INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

RIGIDITY OF MAPS FROM HOPF ALGEBRAS TO GROUP ALGEBRAS

by

Alexandru BUIUM

PREPRINT SERIES IN MATHEMATICS
No.2/1987

BUCURESTI

RIGIDITY OF MAPS FROM HOPF ALGEBRAS TO GROUP ALGEBRAS

by

Alexandru BUIUM*)

January 1987

*) Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania.

0. INTRODUCTION

It is well known from the theory of algebraic groups that one cannot deform maps from a diagonalizable group to an affine algebraic group except in a trivial way i.e. by composing a fixed map with some "variable" inner automorphism of the target. The main effect of the present paper is that it provides a "non-commutative" analog of the above property. With Sweedler's terminology in mind [5] the non-commutative analog of affine algebraic groups are the finitely generated Hopf algebras while the non-commutative analog of diagonalizable groups are the finitely generated group algebras. What we shall prove is roughly speaking that one cannot deform maps from a finitely generated Hopf algebra to a group algebra except in a trivial way i.e. by composing a fixed map with some "variable coinner automorphism" of the source, see Theorem 1.1 below.

Our approach is quite different from the one used in the "commutative case" (i.e. in the case of affine algebraic groups) and has some interest in itself. In particular it provides new informations also in the "commutative case", see Theorems 2.9 and 2.11 below.

1. STATEMENT OF THE MAIN RESULT

First recall some terminology and notations from [5]. Throughout the paper fields will be commutative and will contain a fixed ground field k_o which is supposed to be algebraically closed of characteristic zero. For a field k we denote by $Hom_k(-,-)$, $Alg_k(-,-)$, $Bialg_k(-,-)$ the sets of k-linear maps, k-algebra maps and k-bialgebra maps respectively. Recall from [5]p.81 that any bialgebra map between Hopf algebras automatically is a Hopf algebra map. A k-bialgebra is called finitely generated if it is so as a k-algebra. For any k-bialgebra H we denote by Δ_H and ε_H (or simply by Δ and ε) the comultiplication and the counit of H; if H has an antipode it will be denoted by S_H (or simply by S). For any Hopf k-algebra H we denote by $G(H)$ the group of group-like elements (i.e. of elements $x \in H$, $x \neq 0$ such that $\Delta x = x \otimes x$). H is called a group k-algebra if it is spanned over k by $G(H)$.

Now recall that if H is a Hopf k-algebra then $Hom_k(H,k)$ has a natural structure of k-algebra (with multiplication $*$ induced by Δ called convolution and unit given by ε) while $Alg_k(H,k)$ is a subgroup of it under convolution [5]p.82. For any $u \in Alg_k(H,k)$ define the map $C(u):H \longrightarrow H$ by the formula

$$C(u)(x) = \sum_{(x)} u(Sx_{(1)})u(x_{(3)})x_{(2)}$$

Here we used the "sigma notation" [5]p.10. It is easy to see (cf.(4.3) below) that $C(u)$ is an invertible k-bialgebra map and the map $C:Alg_k(H,k) \longrightarrow Bialg_k(H,H)^{\times}$ is a group homomorphism (where $M^{\times}$ denotes the group of invertible elements of the monoid M). The maps $C(u)$ above are called coinner automorphisms of H; as one expects, in the commutative case i.e. in the case when H is the coordinate ring of an affine algebraic group X, the coinner automorphisms of H correspond precisely to the inner automorphisms of X.

Finally given two Hopf k-algebras H and F and two maps $\varphi, \psi \in \text{Bialg}_k(H,F)$ we say that φ and ψ are conjugate over a field extension $\tilde{k}$ of k if there is a coinner $\tilde{k}$-automorphism σ of $\tilde{H}:=H\otimes_k\tilde{k}$ such that $\tilde{\varphi}\circ\sigma=\tilde{\psi}$ where $\tilde{\varphi}$ and $\tilde{\psi}$ are the $\tilde{k}$-bialgebra maps from $\tilde{H}$ to $\tilde{F}:=F\otimes_k\tilde{k}$ naturally induced by φ, ψ.

Our main result is the following:

THEOREM 1.1. Let H_o and F_o be two Hopf k_o-algebras with H_o finitely generated and F_o a group algebra, let k be a field extension of k_o and put $H=H_o\otimes_{k_o}k$, $F=F_o\otimes_{k_o}k$. Then for any map $\varphi\in\text{Bialg}_k(H,F)$ there is a map $\varphi_o\in\text{Bialg}_{k_o}(H_o,F_o)$ such that φ and $\varphi_o\otimes 1_k$ are conjugate over some field extension of k.

Intuitively we may view the elements of $\text{Bialg}_k(H,F)$ as families of k_o-bialgebra maps from H_o to F_o with parameter space Spec k; so what our theorem says is that any such family is conjugate to a "constant" one.

In the commutative case Theorem 1.1 is well known and at least in the case when Spec F is connected (i.e. a torus) our theorem is essentially a consequence of the conjugacy of maximal tori in Spec H, see [2] p.135. Note also that an infinitesimal version of Theorem 1.1 is proved in the commutative case in [6] p.116. The idea in [6] is to relate infinitesimal deformations of maps between algebraic groups to cohomology of

G-modules.

Our approach to the non-commutative case will be quite different. First step will be to prove an infinitesimal version of (1.1) by a Hopf-theoretic (rather than cohomological) argument; we will show in fact that there is an explicit "natural" way of constructing "trivializing vector fields" for infinitesimal deformations of maps from Hopf algebras to a fixed group algebra (Theorem 2.9 below). The second step will be to integrate our vector fields (Theorem 2.11 below); this will be done by using a slight variation on Kolchin's existence theorem for Picard-Vessiot extensions [3]p.420.

Our method for the "second step" proves itself to be useful also in some other quite different situations, see [1].

The plan of the paper is as follows. In Section 2 we introduce a deformation-theoretic setting suited for our purpose. Section 3 is devoted to the infinitesimal case. Section 4 is devoted to integration. In Section 5 we make some final comments.

2. THE DEFORMATION THEORETIC SETTING

First some more terminology and conventions.

2.1. For any k-linear space V put $V^{*}=\mathrm{Hom}_k(V,k)$ the linear dual of V. For $u\in V^{*}$ and $x\in V$ we write $\langle u,x\rangle$ instead of $u(x)$. For any $\varphi\in \mathrm{Hom}_k(V,W)$ we denote by $\varphi^{*}\in \mathrm{Hom}_k(W^{*},V^{*})$ the usual transpose of φ so $\langle \varphi^{*}u,x\rangle=\langle u,\varphi x\rangle$ for all $x\in V$, $u\in W^{*}$. If $(u_i)_i$ is a family of elements in V^{*} and if for all $x\in V$ there are at most finitely many indices i such that $\langle u_i,x\rangle\neq 0$ then $\sum u_i$ is a well defined element in V^{*}; we shall consider several times such (possibly infinite) sums.

If $\varphi: A \longrightarrow B$ is a k-algebra map and k_1 is a subfield of k then a map $d: A \longrightarrow B$ is called a k_1-φ-derivation from A to B if it is k_1-linear and $d(xy)=d(x)\varphi(y)+\varphi(x)d(y)$ for all $x,y \in A$. If $A=B$ and φ is the identity then we say d is a k_1-derivation on A. Denote by $\mathrm{Der}_{k_1}^{\varphi}(A,B)$ the set of k_1-φ-derivations from A to B and by $\mathrm{Der}_{k_1}(A)$ the set of k_1-derivations on A. Now suppose $\varphi: A \longrightarrow B$ above is a k-bialgebra map. Then define

$$\mathrm{Bider}_k^{\varphi}(A,B)=\{d \in \mathrm{Der}_k^{\varphi}(A,B);\ d^* \in \mathrm{Der}_k^{\varphi^*}(B^*,A^*)\}$$

$$\mathrm{Bider}_k(A)=\mathrm{Bider}_k^{\mathrm{identity}}(A,A)$$

the elements of which we call biderivations. Here A^* and B^* are viewed as algebras under convolution.

2.2. Let H_o, F_o be k_o-bialgebras, k a field extension of k_o and $H=H_o \otimes k$, $F=F_o \otimes k$. For any $\varphi \in \mathrm{Bialg}_k(H,F)$ we shall define a k-linear map

$$K_{\varphi}: \mathrm{Der}_{k_o}(k) \longrightarrow \mathrm{Bider}_k^{\varphi}(H,F)$$

which we shall think of as the "Kodaira-Spencer map" associated to φ. First we define it as a map from $\mathrm{Der}_{k_o}(k)$ to $\mathrm{Hom}_{k_o}(H,F)$ as follows. For any $\delta \in \mathrm{Der}_{k_o}(k)$ denote by $\delta^1, \delta^2, \delta^{11}$ and δ^{22} the unique k_o-derivations on $H, F, H \otimes_k H$ and $F \otimes_k F$ respectively which agree with δ on k and vanish on $H_o, F_o, H_o \otimes H_o$ and $F_o \otimes F_o$. Explicitely we have $\delta^1(ax)=(\delta a)x$ for $a \in k$, $x \in H_o$, $\delta^{11}(ax \otimes y)=(\delta a)x \otimes y$ for $a \in k$, $x,y \in H_o$ and similarily for δ^2 and δ^{22}. Now put $K_{\varphi}(\delta)=\delta^2 \circ \varphi - \varphi \circ \delta^1$. One easily sees that $K_{\varphi}(\delta) \in \mathrm{Der}_k^{\varphi}(H,F)$. We claim that $K_{\varphi}(\delta) \in \mathrm{Bider}_k^{\varphi}(H,F)$. To check this we are forced to do some computations:

LEMMA 2.3. $\delta^{11}\circ\Delta_H=\Delta_H\circ\delta^1$ and $\delta^{22}\circ\Delta_F=\Delta_F\circ\delta^2$.

Proof. Just use definitions.

LEMMA 2.4. $\delta^{ii}(x\otimes y)=(\delta^i x)\otimes y+x\otimes\delta^i y$ for all x,y and $i=1,2$.

Proof. Write $x=\sum a_p x_p$, $y=\sum b_q y_q$ with $a_p,b_q\in k$ and $x_p,y_q\in H_o$ (same arguments will hold for F_o). Then

$$\delta^{11}(x\otimes y)=\delta^{11}(\sum a_p b_q x_p\otimes y_q)=\sum\delta(a_p b_q)x_p\otimes y_q=$$

$$=\sum(\delta a_p)b_q x_p\otimes y_q+\sum a_p(\delta b_q)x_p\otimes y_q=$$

$$=(\delta^1 x)\otimes y+x\otimes\delta^1 y,\quad \text{QED.}$$

LEMMA 2.5. $(K_\varphi(\delta)\otimes\varphi+\varphi\otimes K_\varphi(\delta))\circ\Delta_H=\Delta_F\circ K_\varphi(\delta)$.

Proof. Applying Lemma 2.3 we have

$$\Delta_F\circ K_\varphi(\delta)=\Delta_F\circ\delta^2\circ\varphi-\Delta_F\circ\varphi\circ\delta^1=\delta^{22}\circ\Delta_F\circ\varphi-\Delta_F\circ\varphi\circ\delta^1=$$

$$=\delta^{22}\circ(\varphi\otimes\varphi)\circ\Delta_H-(\varphi\otimes\varphi)\circ\Delta_H\circ\delta^1=$$

$$=(\delta^{22}\circ(\varphi\otimes\varphi)-(\varphi\otimes\varphi)\circ\delta^{11})\circ\Delta_H$$

so we are left to prove that

$$K_\varphi(\delta)\otimes\varphi+\varphi\otimes K_\varphi(\delta)=\delta^{22}\circ(\varphi\otimes\varphi)-(\varphi\otimes\varphi)\circ\delta^{11}$$

Now both members of the above equality are k-$(\varphi\otimes\varphi)$-derivations from $H\otimes_k H$ to $F\otimes_k F$ so it is sufficient to prove that they agree on elements of the form $x\otimes y$ with $x,y\in H_o$. But

$$(K_\varphi(\delta)\otimes\varphi+\varphi\otimes K_\varphi(\delta))(x\otimes y)=K_\varphi(\delta)x\otimes\varphi(y)+\varphi(x)\otimes K_\varphi(\delta)y=$$

$$=\delta^2(\varphi(x))\otimes\varphi(y)+\varphi(x)\otimes\delta^2(\varphi(y))=(\delta^{22}\circ(\varphi\otimes\varphi))(x\otimes y)$$

by Lemma 2.4 and we are done since $\delta^{11}(x\otimes y)=0$.

2.6. Now we are prepared to prove the claim in (2.2) namely that $K_\varphi(\delta)^* \in \mathrm{Der}_k^{\varphi^*}(F^*,H^*)$. Indeed for all $u,v\in F^*$ and $x\in H$ we have, using (2.5):

$$\langle K_\varphi(\delta)^*(u * v),x\rangle = \langle u * v, K_\varphi(\delta)x\rangle = \langle u\otimes v, \Delta_F(K_\varphi(\delta)x)\rangle =$$
$$= \langle u\otimes v, (K_\varphi(\delta)\otimes\varphi + \varphi\otimes K_\varphi(\delta))(\Delta_H x)\rangle =$$
$$= \langle (K_\varphi(\delta)^*u)*(\varphi^*v)+(\varphi^*u)*(K_\varphi(\delta)^*v), x\rangle.$$

The following property of the Kodaira-Spencer map K_φ will play a key role later:

LEMMA 2.7. The following conditions are equivalent:

1) $\varphi=\varphi_o\otimes 1_k$ for some $\varphi_o\in \mathrm{Bialg}_{k_o}(H_o,F_o)$.

2) K_φ is the zero map.

3) There exists a family $(\delta_i)_i$ with $\delta_i\in \mathrm{Der}_{k_o}(k)$ such that $K_\varphi(\delta_i)=0$ for all i and such that $\{x\in k;\ \delta_i x=0$ for all $i\}=k_o$.

Proof. 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) are trivial (note that the characteristic zero assumption is here essential). To prove 3) $\Longrightarrow$ 1) it is sufficient to check that φ maps H_o into F_o. Take $x\in H_o$ let $(y_j)_j$ be a k_o-basis of F_o and write $\varphi(x)=\sum a_j y_j$ with $a_j\in k$. We have:

$$0=\varphi(\delta_i^1 x)=\delta_i^2(\varphi(x))=\sum(\delta_i a_j)y_j$$

for all i, hence $\delta_i a_j=0$ for all i and j. Consequently $a_j\in k_o$ so $\varphi(x)\in F_o$.

2.8. We shall define for any Hopf k-algebras H and F and

and any $\varphi \in \mathrm{Bialg}_k(H,F)$ a k-linear map

$$R_\varphi : \mathrm{Der}_k^{\varepsilon}(H,k) \longrightarrow \mathrm{Bider}_k^{\varphi}(H,F)$$

as follows. First define a k-linear map

$$c : \mathrm{Der}_k^{\varepsilon}(H,k) \longrightarrow \mathrm{Bider}_k(H)$$

by the formula

$$c(\theta)x = \sum_{(x)} (\varepsilon(x_{(1)})\theta(x_{(3)}) - \theta(x_{(1)})\varepsilon(x_{(3)}))x_{(2)}$$

where $\theta \in \mathrm{Der}_k^{\varepsilon}(H,k)$ and $x \in H$. Using standard computations it is an easy exercise to check that $c(\theta)$ is a k-derivation on H. To check that $c(\theta)^*$ is a k-derivation on H^* note that one may write $c(\theta) = (\varepsilon \otimes 1 \otimes \theta - \theta \otimes 1 \otimes \varepsilon) \circ \Delta_2 : H \longrightarrow H$ where $\Delta_2 = (1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta : H \longrightarrow H \otimes_k H \otimes_k H$ and $\varepsilon \otimes 1 \otimes \theta : H \otimes_k H \otimes_k H \longrightarrow H$ takes $x_1 \otimes x_2 \otimes x_3$ into $\varepsilon(x_1)\theta(x_3)x_2$ (similarily for $\theta \otimes 1 \otimes \varepsilon$). Hence for any $u \in H^*$ we have $(\varepsilon \otimes 1 \otimes \theta - \theta \otimes 1 \otimes \varepsilon)^* u = \varepsilon \otimes u \otimes \theta - \theta \otimes u \otimes \varepsilon$ so

$$c(\theta)^* u = u * \theta - \theta * u$$

which shows that $c(\theta)^*$ is a k-derivation on H^* (it is even an inner derivation !). Now define the k-linear map R_φ by the formula

$$R_\varphi(\theta) = \varphi \circ c(\theta) \quad \text{for all } \theta \in \mathrm{Der}_k^{\varepsilon}(H,k)$$

It is easy to see that the maps R_φ behave naturally in the following sense. Consider the category Φ whose objects are Hopf k-algebra maps $\varphi : H \longrightarrow F$, where H is a variable Hopf algebra and F is a fixed Hopf algebra, and whose morphisms are defined in an obvious way. Then we may look at the functors D and B from Φ to $\{$k-linear spaces$\}$ defined by $D(\varphi) = \mathrm{Der}_k^{\varepsilon}(H,k)$ and

$B(\varphi)=\mathrm{Bider}_k^{\varphi}(H,F)$ and remark that the maps R_φ define a natural homomorphism $R:D \longrightarrow B$.

Our infinitesimal rigidity result is the following:

THEOREM 2.9. If F is a group algebra then $R:D \longrightarrow B$ has a right inverse $\theta :B \longrightarrow D$ which may be explicitely described as follows. For any map $\varphi \in \mathrm{Bialg}_k(H,F)$ the map $\theta_\varphi : \mathrm{Bider}_k^{\varphi}(H,F) \longrightarrow \mathrm{Der}_k^{\varepsilon}(H,k)$ is given by the formula

$$\theta_\varphi(\partial) = \sum_{g\in G(F)} (\varphi^* g^*)*(\partial^* g^*)$$

where $\partial \in \mathrm{Bider}_k^{\varphi}(H,F)$ and the elements $g^* \in F^*$ are defined by requiring that

$$\langle g^*,h\rangle = \begin{cases} 1 & \text{if } h=g \\ 0 & \text{if } h\neq g \end{cases}$$

for all $h \in G(F)$.

The defining equality for $\theta_\varphi(\partial)$ in the statement above should be viewed of course as an equality in H^*.

COROLLARY 2.10. If F is a group algebra then the maps R_φ are surjective for all φ.

Note that if we restrict ourselves to commutative Hopf algebras then Corollary 2.10 is a consequence of a result in [6] p.116.

The next theorem shows in particular that "infinitesimal rigidity" and "actual rigidity" are equivalent "up to base change".

THEOREM 2.11. Let H_o be a finitely generated Hopf k_o-algebra, F_o any Hopf k_o-algebra, k a field extension of k_o, and $H=H_o\otimes k$, $F=F_o\otimes k$. Then for any map $\varphi \in \mathrm{Bialg}_k(H,F)$ the following hold:

1) If there exists a map $\varphi_o \in Bialg_{k_o}(H_o, F_o)$ such that φ and $\varphi_o \otimes 1_k$ are conjugate over k then the image of the map $K_\varphi : Der_{k_o}(k) \longrightarrow Bider_k^{\varphi}(H,F)$ is contained in the image of the map $R_\varphi : Der_k^{\varepsilon}(H,k) \longrightarrow Bider_k^{\varphi}(H,F)$.

2) If the image of K_φ is contained in the image of R_φ then there exists a map $\varphi_o \in Bialg_{k_o}(H_o, F_o)$ such that φ and $\varphi_o \otimes 1_k$ are conjugate over some field extension of k.

Clearly our Theorem 1.1 is a consequence of Corollary 2.10 and of the second part of Theorem 2.11.

The rest of the paper is devoted to the proof of the two theorems above.

3. INFINITESIMAL RIGIDITY

In this section we prove Theorem 2.9. Let $G = G(F)$; a trivial computation shows that:

LEMMA 3.1. The following equalities hold in F^*:

$$g^* * g^* = g^* \qquad \text{for all } g \in G$$

$$g^* * h^* = 0 \qquad \text{for all } g,h \in G,\ g \neq h.$$

$$\sum_{g \in G} g^* = 1 \qquad (\text{here } 1 = \varepsilon_F \text{ !})$$

LEMMA 3.2. For any $\varphi \in Bialg_k(H,F)$ and $\partial \in Bider_k^{\varphi}(H,F)$ the sum $\theta_\varphi(\partial) = \sum_{g \in G} (\varphi^* g^*) * (\partial^* g^*)$ is a well defined element in H^* and viewed as map from H to k it is a k-ε-derivation. Moreover the k-linear maps $\partial \longmapsto \theta_\varphi(\partial)$ from $Bider_k^{\varphi}(H,F)$ to $Der_k^{\varepsilon}(H,k)$ behave naturally in the φ-argument.

Proof. For $x \in H$ and $g \in G$ we have

$$\langle (\varphi^* g^*) * (\partial^* g^*), x \rangle = \sum_{(x)} \langle g^*, \varphi x_{(1)} \rangle \langle g^*, \partial x_{(2)} \rangle$$

which vanishes for all but finitely many g's so $\theta_\varphi(\partial)$ is a well defined element in H^*. To prove that $\theta_\varphi(\partial)$ is a k-ε-derivation from H to k note first that for all $x, y \in H$ we have:

$$\langle g^*, \varphi(xy) \rangle = \sum_{fh=g} \langle f^*, \varphi x \rangle \langle h^*, \varphi y \rangle \quad \text{and}$$

$$\sum_{(x)} \langle g^*, \varphi x_{(1)} \rangle \langle h^*, \varphi x_{(2)} \rangle = \langle g^* * h^*, \varphi x \rangle .$$

Using these formulae we get:

$$\langle \theta_\varphi(\partial), xy \rangle = \sum \langle g^*, \varphi(x_{(1)} y_{(1)}) \rangle \langle g^*, \partial(x_{(2)} y_{(2)}) \rangle =$$

$$= \sum_{fh=ab} \langle f^*, \varphi x_{(1)} \rangle \langle h^*, \varphi y_{(1)} \rangle \langle a^*, \partial x_{(2)} \rangle \langle b^*, \varphi y_{(2)} \rangle +$$

$$+ \sum_{fh=ab} \langle f^*, \varphi x_{(1)} \rangle \langle h^*, \varphi y_{(1)} \rangle \langle a^*, \varphi x_{(2)} \rangle \langle b^*, \partial y_{(2)} \rangle =$$

$$= \sum \langle f^*, \varphi x_{(1)} \rangle \langle f^*, \partial x_{(2)} \rangle \langle h^*, \varphi y \rangle +$$

$$+ \sum \langle h^*, \varphi y_{(1)} \rangle \langle h^*, \partial y_{(2)} \rangle \langle f^*, \varphi x \rangle =$$

$$= \langle \theta_\varphi(\partial), x \rangle \langle \varepsilon, y \rangle + \langle \varepsilon, x \rangle \langle \theta_\varphi(\partial), y \rangle .$$

Finally to check that θ_φ behave naturally we must prove that for any map $\pi \in \mathrm{Bialg}_k(J,H)$ the map $\pi^* \circ \theta_\varphi : \mathrm{Bider}_k^{\varphi}(H,F) \longrightarrow \mathrm{Der}_k^{\varepsilon}(H,k) \longrightarrow \mathrm{Der}_k^{\varepsilon}(J,k)$ equals the map $\theta_\psi \circ \pi^* : \mathrm{Bider}_k^{\varphi}(H,F) \longrightarrow \mathrm{Bider}_k^{\psi}(J,F) \longrightarrow \mathrm{Der}_k^{\varepsilon}(J,k)$ where $\psi = \pi^* \varphi$. And indeed we have

$$\pi^* \theta_\varphi(\partial) = \sum (\pi^*(\varphi^* g^*)) * (\pi^*(\partial^* g^*)) = \sum (\psi^* g^*) * ((\pi^* \partial)^* g^*) = \theta_\psi(\pi^* \partial). \quad \text{QED.}$$

The following statement closes the proof of Theorem 2.9:

LEMMA 3.3. $\varphi \circ c(\theta_\varphi(\partial)) = \partial$ for all $\partial \in \mathrm{Bider}_k^{\varphi}(H,F)$.

Proof. It is sufficient to check that ∂^* and $c(\theta_\varphi(\partial))^* \circ \varphi^*$ agree on g^* for all $g \in G$. Now by (3.1) we have $\sum g^* * g^* = 1$ so applying ∂^* we get

$$\sum_g (\partial^* g^*) * (\varphi^* g^*) + \sum_g (\varphi^* g^*) * (\partial^* g^*) = 0$$

Consequently by (2.8) and (3.1) we get for all $g \in G$:

$$(c(\theta_\varphi(\partial))^*)(\varphi^* g^*) = (\varphi^* g^*) * \theta_\varphi(\partial) - \theta_\varphi(\partial) * (\varphi^* g^*) =$$

$$= \sum_h (\varphi^* g^*) * (\varphi^* h^*) * (\partial^* h^*) + \sum_h (\partial^* h^*) * (\varphi^* h^*) * (\varphi^* g^*) =$$

$$= (\varphi^* g^*) * (\partial^* g^*) + (\partial^* g^*) * (\varphi^* g^*) =$$

$$= \partial^*(g^* * g^*) = \partial^* g^* \qquad \text{Q.E.D.}$$

4. INTEGRATION.

This section is devoted to the proof of Theorem 2.11.

4.1. First it will be convenient to adopt a functorial way of looking at coinner automorphisms. Recall from [5] p.80 that for any commutative k_o-algebra A, $\mathrm{Alg}_{k_o}(H_o, A)$ is a subgroup of $\mathrm{Hom}_{k_o}(H_o, A)$ under convolution. The group inverse of any element $u \in \mathrm{Alg}_{k_o}(H_o, A)$ is $u^S := u \circ S$. Moreover the correspondence $A \longmapsto \mathrm{Alg}_{k_o}(H_o, A)$ defines a functor which we call X from {commutative k_o-algebras} to {groups}. Since H_o is finitely generated X is representable by some finitely generated

k_o-algebra which becomes an affine Hopf algebra.

4.2. In what follows we denote by $Hom_A(-,-)$ and $Alg_A(-,-)$ the sets of A-module and A-algebra maps respectively. One can define exactly as in [5] the notion of A-bialgebra and A-bialgebra map; the sets of A-bialgebra maps will be denoted by $Bialg_A(-,-)$. Now consider the functor Y from {commutative k_o-algebras} to {groups} defined by $Y(A)=Bialg_A(H_A,H_A)^{\times}$ where $H_A=H_o\otimes A$ (here Y(A) is viewed as a group under composition of maps). We shall define a natural homomorphism $C:X\longrightarrow Y$ in the following way. First we shall define for any commutative k_o-algebra A a map $C_A:Alg_{k_o}(H_o,A)\longrightarrow Alg_A(H_A,H_A)$ as follows. For any $u\in Alg_{k_o}(H_o,A)$ put $C_A(u)=(u^S\otimes 1\otimes u)\circ\Delta_{2A}$ where $\Delta_{2A}=\Delta_2\otimes 1_A:H_A\longrightarrow H_A\otimes_A H_A\otimes_A H_A$ and $u^S\otimes 1\otimes u:H_A\otimes_A H_A\otimes_A H_A\longrightarrow H_A$ takes $x_1\otimes x_2\otimes x_3$ into $u(Sx_1)u(x_3)x_2$.

LEMMA 4.3. C_A induces a natural group homomorphism from X(A) to Y(A).

Proof. First let's agree to write for any A-module M (respectively for any A-module map φ) $M^V=Hom_A(M,A)$ (respectively $\varphi^V=Hom_A(\varphi,A)$). Now for any $u\in X(A)$ denote by $I_A(u)$ the map from $Hom_{k_o}(H_o,A)=H_A^V$ to itself which takes w into $u^S * w * u$ (so $I_A(u)$ is the inner automorphism of the convolution algebra $Hom_{k_o}(H_o,A)$ determined by u). It is easy to check that we have

$$(C_A(u))^V=I_A(u)$$

for all u. In particular we get for all $u_1,u_2\in X(A)$:

$$(C_A(u_1 * u_2))^V=I_A(u_1 * u_2)=I_A(u_2)\circ I_A(u_1)=(C_A(u_2))^V\circ(C_A(u_1))^V=$$

$$=(C_A(u_1)\circ C_A(u_2))^V$$

Since H_A is a free A-module we conclude that $C_A(u_1 * u_2) = C_A(u_1) \circ C_A(u_2)$.

We are left to prove that for all $u \in X(A)$ we have

$$\Delta_A \circ C_A(u) = (C_A(u) \otimes C_A(u)) \circ \Delta_A : H_A \longrightarrow H_A \otimes_A H_A$$

Again since H_A is free it is sufficient to prove that $(C_A(u))^{\vee} \circ (\Delta_A)^{\vee}$ and $(\Delta_A)^{\vee} \circ (C_A(u) \otimes C_A(u))^{\vee}$ agree when composed with the map

$$(H_A^{\vee}) \otimes_A (H_A^{\vee}) \longrightarrow (H_A \otimes_A H_A)^{\vee}$$

But this is a trivial consequence of the fact that $I_A(u)$ is a ring homomorphism and the lemma is proved.

4.4. Next it will be convenient to slightly extend Kolchin's concept of logarithmic derivative (see [3] p.394 or [4] p.959). Let X be any functor from $\{$commutative k_o-algebras$\}$ to $\{$groups$\}$ (which is not necessarily representable !). One can define then a functor $Lie_X : \{$commutative k_o-algebras$\} \to \{$groups$\}$ by the formula:

$$Lie_X(A) = Ker(X(p) : X(A[z]) \longrightarrow X(A))$$

where $A[z] = A \oplus Az$, $z^2 = 0$ and $p : A[z] \longrightarrow A$ is the reduction modulo z. Moreover for any $\delta \in Der_{k_o}(A)$ one can define a map (which should be called "logarithmic derivative" and which is not a group homomorphism in general):

$$\ell_X \delta : X(A) \longrightarrow Lie_X(A)$$

$$\ell_X \delta g = ((X(i+z\delta))(g))((X(i))(g))^{-1} \quad \text{for } g \in X(A)$$

where $i : A \longrightarrow A[z]$ is the natural inclusion map and $i+z\delta : A \longrightarrow A[z]$ is the map sending $a \in A$ into $a + (\delta a)z$.

If $\varphi: A \longrightarrow B$ is a k_o-algebra map between commutative k_o-algebras and if $\delta_A \in \mathrm{Der}_{k_o}(A)$, $\delta_B \in \mathrm{Der}_{k_o}(B)$ are derivations such that $\delta_B \circ \varphi = \varphi \circ \delta_A$ then one immediately checks that $\mathrm{Lie}_X(\varphi) \circ \ell_X \delta_A = \ell_X \delta_B \circ X(\varphi)$.

Now if $C: X \longrightarrow Y$ is any homomorphism between functors from {commutative k_o-algebras} to {groups} then there is an induced homomorphism $\mathrm{Lie}_C : \mathrm{Lie}_X \longrightarrow \mathrm{Lie}_Y$. Moreover if $\delta \in \mathrm{Der}_{k_o}(A)$ then it is easy to check that $\mathrm{Lie}_C(A) \circ \ell_X \delta = \ell_Y \delta \circ C(A)$.

We shall need the following variation on a result due to Kolchin [3] p.420 :

THEOREM 4.5. Suppose we are given a functor X from {commutative k_o-algebras} to {groups} representable by a finitely generated k_o-algebra and suppose we are given a field extension k of k_o and a family of derivations $(\delta_i)_i$, $\delta_i \in \mathrm{Der}_{k_o}(k)$ such that $\{x \in k;\ \delta_i x = 0 \text{ for all } i\} = k_o$. Suppose moreover that we are given a family $(\theta_i)_i$ with $\theta_i \in \mathrm{Lie}_X(k)$. Then there exist a field extension $\tilde{k}$ of k a family of k_o-derivations $(\tilde{\delta}_i)_i$ on $\tilde{k}$ extending the derivations $(\delta_i)_i$ and an element $t \in X(\tilde{k})$ such that if we put $\tilde{\theta}_i = (X(\varphi))(\theta_i)$ where $\varphi: k \longrightarrow \tilde{k}$ is the natural inclusion then the following hold:

1) $\{x \in \tilde{k};\ \tilde{\delta}_i x = 0 \text{ for all } i\} = k_o$ and

2) $\ell_X \tilde{\delta}_i t = \tilde{\theta}_i$ for all i.

A proof of the above theorem is given in [1]. Note that Kolchin's original statement in [3] requires certain commutation relations satisfied by the θ_i's (which will not be satisfied in general in our specific situation). Note also that there is a weaker version of (4.5) in [4] p.982 which does not provide our conclusion 1) (which will be essential to our proof of Theorem 2.11.)

For convenience we sketch a proof for (4.5) above. We may suppose k is algebraically closed. Let A represent X; then Spec A is an affine algebraic group over k_o. We view $X(k)$ as a matrix group i.e. as a closed subgroup of $GL_n(k)$ for some n. Moreover we view $Lie_X(k)$ as a subspace of $gl_n(k)$; then the map $\ell_X\delta_i$: $X(k) \longrightarrow Lie_X(k)$ identifies with the "usual" logarithmic derivative $\ell_X\delta_i g=(\delta_i g)g^{-1}$. Define derivations D_i on $k[T]=$ $=k[T_{pq}, 1\leq p,q\leq n]$ by putting $D_i x=\delta_i x$ for $x\in k$ and

$$D_i T_{pq}=\sum_r \theta_{ipr}T_{rq}$$

for all i,p,q where $\theta_i=(\theta_{ipq})_{pq}$, $\theta_{ipq}\in k$. One checks that the defining prime ideal $P\subset k[T]$ of the identity component of $X(k)$ is stable under all D_i's. Now choose a maximal element in the set $\{Q\in \text{Spec } k[T];\ \det(T)\notin Q,\ P\subset Q,\ D_i(Q)\subset Q$ for all $i\}$. It is easy to check that if we put $\tilde{k}=k[T]_Q/Qk[T]_Q$ and $\tilde{\delta}_i=D_i$ mod Q then $\tilde{k}$ and $\tilde{\delta}_i$ satisfy conclusion 1) in (4.5). Then we are done by putting $t=(t_{pq})$, $t_{pq}=T_{pq}$ mod Q.

4.6. Now we come back to our specific homomorphism $C:X \longrightarrow Y$ from (4.3) and consider a field extension k of k_o. First it is quite clear that $Lie_X(k)$ and $Lie_Y(k)$ can be identified with $Der_k^{\varepsilon}(H,k)$ and $Bider_k(H)$ respectively; moreover under this identification the map $Lie_C(k):Lie_X(k) \longrightarrow Lie_Y(k)$ corresponds precisely to the map $c:Der_k^{\varepsilon}(H,k) \longrightarrow Bider_k(H)$ defined at (2.8). Let $\delta\in Der_{k_o}(k)$; we claim that under the above identification the map $\ell_Y\delta:Y(k) \longrightarrow Lie_Y(k)$ corresponds to the map $Bialg_k(H,H)^{\times} \longrightarrow Bider_k(H)$ sending $\sigma\in Bialg_k(H,H)^{\times}$ into $\sigma\circ\delta^1\circ\sigma^{-1}-\delta^1$. Indeed by its very definition $\ell_Y\delta$ sends σ into the map

$$\ell_Y\delta\sigma : H[z] \xrightarrow{\sigma_z^{-1}} H[z] \xrightarrow{1+z\delta^1} H[z] \xrightarrow{\sigma_z} H[z] \xrightarrow{1-z\delta^1} H[z]$$

where $H[z]=H\otimes_k k[z]$, σ_z and σ_z^{-1} are the $k[z]$-bialgebra maps naturally induced by σ and σ^{-1} and $1+z\delta^1$, $1-z\delta^1$ are the $k_o[z]$-algebra maps which send any $x\in H$ into $x+(\delta^1 x)z$ and $x-(\delta^1 x)z$ respectively. Consequently for all $x\in H$ we have

$$(\ell_Y\delta\sigma)(x)=x+(\sigma\circ\delta^1\circ\sigma^{-1}x-\delta^1 x)z$$

which proves our claim.

So far we obtained for each $\delta\in \mathrm{Der}_{k_o}(k)$ a commutative diagram

$$\begin{array}{ccc}
\mathrm{Alg}_k(H,k)=X(k) & \xrightarrow{\ \ell_X\delta\ } & \mathrm{Lie}_X(k)=\mathrm{Der}_k^{\varepsilon}(H,k)\\
\Big\downarrow C & & \Big\downarrow C\\
\mathrm{Bialg}_k(H,H)^{\times}=Y(k) & \xrightarrow{\ \ell_Y\delta\ } & \mathrm{Lie}_Y(k)=\mathrm{Bider}_k(H)
\end{array}$$

with $\ell_Y\delta\sigma=\sigma\circ\delta^1\circ\sigma^{-1}-\delta^1$.

4.7. Now we are in a position to prove Theorem 2.11. Start with the first statement so suppose there is a map $\varphi_o\in\mathrm{Bialg}_{k_o}(H_o,F_o)$ and a coinner automorphism $\sigma=C(u)$, $u\in\mathrm{Alg}_k(H,k)$ such that $\varphi\circ\sigma=\varphi_o\otimes 1_k$. Then by (2.7) $\delta^2\circ\varphi\circ\sigma=\varphi\circ\sigma\circ\delta^1$ for all $\delta\in\mathrm{Der}_{k_o}(k)$ hence

$$K_\varphi(\delta)=\delta^2\circ\varphi-\varphi\circ\delta^1=\varphi\circ(\sigma\circ\delta^1\circ\sigma^{-1}-\delta^1)=\varphi\circ(\ell_Y\delta)(C(u))=$$

$$=\varphi\circ C(\ell_X\delta u)=R_\varphi(\ell_X\delta u)$$

so $\mathrm{Im}\,K_\varphi\subset\mathrm{Im}\,R_\varphi$.

To prove the second statement in (2.11) choose a family $(\delta_i)_i$ of k_o-derivations on k such that $\{x \in k;\ \delta_i x = 0 \text{ for all } i\} = k_o$. By hypothesis there exist derivations $\theta_i \in \mathrm{Der}_k^{\varepsilon}(H,k)$ such that $\delta_i^2 \circ \varphi - \varphi \circ \delta_i^1 = \varphi \circ c(\theta_i)$. Apply (4.5) to our specific X and θ_i and let $\tilde{k}$, $\tilde{\delta}_i$, $\tilde{\theta}_i$ and t be as in (4.5). We claim that

$$\tilde{\delta}_i^2 \circ \tilde{\varphi} - \tilde{\varphi} \circ \tilde{\delta}_i^1 = \tilde{\varphi} \circ \tilde{c}(\tilde{\theta}_i)$$

where $\tilde{\delta}_i^1$, $\tilde{\delta}_i^2$ are the unique k_o-derivations on $\tilde{H} = H \otimes_k \tilde{k}$ and $\tilde{F} = F \otimes_k \tilde{k}$ respectively which agree with $\tilde{\delta}_i$ on $\tilde{k}$ and vanish on H_o and F_o respectively; moreover $\tilde{\varphi} = \varphi \otimes 1_{\tilde{k}} \in \mathrm{Bialg}_{\tilde{k}}(\tilde{H},\tilde{F})$ and $\tilde{c} = \mathrm{Lie}_C(\tilde{k})$. Indeed both members of the above equality are $\tilde{k}$-$\tilde{\varphi}$-derivations from $\tilde{H}$ to $\tilde{F}$ and agree on H hence they agree everywhere. Now if $\sigma := C_{\tilde{k}}(t) \in \mathrm{Bialg}_{\tilde{k}}(\tilde{H},\tilde{H})^{\times}$ we get

$$\tilde{\delta}_i^2 \circ \tilde{\varphi} - \tilde{\varphi} \circ \tilde{\delta}_i^1 = \tilde{\varphi} \circ \tilde{c}(\ell_X \tilde{\delta}_i t) = \tilde{\varphi} \circ (\ell_Y \tilde{\delta}_i \sigma) = \tilde{\varphi} \circ (\sigma \circ \tilde{\delta}_i^1 \circ \sigma^{-1} - \tilde{\delta}_i^1)$$

hence $\tilde{\delta}_i^2 \circ \tilde{\varphi} \circ \sigma = \tilde{\varphi} \circ \sigma \circ \tilde{\delta}_i^1$. In other words $K_{\tilde{\varphi}\circ\sigma}(\tilde{\delta}_i) = 0$ where $K_{\tilde{\varphi}\circ\sigma} : \mathrm{Der}_{k_o}(\tilde{k}) \longrightarrow \mathrm{Bider}_{\tilde{k}}^{\tilde{\varphi}\circ\sigma}(H,F)$ is the "Kodaira-Spencer" map associated to $\tilde{\varphi}\circ\sigma$. By (2.7) $\tilde{\varphi} \circ \sigma = \varphi_o \otimes 1_{\tilde{k}}$ for some $\varphi_o \in \mathrm{Bialg}_{k_o}(H_o,F_o)$ and we are done.

5. COMMENTS

5.1. Antipodes are not at all essential to our work. Indeed everything holds if one replaces Hopf algebras by bialgebras and group algebras by cancellative monoid algebras. Recall that a monoid M is called cancellative if either $ab=ac$ or $ba=ca$ with $a,b,c \in M$ imply $b=c$. In this more general context coinner automorphisms are defined as follows. If H is a bialgebra then

$Alg_k(H,k)$ is a multiplicative submonoid of the convolution algebra $Hom_k(H,k)$ (but possibly not a group). By a coinner automorphism of H we understand a map $H \longrightarrow H$ of the form

$$x \longmapsto \sum_{(x)} u^{-1}(x_{(1)})u(x_{(3)})x_{(2)}$$

with $u \in Alg_k(H,k)^{\times}$. We should emphasize that cancellativity is needed to go through the computations in (3.2).

5.2. Along the lines of Theorem 1.1 it would be interesting to dispose of a non-commutative analog for the conjugacy of maximal tori in affine algebraic groups. The non-commutative analog of maximal tori in a Hopf k-algebra H should be perhaps the minimal elements in the set

$$\Sigma = \{\text{prime Hopf ideals } P \text{ in } H \text{ with } H/P \text{ a group algebra}\}$$

One might conjecture that if H is finitely generated and k is algebraically closed then for any minimal elements P_1 and P_2 in Σ there is a coinner k-automorphism σ of H with $\sigma P_1 = P_2$.

REFERENCES

1. A.Buium, Differential Function Fields and Moduli of Algebraic Varieties, Lecture Notes in Math. 1226, Springer Verlag 1986.
2. J.E.Humphreys, Linear Algebraic Groups, Springer Verlag 1975.
3. E.R.Kolchin, Differential Algebra and Algebraic Groups, Academic Press, New York 1973.
4. W.Nichols, B.Weisfeiler, Differential formal groups of J.F. Ritt, Amer.J.Math. 104,5(1982),943-1003.
5. M.Sweedler, Hopf Algebras, Benjamin, New York 1969.
6. M.Demazure, A.Grothendieck, Seminaire de Géometrie Algébrique, Schémas en Groupes I, Lecture Notes in Math. 151, Springer Verlag 1970.