ldots, e_{i_k}, \Theta)\ (\underbrace{0, \ldots, 0}_{k\text{-times}}, g)^t$$

("t" standing for the transposed matrix).

and we obtain $g = \Theta_\sigma^{-1} h_\sigma$, $h_\sigma \in H^2_{n-p}$. Thus, on account of (2.4) and applying a lemma of [5] we deduce that $g \in H^2_{n-p}$. This completes the proof.

LEMMA 2. 2. *If* $T = T_\Theta \in C_{.0}$ *is such that* $d_{T^*} < \infty$ *and* $\operatorname{ind} T \neq 0$ *then* $\sigma_p(T^*) \supset \mathbf{D}$

(σ_p *stands for the point spectrum*).

PROOF. Setting $d_{T^*} = n$ and $\operatorname{ind} T = -k$ $(k = 1,2,\ldots)$ we have as in the previous lemma that there exists $\sigma = \{i_1, \ldots, i_k\} \in \Sigma^n_k$ such that $\Theta_\sigma \not\equiv 0$. Put $V = (V_1, \ldots, V_n)$, where $V_i = \det(e_i, e_{i_2}, \ldots, e_{i_k}, \Theta)$, $i = 1,2,\ldots, n$. Obviously $V \not\equiv 0$. Let us show that for each $\lambda \in \mathbf{D}$ the function $f(e^{it}) = (1 - \lambda e^{it})^{-1} V(\bar{\lambda})^*$ is an eigenvector corresponding to the eigenvalue λ of T^*_Θ.

It is sufficient to prove that $f \perp \Theta H^2_{n-k}$. For any $h \in H^2_{n-k}$ and $\lambda \in \mathbf{D}$ we have

$$\Theta h \perp (1 - \bar{\lambda} e^{it})^{-1} V(\lambda)^* \Longleftrightarrow (\Theta h)(\lambda) \perp V(\lambda)^* \Longleftrightarrow \det(\Theta(\lambda) h(\lambda), e_{i_2}, \ldots, e_{i_k}, \Theta(\lambda)) = 0$$

which is obviousl y. The proof is complete.

Before stating our main result, let us define for the model contraction $T_\Theta \in C_{-k}$ $(d_{T^*_\Theta} = n)$ the following set of inner functions:

$$A_\Theta = \{\det(f_1, \ldots, f_k, \Theta)^i;\ f_1, \ldots, f_k \in H^2_n\}. \tag{2.6}$$

Considering $T_\Phi \in C_{-k}$ $(d_{T^*_\Phi} = m)$ we set $A_\Theta \overset{Y}{\subset} A_\Phi$ if there exists an $m \times n$ matrix over H^∞ such that for any $f_1, \ldots, f_k \in H^2_n$

$$\det(f_1, \ldots, f_k, \Theta)^i = \rho(f_1, \ldots, f_k) \det(Yf_1, \ldots, Yf_k, \Phi)^i, \tag{2.7}$$

where $\rho(f_1, \ldots, f_k)$ is a non-zero inner function. Here we admit that $0^i = 0$.

The main result of this paper is the following

THEOREM 2. 3. *If* $T_\Theta, T_\Phi \in C_{-k}$ *then*

$$T_\Theta \overset{d}{\prec} T_\Phi \Longleftrightarrow T_\Theta \prec T_\Phi \Longleftrightarrow A_\Theta \overset{Y}{\subset} A_\Phi.$$

The proof of this theorem will be done in two lemmas.

LEMMA 2. 4. *If* $T_\Theta, T_\Phi \in C_{-k}$ *then*

$$T_\Theta \overset{d}{\prec} T_\Phi \Longrightarrow T_\Theta \prec T_\Phi \Longrightarrow A_\Theta \overset{Y}{\subset} A_\Phi.$$

PROOF. First we note that if $f_1 \in H(\Theta)$, $f_1 \not\equiv 0$, then there exist $f_2, \dots, f_k \in H(\Theta)$ with $\det(f_1, \dots, f_k, \Theta) \not\equiv 0$.

Indeed, if $\det(f_1, g_2, \dots, g_k, \Theta) \equiv 0$ for every $g_2, \dots, g_k \in H^2_n$ then, according to Lemma 2.1, we infer that $f_1 \in \Theta H^2_{n-k}$, contradiction. Thus, there exist $g_2, \dots, g_k \in H^2_n$ such that $\det(f_1, g_2, \dots, g_k, \Theta) \not\equiv 0$, whence $\det(f_1, f_2, \dots, f_k, \Theta) \not\equiv 0$ for $f_i = P_{H(\Theta)} g_i$, $i = 2, \dots, n$.

Now let $X : H(\Theta) \to H(\Phi)$ be an operator with dense range intertwining T_Θ and T_Φ.

For $f_1, \dots, f_k \in H(\Theta)$ with $\det(f_1, \dots, f_k, \Theta) \not\equiv 0$ let us consider the following subspaces:

$$H_{f_1, \dots, f_k} = \operatorname{span}\{T_\Theta^{p_1} f_1, \dots, T_\Theta^{p_k} f_k;\ p_1, \dots, p_k \geq 0\}$$

$$H_{Xf_1, \dots, Xf_k} = \operatorname{span}\{T_\Phi^{p_1} Xf_1, \dots, T_\Phi^{p_k} Xf_k;\ p_1, \dots, p_k \geq 0\}.$$

Obviously $H_{f_1, \dots, f_k}$ (resp. $H_{Xf_1, \dots, Xf_k}$) is an invariant subspace of T_Θ (resp. T_Φ). Note that

$$X^*(H(\Phi) \ominus H_{Xf_1, \dots, Xf_k}) \subset H(\Theta) \ominus H_{f_1, \dots, f_k}. \tag{2.8}$$

By a theorem of Beurling we deduce that

$$H_{f_1, \dots, f_k} \oplus \Theta H^2_{n-p} = \operatorname{clos}\{(f_1, \dots, f_k, \Theta) P(\mathbf{C}^n)\} = (f_1, \dots, f_k, \Theta)^i H^2_n,$$

where $P(\mathbf{C}^n)$ denotes the set of all polynomials with values in $\mathbf{C}^n$. Hence we obtain that

$$H(\Theta) \ominus H_{f_1, \dots, f_k} = H^2_n \ominus (f_1, \dots, f_k, \Theta)^i H^2_n = H((f_1, \dots, f_k, \Theta)^i)$$

and analogously that

$$H(\Phi) \ominus H_{Xf_1, \dots, Xf_k} = H^2_m \ominus (Xf_1, \dots, Xf_k, \Phi)^i H^2_m = H((Xf_1, \dots, Xf_k, \Phi)^i).$$

We have also that

$$T^*_\Theta \,|\, H((f_1, \dots, f_k, \Theta)^i) = T^*_{(f_1, \dots, f_k, \Theta)^i} \quad ; \quad T^*_\Phi \,|\, H((Xf_1, \dots, Xf_k, \Phi)^i) = T^*_{(Xf_1, \dots, Xf_k, \Phi)^i}.$$

The relation (2.8) shows that the operator

$$W_{f_1, \dots, f_k} := X^* \,|\, H((Xf_1, \dots, Xf_k, \Phi)^i)$$

acts from $H((Xf_1,\dots,Xf_k,\Phi)^i)$ to $H((f_1,\dots,f_k,\Theta)^i)$ and

$$W_{f_1,\dots,f_k} T^*_{(Xf_1,\dots,Xf_k,\Phi)^i} = T^*_{(f_1,\dots,f_k,\Theta)^i} W_{f_1,\dots,f_k}. \tag{2.9}$$

Since $\det(f_1,\dots,f_k,\Theta) \not\equiv 0$ it follows that $(f_1,\dots,f_k,\Theta)^i$ is inner from both sides. Therefore $T_{(f_1,\dots,f_k,\Theta)^i} \in C_{00}$ and hence to the class C_0 (cf. Thm. 5.2 in [6, Chap. VI]).

We prove now that $\operatorname{Ker} X = \{0\}$. Indeed, otherwise there exists $f_1 \in H(\Theta)$, $f_1 \neq 0$ such that $Xf_1 = 0$. Then the injective operator $W_{f_1,\dots,f_k}$ intertwines the operators $T^*_{(f_1,\dots,f_k)^i} \in C_0$ and $T^*_{(Xf_2,\dots,Xf_k,\Phi)^i}$. This is impossible since by virtue of Lemma 2.2, $\sigma_p(T^*_{(Xf_2,\dots,Xf_k,\Phi)^i}) \supset \mathbb{D}$ while $\sigma_p(T^*_{(f_1,\dots,f_k,\Theta)^i}) \not\supset \mathbb{D}$ (see Thm. 5.1 in [6, Chap.III]). Thus X is injective. In particular for $f_1,\dots,f_k \in H(\Theta)$ with $\det(f_1,\dots,f_k,\Theta) \not\equiv 0$ it follows that $\det(Xf_1,\dots,Xf_k,\Phi) \not\equiv 0$ and as above $T^*_{(Xf_1,\dots,Xf_k,\Phi)^i} \in C_0$.

Let us denote by $K_{f_1,\dots,f_k}$ the space $\operatorname{clos} X^* H((Xf_1,\dots,Xf_k,\Phi)^i)$.

Taking into account (2.9) it is easy to see that $K_{f_1,\dots,f_k}$ is an invariant subspace of $T^*_{(f_1,\dots,f_k,\Theta)^i}$ and

$$T^*_{(Xf_1,\dots,Xf_k,\Phi)^i} \prec T^*_{(Xf_1,\dots,Xf_k,\Theta)^i} | K_{f_1,\dots,f_k}.$$

Since both these operators belong to the class C_0, they are quasi-similar (cf. [7, Corollary 1]) and the determinants of their characteristic functions are equal.

Therefore, we have proved that

$$\det(Xf_1,\dots,Xf_k,\Phi)^i \text{ divides } \det(f_1,\dots,f_k,\Theta)^i. \tag{2.10}$$

On the other hand since $XT_\Theta = T_\Theta X$, by the lifting theorem [6; p.258], there exists an $m \times n$ matrix Y over H^∞ such that

$$X = P_{H(\Theta)} Y | H(\Theta) \quad \text{and} \quad Y\Theta H^2_{n-k} \subseteq \Phi H^2_{m-k}. \tag{2.11}$$

For $f_1,\dots,f_k \in H^2_n$ with $\det(f_1,\dots,f_k,\Theta) \not\equiv 0$ we have

$$\det(f_1,\dots,f_k,\Theta)^i = \det(P_{H(\Theta)}f_1,\dots,P_{H(\Theta)}f_k,\Theta)^i \overset{(2.10)}{=}$$

$$= \rho(f_1,\dots,f_k) \det(XP_{H(\Theta)}f_1,\dots,XP_{H(\Theta)}f_k,\Phi)^i \overset{(2.11)}{=}$$

$$= \rho(f_1,\dots,f_k) \det(Yf_1,\dots,Yf_k,\Phi)^i,$$

where $\rho(f_1, \dots, f_k) \not\equiv 0$ is an inner function.

For $f_1, \dots, f_k \in H^2_n$ with $\det(f_1, \dots, f_k, \Theta) \equiv 0$ it follows that $\det(P_{H(\Theta)} f_1, \dots, P_{H(\Theta)} f_k, \Theta) \equiv 0$. Since X is injective we infer that $\det(XP_{H(\Theta)} f_1, \dots, XP_{H(\Theta)} f_k, \Phi) \equiv 0$ whence $\det(Yf_1, \dots, Yf_k, \Phi) \equiv 0$. Therefore $A_\Theta \overset{Y}{\subset} A_\Phi$ and the proof is complete.

COROLLARY 2. 5. *If* $T_\Theta, T_\Phi \in C_{-k}$ *and* $T_\Theta \sim T_\Phi$ *then* $A_\Theta = A_\Phi$.

We are going to prove a statement which completes the proof of Theorem 2.3.

LEMMA 2.6. *If* $T_\Theta, T_\Phi \in C_{-k}$ *is such that* $A_\Theta \overset{Y}{\subset} A_\Phi$, *then* $T_\Theta \prec T_\Phi$.

PROOF. As (2.7) holds, we infer that for any $\sigma = \{i_1, \dots, i_k\}, \sigma \in \Sigma^n_m$,

$$\Theta^i_\sigma = \det(e_{i_1}, \dots, e_{i_k}, \Theta)^i = \rho(e_{i_1}, \dots, e_{i_k}) \det(Ye_{i_1}, \dots, Ye_{i_k}, \Phi)^i.$$

From (2.4) it follows that

$$\bigwedge_{\substack{\sigma = \{i_1, \dots, i_k\} \\ \sigma \in \Sigma^n_m}} \det(Ye_1, \dots, Ye_k, \Phi)^i = 1$$

and according to the Theorem on Outer Function [3, p.21] the matrix (Y, Φ) is outer, whence we deduce that

$$P_{H(\Phi)} Y H^2_n \text{ is dense in } H(\Phi). \tag{2.12}$$

Now let $f_1 = \Theta h$, $h \in H^2_{n-k}$. From (2.7) and (2.12) it follows that $\det(Y\Theta h, g_2, \dots, g_k, \Phi) \equiv 0$ for any $g_2, \dots, g_k \in H^2_m$.

Using Lemma 2.1 we obtain $Y\Theta h \in \Phi H^2_{m-k}$, therefore

$$Y\Theta H^2_{m-k} \subseteq \Phi H^2_{m-k}. \tag{2.13}$$

Let us show that

$$\operatorname{Ker}(Y, \Phi) \subseteq \Theta H^2_{n-k} \oplus H^2_{m-k}. \tag{2.14}$$

For this, let $f_1 \in H^2_n$ such that $Yf_1 \in \Phi H^2_{m-k}$. From (2.7) we have $\det(f_1, f_2, \dots, f_k, \Theta) \equiv 0$ for any $f_2, \dots, f_k \in H^2_n$.

Using Lemma 2.1 we deduce $f_1 \in \Theta H^2_{n-k}$, therefore (2.14) holds.

Let us define X to be $P_{H(\Phi)} Y|_{H(\Theta)}$. Since the relations (2.12), (2.13), (2.14) hold, the argument that X is a quasi-affinity and satisfies $XT_{\Theta} = T_{\Phi}X$ is straightforward.

The proof is complete.

COROLLARY 2.7. *Let* $T_{\Phi} \in C_{-k}$ *be such that* $d_{T^*_{\Phi}} = m$.

Then $S_k \prec T_{\Phi}$ *iff there exists an* $m\times k$ *matrix* Y *over* H^{∞} *such that* $\det(Y,\Phi)$ *is an outer function.*

PROOF. *Setting* $\Theta \equiv 0$ and $n = k$ in Theorem 2.3 we find that $S_k \prec T_{\Phi}$ if there exists Y as above with the property that for any $f_1, \ldots, f_k \in H^2_k$

$$\det(f_1, \ldots, f_k)^i = \rho(f_1, \ldots, f_k) \det(Yf_1, \ldots, Yf_k, \Phi)^i. \tag{2.15}$$

Since $\det(Yf_1, \ldots, Yf_k, \Phi) = \det(Y,\Phi) \det\begin{pmatrix} f_1, \ldots, f_k, 0 \\ 0, \ldots, 0, 1 \end{pmatrix} = \det(Y,\Phi)\det(f_1, \ldots, f_k)$ the relation (2.15) is equivalent to $\rho(f_1, \ldots, f_k)\det(Y,\Phi)^i = 1$ that is $\rho(f_1, \ldots, f_k)$ is a constant inner function and $\det(Y,\Phi)$ is an outer function.

In what follows we shall define the i-spectrum ($i = 1,2,\ldots$) of an operator T on $\mathcal{H}$ as being the compact set

$$\sigma_i(T) = \bigcap_{(f)_i} \sigma(P_{\mathcal{H}^{\perp}_{f_1,\ldots,f_i}} T|\mathcal{H}^{\perp}_{f_1,\ldots,f_i}),$$

where $(f)_i = (f_1, \ldots, f_i)$ runs over all i-tuples $(f_1, \ldots, f_i)$ of linearly independent vectors in $\mathcal{H}$ and $\mathcal{H}_{f_1,\ldots,f_i}$ is defined as in the proof of Lemma 2.4.

For $i = 1$ we find again the formula for σ_1 given in [1]. It is easy to see that if $\mu_T = k$ then $\sigma_k(T) = \emptyset$. Now if $T = T_{\Theta} \in C_{-k}$ then taking into account the proof of Lemma 2.4 we get

$$\sigma_k(T) = \bigcap_{\alpha \in A_{\Theta}} \sigma(\alpha), \tag{2.16}$$

where $\sigma(\alpha)$ stands for the spectrum of the inner function α (see [3, Lecture III])

From Corollary 2.5 and (2.16) we infer that σ_k is an invariant to the quasi-similarity of C_{-k} contractions.

Now, using this invariant we can give some examples of C_{-k} operators which are not quasi-similar. (For the class C_{-1} see [4, 8, 1].)

Let us consider $T_\Theta \in C_{-k}$ $(k = 1,2,\dots)$ with Θ given by

$$\Theta = (k+1)^{-\frac{1}{2}}(A, \underbrace{B,\dots,B}_{k\text{-times}})^t \tag{2.17}$$

where A,B are some inner functions such that $A \wedge B = 1$.

EXAMPLE 2. 8. Let $\xi \in \partial\mathbb{D}$, Θ as above, where A is the singular inner function $A(\lambda) = \exp\frac{\lambda+\xi}{\lambda-\xi}$ and B is an infinite Blaschke product with zeros $\lambda_n \to \xi$ (non-tangential). Then $\sigma_k(T_\Theta) = \{\xi\}$. In particular for distinct points ξ we obtain C_{-k} operators which are not quasi-similar.

EXAMPLE 2. 9. Let $\omega \subset \partial\mathbb{D}$ a closed subset. Then there exists $T \in C_{-k}$ such that $\sigma_k(T) = \omega$.

These proofs essentially follow the same line of arguments as given in [1, Ex. 1,2] for the case $k = 1$. We leave the verification to the reader.

3. THE QUASI-SIMILARITY TO A UNILATERAL SHIFT

In this Section we shall provide some new necessary and sufficient conditions for a contraction T with finite defect indices in order to be quasi-similar to a unilateral shift.

The following theorem generalizes [8, Proposition 2] and [9, Theorem 3.1].

THEOREM 3. 1. *Let T_Θ be a C_{-k} contraction with $d_{T^*_\Theta} = n$ and let S_k be the unilateral shift on H^2_k. Then the following are equivalent:*

(i) $T_\Theta \sim S_k$

(ii) *There exists an $n \times k$ matrix Y over H^∞ such that* $\det(Y,\Theta)$ *is outer.*

(iii) $\mu_{T_\Theta} = -\mathrm{ind}\, T_\Theta$.

PROOF. (i) $\Longrightarrow$ (ii) follows from Corollary 2.7.

(ii) $\Longrightarrow$ (i) If (ii) holds then from the same corollary we have $S_k \prec T_\Theta$ and since

$S_k \succ T_\Theta$ (cf. [8, Corollary 2]) we infer that $S_k \sim T_\Theta$.

(i)$\Longrightarrow$(iii) is obviously.

(iii)$\Longrightarrow$(ii). For this let $f_i = (f_i^1, \ldots, f_i^n)^t \in H_n^2$ $(i = 1,2,\ldots,k)$ such that $H_{f_1,\ldots,f_k} = H(\Theta)$ (for notations see the proof of Lemma 2.4). This means that

$$\mathrm{span}\,\{P_{H(\Theta)} e^{im_1 t} f_1, \ldots, P_{H(\Theta)} e^{im_k t} f_k;\ m_1, \ldots, m_k \geq 0\} = H(\Theta)$$

hence

$$\mathrm{span}\,\{e^{im_1 t} f_1, \ldots, e^{im_k t} f_k, \Theta H_{n-k};\ m_1, \ldots, m_k \geq 0\} = H_n^2,$$

that is,

$$\mathrm{clos}\,(f_1, \ldots, f_k, \Theta)\, P(\mathbb{C}^n) = H_n^2. \tag{3.1}$$

Considering the function

$$q(e^{it}) = \Big[\sum_{\substack{i=1,\ldots,k \\ p=1,\ldots,n}} |f_i^p(e^{it})|^2 + 1\Big]^{-\frac{1}{2}}$$

we deduce that the outer function

$$h(\lambda) = \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} \log q(e^{it})\,dt \qquad (\lambda \in \mathbb{D})$$

belongs to H^∞ and $|h(e^{it})| = q(e^{it})$ a.e. on $\partial\mathbb{D}$.

Let $Y_i^p = f_i^p h \in H^\infty$ and $Y_i = (Y_i^1, \ldots, Y_i^n)^t$ $(i = 1,2,\ldots,k)$.

On account of (3.1) we get:

$\mathrm{clos}\,(Y_1, \ldots, Y_k, \Theta) P(\mathbb{C}^n) = \mathrm{clos}\,(f_1 h, \ldots, f_k h, \Theta) P(\mathbb{C}^n) = \mathrm{clos}\,(f_1, \ldots, f_k, \Theta) P(\mathbb{C}^n) = H_n^2$.

Hence (Y,Θ) is outer, where $Y = (Y_1, \ldots, Y_k)$. Therefore $\det(Y,\Theta)$ is outer and the proof is complete.

Using this theorem and [10, Lemma 1] one can easily prove the following

THEOREM 3. 2. *Let* T *be a contraction with finite defect indices. Then the following are equivalent:*

(i) T *is quasi-similar to a unilateral shift.*

(ii) $T \in C_{10}$ *and* $\mu_T = -\mathrm{ind}\,T$.

REMARK 3. 3. If we set Θ as in (2.17), where $A(\lambda) = \exp \frac{\lambda + 1}{\lambda - 1}$ and B is an infinite Blaschke product with zeros $a_n = 1 - b_n$ $(0 \leq b_n < 1, \ \sum b_n < \infty)$, then $T_\Theta \not\prec S_k$ (see [8, Proposition 2]).

Indeed, for every $(k + 1) \times k$ matrix Y over H^∞ we have $\det(Y, \Theta) = \alpha A + \beta B$ for some $\alpha, \beta \in H^\infty$. Since $A \wedge B = 1$ it follows that $\alpha A + \beta B$ will not be outer at any choice of $\alpha, \beta \in H^\infty$ (see [4]). According to Theorem 3.1 we have $T_\Theta \not\prec S_k$.

At the end, let us notice that Theorem 3.1 gives us the possibility to state Theorem 3 from [11] and Theorem2 from [12] in terms of the multiplicity, as follows:

THEOREM 3. 4. *Let* T *be a contraction with finite defect indices and let* $T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$ *be the triangulation of type* $\begin{pmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{pmatrix}$

Then the following statements are equivalent:

(i) T *is quasi-similar to an isometry*

(ii) T_1 *is quasi-similar to a unitary operator and* $\mu_{T_2} = -\text{ind}\, T_2$

THEOREM 3. 5. Let T *be a* $C_{\cdot 0}$ *contraction with finite defect indices and let* $T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$ *be the triungulation of type* $\begin{pmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{pmatrix}$. *Then the following statemnts are equivalent:*

(i) T *is quasi-similar to its Jordan model.*

(ii) $\mu_{T_2} = -\text{ind}\, T_2$.

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