INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

ON RESTRICTED PERTURBATIONS IN INVERSE IMAGES AND A DESCRIPTION OF NORMALIZER ALGEBRAS IN C^*-ALGEBRAS

by

Eberhard KIRCHBERG (Berlin)

PREPRINT SERIES IN MATHEMATICS
No.31/1987

BUCURESTI

ON RESTRICTED PERTURBATIONS IN INVERSE IMAGES AND A DESCRIPTION OF NORMALIZER ALGEBRAS IN C^*-ALGEBRAS

by

Eberhard KIRCHBERG *) (Berlin)

August 1987

*) Otto - Buchwitz - Str. 63, 1140 Berlin

ON RESTRICTED PERTURBATIONS IN INVERSE IMAGES AND A DESCRIPTION OF NORMALIZER ALGEBRAS IN C*-ALGEBRAS

by Eberhard Kirchberg (Berlin)

SUMMARY:
Let E and F be Banach spaces and T a bounded linear map from E into F. Initiating a certain perturbation function $f(.,T)$ we find a useful sufficient criteria that T maps the closed unit ball onto a closed set. Applying this to the quotient map from a C*-algebra A onto its quotient Banach space A/(L+R) by the sum L+R of closed left- and rightideals L and R of A we obtain that the closed unit ball of A maps onto the closed unit ball of A/(L+R). It results an independent description of the images of the rightnormalizer algebra $N_r(D)$ and the normalizer algebra $N(D)= N_l(D) \cap N_r(D)$ of a hereditary C*-subalgebra D of A by the quotient map from A onto A/(cl(AD)+cl(DA)). We use this to prove a necessary and sufficient criteria for a C*-algebra B to be a C*-quotient-algebra of a C*-subalgebra of a C*-algebra A. The latter criteria will be applied in situation A equals the CAR-algebra in some forthcoming papers (cf. sec.6 for more details).

CONTENT:

1. Introduction and formulation of main results.

Let E and F be Banach spaces and T a bounded linear map from E into F. If X and Y are subsets of E, cl(X) means the (norm-) closure of X, $wcl(X) \subseteq E^{**}$ is the $\sigma(E^{**},E^{*})$-closure of X in the second conjugate space E^{**} of E and H-dist(X,Y) denotes sup{ dist(a,Y),dist(b,X): $a \in X$, $b \in Y$ }, the Hausdorff distance between X and Y. By S we denote the open unit ball of E. We put
$f(x,T)=\sup\{\text{H-dist}(S \cap T^{-1}(T(a)), S \cap T^{-1}(T(b))): a,b \in S, \|T(a-b)\| \leq x\}$
where $x>0$. Let per(T) be the limit of f(x,T) if x trends to 0.

Proposition 1.1

(i) f(x,T) *is a continuous increasing function on*] 0, oo[.
(ii) per(T) = 0 *implies* T(cl(S)) = cl(T(S)).
(iii) $f(x,T) = f(x,T^{**})$ *for every* $x > 0$.

Here T^{**} means the second adjoint operator of T. f(x,T) is the infimum of all numbers $y>0$ such that given b S, d T(S) with $\|T(b)-d\| \leq x$ there exists a perturbation b+h of b *inside* S such that $\|h\| \leq y$ and $T(b+h)=d$. Thus per(T)=0 says that a small perturbation inside T(S) can be realized by a small perturbation inside S. The proofs and further results we shall give in section 2.

Now let A be a C*-algebra and L,R closed left- and rightideals with support projections l and r in A^{**} respectively, i.e. l,r are the open projections in A^{**} satisfying $A^{**}l=wcl(L)$ and $rA^{**}=wcl(R)$, cf. [TAK,3.4.2]. Put $q=1-l$ and $p=1-r$. Let b,d be elements of A^{**} in the multiplier algebra M(A) of A such that $pbb^{*}p$ is invertible in $pA^{**}p$ with inverse g and $qd^{*}dq$ is invertible in $qA^{**}q$ with inverse h. R+L is a closed linear subspace of A (cf. sec.4). We denote by $\pi=\pi_{L,R}$ the quotient map A --> A/(L+R) given by c --> c+L+R and denote by $\pi(b(.)d)$ the map given by c --> bcd+L+R, ($c \in A$).

Proposition 1.2

$$f(x,\pi(b(.)d)) \leq x.(\|g\|.\|h\|)^{1/2} + (2x.(\|g\|.\|h\|)^{1/2})^{1/2}$$

The Proof is given in section 4. From Proposition 1.1(ii) and Proposition 1.2 we immediately obtain the following.

Corollary 1.3

Under the above assumptions concernig A,L,R,b *and* d *the quotient map* A --> A/(L+R) *maps* b(cl(S))d *onto a closed set.*

Especially the quotient map maps the closed unit ball of A onto the closed unit ball of A/(L+R).

For every positive integer n, the map $[b_{j,k}]_n$ --> $[pb_{j,k}q]_n$ from $Mat_n(A^{**}) \cong Mat_n(A)^{**}$ onto $Mat_n(pA^{**}q)$ is a contraction and has kernel $Mat_n(A^{**}l+rA^{**})=Mat_n(wcl(L+R)) \cong wcl(Mat_n(L+R))$ and its factorization defines an isometrical Mat_n-bimodul isomorphism from $(Mat_n(A)/Mat_n(L+R))^{**} \cong Mat_n(A^{**})/Mat_n(wcl(L+R))$ onto $Mat_n(pA^{**}q)$. On the other hand there is a canonical Mat_n-bimodul isomorphism from $Mat_n(A/(L+R))$ onto $Mat_n(A)/Mat_n(L+R)$. The matrix norms induced by this isomorphisms give A/(L+R) the structure of a *matrix normed space* in the sense of Effros [EF2] such that the second conjugate matrix normed space is comletely isometric isomorphic to $pA^{**}q$, an operator subspace of A^{**} and C*-triple system (cf. sec.4). If moreover A is

unital and $L=\{b^*: b\in R\}$ then $p=q$ and under the above identifications $A/(L+R)=pAp\subseteq pA^{**}p$ becomes a matrix order unit space in the sense of Effros [EF2] with matrix order unit $p\in A/(L+R)$ such that the second conjugate matrix order unit space is just the unital C*-algebra $pA^{**}p$. More general we call a matrix order unit space X *C*-system* if its second conjugate matrix order unit space X^{**} is unital matrix order isomorphic to a unital C*-algebra. Our definition especially says that X is an operator system in the sense of [C/E2], i.e. a closed unital and selfadjoint linear subspace of a C*-algebra together with the matrix order inherited from this containment. The C*-algebra structure on X^{**} is uniquely defined by the second conjugate matrix order unit structure of the given one on X and we can define the *leftmultiplier algebra* $M_l(X)$, the *rightmultiplier algebra* $M_r(X)$ and the *multiplier algebra* $M(X)$ of X as follows:
$M_l(X):=\{b\in X^{**}: bX\subseteq X\}$, $M_r(X):=\{b\in X^{**}: Xb\subseteq X\}$, $M(X):=M_l(X)\cap M_r(X)$
Here we identify X with its canonical and isometrical image in X^{**} by the evaluation map $ev_X: X \to X^{**}$. $M_l(X)$ and $M_r(X)$ are closed subalgebras of X^{**} contained in X (more precisely in the image of ev_X), $M_l(X)=\{b^*: b\in M_r(X)\}$ and the multiplier algebra $M(X)$ of X is a unital C*-algebra which is unital completely positive and completely isometric contained in the operator system X. In our special case $A/(L+R)$ is identified with pAp ($\subseteq pA^{**}p\subseteq A^{**}$) and $M_r(A/(L+R))=\{pbp: b\in A, pApbp\subseteq pAp\}$, $M(A/(L+R))=\{pbp: b\in A, pbpAp+pApbp\subseteq pAp\}$.
Now let D be a hereditary C*-subalgebra of A (i.e. D closed, selfadjoint and $DAD\subseteq D$). By a *rightnormalizer (leftnormalizer, normalizer)* of D in A we understand b A satisfying $Db\subseteq D$ ($bD\subseteq D$, $bD+Db\subseteq D$). The rightnormalizers, leftnormalizers, normalizers obviously form closed operator algebras $N_r(D)$, $N_l(D)$ and $N(D)$ respectively, $N_r(D)=\{b^*: b\in N_l(D)\}$ and $N(D)=N_l(D)\cap N_r(D)$ is a C*-subalgebra of A. It turns out that $L=cl(AD)$ and $R=cl(DA)$ are closed left- and rightideals of A respectively whose support projections in A^{**} equals that of D (i.e. equals the unit of $D^{**}=wcl(D)\subseteq A^{**}$). From definitions we see that $L=cl(AD)\subseteq N_r(D)$, $R=cl(DA)\subseteq N_l(D)$ and $D\subseteq N(D)$ are closed ideals of $N_r(D)$, $N_l(D)$ and $N(D)$ respectively.
We define $A//D:=A/(cl(AD)+cl(DA))$, the (unital) *quotient-C*-system* of the unital C*-algebra A *with respect to the hereditary C*-subalgebra* D of A. We again denote by $\pi_D: A \to A//D$ the quotient map $b \to b+cl(AD)+cl(DA)$. Now we are in position to formulate the main result of this paper.

Theorem 1.4

Let A be a unital C-algebra, D a hereditary C*-subalgebra of A, A//D the quotient-C*-system of A with respect to D and π_D: A --> A//D the quotient map. Then*

(i) *the restriction of* π_D *to* $N_r(D)$ *(resp. to* $N_l(D)$ *) is a Banach algebra epimorphism onto* $M_r(A//D)$ *(resp. onto* $M_l(A//D)$ *) with kernel* $cl(AD)$ *(resp.* $cl(DA)$ *),*

(ii) *for every positive integer* n, $\pi_D\otimes id_n$ *maps the closed unit ball of* $N_r(Mat_n(D))=Mat_n(N_r(D))\subseteq Mat_n(A)$ *onto the closed unit ball of* $Mat_n(M_r(A//D)) \subseteq Mat_n(A//D) \cong Mat_n(A)//Mat_n(D)$,

(iii) $\pi_D|N(D)$ *is a C*-algebra epimorphism from the normalizer algebra* $N(D)\subseteq A$ *of* D *onto the multiplier algebra* $M(A//D)$ *of* A//D *with kernel ideal* $\ker(\pi_D|D) = D$.

As a corollary we get the following which is in turn an other formulation of Theorem 1.4 in view of Proposition 1.3.

Corollary 1.5

Let A and C be unital C-algebras, B a unital closed subalgebra of C and $V: A//D \longrightarrow C$ a unital completely isometric map such that $B \cup \{b^*b : b \in B\} \subseteq Im(V)$. Then there exists a unital closed subalgebra E of A such that*

(i) $E \cap (cl(AD)+cl(DA)) = cl(AD)$,

(ii) $V \circ \pi_D | E$ *is a Banach algebra epimorphism from E onto B with kernel* $cl(AD)$ *and*

(iii) *the induced map* $[V \circ \pi_D]^{\circ}: E/cl(AD) \longrightarrow B$ *is completely isometric where B is equipped with the matrix norms induced by C and* $E/cl(AD)$ *is equipped with the matrix norms induced by* $A/cl(AD) \subseteq A^{**}p$.

If moreover B is a C-subalgebra of C then there exists a unital C*-subalgebra F of A such that*

(iv) $F \cap (cl(DA)+cl(AD)) = D$ *and*

(v) $V \circ \pi_D | F$ *is a C*-algebra epimorphism from F onto B with kernel D.*

With other words under assumptions of Corollary 1.5 B is a C*-quotientalgebra of a C*-subalgebra of A if B is a C*-subalgebra of C. Theorem 1.4 and Corollary 1.5 are proven in section 5. In two separate forthcoming papers we shall show: *The assumptions of Corollary 1.5 are satisfied with the CAR-algebra in place of A and with D,C,V suitable chosen if and only if B is separable and exact in the sense of* [KI2], cf. section 6 for more details.

2. Restricted perturbations in inverse images

Let E,F be real or complex Banach spaces, $T: E \longrightarrow F$ a bounded linear map from E into F and $K \subseteq E$ a convex subset of E. If $a,b \in K$ by $\rho(T(a),T(b)) = \rho_{K,T}(T(a),T(b))$ we denote the Hausdorff distance between $K \cap (a+kerT)$ and $K \cap (b+kerT)$, i.e. $\rho(c,d) =$ H-dist$(K \cap T^{-1}(c), K \cap T^{-1}(d))$ if $c,d \in T(K)$ (cf. sec.1). In this way T(K) becomes a metric space $(T(K),\rho)$ with distance ρ. On the other hand T defines a Lipschitz map from K onto T(K) with Lipschitz constant $\|T\|$ where K and T(K) are equipped with the usual distance between points in Banach spaces. We have $T(cl(K)) \subseteq cl(T(K))$. As we shall see it is of importence to know if $T(cl(K)) = cl(T(K))$ in special situations we are concerned with. In general this is not the case, e.g. let K be the open unit ball of c_0 and let T be the functional $(t_1,t_2,\ldots) \longrightarrow t_1 2^{-1} + t_2 2^{-2} + \ldots + t_n 2^{-n} + \ldots$ then $T(cl(K)) = T(K)$ is the open unit disk. In terms of metric spaces following situation appears: Let (R_1,r_1), (R_2,r_2) be metric spaces and σ a Lipschitz map from R_1 onto R_2 with Lipschitz constant $k < \infty$ and let (R_2,ρ) be the metric space with distance $\rho(a,b) =$ H-dist$(\sigma^{-1}(a), \sigma^{-1}(b))$ then $r_2 \leq k\rho$. We denote by $(R_1\hat{}, r_1\hat{})$ and $(R_2\hat{}, r_2\hat{})$ the Cauchy completions of (R_1,r_1) and (R_2,r_2) respectively. Because σ is a Lipschitz map it uniquely extends to a Lipschitz map $\sigma\hat{}$ from $(R_1\hat{}, r_1\hat{})$ *into* $(R_2\hat{}, r_2\hat{})$ with Lipschitz constant k.

Lemma 2.1

Assume there exists a continuous function $g(x)$, $x \geq 0$, *such that* $g(0)=0$ *and* $\rho(a,b) \leq g(r_2(a,b))$ *then* $\sigma\hat{}$ *maps* $(R_1\hat{}, r_1\hat{})$ *onto* $(R_2\hat{}, r_2\hat{})$.

Proof: Let be $d \in R_2\hat{}$ and let $(d_1,d_2,\ldots)$ be a Cauchy sequence in (R_2,r_2) representing d. From $\varrho(d_i,d_j)\le g(r_2(d_i,d_j))$ and $g(0)=0$ we obtain that $(d_1,d_2,\ldots)$ is also a Cauchy sequence in (R_2,ϱ). We select a subsequence $(e_1,e_2,\ldots)$ of $(d_1,d_2,\ldots)$ such that $\varrho(e_n,e_{n+1})<2^{-n}$. If $c_n\in\sigma^{-1}(e_n)$ we find $c_{n+1}\in\sigma^{-1}(e_{n+1})$ with $r_1(c_n,c_{n+1})<2^{-n}$ because $\mathrm{dist}(c_n,\sigma^{-1}(e_{n+1}))\le g(e_n,e_{n+1})$. By induction we get a Cauchy sequence $(c_1,c_2,\ldots)$ in (R_1,r_1) with $c_n\in\sigma^{-1}(e_{n+1})$. Let c be the point of $(R_1^\wedge,r_1^\wedge)$ represented by $(c_1,c_2,\ldots)$. By continuity of $\sigma^\wedge$, $d=\sigma^\wedge(c)$. q.e.d.

Now following considerations are motivated. We define for $x>0$ $f(x):=f(x,K,T):=\sup\{\varrho(T(a),T(b)):\ a,b\in K,\ \|T(a-b)\|\le x\}$. If $K=S$ is the open unit ball of E we also shall write $f(x,T)$ instead of $f(x,S,T)$. Obviously $f(x)$ is an increasing nonnegative function on $]0,\infty[$ and $\mathrm{dist}(a,T^{-1}(T(b))\cap K)\le$ $\le\varrho(T(a),T(b))\le f(\|T(a-b)\|)$ if $a,b\in K$. So there exists a continuous function $g(x)\ge f(x)$ such that $g(0)=\inf\{f(x):x>0\}$. We put $\mathrm{per}(K,T):=\inf\{f(x,K,T):x>0\}$ and $\mathrm{per}(T):=\mathrm{per}(T,S)$. From the above Lemma 2.1 applied to $(R_1,r_1)=$ (K, norm distance) and $(R_2,r_2)=$ (T(K), norm distance) we get the following.

Proposition 2.2

$\mathrm{per}(K,T) = 0$ *implies* $T(\mathrm{cl}(K)) = \mathrm{cl}(T(K))$.

In our applications of Proposition 2.2 we need some technics involving upper estimates for $f(x,K,T)$.

Lemma 2.3

(i) $f(x,K,T)$ *is the minimal number* y *such that* (~) *holds:*
(~) { *For any couple* $a,b\in K$ *with* $\|T(a-b)\|<x$ *and every*
{ $t>0$ *there exists* $c\in K$ *with* $T(c)=T(b)$ *and*
{ $\|a-c\|\le y+t$ (*i.e.* $\mathrm{dist}(a,K\cap T^{-1}(T(b)))\le y$ *if* $a,b\in K$
{ *and* $\|T(a-b)\|<x$).

(ii) $f(x,K,T) = \sup\{f(w,K,T):\ 0<w<x\}$ *and*
$f(x,K,T)\le f(w,K,T)+(1-(w/x))\sup\{\|a-b\|:a,b\in K\}$ *if* $0<w\le x$.
Especially $f(x,K,T)$ *is continuous if* K *is bounded.*

(iii) *If* G *is a Banach space and* $U: F\to G$ *is a bounded linear map then* $f(x,K,UT)\le f^\sim(f(x,K,T),T(K),U)$ *where* $f^\sim(x,\ldots)=$ $\inf\{f(x+t,\ldots):\ t>0\}$.

(iv) $\mathrm{per}(K,UT)\le \mathrm{per}(T(K),U)$ *if* $\mathrm{per}(K,T)=0$.

Proof: Ad(i): Let y be a number satisfying (~) and let be $\|T(a-b)\|\le x$ and $d\in T^{-1}(T(a))\cap K$, $t>0$. Then $\|T(d-b)\|\le x$, $b+s(d-b)\in K$ and $\|T(b+s(d-b))-T(b)\|\le sx<x$ if $0<s<1$. By (~) there exists $c\in K$ with $T(c)=T(b)$ such that $\|b+s(d-b)-c\|\le y+t$. It follows $\mathrm{dist}(b+s(d-b),T^{-1}(T(b))\cap K)\le y+t$ if $0<s<1$ and thus $\mathrm{dist}(d,T^{-1}(T(b))\cap K)\le y+t$. This holds for any couple $a,b\in K$ with $\|T(a-b)\|\le x$, every $t>0$ and every $d\in K\cap T^{-1}(T(a))$, i.e. $f(x,K,T)\le y$. Conversely let be given $a,b\in K$ with $\|T(a-b)\|<x$ then $\mathrm{dist}(a,T^{-1}(T(b))\cap K)\le\varrho(T(a),T(b))\le f(\|T(a-b)\|)\le f(x,K,T)$. Thus for any $t>0$ there exists $c\in K$ with $T(a)=T(b)$ and $\|a-c\|\le f(x,K,T)+t$, i.e. $f(x,K,T)$ satisfies (~).

Ad(ii): In other words (i) means
$f(x,K,T) = \sup\{\mathrm{dist}(a,T^{-1}(T(b))\cap K):\ a,b\in K,\ \|T(a-b)\|<x\}$.
Let R(t) be the set $\{\mathrm{dist}(a,T^{-1}(T(b))\cap K):\ a,b\in K, \|T(a-b)\|<t\}$.
From $R(x)=\bigcup_{0<w<x}R(w)$ we get $f(x,K,T)=\sup\{f(w,K,T):\ 0<w<x\}$ and $f(x,K,T)=\sup\{\varrho(T(a),T(b)):\ a,b\in K,\ \|T(a-b)\|<x\}$.
Now let be given $w>x>0$ and put $t=x/w$, $c=b+t(a-b)$. Then $\|T(c-b)\|=t\|T(a-b)\|\le tw=x$ if $a,b\in K$ and $\|T(a-b)\|\le w$. Thus $\mathrm{dist}(a,T^{-1}(T(b))\cap K)\le\|a-c\|+\mathrm{dist}(c,T^{-1}(T(b))\cap K)\le$ $\le(1-t)\|a-b\|+f(x,K,T)$. By (i) we get $f(w)\le f(x)+(1-t)\mathrm{diam}(K)$.
Ad(iii): Let be $a,b\in K$, $\|UT(a-b)\|<x$, $t>0$. We choose $s>0$ such

that $f(f(x,T(K),S)+s,K,T) \le f^{\sim}(f(x,T(K),U),K,T)+t/2$. Then T(a) and T(b) are in T(K), $\|U(T(a)-T(b))\|<x$ and by (i) there is c in T(K) with U(c)= U(T(b)) and $\|T(a)-c\| < f(x,T(K),U)+ s$. We find g in K such that T(g)=c. Again by (i) there exists d in K satisfying T(d)=T(g)=c and $\|a-d\| \le f(f(x,T(K),U)+s,K,T)+t/2$. Thus we have $d\in K$, UT(d)= UT(b), $\|a-d\| \le f^{\sim}(f(x,T(K),U),K,T)+t$. By (i) we get (iii).
Ad(iv): By (iii) it holds $f(x,K,UT) \le f(f(x,K,T)+x,T(K),U)$. Thus $per(K,UT) = \lim_{x\to 0} f(x,K,UT) \le \lim_{x\to 0} f(f(x,K,T)+x,T(K),U) =$ $=per(T(K),U)$ if $\lim_{x\to 0} (f(x,K,T)+x) = per(K,T) = 0$. q.e.d.

If L is a closed linear subspace of E and $e\in E$ such that $K\subseteq e+L$ then $K-d \subseteq (e+L)-d \subseteq L$ if $d\in K$, d= e+l with $l\in L$, i.e. the closed real linear span clspan(K-d) of K-d is contained in L. This shows that $K-K \subseteq$ clspan(K-d), that d+clspan(K-d) is the closed real affine span of K and clspan(K-c)= clspan(K-d) if c and d are in K. We define the relative interior of K (relatively to its closed real affine span d+clspan(K-d) with d in K):
rint(K):= d+int(K-d), where the interior of K-d is taken relatively to clspan(K-d).
From $d-c \in$ clspan(K-c) and (d-c)+(K-d)= K-c $\subseteq$clspan(K-c) we get (d-c)+int(K-d)=int(K-c) in clspan(K-c). Thus the definition of rint(K) is independent of the choice of d in K.
Via canonical isometric inclusion we identify E with its canonical image (by the evaluation map ev_E) in its second dual E** and denote by cl(K) the (norm-)closure and by wcl(K) the σ(E**,E*)-closure of T. Now we are in position for the main result of this section.

Theorem 2.4

Let $K\subseteq E$ be bounded convex set such that rint(K) *is nonvoid and the image* T(d+clspan(K-d)) *of the closed real affine span of* K *by* T *is closed in* F. *Then*

(i) $g_{wcl(K),T^{**}}(T^{**}(a), T^{**}(b)) = g_{rint(K),T}(T(a),T(b)) =$
$= g_{cl(K),T}(T(a),T(b)) = g_{K,T}(T(a),T(b))$ *if* $a,b \in$ rint(K),

(ii) f(x,rint(K),T) = f(x,wcl(K),T**) $\le$
$\le$ min(f(x,cl(K),T),f(x,K,T)),

(iii) f(x, rint(K), T) = f(x, cl(K), T) *if* per(rint(K),T)= 0.

To prove Theorem 2.4 we need two preliminary lemmata.

Lemma 2.5

Let Y *be a convex subset of* E *such that* 0 intY *and let* L *be a closed linear subspace of E. Then*

(i) wcl($L\cap$ int(Y)) = wcl(L) $\cap$ wcl(Y),

(ii) cl($L\cap$ int(Y)) = $L\cap$cl(Y) $\supseteq L\cap Y \supseteq L\cap$ int(Y).

<u>Proof:</u> We use the bipolar theorem: If $X\subseteq E$ and $X^{\circ}:=\{f \in E^*: Re(f(e)) \le 1$ if $e\in X\} \subseteq E^*$ is the σ(E,E*)-polar of X in E* and $X^{\circ\circ} \subseteq E^{**}$ is the σ(E*,E**)-polar of X° in E** (i.e. the bipolar of X) then $X^{\circ\circ}$ is the σ(E**,E*)-closed convex hull of $X\cup\{0\} \subseteq E\subseteq E^{**}$ by Hahn-Banach separation theorem, cf. th. IV.1.5 of [SCHA]. Especially $X^{\circ\circ}$= wcl(X) if $0 \in X$ and X is convex.
The nontrivial inclusions we have to show are $L\cap$cl(Y) $\subseteq$
$\subseteq$ cl($L\cap$int(Y)) and wcl(L)$\cap$wcl(Y)$\subseteq$wcl($L\cap$int(Y)). Now if $X\subseteq E$ is a convex set, $e\in E$ and $e\in$wcl(X) then $e\in$cl(X) by Hahn-Banach separation theorem, i.e. $E\cap$wcl(X)=cl(X). Thus it suffices to show $L^{\circ\circ}\cap Y^{\circ\circ} \subseteq (L\cap int(Y))^{\circ\circ}$ if $tS\subseteq Y$ for some t>0 where S is the open unit ball of E. From $tS\subseteq Y$ and $Y^{\circ\circ}$= wcl(Y)= wcl(cl(Y)) one gets cl(Y)= cl(int(Y)) and $Y^{\circ\circ}= (int(Y))^{\circ\circ}$. Thus we may

assume $Y = \mathrm{int}(Y)$.
We have $L^\circ = L^\perp = \{g \in E^*: g|L = 0\}$ and $Y^\circ \subseteq (tS)^\circ = (1/t)S^\circ$ is a $\sigma(E^*,E)$-closed subset of the $\sigma(E^*,E)$-compact set $(1/t)S^\circ$ (=closed 1/t-ball of E*). By cor. 2 of th. IV.1.5 of [SCHA], $(Y \cap L)^\circ$ is the $\sigma(E^*,E)$-closed convex combination of $Y^\circ \cup L^\circ$ but $Y^{\circ\circ} \cap L^{\circ\circ}$ is the polar of the convex combination of $Y^\circ \cup L^\circ$. All we need is that the convex combination of Y° and L° in E* is $\sigma(E^*,E)$-closed. But this follows from the $\sigma(E^*,E)$-compactness of $Y^\circ \subseteq (1/t)S^\circ$. q.e.d.

<u>Lemma 2.6</u>

Let K_1 and K_2 be convex subsets of E. Then
$\text{H-dist}(K_1,K_2) = \text{H-dist}(\mathrm{cl}(K_1),\mathrm{cl}(K_2)) = \text{H-dist}(\mathrm{wcl}(K_1),\mathrm{wcl}(K_2))$.

<u>Proof:</u> If X,Y are subsets of a Banach space E with open unit ball S then by definition we have $\text{H-dist}(X,Y) = \inf\{ t>0: X \subseteq Y+tS,\ Y \subseteq X+tS\} = \inf\{t>0: X \subseteq Y+t\,\mathrm{cl}S),\ Y \subseteq X+t\,\mathrm{cl}(S)\}$. H-dist(.,.) is a semimetric on subsets of E such that $\text{H-dist}(X,\mathrm{cl}(X))=0$ and H-dist restricted to the system of closed subsets of E is a metric (with possible infinite distances if we also consider unbounded closed sets). By the triangular inequality we get $\text{H-dist}(X,Y) = \text{H-dist}(\mathrm{cl}(X),\mathrm{cl}(Y))$ for all subsets $X,Y \subseteq E$. The closed unit ball $S^{\circ\circ}$ of the second conjugate E** of E is the weak (i.e. $\sigma(E^{**},E^*)$-) closure of S and $S^{\circ\circ}$ is weakly compact. Thus $\mathrm{wcl}(Y)+tS^{\circ\circ}$ is weakly closed and is the weak closure of Y+tS. We obtain $X \subseteq Y+tS \subseteq \mathrm{wcl}(Y)+tS^{\circ\circ}$, $Y \subseteq \mathrm{wcl}(X)+tS^{\circ\circ}$ if $t > \text{H-dist}(X,Y)$. Hence $\mathrm{wcl}(X) \subseteq \mathrm{wcl}(Y)+tS^{\circ\circ}$ and $\mathrm{wcl}(Y) \subseteq \mathrm{wcl}(X)+tS^{\circ\circ}$, i.e. $t \geq \text{H-dist}(\mathrm{wcl}(X),\mathrm{wcl}(Y))$, if $t > \text{H-dist}(X,Y)$ or $\text{H-dist}(X,Y) \geq \text{H-dist}(\mathrm{wcl}(x),\mathrm{wcl}(Y))$ if X and Y are subsets of E.
Now let be given convex subsets K_1 and K_2 of E and $s > \text{H-dist}(\mathrm{wcl}(K_1),\mathrm{wcl}(K_2))$. Put $r = (\text{H-dist}(\mathrm{wcl}(K_1),\mathrm{wcl}(K_2))+s)/2$ and $t = (s-r)/2$. Then $K_1 \subseteq \mathrm{wcl}(K_1) \subseteq \mathrm{wcl}(K_2)+rS^{\circ\circ} = \mathrm{wcl}(K_2+rS)$ and $K_2 \subseteq \mathrm{wcl}(K_1+rS)$. By separation theorem, $E \cap \mathrm{wcl}(Y) = \mathrm{cl}(Y)$ if $Y \subseteq E$ is convex. We get $K_1 \subseteq \mathrm{cl}(K_2+rS) \subseteq (K_2+rS)+tS \subseteq K_2+s.S$ and similary $K_2 \subseteq K_1+s.S$, i.e. $s > \text{H-dist}(K_1,K_2)$.
Thus $\text{H-dist}(\mathrm{wcl}(K_1),\mathrm{wcl}(K_2)) \geq \text{H-dist}(K_1,K_2)$. q.e.d.

<u>Proof of Theorem 2.4:</u> By definitions of $\varrho_{K,T}$, $\varrho_{\mathrm{wcl}(K),T^{**}}$, $f(x,\ldots)$ we may replace K by K-d and E by clspan(K-d) if $d \in K$, then T by its restriction to clspan(K-d) and F by cl(T(clspan(K-d)) because $\mathrm{wcl}(K-d)=\mathrm{wcl}(K)-d$, $(T|\mathrm{clspan}(K-d))^{**} = T^{**}|(\mathrm{clspan}(K-d))^{**}$, where we canonically identify the second dual of a closed linear subspace of E (resp. F) with the weak closure of this subspace in the second dual of E (resp. F) as a consequence of Hahn-Banach extension theorem. By our assumptions d can be chosen in rint(K). That is w.l.o.g. we may assume E *is the closed real linear span of* K, K *has nonvoid interior* int(K) = rint(K) *in* E, $0 \in \mathrm{int}(K)$ *and* T(E)= F. The equality T(E)= F follows from our assumtion T(d+clspan(K-d))= cl(T(d+clspan(K-d))).
<u>Ad(i):</u> Let be given $a,b \in \mathrm{rint}(K)=\mathrm{int}(K)$. Then Lemma 2.5 applies to K-a (resp. K-b) and L:=ker(T):
$\mathrm{cl}((a+L) \cap \mathrm{int}K) = a+\mathrm{cl}(\mathrm{int}(K-a) \cap L) = a+(\mathrm{cl}(K-a) \cap L) = (a+L) \cap \mathrm{cl}(K)$, similary $\mathrm{cl}((b+L) \cap \mathrm{int}K) = (b+L) \cap \mathrm{cl}(K)$, $\mathrm{wcl}((a+L) \cap \mathrm{int}K) = (a+\mathrm{wcl}(L)) \cap \mathrm{wcl}(K)$, $\mathrm{wcl}((b+L) \cap \mathrm{int}K) = (b+\mathrm{wcl}(L)) \cap \mathrm{wcl}(K)$. Because $\varrho_{K,T}(T(a),T(b)) = \text{H-dist}(K \cap (a+L), K \cap (b+L))$ and $(a+L) \cap \mathrm{int}K \subseteq$ $\subseteq (a+L) \cap K \subseteq (a+L) \cap \mathrm{cl}(K)$ by Lemma 2.6 we obtain
$\varrho_{K,T}(T(a),T(b)) = \varrho_{\mathrm{cl}(K),T}(T(a),T(b)) = \varrho_{\mathrm{rint}K,T}(T(a),T(b)) =$ $= \text{H-dist}((a+\mathrm{wcl}(L)) \cap \mathrm{wcl}(K), (b+\mathrm{wcl}(L)) \cap \mathrm{wcl}(K))$. On the other hand $\text{H-dist}((a+\ker(T^{**})) \cap \mathrm{wcl}(K), (b+\ker(T^{**})) \cap \mathrm{wcl}(K)) =$ $= \varrho_{\mathrm{wcl}K,T^{**}}(T^{**}(a),T^{**}(b))$. It suffices to show that $\mathrm{wcl}(L) =$

$\ker(T^{**})$ if $T(E)=F$. By the inverse mapping theorem there exists an isomorphism I from E/L onto F such that $T=IP$ where P is the quotient map from E onto E/L. Thus $T^{**}=I^{**}P^{**}$ with I^{**} invertible and $\ker(T^{**})= \ker(P^{**})$. By canonical identification of $(E/L)^*$ with $L^{\perp}\subseteq E^*$ and of $E^{**}/\mathrm{wcl}(L)$ with $(L^{\perp})^*\cong(E/L)^{**}$ (Hahn-Banach theorem) P^* becomes the injection $L^{\perp} \dashrightarrow E^*$ and P^{**} becomes the quotient map $E^{**}\dashrightarrow E^{**}/\mathrm{wcl}(L)$. Thus $\ker(T^{**})=$ $= \ker(P^{**})= \mathrm{wcl}(L)$.

Ad(ii): Let be $X(K,T):=\{\rho_{K,T}(T(a),T(b)): \|T(a-b)\|\le x,\ a,b\in K\}$. By part (i), $X(\mathrm{rint}(K),T)$ is contained in the intersection of $X(K,T)$, $X(\mathrm{cl}(K),T)$ and $X(\mathrm{wcl}(K),T^{**})$, thus $f(x,\mathrm{rint}(K),T)=$ $\sup X(\mathrm{rint}(K),T) \le \min(f(x,K,T),f(x,\mathrm{cl}(K),T),f(x,\mathrm{wcl}(K),T^{**}))$. Our w.l.o.g.-assumptions say that 0 is in $\mathrm{int}(K)$. Now we put $L=\{(a,b,T(a-b)):a,b\in E\}\subseteq E\oplus E\oplus F$ and $Y=\{(a,b,c): a,b\in \mathrm{int}K,\ c\in xS_F\}$ where S_F is the open unit ball of F. Then L is a closed subspace of $E\oplus E\oplus F$ (with l_1 sum norm) isomorphic to $E\oplus E$, $\mathrm{wcl}(L)=$ $= \{ (a,b,T^{**}(a-b)):\ a,b\in E^{**}\} \subseteq E^{**}\oplus E^{**}\oplus F^{**}$, $\mathrm{cl}(Y) =$ $= \{(a,b,c): a,b\in \mathrm{wcl}(K),\ c\in x.\mathrm{wcl}(S_F)\}$ where $E^{**}\oplus E^{**}\oplus F^{**}$ and the second conjugate of $E\oplus E\oplus F$ are canonically identified. Because $0\in Y=\mathrm{int}Y$, by Lemma 2.5 it holds $Y\cap L= \{(a,b,T(a-b)):\ a,b\in \mathrm{int}K,$ $\|T(a-b)\| < x \}$ is weakly dense in $\mathrm{wcl}(Y)\cap \mathrm{wcl}(L) =$ $= \{(a,b,T^{**}(a-b)):a,b\in \mathrm{wcl}(K),\|T^{**}(a-b)\|\le x\}$ and $\mathrm{cl}(Y\cap L) =$ $= \mathrm{cl}(Y)\cap L= \{(a,b,T(a-b)):a,b\in \mathrm{cl}(K),\|T(a-b)\|\le x\}$. With other words there exists a net $\{(a_\gamma,b_\gamma)\}_\gamma$ in $E\oplus E$ such that a_γ and b_γ are in $\mathrm{int}K$, $a_\gamma\dashrightarrow a$ and $b_\gamma\dashrightarrow b$ weakly and $\|T(a_\gamma-b_\gamma)\|<x$ if a and b are in $\mathrm{wcl}(K)$ and $\|T^{**}(a-b)\|\le x$. By definition of $f(x)$, $\mathrm{dist}(a_\gamma,T^{-1}(T(b_\gamma))\cap \mathrm{int}K)\le f(x,\mathrm{int}K,T)$. Now let be given $t>0$. We choose $c_\gamma\in T^{-1}(T(b_\gamma))\cap \mathrm{int}K$ such that $\|a_\gamma-c_\gamma\| \le f(x,\mathrm{int}K,T)+t$. Taking otherwise a suitable subnet by weak compactness of $\mathrm{wcl}(K)$ we may assume that $\{(a_\gamma,c_\gamma)\}_\gamma$ is weakly convergent in $(E\oplus E)^{**}\cong E^{**}\oplus E^{**}$. Let c be the weak limit point of $\{c_\gamma\}$. Then $c \in \mathrm{wcl}(K)$, $\|a-c\|\le\sup\{\|a_\gamma-c_\gamma\|\}\le f(x,\mathrm{int}K,T)+t$ and $T^{**}(c)=$ $=\text{w-lim}\ T^{**}(c_\gamma)=\text{w-lim}\ T^{**}(b_\gamma)= T^{**}(b)$ because $c_\gamma\in \mathrm{int}K$, $T^{**}(c_\gamma)=$ $=T(c_\gamma)= T(b_\gamma)= T^{**}(b_\gamma)$ and T^{**} is weakly continuous. By Lemma 2.3(i), $f(x,\mathrm{wcl}(K),T^{**}) \le f(x,\mathrm{int}(K),T)= f(x,\mathrm{rint}(K),T)$.

Ad(iii): As we have seen in the proof of part (ii) $\{(a,b):$ $a,b\in \mathrm{cl}(K),\ \|T(a-b)\|\le x\}$ is the closure of $\{(a,b):\ a,b\in \mathrm{int}K,$ $\|T(a-b)\|<x\}$ with respect to the l_1-sum norm on $E\oplus E$. At this moment for brevity let be $f(x):= f(x,\mathrm{int}K,T)$. Now assume $\mathrm{per}(\mathrm{int}K,T)= \inf\{f(x):t>0\}= 0$. Then for every fixed $t>0$ there exists a sequence $x_1>x_2>\ldots>0$ such that $f(x_n)+x_n \le t/2^{n+1}$. Let be given $x>0$, $a,b\in \mathrm{cl}(K)$ with $\|T(a-b)\|\le x$ and $t>0$. With $x_1,x_2,\ldots$ as above there are sequences (a_n) and (b_n) in $\mathrm{int}K$ such that $\|a_n-a\|\le x_n$, $\|b_n-b\|\le x_n/(2\|T\|)$ and $\|T(a_n-b_n)\|\le x$. It follows $\|T(b_n-b_{n+1})\| \le x_n$, $\rho_{\mathrm{int}K,T}(T(b_n),T(b_{n+1})) \le f(x_n)$, $\mathrm{dist}(a,T^{-1}(T(b_n))\cap \mathrm{int}K) \le \|a_n-a\|+ \mathrm{dist}(a_n,T^{-1}(T(b_n))\cap \mathrm{int}K) \le$ $\le x_n + f(x) \le f(x) + t/2$. We find c_1 in $T^{-1}(T(b_1))\cap \mathrm{int}K$ with $\|c_1-a\|\le t/2+f(x)$ and then by induction $c_{n+1}\in T^{-1}(T(b_{n+1}))\cap \mathrm{int}K$ with $\|c_n-c_{n+1}\|\le f(x_n)+x_n\le t/2^{n+1}$. Then $\|c_1-c_2\|+\|c_2-c_3\|+\ldots$ $\le t/2$ and $c=\lim(c_n)$ satisfies $c \in \mathrm{cl}(K)$, $\|c_1-c\|\le t/2$, $T(c)=$ $= \lim T(c_n)= \lim T(b_n)= T(b)$ and $\|a-c\|\le \|a-c_1\|+\|c_1-c\| \le$ $\le f(x)+t$, hence $\mathrm{dist}(a,T^{-1}(T(b))\cap \mathrm{cl}(K)) \le f(x)+t$. By Lemma 2.3(i) we get $f(x,\mathrm{cl}(K),T) \le f(x)= f(x,\mathrm{int}(K),T)$. q.e.d.

Remark 2.7

Under assumptions of Theorem 2.4, $\mathrm{rint}(K) \subseteq \mathrm{rint}(\mathrm{wcl}(K))$ *and* $f(x,\mathrm{rint}(\mathrm{wcl}(K)),T^{**}) = f(x,\mathrm{wcl}(K),T^{**}) = f(x,\mathrm{rint}(K),T)$.

As in the proof of Theorem 2.4 one can restrict to the case $0\in \mathrm{int}(K)$ and $T(E)=F$. Then $0\in \mathrm{int}(\mathrm{wcl}(K))=\mathrm{rint}(\mathrm{wcl}(K))$, $\mathrm{wcl}(K)=$ $=\mathrm{cl}(\mathrm{rint}(\mathrm{wcl}(K)))$ and Theorem 2.4 also applies to $\mathrm{wcl}(K)$, T^{**}, E^{**} and F^{**}. From $\mathrm{rint}(K)=\mathrm{int}(K)\subseteq \mathrm{int}(\mathrm{wcl}(K))=\mathrm{rint}(\mathrm{wcl}(K))$ by

Theorem 2.4 (i) one gets $\varrho_{rint(wcl(K)),T^{**}}(T^{**}(a),T^{**}(b)) = \varrho_{wcl(K),T^{**}}(T^{**}(a),T^{**}(b)) = \varrho_{rint(K),T}(T(a),T(b))$ if a,b are in rint(K). It follows $f(x,rint(K),T) \leq f(x,rint(wcl(K)),T^{**})$, cf. proof of Th.2.4(ii). On the other hand by Theorem 2.4(ii), $f(x,rint(wcl(K)),T^{**}) \leq f(x,wcl(K),T^{**}) = f(x,rint(K),T)$.

Lemma 2.3(ii), Proposition 2.2 and Remark 2.7 together prove Theorem 1.1. We need some corollaries.

<u>Lemma 2.8</u>

(i) $f(x,t(K+a),T) = t^{-1}f(tx,K,T)$,

(ii) $f(x,K,T) \leq x\|T^{-1}|T(E)\|$ *if* T *is invertible on* T(E),

(iii) $f(x,P(K),T) \leq f(x,K,T)$ *if* P *is a real linear contraction on* E *such that* $P(K) \subseteq K$, $P^2 = P$ *and there exists a real linear contraction* Q *on* F *such that* $Q^2=Q$ *and* TP=QT.

<u>Proof:</u> (i) and (ii) follow from Lemma 2.3 (iii) and definition of f(x,...). Ad(iii): Let be $a \in P(K) \subseteq K$ and $d \in TP(K) = QT(K)$ with $\|T(a)-d\| < x$. Then Q(d)=d and there exists $b \in K$ with $\|T(a-b)\| \leq f(x,K,T)$ and T(b)=d. Let be e=P(b). Then $T(a-e) = TP(a-b) = QT(a-b)$, $\|T(a-e)\| \leq \|Q\|.\|T(a-b)\| \leq f(x,K,T)$, $e,a \in P(K)$ and T(e)=TP(b)=QT(b)=Q(d)=d. Use Lemma 2.3(i). q.e.d.

Let E and F be Banach spaces and T: E --> F a bounded linear map. Assume there exists an isometry I from E** onto a C*-algebra B, projections p,q in B, operators b,d in B and an isometry J from pBq onto F** such that

(i) $T^{**}(a) = J(pb(I(a))dq)$ if $a \in E^{**}$ and

(ii) there exist positive selfadjoint operators $g,h \in B$ with pbb*pg=p and hqd*dq=q (i.e. pbb*p and qd*dq are invertible in pBp and qBq respectivly).

Put p′= b*pgpb, q′= dqhqd* and let T′ be the map c--->p′cq′ from B into $p'Bq' \subseteq B$.

<u>Corollary 2.9</u>

Under the above assumptions, p′,q′ *are projections in* B *and* $f(x,T) \leq f((\|g\|.\|h\|)^{1/2}x,T')$.

<u>Proof:</u> By (ii) p′ and q′ are projections. Let T" be the map from pBq into p′Bq′ given by T"(pcq)=b*pgpcqhqd*. Then $\|T''\| \leq (\|g\|.\|h\|)^{1/2}$, T" is invertible, $(T'')^{-1}(p'cq') = pbp'cq'dq = pbcdq$ if $c \in B$. Thus $T^{**} = J(T'')^{-1}T'I$. It follows $f(x,T^{**}) = f(x,(T'')^{-1}T')$ because I and J are isometries. T′ maps the open unit ball of B onto that of p′Bq′. By Lemma 2.3 (iii), Lemma 2.8 (iii) and Remark 2.7 we get $f(x,T) \leq f(\|T''\|x,T')$. q.e.d.

Now assume moreover p=q,b=d,g=h and that on E and F there are involutions e--->e* and in E is a "unit" 1_E such that

(iii) $I(1_E)=1_B$, $I(e^*)=I(e)^*$ if $e \in E^{**}$ and $J(pb^*p) = J(pbp)^*$ if $b \in B$ where e--->e* ($e \in E^{**}$ or $e \in F^{**}$) mean the second adjoints of the involutions defined in E and F.

Let be s<t and spec(s,t)= $\{e \in E:\ e=e^*,\ spec(I(e)) \subseteq]s,t[\ \}$. Put $P(e)=(e^*+e)/2$ if $e \in E$ and $Q(c)=(c^*+c)/2$ if $c \in F$. Then spec(s,t)= $= ((t-s)/2)P(S) + ((t+s)/2)1_E$ and P,Q,K=S,T satisfy the assumptions of Lemma 2.8(iii). By Lemma 2.8(i), Lemma 2.8(iii) and Corollary 2.9, under the assumptions above we obtain:

<u>Corollary 2.10</u>

$f(x,spec(s,t),T) \leq (2/(t-s))f(x\|g\|(t-s)/2,T')$.

3. Perturbation of unitaries

If b is a contraction on a Hilbert space we denote by $U(b)$ the unitary matrix

$$\begin{pmatrix} b, & (1-bb^*)^{1/2} \\ (1-b^*b)^{1/2}, & -b^* \end{pmatrix}.$$

Theorem 3.1

Let A be a unital C-algebra (real or complex), u a unitary in A and p,q a couple of projections in A such that* $\|puq\|<1$. *Then for any contraction* b *in* pAq *there exists a unitary* α *in* A *such that* $p\alpha q = b$ *and* $\|u-\alpha\| = \|U(puq)-U(b)\|$.

Corollary 3.2

Let be A *a unital C*-algebra (real or complex),* p, q *nonzero projections in* A *and* T *the map from* A *onto* pAq *given by* $T(a)= paq$ (a in A). *Then* $f(x,T) \leq x + (2x)^{1/2}$.

We need some preliminary lemmata. To simplify notations let $diag(a,b,\ldots)$ be the diagonal matrix with diagonal elements a, b,... , let $M(c)$ be the matrix

$$\begin{pmatrix} c, & (p-cc^*)^{1/2} \\ (q-c^*c)^{1/2}, & -c^* \end{pmatrix}$$

if c is a contraction in pAq and let us denote by Z the matrix

$$\begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}.$$

Lemma 3.3

Let be given a C-algebra A, projections* p,q *in* A, a *in* pAq, d *in* qAp *such that* a *and*

$$N:= \begin{pmatrix} a, & (p-aa^*)1/2 \\ (q-a^*a)1/2, & d \end{pmatrix}$$

are contractions in $M_2(A)$. *Then* $M(a)$ *satisfies* $M(a)^*M(a)= = diag(q,p)$, $M(a)M(a)^*= diag(p,q)$.
If moreover $\|a\|<1$ *then* $d= -a^*$.

Proof: $(1-aa^*)^{1/2}p= p(1-aa^*)^{1/2}= (p-aa^*)^{1/2}$ and $q(1-a^*a)^{1/2}= = (1-a^*a)^{1/2}q= (q-a^*a)^{1/2}$ because $aa^*\leq p$, $aa^*\leq q$ and $paq=a$. It follows $a^*(p-aa^*)^{1/2}= (q-a^*a)^{1/2}a^*$, $(p-aa^*)^{1/2}a= a(q-a^*a)^{1/2}$, and $(q-a^*a)^{1/2}d= (1-a^*a)^{1/2}d$. Put $M:=M(a)$. Using this identities straightforward computations show $M^*M= diag(q,p)$ and $MM^*= diag(p,q)$. Moreover it holds
$M^*N = diag(q,p-aa^*-ad) + diag(c,0)Z$ where
$c= a^*(p-aa^*)^{1/2}+ (q-a^*a)^{1/2}d$. But $\|M^*N\| \leq \|M^*\|.\|N\| \leq 1$ by assumptions on N and the above observations concerning M. Looking to the (1,1)-element of $(M^*N)(M^*N)^*$ we obtain $\|q+cc^*\| \leq 1$. On the other hand $c= qc$ by the above identities. Thus $q+cc^* = q(q+cc^*)q \leq q$, i.e. $c = 0$, $(1-a^*a)^{1/2}(-a^*) = = -a^*(p-aa^*)^{1/2} = (q-a^*a)^{1/2}d = (1-a^*a)^{1/2}d$. Now if $\|a\| < 1$ then $(1-a^*a)^{1/2}$ is invertible and $d= -a^*$. q.e.d.

Lemma 3.4

Let be A, a, p, q *as in Lemma 3.3 and* $\|a\| < 1$. *Assume that*

$$V := \begin{bmatrix} a, & (p-aa^*)^{1/2}, & g \\ (q-a^*a)^{1/2}, & d, & h \\ f, & k, & e \end{bmatrix}$$

is a partial isometry in $M_3(A)$ *such that* $V^*V = \mathrm{diag}(q,p,s)$ *and* $VV^* = \mathrm{diag}(p,q,r)$ *then* $f=g=h=k=0$, $d=-a^*$, $e^*e=s$ *and* $ee^*=r$.

Proof: The upper left 2x2-submatrix of V must be a contraction in $M_2(A)$. By Lemma 3.3 $d=-a^*$ because $\|a\|<1$. Using now the equality $VV^*=\mathrm{diag}(p,q,r)$ from $d=-a^*$ we get $gg^*=0$, $hh^*=0$ and using $V^*V=\mathrm{diag}(q,p,s)$ we obtain $f^*f=0$, $k^*k=0$. From $f=g=h=k=0$ it follows $e^*e=s$ and $ee^*=r$. q.e.d.

Lemma 3.5

Let p,q,r,s *be projections in a unital C*-algebra* A *and* $v,w \in A$ *such that* $v^*v=q$, $ww^*=p$, $p+r+vv^*=1$ *and* $q+s+w^*w=1$. *Put*

$$D := \begin{bmatrix} p,v,r \\ 0,0,0 \\ 0,0,0 \end{bmatrix}, \quad E := \begin{bmatrix} q,w^*,s \\ 0,0,0 \\ 0,0,0 \end{bmatrix}, \quad F(y) := \begin{bmatrix} y,0,0 \\ 0,0,0 \\ 0,0,0 \end{bmatrix},$$

$G(y) := D^*F(y)D$, $H(y) := E^*F(y)E$, $T(y) := D^*F(y)E$ *if* y *is in* A. *Then*

(i) $T(y) = \begin{bmatrix} pyq, & pyw^*, & pys \\ v^*xq, & v^*yw^*, & v^*ys \\ ryq, & ryw^*, & rys \end{bmatrix}$ *for* y *in* A.

(ii) G *and* H *are faithful *-representations from* A *into* $M_3(A)$ *such that* $G(1)= \mathrm{diag}(p,q,r)$ *and* $H(1)= \mathrm{diag}(q,p,s)$.

(iii) T *is an isometry from* A *into* $M_3(A)$ *such that* $T(y)T(y)^* = G(yy^*)$ *and* $T(y)^*T(y)= H(y^*y)$ *if* y *is in* A.

(iv) $T(a+vb+cw+vdw+e) = \begin{bmatrix} a,c,0 \\ b,d,0 \\ 0,0,e \end{bmatrix}$

if $e \in rAs$, $a \in pAq$, $b \in qAq$, $c \in pAp$, $d \in qAp$.

(v) *If* $y \in A$ *then the equality* $T(y) = \begin{bmatrix} a,c,0 \\ b,d,0 \\ 0,0,e \end{bmatrix}$ *implies*

$y=a+vb+cw+vdw+e$ *with* $e \in rAs$, $a \in pAq$, $b \in qAq$, $c \in pAp$, $d \in qAp$.

(vi) *An element* $y= a+vb+cw+vdw+e$ *satisfying the conditions of (iv) is a unitary of* A *if and only if the upper left 2x2-submatrix is a partial isometry (say W) in* $M_2(A)$ *with* $W^*W=\mathrm{diag}(q,p)$ *and* $WW^*=\mathrm{diag}(p,q)$ *and* e *is a partial isometry with* $e^*e=s$ *and* $ee^*=r$.

Proof: (i) is obvious. Ad(ii)+(iii): We denote $M_3(A)$ by B and the projection $\mathrm{diag}(1,0,0)=F(1)$ by P. Then F defines a unital C*-algebra isomorphism from A onto PBP. By our assumtions $DD^* = P = EE^*$. Thus $z \to D^*zE$, $z \to D^*zD$ and $z \to E^*zE$ define linear isometries from PBP onto D^*DBE^*E, D^*DBD^*D and E^*EBE^*E respectively. Moreover the latter two are unital C*-algebra isomorphisms. $T(y)T(y)^* = D^*F(y)EE^*F(y^*)D = D^*F(yy^*)D = G(yy^*)$ and $T(y)^*T(y) = H(y^*y)$. Hence T is an isometry from A onto D^*DBE^*E and G,H are unital *-isomorphisms from A onto D^*DBD^*D and E^*EBE^*E respectively. It holds $G(1)= D^*D= \mathrm{diag}(p,q,r)$ and $H(1)= E^*E= \mathrm{diag}(q,p,s)$ as computation shows.
Ad(iv): By assumptions on p,q,r,s,w,v it holds $ws=qs=rv=rp=pv=$

$=wq=0$. Put $y=a+vb+cw+vdw+e$. Then by assumtions on a,b,c,d,e we have $ry=e=ys=rys$. With $g=y-e$ by (i) it follows

$$T(y)= \begin{bmatrix} pgq, & pgw^*, & 0 \\ v^*gq, & v^*gw^*, & 0 \\ 0, & 0, & e \end{bmatrix}.$$

Using $pv=wq=0$ we get $pgq=paq=a$, $pgw^*=cww^*=cp=c$, $v^*gq=v^*vb=qb=b$ and $v^*gw^*=v^*vdww^*=qdp=d$.

Ad(v): As we have seen in the proof of (ii) and (iii) T defines an isometry from A onto $\mathrm{diag}(p,q,r)B\mathrm{diag}(q,p,s)$. From $T(y)= \mathrm{diag}(p,q,r)T(y)\mathrm{diag}(q,p,s)$ we see that $a=paq$, $b=qbq$,... Put $z=a+vb+cw+vdw+e$. Then by (iv), $T(z)= T(y)$. But $\ker(T)= 0$, i.e. $y=z$.

Ad(vi): By (ii), (iii) and (iv),

$W^*W=\mathrm{diag}(q,p)$, $WW^*=\mathrm{diag}(p,q)$, $e^*e=s$ and $ee^*=r$ if and only if

$T(y)^*T(y) = H(1)$ and $T(y)T(y)^* = G(1)$ if and only if

$H(1-y^*y) = 0$ and $G(1-yy^*) = 0$ if and only if

$y^*y = 1 = yy^*$. q.e.d.

<u>Proof of Theorem 3.1</u>: We put $a=puq$. Then $\|a\|<1$, $(1-a^*a)^{-1/2}$ and $(1-aa^*)^{-1/2}$ exist. Let be $v= (1-p)uq(1-a^*a)^{-1/2}$ and $w=(1-aa^*)^{-1/2}pu(1-q)$. We have

$v^*v= (1-a^*a)^{-1/2}qu^*(1-p)uq(1-a^*a)^{-1/2}=$

$= (1-a^*a)^{-1/2}(q-a^*a)(1-a^*a)^{-1/2} = q$ because $qa^*a=a^*a=a^*aq$. Similary we obtain $ww^*=p$. Especially w and v are partial isometries. By definition $(1-p)v=v$, $w(1-q)=w$, $(1-p)uq=$ $= v(1-a^*a)^{1/2}= vq(1-a^*a)^{1/2}= v(q-a^*a)^{1/2}$, $pu(1-q)=(p-aa^*)^{1/2}w$. Thus $vv^*\leq 1-p$, $w^*w\leq 1-q$, $v^*uq= (q-a^*a)^{1/2}$ and $puw^*= (p-aa^*)^{1/2}$. Put $r=1-p-vv^*$ and $s=1-q-w^*w$. Then p,v,r,q,w,s satisfy the assumptions of Lemma 3.5 and $T(u)$ defined there satisfies $T(u)^*T(u)=H(u^*u)=H(1)=\mathrm{diag}(q,p,s)$, $T(u)T(u)^*=G(1)=\mathrm{diag}(p,q,r)$ by Lemma 3.5 (ii) and (iii). Moreover by Lemma 3.5(i) and the above equations Lemma 3.4 applies to $T(u)$:

$$T(u)= \begin{bmatrix} a, & (p-aa^*)^{1/2}, & 0 \\ (q-a^*a)^{1/2}, & -a^*, & 0 \\ 0, & 0, & e \end{bmatrix}$$

with $e^*e=s$, $ee^*=r$. We put $\alpha= b+v(q-b^*b)^{1/2}+(p-bb^*)^{1/2}w-vb^*w+e$.
By Lemma 3.5(iv),

$$T(\alpha)= \begin{bmatrix} b, & (p-bb^*)^{1/2}, & 0 \\ (q-b^*b)^{1/2}, & -b^*, & 0 \\ 0, & 0, & e \end{bmatrix}.$$

By Lemma 3.3 and Lemma 3.5(vi), α is a unitary element of A.
We have $\|u-\alpha\|= \|T(u-\alpha)\|= \|M(a)-M(b)\|$.
Now if $c \in pAq$ then $U(c) = M(c)+\mathrm{diag}(1-p,1-q)Z$.
Thus $\|u-\alpha\|= \|U(a)-U(b)\|= \|U(puq)-U(b)\|$. q.e.d.

<u>Lemma 3.6</u>

Let be a,b contractions and h,k positive selfadjoint operators on a Hilbert space. Then

(i) $\|h^{1/2}-k^{1/2}\| \leq \|h-k\|^{1/2}$,

(ii) $\|a^*a-b^*b\| \leq 2\|a-b\|$,

(iii) $\|U(a)-U(b)\| \leq \|a-b\| + (2\|a-b\|)^{1/2}$.

<u>Proof</u>: Ad(i): Let $t=\|h-k\|$. Then $h+t\leq(h^{1/2}+t^{1/2})^2$ and $k\leq h+t$. The function $g(t)=t^{1/2}$ is operator monotone on $[0,\infty[$, cf. [TAK,I.6.3]. Thus $k^{1/2} \leq (h+t)^{1/2} \leq h^{1/2}+t^{1/2}$ and we can interchange k and h in this inequality.

Ad(ii): $a^*a-b^*b = a^*(a-b) + (a-b)^*b$.

Ad(iii): With $c:= \mathrm{diag}((1-aa^*)^{1/2}-(1-bb^*)^{1/2}, (1-a^*a)^{1/2}-(1-b^*b)^{1/2})$ by (i) and (ii) we get $\|c\|\leq(2\|a-b\|)^{1/2}$.
We have $U(a)-U(b)= \mathrm{diag}(a-b,(a-b)^*)+cZ$. q.e.d.

Proof of Corollary 3.2: Let be $a,b \in A$ and $x>0$ such that $pbq=b$, $||a||<1$, $||b||<1$ and $||paq-b||<x$. Put $P=\mathrm{diag}(p,0)$, $Q=\mathrm{diag}(q,0)$, $B=\mathrm{diag}(b,0)$ and $u=U(a)$. Then u is a unitary in $M_2(A)$ such that $PuQ=\mathrm{diag}(paq,0)$, $||PuQ|| \leq ||a|| <1$ and $||PuQ-B||=||paq-b||<x$. Moreover $PBQ=B$ and $||B||=||b||<1$. By Theorem 3.1 there exists a unitary $\alpha \in M_2(A)$ such that $||u-\alpha||= =||U(PuQ)-U(B)||$ and $\mathrm{diag}(pcq,0)=P\alpha Q=B=\mathrm{diag}(b,0)$ where c means the $(1,1)$-element of the unitary $2x2$-matrix α. Looking at the $(1,1)$-element of $u-\alpha$ by Lemma 3.6(iii) we obtain $||a-c|| \leq \leq ||u-\alpha|| \leq ||U(PuQ)-U(B)||<x+(2x)^{1/2}$. On the other hand $||c|| \leq \leq ||\alpha|| = 1$ and $pcq=b$. Now let be $0<t<2$. Put $s=1-(t/2)$ and $e=(p+s(1-p))c(q+s(1-q))$. Then $peq=pcq=b$, $||e-c|| \leq 2(1-s)=t$, $||a-e|| < t+x+(2x)^{1/2}$, $||(p+s(1-p))cq|| \leq (1-s)||b||+s||cq|| \leq \leq 1-(1-s)(1-||b||)$ and $||e|| \leq 1-(1-s)^2(1-||b||) < 1$. Thus $f(x,T) \leq x+(2x)^{1/2}$ by Lemma 2.3(i). q.e.d.

4. On C*-spaces and C*-systems

As in [C/E2] and [WI] a *(complete) matrix normed space* or *matrix Banach space* X is a Banach space together with a system of norms $(||.||_n)_{n>0}$ on the spaces $\mathrm{Mat}_n(X)$ of nxn-matrices over X such that $(\mathrm{Mat}_n(X),||.||_n)$ becomes for every positive integer n a uniform C*-bimodul with respect to the action of $\mathrm{Mat}_n=\mathrm{Mat}_n(K)$ on $\mathrm{Mat}_n(X)$ where K means the real or complex field and such that $||b \oplus 0_m||_{n+m}=||b||_n$ if $b \in \mathrm{Mat}_n(X)$ and 0_m is the zero of $\mathrm{Mat}_m(X)$. If X and Y are matrix normed spaces and if T is a linear map from X into Y such that the matricial extensions $T_n : [a_{i,k}] \in \mathrm{Mat}_n(X) \dashrightarrow [T(a_{i,k})] \in \mathrm{Mat}_n(Y)$ of T are contractions (resp. isometries) then T is *completely contractive* {c.c.} (resp. is *completely isometric* {c.i.}). X and Y are *completely isometric isomorphic* (c.i.i.) if there exists a c.i. map from X *onto* Y. Any C*-algebra A is a *matrix normed space* in a natural way if we use the C*-algebra norms on $\mathrm{Mat}_n(A)$. X is an *operator space* if there is a complete isometry from X into a C*-algebra (i.e. X is c.i.i. to a closed linear subspace of $\mathcal{L}(H)$ for some Hilbert space H). If $Y \subseteq X$ is a closed subspace of X and if we consider Y as a matrix normed space with matrix norms inherited from X we call Y a *matrix subspace* of X (or *operator subspace* of X if moreover X is an operator space). If $Y \subseteq X$ is a matrix subspace of X the algebraic Mat_n-bimodul isomorphies $\mathrm{Mat}_n(X/Y) \simeq \simeq \mathrm{Mat}_n(X)/\mathrm{Mat}_n(Y)$ define the structure of a matrix normed space on X/Y which is called *quotient matrix space* of X by Y. We say *quotient operator space* if X is an operator space. If $T:X \dashrightarrow Z$ is a c.c. map from X into a matrix normed space Z with $\ker(T)=Y$ then the maps I,π_Y given by the Banach space theory decomposition $T=I\circ\pi_Y$ where $I:X/Y \dashrightarrow Z$ and π_Y is the quotient map are c.c. maps. $(\pi_Y)_n$ maps the open unit ball of $\mathrm{Mat}_n(X)$ onto the open unit ball of $\mathrm{Mat}_n(X/Y)$ (i.e. $((\pi_Y)_n)^*$ is an isometry for $n=1,2,...$) and I is a c.i. isomorphism if and only if T_n maps the open unit ball of $\mathrm{Mat}_n(X)$ onto open unit ball of $\mathrm{Mat}_n(Z)$ for $n=1,2,...$

A closed linear subspace X of a C*-algebra which is invariant under triple-products $(a,b,c) \dashrightarrow ab^*c$ is called *C*-triple system* (cf. [YO]). There is an axiomatic definition we won't bore the reader with. In our definition at the same time a C*-triple system is an operator space and hence is matrix normed. If p and q are projections in a C*-algebra C then pCq is a simple example of a C*-triple system. Two C*-triple systems are isomophic if there is a triple product preserving linear isomorphy between them. Simple calculations verify Lemma 4.1.

Lemma 4.1

Let X be a C-triple subsystem of a unital C*-algebra B.*

(i) *Y=X*+X+span(X*X+XX*) is a *-subalgebra of B, where X*={x*: x∈X}.*

(ii) *If p is a projection in B such that px(1-p)=x for x in X then p(cl(Y))(1-p)=X.*

Proposition 4.2

If B is a C-algebra and X⊆B is a C*-triple system then there exists a unital C*-algebra C and a projection p in C such that X is C*-triple system isomorphic to pC(1-p). Moreover this isomorphy is completely isometric.*

Proof: We may assume B to be unital. The map T(b)=diag(b,0)Z, where we adopt the notations of sec. 3, defines a completely isometric triple product preserving linear map from B onto the upper triangular matrices of $M_2(B)$. T(X) and p=diag(1,0) satisfy the assumptions of Lemma 4.1. If C is the closure of T(X)*+T(X)+span(T(X)*T(X)+T(X)T(X)*) we get the result. q.e.d.

Combining Proposition 4.2 with Youngson's result [YO] we get:

Corollary 4.3

The range of a completely contractive projection on C-triple system is completely isometric isomorphic to a C*-triple system.*

Now let E be a Banach space and ev_E: E --> E** the canonical isometric inclusion of E into its second conjugate space (second dual) given by evaluation $ev_E(e)(g)=g(e)$ if e∈E, g∈E*. Then $(ev_E)^* ev_{E^*} = id_{E^*}$ and $(ev_E)^{**}$: E**-->E**** is a σ(E**,E*)--σ(E****,E**)-continuous isometric inclusion. In general $ev_{E^{**}} \neq (ev_E)^{**}$. $(ev_{E^*})^*$: E**** --> E** is a σ(E****,E***)--σ(E**,E*)-continuous contraction from E**** onto E** such that $(ev_{E^*})^*(ev_E)^{**} = ((ev_E)^* ev_{E^*})^* = id_{E^{**}}$. If we consider E* as a subspace of E*** via inclusion by ev_{E^*} then $(ev_{E^*})^*$ is just the restriction of elements of E**** to E*. We are interested in the study of $P_E := (ev_E)^{**} \circ (ev_{E^*})^* = (ev_{E^*} \circ (ev_E)^*)^*$. As we have seen P_E *is a σ(E****,E***)-continuous projection of norm one on* E**** *with range isometrical isomorphic to* E**. P_E fixes the points of E in E**** given by ev-inclusions E⊆E**⊆E**** but in general not those of E**(⊆E****) and *the range of* P_E *is the σ(E****,E***)-closure of* E *in* E****. If E moreover is a matrix normed space the is an algebraic Mat_n-bimodul isomorphism from $Mat_n(E^*)$ onto $Mat_n(E)^*$ given by the definition $[f_{j,k}]([e_{j,k}])=\sum_{1\leq j,k\leq n} f_{j,k}(e_{j,k})$ if $[f_{j,k}]$ is in $Mat_n(E^*)$ and $[e_{j,k}]$ is in $Mat_n(E)$. The dual (i.e. polar) norms on $Mat_n(E^*)$ of the matrix norms on $Mat_n(E)$ define the structure of a matrix space on E*, the *conjugate matrix normed space* of E. Iterating this constructions we get matrix normed spaces E*, E**, E***, E**** such that the injections E -->E**, E**-->E**** and E*-->E*** given by the evaluation maps become complete isometries. If E and F are matrix Banach spaces and T:E-->F is a bounded linear map then under the above identification of $Mat_n(E^*)$ with $Mat_n(E)^*$ the adjoint map $(T_n)^*$ of the matricial extension T_n of T becomes the matricial extension $(T^*)_n$ of the adjoint map T*:F*-->E*. From theory of Banach spaces applied to the T_n and $(T^*)_n$, n=1,2,..., we see that T *is c.c. if and only if* T* *is c.c.,* T_n *maps the open*

unit ball of $Mat_n(E)$ *onto the open unit ball of* $Mat_n(F)$ *for* $n=1,2,...$ *(resp.* T *is c.i.) if and only if* T* *is c.i. (resp.* $(T*)_n$ *maps the open unit ball of* $Mat_n(F*)$ *onto the open unit ball of* $Mat_n(E*)$ *for* $n=1,2,...$*).* Thus T is c.i. if and only if T** is a c.i. map from E** into F** and T is c.i. if and only if there is a completely isometric isomorphism I from E* onto the quotient matrix space F*/(T(E))° of F* by the orthogonal space ker(T*)=(T(E))° of Im(T) such that $I(T*)=\pi_{ker(T*)}$.
The second conjugate matrix normed space of a C*-algebra A is the matrix normed space defined on A** by the C*-algebra matrix norms. By straightforward calculations one gets:

Lemma 4.4

(i) *If E is a *-space and E**** is equipped with the fourth adjoint of the antilinear isometric involution * then* P_E *is *-invariant and* $P_E(E_{s.a.}****)\subseteq E_{s.a.}****=(E****)_{s.a.}$

(ii) $P_E|E_{s.a.}****$ *is unital and positive if* $E_{s.a.}$ *is moreover an order unit space.*

(iii) *If E is a matrix normed space then* P_E *is a completely contractive projection on the fourth conjugate matrix normed space E**** of E.*

(iv) *The second conjugate matrix normed space X** of an operator space X is an operator space.*

(v) *If X is matrix Banach space and* $Y\subseteq X$ *is a matrix subspace then there is a unique c.i. map I from the second conjugate matrix normed space (X/Y)** of the quotient matrix space X/Y onto the quotient matrix space X**/wcl(Y) of X** by the bipolar* $Y^{\circ\circ}$*=wcl(Y) of Y such that* $I.(\pi_Y)**=\pi_{wcl(Y)}$.

In view of Corollary 4.3 we obtain:

Corollary 4.5

*E**** is c.i.i. to a C*-triple-system if and only if E** is c.i.i. to a C*-triple-system.*

Let X and Y C*-triple systems and T:X-->Y a linear map. Then T is called *completely decomposable* (*c.d.* map) if there exist unital C*-algebras A,B, a unital completely positive map V:A-->B, projections p in A and q in B and completely isometric triple system isomorphisms I: X** -->pA(1-p), J:qB(1-q)-->Y** such that V(p)=q and J∘(V|pA(1-p))∘I=T**.
We call an operator space X *C*-space* if the second conjugate operator space X** is completely isometric isomorphic to a C*-triple system *and* P_X is a c.d. map.
As in [EF2] by an *operator system* X we mean a closed unital selfadjoint linear subspace of a unital C*-algebra B together with the involution, unit and the order unit structures on $Mat_n(X)_{s.a.}$ inherited from $Mat_n(B)_{s.a.}$, or X is an (involutive) matrix order unit space complete in its matricial order unit norms given by $||b||_n\leq 1$ <=> diag(1,1)+diag(b,b*)≥ (notation cf.sec.3) if b is in $Mat_n(X)$. By our definition every operator system is an operator space with respect to its matricial norms.
If X and Y are operator systems then a map T:X --> Y is a *unital completely positive (u.c.p.)* map if $T(1_X)=1_Y$ and T_n: $Mat_n(X)$ --> $Mat_n(Y)$ is positive for every n. From the corresponding properties of order unit spaces one gets that T is u.c.p. if and only if T is unital and c.c. considered as a map between operator spaces, i.e. u.c.c.<=>u.c.p. on operator systems. Especially every u.c.i. map is an u.c.p. map. X is

unital completely isometric isomorphic (u.c.i.i.) to Y if there is a u.c.i. map from X onto Y. If $X\subseteq A$, $X^{**}\cong wcl(X)\subseteq A^{**}$ is the *bidual operator system* of X. The inclusion $X^{**}\subseteq A^{**}$ equips $Mat_n(X^{**})\cong Mat_n(X)^{**}$ with bidual involution and bidual order unit structure on $(Mat_n(X^{**}))_{s.a.}\cong(Mat_n(X)_{s.a.})^{**}$.
X/Y is a *quotient operator systems* of X if there is a c.i. map T from X/Y into a unital C*-algebra B such that $T\pi_Y$ is a u.c.p. map from X into B and $(\pi_Y)_n(Mat_n(X)_+)$ is dense in the positive cone of $Mat_n(X/Y)$ induced by $X/Y\cong T(X/Y)\subseteq B$.
An operator system X whose second conjugate operator system X** is u.c.i.i. to a C*-algebra we call *(unital) C*-system*. The unital complete isometry V from X** onto a C*-algebra A induces on X** the structure of a C*-algebra (X**,.,*) such that the given matrix order unit structure and the matrix order unit structure of the C*-algebra (X**,.,*) coincide.

Lemma 4.6

(i) *The C*-algebra structure on X** is uniquely determined by the matrix order unit space X if X is a unital C*-system.*
(ii) *Let X be a matrix Banach space, e an element of X and I a c.i. map from the second conjugate matrix normed space X** onto a C*-algebra B such that I(e) is the unit of B then X is a C*-system with unit e and involution and matrix order induced by the inclusion* $I:X\subseteq X^{**}\longrightarrow B$.

Proof: By Kadison theorem [KA] we get (i). Ad(ii): I^{-1} induces on X** the structure of a C*-algebra with unit e such that the second conjugate matrix norms and the C*-algebra matrix norms on X** coincide. P_X is a u.c.c. map (i.e. a u.c.p. projection) on X**** by Lemma 4.4(iii) because $e\in X\subseteq Im(P)$. Thus X must be a selfadjoint unital subspace of the C*-algebra induced by I on X** such that the second dual operator system $X^{**}\cong X^{oo}=Im(P)$ is the range of a u.c.p. projection on a C*-algebra. By [C/E2], Im(P) is u.c.i.i. to a unital C*-algebra. q.e.d.

Let X be a unital selfadjoint linear subspace of a unital C*-algebra A. We put $M_r(X)=M_r(X,A)=\{a\in A: aX\subseteq X \text{ in } A\}$ and $M_l(X)=\{a\in A: Xa\subseteq X\}$, $M(X)=M_r(X)\cap M_l(X)$. If X is a C*-system we use $A=X^{**}$. Now let be A,B unital C*-algebras and $T:A\longrightarrow B$ a u.c.p. map. The *right multiplicative domain* $M_r(T)$ of T is defined as $M_r(T)=\{a\in A: T(ba)=T(b)T(a) \text{ for all } b\in A\}$. Similary one defines the left multiplicative domain $M_l(T)$ of T and the multiplicative domain $M(T)=M_l(T)\cap M_r(T)$.

Lemma 4.7

(i) $T(a)^*T(a)\leq T(a^*a)$ *if* $a\in A$,
(ii) $M_l(T)=\{a^*: a\in M_r(T)\}$. $M_r(T)$ *is a closed subalgebra of A and M(T) is a C*-subalgebra of A. The restrictions of T are algebra homomorphisms.*
(iii) $a\in M_r(T) \iff T(a)^*T(a)=T(a^*a)$,
$a\in M_l(T) \iff T(a)T(a)^*=T(aa^*)$.
(iv) *If* T(b) *is unitary and* $||b||\leq 1$ *then* b *is in* M(T).
(v) $b\in M_r(T) \iff (b,T(b))$ *is in* $M_r(Graph(T))\subseteq A\oplus B$.
(vi) $b\in M_r(T)$ *then* $T(b)\in M_r(cl(T(A)))$.
(vii) *If* $A=B$ *and* $T^2=T$ *then* $Im(T)\cap M_r(T)=M_r(Im(T))$ *and then* b $M_r(T)$ *if* $\{b,b^*b\}\subseteq Im(T)$.

Proof: (i), (ii), (iii) cf. [CH]. (v) is just the definition and (v) implies (vi). Ad(iv): $1=T(b)^*T(b)\leq T(b^*b)\leq 1$ by (i) because T is unital and positive. b is in $M_r(T)$ by (iii). Replacing b by b* we obtain $b\in M(T)$. Ad(vii): If $Im(T)b\subseteq Im(T)$

or if $b\in Im(T)$ and $b\in M_r(T)$ then b and b*b are in Im(T). But $\{b,b*b\}\subseteq Im(T)$ implies $T(b*b)=b*b=T(b)*T(b)$. By (iii), $b\in M_r(T)$.

q.e.d.

Lemma 4.8

Let X be a unital C-system. Then*

(i) P_X *is u.c.p. and* $M_r(X)=X\cap M_r(P_X)$ *in* X****.

(ii) $b\in M_r(X)$ *(resp.* $b\in M_l(X)$*) if and only if* $b\in X$ *and* $b*b\in X$ *(resp.* $b\in X$ *and* $bb*\in X$*).*

(iii) $M_l(X)=\{x*:x\in M_r(X)\}$. $M_r(X)$ *is a closed subalgebra of X** contained in X.* M(X) *is a C*-subalgebra of X**.*

(iv) *If q is a projection in* M(X) *then the operator system* qX(1-q) *is a C*-space.*

(v) $Mat_n(X)$ *is a unital C*-system*

(vi) $M_r(Mat_n(X))=Mat_n(M_r(X))\subseteq Mat_n(X)$

(vii) $M(Mat_n(X))=Mat_n(M(X))\subseteq Mat_n(X)$

Proof: Let be b X** and let $P=P_X$. Then P is a σ(X****,X***)-continuous unital completely positve projection from the C*-algebra X**** onto the σ(X****,X***)-closure of X in X****. Thus $Xb\subseteq X$ in X** if and only if $Im(P)ev_{X**}(b)\subseteq Im(P)$ in X**** by Hahn-Banach separation theorem. Thus (i) follows from Lemma 4.7(vii) because X is unital.
Ad(ii): $\{b,b*b\}\subseteq X$ implies $\{b,b*b\}\subseteq Im(P)$ and $b\in M_r(P)\cap X=M_r(X)$ by (i) and Lemma 4.7(iii). Conversely $\{b, b*b\}\subseteq Xb$ if $Xb\subseteq X$ because X is a unital selfadjoint subspace of the C*-algebra X**. The other case is similar.
(iii) follows from definitions.
Ad(iv): Let q be a projection in M(X) and let us denote the operator space qX(1-q). Then $r=ev(q)\in Im(P)$ and $rX****(1-r)\cong$ $\cong Y****$ such that P_Y becomes P|(rX****(1-r)).
Ad(iv): The u.c.i. map j from X** onto a C*-algebra B defines a u.c.i. map $(j)_n$ from $Mat_n(X)**\cong Mat_n(X**)$ onto the C*-algebra $Mat_n(B)$.
(v) and (vi) are straightforward because the scalar matrices are contained in $M(Mat_n(X))\subseteq M_r(Mat_n(X))$ and in $Mat_n(M(X))\subseteq Mat_n(M_r(X))$.

q.e.d.

In the remaining part of this section let A be a C*-algebra, R and L closed right and left ideals of A respectively and D a hereditary C*-subalgebra of A. l=supp(L),r=supp(R) and supp(D) are the projections in A** defined as in sec.1 such that A**l= =wcl(L), rA**=wcl(R) and wcl(D)=supp(D)A**supp(D), cf. [TAK].

Lemma 4.9

(i) wcl(L+R)=wcl(L)+wcl(R)=A**l+rA**,

(ii) dist(a,wcl(L)+wcl(R)) = ||paq|| *where* q=1-supp(L) *and* p=1-supp(R) *if* $a\in A**$, dist(a,L+R)=||paq|| *if* $a\in A$,

(iii) $wcl(L\cap R)=wcl(L)\cap wcl(R)=rA**l$,

(iv) $dist(a,R)=dist(a,L\cap R)=||pa||$ *if* $a\in L$, *i.e. the canonical map* $L/L\cap R \dashrightarrow A/R$ *is isometric and L+R is closed in A,*

(v) L=cl(AD) *is a leftideal and* supp(cl(DA))=supp(L)=supp(D)

Proof: Ad(i)+(ii): L+R is weakly dense wcl(L)+wcl(R)=A**l+rA** and a-->paq=(1-r)a(1-l) is a weakly continuous completely contractive projection on A** with kernel A**l+rA**. dist(a,L+R)=dist(a,wcl(L+R)) if $a\in A$ by Hahn-Banach theorem.
Ad(iii)+(vi): $RL\subseteq L\cap R$, $wcl(L)\cap wcl(R)=wcl(R)wcl(L)\subseteq wcl(RL)$ by partial weak continuity of multiplication in A**.
By Hahn-Banach theorem, $dist(a,L\cap R)=dist(a,rA**l)$ and $a\in A**l$

if $a \in A$. But $\mathrm{dist}(al, rA^{**}l) \leq \mathrm{dist}(a, rA^{**}) = \mathrm{dist}(a,R) \leq \mathrm{dist}(a, L \cap R)$ and $\mathrm{dist}(a,R) = \|pa\|$ if $a \in A$ by (ii).
Ad(v): If $a,b \in A$, $d,e \in D$, $c = 2(\|a\|^2 + \|b\|^2)(d^*d + e^*e)$, $x = ad + be$ then $x^*x \leq c \in D$ and $x \in \mathrm{cl}(Ac^{1/2}) \subseteq \mathrm{cl}(AD)$, i.e. $AD + AD \subseteq \mathrm{cl}(AD)$. $\mathrm{wcl}(L) = \mathrm{wcl}(AD) = A^{**}\mathrm{supp}(D)$ and $\mathrm{wcl}(DA) = \mathrm{supp}(D)A^{**}$. q.e.d.

Lemma 4.10

(i) $\mathrm{wcl}(\mathrm{Mat}_n(L+R)) = \mathrm{Mat}_n(\mathrm{wcl}(L)) + \mathrm{Mat}_n(\mathrm{wcl}(R))$ *and* $\mathrm{supp}(\mathrm{Mat}_n(L)) = \mathrm{diag}(l,\dots,l)$ *where* $l = \mathrm{supp}(L)$,

(ii) $\mathrm{dist}(a, \mathrm{wcl}(\mathrm{Mat}_n(L+R))) = \|PaQ\|$ *if* a $\mathrm{Mat}_n(A^{**})$ *and if* $P = \mathrm{diag}(p,\dots,p)$, $Q = \mathrm{diag}(q,\dots,q)$ *where* $p = 1-r$, $q = 1-l$.

(iii) *There is a unique isomorphism* j *from* $(A/(L+R))^{**}$ *onto the operator space* $pA^{**}q \subseteq A^{**}$ *such that* $j((\pi_{L,R})^{**}(b)) = pbq$ *if* $b \in A^{**}$. j *is a completely isometric isomorphism from the second conjugate operator space of the quotient operator space* $A/(L+R)$ *of* A *onto the operator space* $pA^{**}q \subseteq A^{**}$. *Especially* $A/(L+R)$ *is an operator space.*

(iv) $\mathrm{Mat}_n(D)$ *is a hereditary C*-subalgebra of* $\mathrm{Mat}_n(A)$, $\mathrm{supp}(\mathrm{Mat}_n(D)) = \mathrm{diag}(\mathrm{supp}(D),\dots,\mathrm{supp}(D))$, $\mathrm{cl}(\mathrm{Mat}_n(A)\mathrm{Mat}_n(D)) = \mathrm{Mat}_n(\mathrm{cl}(AD))$ *and* $\mathrm{cl}(\mathrm{Mat}_n(A)\mathrm{Mat}_n(D)) + \mathrm{cl}(\mathrm{Mat}_n(D)\mathrm{Mat}_n(A)) = \mathrm{Mat}_n(\mathrm{cl}(AD) + \mathrm{cl}(DA))$.

Proof: Ad(i): $\mathrm{wcl}(\mathrm{Mat}_n(L)) = \mathrm{Mat}_n(\mathrm{wcl}(L)) = \mathrm{Mat}_n(A^{**}l) = \mathrm{Mat}_n(A^{**})\mathrm{diag}(l,\dots,l)$ under the identification of $\mathrm{Mat}_n(A^{**})$ and $\mathrm{Mat}_n(A)^{**}$ as above, i.e. $\mathrm{supp}(\mathrm{Mat}_n(L)) = \mathrm{diag}(l,\dots,l)$. Similary $\mathrm{wcl}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(A^{**})\mathrm{diag}(r,\dots,r)$. Now (ii) follows from (i) and Lemma 4.9(ii). Ad(iii): Let be $Y = \mathrm{wcl}(L+R)$. If $J\pi_Y$ is the factorization of the c.c. projection $a \dashrightarrow paq$ on A^{**} then J is a c.i. map from A^{**}/Y onto the operator space $pA^{**}q$ by (ii). Now if I is the c.i. isomorphism from $(A/(L+R))^{**}$ onto A^{**}/Y then $j = JI$ is a c.i. map from $(A/(L+R))^{**}$ onto $pA^{**}q$ such that $j((\pi_{L,R})^{**}(b)) = pbq$ if $b \in A^{**}$.
Ad(iv): Let be $d = \mathrm{supp}(D)$ then $\mathrm{wcl}(\mathrm{Mat}_n(D)) = \mathrm{Mat}_n(\mathrm{wcl}(D)) = \mathrm{Mat}_n(dA^{**}d) = \mathrm{diag}(d,\dots,d)\mathrm{Mat}_n(A^{**})\mathrm{diag}(d,\dots,d)$. Thus $\mathrm{Mat}_n(D)$ is hereditary in $\mathrm{Mat}_n(A)$ with support $\mathrm{diag}(d,\dots,d)$. $\mathrm{Mat}_n(DA) \subseteq \mathrm{span}(\mathrm{Mat}_n(A)\mathrm{Mat}_n(D)) \subseteq \mathrm{Mat}_n(\mathrm{span}(AD))$, apply Lemma 4.9(v). q.e.d.

From now on we use the identifications as above and the abrevations as at the end of section 1.

Proposition 4.11

(i) *Assume that* $c \in A//D$, $d \in A$, $0 \leq c \leq \pi_D(d^*d)$ *and* $\pi_D(d^*d)$ *is invertible in* $(A//D)^{**}$. *If* A *is unital then there exists* e *in* A *such that* $\pi_D(d^*e^*ed) = c$ *and* $\|e\| \leq 1$.

(ii) *If* $c \in A//D$, $c \geq 0$ *in* $(A//D)^{**}$ *and* A *is unital then there is* $a \geq 0$ *in* A *such that* $\pi_D(a) = c$.

(iii) *Let be* $f \in A//D$ *and* $d \in M(A) \subseteq A^{**}$ *such that* $f^*f \leq (\pi_D)^{**}(d^*d)$ *and* $(\pi_D)^{**}(d^*d)$ *is invertible in* $(A//D)^{**}$. *Then there is* e *in* A *such that* $\pi_D(ed) = f$ *and* $\|e\| \leq 1$.

Proof of Prop. 1.2 and of Prop. 4.11: By Lemma 4.10 there is an isometry j from $(A/(L+R))^{**}$ onto $pA^{**}q$ such that $j \circ (\pi_{L,R})(a) = paq$. $E = A$, $T = \pi_{L,R}(b(.)d)$, $B = A^{**}$, $J = j$, $I = \mathrm{id}_B$ and b,d in $M(A) \subseteq B$ satisfy the assumptions of Cor.2.9. Proposition 1.2 now follows from Cor.3.2.
At this stage also the proof of Corollary 1.3 is complete.
Proof of Prop.4.11: Ad(i): The situation above appears in case $b = d^*$, $L = \mathrm{cl}(AD)$, $R = \mathrm{cl}(DA)$ and $p = q = 1 - \mathrm{supp}(D)$. If moreover A is unital then by Cor.2.10, Cor.3.2 and Prop. 2.2,

$\pi_D(d^*(cl(spec(0,1)))d)$ is closed in A//D. By Hahn-Banach separation theorem, it suffices to show that j(c) is in $\{pd^*fdp: 0 \leq f \leq 1, f \in A^{**}\} \subseteq wcl(j(\pi_D(d^*(spec(0,1))d))$. But this follows from $j(c) \leq pd^*dp = j(\pi_D(d^*d))$.
(ii) is a special case of (i).
Ad(iii): b=1,d, L=cl(AD) and R=cl(DA) satisfy Cor.1.3, i.e. $\pi(cl(S)d)$ is closed in A//D. By Hahn-Banach separation theorem it suffices to check that j(f) is in the weak closure of $j(\pi(Sd))$. But $j(f)^*j(f) \leq j(\pi^{**}(d^*d)) = pd^*dp$ and $j(f) = pj(f)p$, i.e. $j(f) \in p(wcl(S)d)p \subseteq j(wcl(\pi(Sd)))$. q.e.d.

In the remaining part of this section let A be a unital.

Proposition 4.12

The quotient operator system A//D=A/(cl(AD)+cl(DA)) is a unital C-system with unit $\pi_D(1)$. There is a unique c.i. isomorphism j from the bidual operator system (A//D)** of A//D onto the W*-algebra pA**p where p=1-supp(D) such that $j(\pi_D^{**}(a))=pap$ if $a \in A^{**}$. Especially π_D defines a unital completely positive map from A into (A//D)**.*

Proof: Let j be the c.i. map from the operator space (A//D)** onto pA**p as defined in Lemma 4.10(iii) such that $j(\ _Y^{**}(a))=$ $=pap$ where Y=cl(AD)+cl(DA), p=1-supp(D). Then π_D is the abridgement of π_Y, i.e. $\pi_D(1)=\pi_Y(1)$ and $j(\pi_D(1))=p$ is the unit of pA**p. By Lemma 4.6(iii), j: A//D$\subseteq$(A//D)** --> pA**p defines on A//D an involution and a matrix order structure such that $\pi_D(1)$ is the matrix order unit and with the second conjugate order structure, second conjugate involution and matrix order unit $\pi_D(1)$ the operator space (A//D)** becomes an operator system such that j is a u.c.i. isomorphism. Thus with this matrix order unit space structure A//D is a C*-system with unit $\pi_D(1)$. The uniqueness of j follows from uniqueness of factorizations. The images of the positive matrix cones of A are weakly dense in the positive matrix cones of pA**p by the matricial extensions of the map a-->pap. Using the inverse map of j and Hahn-Banach separation theorem we obtain that the matricial extensions of π_D map the the positive matrix cones of A onto cones dense in the positive matrix cones of the operator system A//D, i.e. A//D is a quotient operator system in the sense of the definition given above. q.e.d.

Lemma 4.13

(i) *There is a unique unital c.i. isomorphism $J=J_n$ from* $Mat_n(A)//Mat_n(D)$ *onto* $Mat_n(A//D)$ *such that*

$$J_n\left(\pi_E\left(\begin{bmatrix} a_{11},\dots,a_{1n} \\ \dots\dots\dots \\ a_{n1},\dots,a_{nn}\end{bmatrix}\right)\right) = \begin{bmatrix} \pi_D(a_{11}),\dots,\pi_D(a_{1n}) \\ \dots\dots\dots \\ \pi_D(a_{n1}),\dots,\pi_D(a_{nn})\end{bmatrix}$$

where $E=Mat_n(D)$.
(ii) $J_n(M_r(Mat_n(A)//Mat_n(D))) = Mat_n(M_r(A//D))$
(iii) $J_n(M(Mat_n(A)//Mat_n(D)) = Mat_n(M(A//D))$

Proof: X=A//D, $Y=Mat_n(A)//Mat_n(D)$ and $Mat_n(X)$ are C*-systems by Prop.4.14, Lemma 4.10(i) and Lemma 4.8(v). Let be Q= =diag(p,...,p) where p=1-supp(D) and let k be the u.c.i. isomorphism from Y** onto $Mat_n(pA^{**}p)=QMat_n(A^{**})Q$ such that $k(\pi_E(b))=QbQ$ and j:X**-->pA**p such that $j(\pi_D^{**}(a))=pap$ then $j_n([\pi_D(a_{i,k})])=k(\pi_E([a_{i,k}]))$ and j_n is a u.c.i. map from $Mat_n(X^{**})$ onto $Mat_n(pA^{**}p)$. Thus $J_n=(j_n)^{-1}k|Y$ is a u.c.i.

map from Y onto $Mat_n(X)$.

Ad(ii)+(iii): $M_r Mat_n(X) = Mat_n(M_r(X))$ (resp. $M(Mat_n(X)) = Mat_n(M(X))$) by Lemma 4.10. By Lemma 4.6, J_n extends to a C*-algebra isomorphism between the second conjugate C*-algebras of Y and of $Mat_n(X)$. q.e.d.

Corollary 4.14

Let A be a C-algebra not necessarily unital and R and L closed right and left ideals of A respectively. Then there exists a unital C*-algebra B, a hereditary C*-subalgebra D of B and a projection q in M(B//D) such that A/(L+R) is completely isometric isomorphic to q(B//D)(1-q). Especially A/(L+R) is a C*-space.*

Proof: Let C be the unitization of A, D the hereditary C*-subalgebra of $Mat_2(C)$ generated by $EMat_2(C)E$ with $E = cl(RR^*) \oplus cl(L^*L)$ and denote by B the unital C*-subalgebra of $Mat_2(C)$ generated by diag(1,0), diag(0,1) and $Mat_2(A)$. Then $D \subseteq B$, $p = \pi_D(\mathrm{diag}(1,0)) \in M(B//D)$ and A/(L+R) is c.i.i. to p(B//D)(1-p) by the map defined by the factorization of $T: c \in A \to p(\pi_D(\mathrm{diag}(c,0)Z))(1-p) \in p(A//D)(1-p)$. (Consider T**, use Lemma 4.10 and Lemma 4.8(iii).) q.e.d.

Remarks: A matrix Banach space X is c.i.i. to an operator space if and only if $\|b \oplus c\|_{n+m} = \max(\|b\|_n, \|c\|_m)$ for b in $Mat_n(X)$ and c in $Mat_m(X)$. This implies that quotient operator spaces again are operator spaces. We need only a very special case, cf. Lemma 4.12.

It is not hard to see that triple product morphisms between C*-triple systems are contractive (it suffices to consider the situation of "abelian" C*-triple systems satisfying $ab^*c = cb^*a$ which are isometric and triple isomorphic to abelian C*-algebras). The matricial extensions of triple system morphisms are again triple system morphisms. Thus triple system morphisms are completely contractive and triple system isomorphy of C*-triple systems implies completely isometric isomorphy between them (the detailed proof is left to the reader). A variant of Kadison's result on isometries of C*-algebras holds also for triple systems like pBq by some modifications of the fundamental lemmata of his original proof [KA]. The above observations then imply that two C*-triple systems are triple system isomorphic if and only if they are completely isometric isomorphic as operator systems. By Prop.4.2 the maximal C*-algebra tensor product suitable extends to C*-triple systems. Then T is c.d. if and only if $T \otimes id_A$ is contractive with respect to the maximal C*-triple system tensor product norms on the algebraic tensor products $X \odot A$ and $Y \odot A$ for every C*-algebra A (An Application of Wittstock extension theorem [WI2] and of Prop. 4.2 again). The C*-triple system c.i.i. to X** is unique up to triple system isomorphy and the triple product induced on X** is well-defined if X is a C*-space. If X is an operator system and X** (resp.X) is c.i.i. to a C*-triple system A then A is a unital C*-algebra and X** (resp. X) is u.c.i.i. to A by an other u.c.i. map, i.e. X is a unital C*-system (resp. X is a C*-algebra). There exists an operator space X such that X** is c.i.i. to a C*-algebra but X is not a C*-space.

5. Normalizer algebras

5.1. Preliminaries and Proof of parts (i),(ii) of Theorem 1.4

Let be A a unital C*-algebra, D a hereditary C*-subalgebra of A, q=supp(D), p=1-q and π_D: A --> A//D = A/(cl(AD)+cl(DA)) the quotient map. Considered as a map from A into the W*-algebra (A//D)**, π_D is a unital completely positive map by Proposition 4.12. The algebras $M_r(\pi_D)$, $M_l(\pi_D)$, $M(\pi_D)$, $M_r(A//D)$, $M_l(A//D)$ and M(A//D) are defined in section 4 and the (right-, left-) normalizer algebras $N_r(D)$, $N_l(D)$ and N(D) are as in section 1.

Lemma 5.1

(I) *If a∈A following properties are equivalent.*
   (i) a∈N_r(D) *(resp.* a∈N_l(D)*, resp.* a∈N(D) *),*
   (ii) (1-p)ap=0 *(resp.* pa(1-p)=0*, resp.* pa=ap *),*
   (iii) $\pi_D(a)^*\pi_D(a)=\pi_D(a^*a)$ *(resp.* $\pi_D(a)\pi_D(a)^*=\pi_D(aa^*)$*, resp.* $\pi_D(a)^*\pi_D(a)=\pi_D(a^*a)$ *and* $\pi_D(a)\pi_D(a)^*=\pi_D(aa^*)$ *),*
   (iv) a∈$M_r(\pi_D)$ *(resp.* a∈$M_l(\pi_D)$*, resp.* a∈$M(\pi_D)$ *).*

(II) $\pi_D|N_r(D)$ *and* $\pi_D|N_l(D)$ *are algebra homomorphisms into* $M_r(A//D)\subseteq(A//D)^{**}$ *and* $M_l(A//D)\subseteq(A//D)^{**}$ *respectively.*

(III) *If* $\pi_D(a)$ *is unitary in* (A//D)** *and* ||a||≤1 *then* a∈N(D).

Proof: That $\pi_D|M_r(\pi_D)$ and $\pi_D|M_l(\pi_D)$ are algebra homomorphisms into $M_r(Im(\pi_D))=M_r(A//D)$ and $M_l(A//D)$ respectively and the equivalence of (iii) and (iv) follow from Lemma 4.7. wcl(D)= = qA**q. By partial weak continuity of the multiplication in A** and by separation theorem, Da⊆D <=> qA**qa⊆qA**q <=> qa=qaq <=> (1-p)ap=0, aD⊆D <=> pa(1-p)=0. Thus (i) <=> (ii). By Prop.4.12 there is a W*-algebra isomorphism j from (A//D)** onto pA**p such that $j(\pi_D(a))$=pap if a∈A. But pa*pap=pa*ap <=> (1-p)ap=0, papa*p=paa*p <=> pa(1-p)=0, i.e.(ii)<=>(iii). (III) follows from (I) and Lemma 4.7. q.e.d.

Lemma 5.2

Mat_n(N_r(D)) = N_r(Mat_n(D)), Mat_n(N_l(D)) = N_l(Mat_n(D)), *and* Mat_n(N(D)) = N(Mat_n(D)).

Proof: supp(Mat_n(D))= diag(supp(D),...,supp(D)) by Lemma 4.10 and diag(1-p,...,1-p)$[a_{i,j}]$diag(p,...,p) = 0 if and only if (1-p)$a_{i,j}$p=0 for i,j=1,...,n. Use Lemma 5.1(I). q.e.d.

Lemma 5.3

dist(c,Mat_n(cl(AD)))= dist(c,Mat_n(cl(AD)+cl(DA))) *if* c *is in* Mat_n(N_r(D)) *and* n=1,2,..., *i.e. the restriction to* N_r(D)/cl(AD) *of the quotient map from* A/cl(AD) *onto* A//D *is completely isometric.*

Proof: By Lemma 5.2 we can restrict to the case n=1. Put p= =1-supp(D) and let be c in N_r(D). Then cp=pcp by Lemma 5.1 and dist(c,cl(AD)) = dist(c,A**(1-p)) = ||cp|| = ||pcp|| = =dist(c,A**(1-p)+(1-p)A**)=dist(c,cl(AD)+cl(DA)) by Lemma 4.9. q.e.d.

Proof of Theorem 1.4 (i) and (ii): By Lemma 5.1(II) π_D defines a Banach algebra homomorphism h from N_r(D) into M_r(A//D). Obviously cl(AD) is contained in N_r(D) and by Lemma 5.3 it is the kernel ideal of h. Now let be c∈A//D such that

$c^*c \in (A//D)^{**}$ is in $A//D$. Put $t=2\|c\|$ and $d=t+c$ in $(A//D)^{**}$. Then d and $d^*d=t^2+tc^*+tc+c^*c$ are in $A//D$ and d^*d is invertible in $(A//D)^{**}$. Using Proposition 4.11(i) we find b in A such that $\pi_D(b^*b)=d^*d$. By Proposition 4.11(ii) there exists a contraction e in A such that $\pi_D(eb)=d$. Considering π_D as a map from A into $(A//D)^{**}$ from the complete positivity of π_D by Choi's inequality ([CH],[C/E2]) we get $d^*d=\pi_D(eb)^*\pi_D(eb)\leq \pi_D(b^*e^*eb)\leq$ $\leq\pi_D(b^*b)=d^*d$. By Lemma 5.1, eb and $eb-2\|c\|$ are in $N_r(D)$ and $c =\pi_D(eb-2\|c\|)$. Thus π_D maps $N_r(D)$ *onto* $M_r(A//D)$. By Corollary 1.2 the quotient map from $Mat_n(A)$ onto the quotient C*-system by $Mat_n(D)$ maps the closed unit ball of $Mat_n(A)$ onto the closed unit ball of the quotient C*-system. $Mat_n(cl(AD))$ is contained in $Mat_n(N_r(D))$. Thus the restriction of the quotient map to the closed unit balls of $Mat_n(N_r(D))$ and its quotient by $Mat_n(cl(AD))$ is again surjective. Now Lemma 5.3 yields (i) and (ii) in case of N_r and M_r. To obtain the result concerning $N_l(D)$ and $M_l(A//D)$ one has to replace A by A^{op} (=A with opposite multiplication). q.e.d.

<u>5.2. Proof of part (iii) of Theorem 1.4:</u> $D\subseteq N(D)=N_l(D)\cap N_r(D)$ by definiton of $N(D)$. Thus π_D is a positive unital algebra homomorphism from $N(D)$ into $M_l(A//D)\cap M_r(A//D) = M(A//D)$. It remains to show that π_D is an epimorphism. Let m be a contraction in $M(A//D)$. Then $U(m)$ is unitary in $Mat_2(M(A//D))$, cf. sec.3. If J is the defining isometry from $Y:=Mat_2(A)//Mat_2(D)$ onto $Mat_2(A//D)$ then J is a C*-algebra isomorphism from $M(Y)$ onto $Mat_2(M(A//D))$ by Lemma 4.18. Applying Corollary 1.3 we get a contraction f in $Mat_2(A)$ such that $J(\,''(f))= U(m)$ where $''$ denotes the quotient map from $Mat_2(A)$ onto Y. Then f is in $Mat_2(N(D))= N(Mat_2(D))$ by Lemmata 5.1 and 5.2. Let c be the (1,1)-element of f. Then $c\in N(D)$ and $\pi_D(c)$ is the (1,1)-element of $U(m)$, i.e. $\pi_D(c)= m$.

q.e.d.

<u>5.3. Proof of Corollary 1.5.</u>

<u>Lemma 5.4</u>

Let X be a unital C-system, C a unital C*-algebra, $V:X\to C$ a unital completely isometric map from X into C then there exists a W*-algebra epimorphism P from the W*-subalgebra of the second conjugate W*-algebra C^{**} of C generated by $V(X)\subseteq C\subseteq C^{**}$ onto the second conjugate W*-algebra X^{**} such that $P\circ(V^{**}) = id_{X^{**}}$. Especially $PV=id_X$.*

<u>Proof:</u> Let $h:X^{**}\to\mathcal{L}(K)$ be a faithful normal *-representation of the W*-algebra X^{**} onto a von Neumann algebra $N=h(X^{**})\subseteq\mathcal{L}(K)$ acting on some Hilbert space K. Because V is unital and completely isometric by Arveson extension theorem [ARV1] there exists a unital completely positive map $W:C\to\mathcal{L}(K)$ with $WV= h|X$. If $T:C^{**}\to\mathcal{L}(K)$ is the normal extension of W then T is a completely positive map satisfying $T\circ(V^{**}) = h$ and $h(a^*)h(a)\leq T((V^{**}(a))^*(V^{**}(a)))\leq T(V^{**}(a^*a))=h(a^*a)$ if $a\in X^{**}$ by Choi's generalized Kadison inequality. By Lemma 4.7 $V^{**}(X^{**})$ is contained in $M(T)$, $M(T)$ is a C*-algebra and $T|M(T)$ is a *-representation of $M(T)$. $M(T)$ is moreover a W*-subalgebra of C^{**} because T is ultraweakly continuous. Let N be the W*-subalgebra of C^{**} generated by $V^{**}(X^{**})$. Then $T(N)=h(X^{**})$ and T is a W*-algebra epimorphism. $P= h^{-1}(T|N)$ is as desired. q.e.d.

Lemma 5.5

Let be X, C, V: X––> C as in Lemma 5.4 and let $B \subseteq C$ be a unital subalgebra of C such that $B \cup \{b^*b : b \in B\}$ (resp. $B \cup \{bb^* : b \in B\}$) is contained in V(X). Then P(B) is a unital subalgebra of $M_r(X)$ (resp. of $M_l(X)$) and V defines a unital algebra isomorphism from P(B) onto B.

Proof: $B \subseteq V(X)$ and $P|V(X)$ is invertible with inverse V. We get $P(B) \subseteq P(V(X)) = X$, $\ker(P|B) = 0$, $(P|B)^{-1} = V|P(B)$. On the other hand $\{c^*c : c \in P(B)\} = P(\{b^*b : b \in B\}) \subseteq P(V(X)) = X$ and P is a unital algebra homomorphism from B onto P(B) by Lemma 5.4. Thus $P(B) \subseteq M_r(X)$ by Lemma 4.8. q.e.d.

Proof of Corollary 1.5: By Proposition 4.12, A//D is a unital C*-system. Using Theorem 1.4 and Lemma 5.5 we put $E = (\pi_D|N_r(D))^{-1}(P(B))$ (resp. $F = (\pi_D|N(D))^{-1}(P(B))$ if B is moreover a C*-subalgebra of C). Then E and F have the desired properties (i),(ii) and (iv),(v) of Corollary 1.5 by Theorem 1.4 (i) and (iii). Property (iii) of 1.5 follows from 1.4 (ii) by Lemma 5.3. q.e.d.

6. Outline of further results and applications

6.1. Let $\mathcal{L}(H)$ be the C*-algebra of all bounded operators on a separable Hilbert space H of infinite dimension. An operator space X, i.e. a closed subspace of $\mathcal{L}(H)$ together with the matrix norms inherited from $\mathcal{L}(H)$, is called *nuclear* if the identity map on X has a semidiscrete approximation in the sense of Effros [C/E3], i.e. the identity map on X is in the strong closure of maps T=VW where V: X––> Mat_n, W: Mat_n––> X are complete contractions and n depends from the chosen decomposition of T of this kind. It turns out that the maps T, V, W may be chosen moreover unital if X is unital and that X is nuclear if and only if it is a C*-space and the second conjugate C*-triple system X** is an injective operator system. *A separable operator space X is nuclear if and only if X is completely isometric isomorphic to a quotient-C*-system of the CAR-algebra $B := M_2 \otimes M_2 \otimes \ldots$ by a sum L+R of a closed leftideal L and a closed rightideal R of B.* The Lindenstrauss spaces [LI] are just the abelian C*-spaces, i.e. the C*-spaces whose second conjugate spaces are isometric isomorphic to abelian W*-algebras. The Choquet simplices are just the state spaces of the unital abelian C*-systems. If X is a unital separable nuclear C*-system then there is a hereditary C*-subalgebra D of the CAR-algebra B such that X is completely isomorphic to the quotient-C*-system B//D of B by D. By the results of this paper this implies that $M(X) \cong N(D)/D$. This applies to separable unital nuclear C*-algebras X=M(X)=A. Combined with the lifting theorem of Choi and Effros [C/E1], [ARV2,th.7] and with their results on ranges of completely positive unital projections on C*-algebras [C/E2] one gets: Up to C*-algebra isomorphy the unital separable nuclear C*-algebras are the ranges P(B) of unital completely positive projections P on the CAR-algebra B with P-compressed multiplication $P(a) \circ P(b) := P(P(a)P(b))$ on the range P(B). An analyse of Glimm's result on C*-algebras of type I [DIX,Chap.9] shows that Glimm actually proved that conversely in every separable C*-algebra A not of type I there exists a hereditary C*-subalgebra D such that $A//D = N(D)/D \cong B$. Using the method of comparisation of Elliot [EL] one obtains very easy

from the latter both results that the infinite parts of A^{**} and of B^{**} are isomorphic if A is a separable unital nuclear C*-algebra not of type I and B is the CAR-algebra.
Every *nuclear* C*-space (resp. unital C*-system, C*-algebra) is an inductive limit of its *separable and nuclear* C*-subspaces (resp. unital C*-subsystems, C*-subalgebras).

6.2. If M is a von Neumann algebra with separable predual then by an inductive limit construction one can easy show that M is semidiscrete if and only if there exists a unital separable nuclear C*-system X such that M is isomorphic to $z.X^{**}$ for some central projection z of X^{**}. The above described results on nuclear operator spaces then imply: *M is injective if and only if* $M\bar{\otimes}\mathcal{L}(H)$ *contains a weakly dense C*-subalgebra isomorphic to the CAR-algebra.*

6.3. Let X be a separable unital C*-system (resp. separable C*-space) then there exists a separable unital C*-algebra A and a hereditary C*-subalgebra D_A (resp. closed left- and rightideals L and R) such that $X\cong A//D$ (resp. $X\cong A/(L+R)$).
Every unital C*-system (resp. C*-space) is the inductive limit of separable unital C*-systems (resp. of separable C*-spaces) in the corresponding categories.

6.4. Let $\mathcal{FD}$ be the set of maps V: $\mathcal{L}(H)\text{-->}\mathcal{L}(H)$ decomposable in the sense described in sec. 6.1 and let $Z\subseteq Y\subseteq\mathcal{L}(H)$ be operator spaces. We define fin(Z):= inf{ || (V-id)|Z ||: V in $\mathcal{FD}$} and locfin(Y):= sup{ fin(Z): $Z\subseteq Y$, dim(Z)<oo }. By $m(\mathcal{L}(H))\cong l_{\infty}\otimes\mathcal{L}(H)$ and $c_{o}(\mathcal{L}(H))\cong c_{o}\otimes\mathcal{L}(H)$ we denote the bounded sequences and zero sequences of operators respectively. $c_{o}(\mathcal{L}(H))$ is an ideal of $m(\mathcal{L}(H))$ and the diagonal map d_{∞}: $b\in\mathcal{L}(H)$ --> $(b,b,...)+c_{o}(\mathcal{L}(H))$ defines a C*-algebra monomorphism from $\mathcal{L}(H)$ into $m(\mathcal{L}(H))/c_{o}(\mathcal{L}(H))$. In a forthcoming paper we shall show (by an inductive limit construction): *If B is a separable unital Banach subalgebra of* $\mathcal{L}(H)$ *such that* locfin(B*B)=0 *then there exist a separable nuclear unital C*-system X and a unital complete isometry V from X into* $m(\mathcal{L}(H))/c_{o}(\mathcal{L}(H))$ *such that* $d_{\infty}(B)\subseteq V(X)$. By Corollary 1.5 of the present paper and the forthcoming results on nuclear operator spaces announced in sec.6.1, B is a C*-quotientalgebra of a C*-subalgebra of the CAR-algebra if B is *moreover* selfadjoint.

6.5. We denote by $\odot$ the algebraic tensor product of vector spaces, by $\otimes$ the minimal C*-algebra tensor product, by $\mathcal{LC}(H)$ the C*-algebra of compact operators on H and by C(H) the Calkin algebra $\mathcal{L}(H)/\mathcal{LC}(H)$. The minimal C*-algebra tensor product $\otimes$ is a bifunctor on the category of C*-algebras. Thus there is a (unique) C*-algebra epimorphism T from the quotient algebra $(\mathcal{L}(H)\otimes\mathcal{L}(H))/(\mathcal{L}(H)\otimes\mathcal{LC}(H))$ onto $\mathcal{L}(H)\otimes C(H)$. If X is a a closed subspace of $\mathcal{L}(H)$ and A is a C*-algebra $X\odot A\subseteq\mathcal{L}(H)\otimes A$ defines a dualisable crossnorm $||.||_{*}$ on $X\odot A$ such that the completion $X\otimes A$ with respect to $||.||_{*}$ canonically and isometrically identifies with cl($X\odot A$) in $\mathcal{L}(H)\otimes A$. It turns out that the canonical map from $(X\otimes\mathcal{L}(H))/(X\otimes\mathcal{LC}(H))$ into $(\mathcal{L}(H)\otimes\mathcal{L}(H))/(\mathcal{L}(H)\otimes\mathcal{LC}(H))$ defines an isometric inclusion and T maps $(X\otimes\mathcal{L}(H))/(X\otimes\mathcal{LC}(H))$ into $X\otimes C(H)$. Let T_{x} be the restriction of T to $(X\otimes\mathcal{L}(H))/(X\otimes\mathcal{LC}(H))$ and $X\otimes C(H)$. We define ex(X):= $||T_{x}^{-1}||^{-1}$. Using Wittstock's extension theorem [WI1,2.3.1] (cf. also [PAU] for a more convenient idea of proof) in a separate forthcoming paper we obtain:
ex(X)= 1 *if and only if* locfin(X)= 0.
For C*-algebras B, ex(B) takes only the values 0 and 1. ex(B)=1 if and only if B is exact in the sense of [KI2]. We

get the result formulated at the end of sec. 1. In other forthcoming papers we shall refine this result essentially: *For every unital separable exact C*-algebra B there exists a unitary u in the CAR-algebra such that B is a C*-quotient-algebra by an AF-ideal of the (relative) commutante of u in the CAR-algebra.*

One gets several corollaries, e.g. *a C*-algebra is exact if and only if it satisfies the property (C) of Archbold and Batty [A/B], C*-quotientalgebras of exact C*-algebras are exact, every unital separable exact algebra has a unital completely isometric (linear) embedding into the CAR-algebra,...*

6.6. In general a *nuclear* separable *C*-system* does not admit a completely isometric embedding into an *exact C*-algebra*. We do not know if any separable exact C*-algebra is isomorphic to a C*-subalgebra of a nuclear C*-algebra (a long outstanding nontrivial open question).

Let A be the unitization of $C_o(]0,1])\otimes C^*_{reg.}(SL_2(Z))$. By a result of de Canniere and Haagerup [dC/H], A is an exact separable unital C*-algebra. *In* Ext(A) *there exists an element* [B] *such that the C*-algebra* B *corresponding to the class* [B] *with exact sequence* 0 --> *LC*(H) --> B --> A --> 0 *satisfies* $B^{op}\otimes B \doteq B^{op}\otimes^{max} B$ *where* B^{op} *means* B *with opposite multiplication.* This implies that B has the weak expection property of Lance [LA1]. B can not be exact because otherwise B is nuclear and therefore A and $C^*_{reg.}(SL_2(Z))$ are nuclear. But this is impossible by the result of Lance [LA1,th.4.2] because $SL_2(Z)$ is not amenable. Thus the category of exact C*-algebras is not closed under extensions.

References

[A/B] Archbold, R.J., Batty, J.K., C*-tensor norms and slice maps, J.London Math.Soc. 22(1980), 127-138.

[ARV1] Arveson, W., Subalgebras of C*-algebras, Acta Math. 123(1969), 141-224.

[ARV2] --, Notes on extensions of C*-algebras, Duke Math.J. 44 (1977), 329-355.

[C/E1] Choi, M.D., Effros, E.G., The completely positive lifting problem for C*-algebras, Ann.Math. 104 (1976), 585-609.

[C/E2] --, Injectivity and operator spaces, J.Funct.Anal., 24 (1977), 156-209.

[C/E3] --, Nuclear C*-algebras and the approximation property, Amer.J.Math., 100(1978), 61-79.

[CH] Choi, M.D., A Schwarz inequality for positive linear maps on C*-algebras, Ill.J.Math. 18(1974), 565-574.

[CON1] Connes, A., Classification of injective factors, Ann.Math. 104(1976), 73-116.

[CON2] --, On the equivalence between injectivity and semi-discreteness for operator algebras, Algèbres d'operateurs et leurs applications en physique mathematique (Proc. Colloq., Marseille, 1977), pp.107-112, Colloques Internat. CNRS, 274, CNRS, Paris, 1979.

[dC/H] De Cannierè, J., Haagerup, U., Multipliers of the Fourier algebras on some simple Lie groups and their discrete subgroups, Amer.J.Math. to appear ?

[DIX] Dixmier, J., Les C*-algèbres et leurs représentations, Gauthier-Villars, Paris 1969.

[EF1] Effros, E., Order ideals in a C*-algebra and its dual, Duke Math.J. 30(1963), 391-412.

[EF2] --, Aspects of noncommutative order, Lecture Notes in Math. 650: C*-algebras and applications to physics, Springer, Berlin Heidelberg New York 1978.

[EF3] --, On the structure theory of C*-algebras: Some old and new problems, Operator algebras and applications, Part 1 (Kingston, Ont., 1980), pp.19-34, Proc. Sympos. Pure Math.,38, Amer. Math. Soc., Providence, R.I.,1982.

[E/H] Effros, E., Haagerup, U., Lifting problems and local reflexivity for C*-algebras, Duke Math. J. 52 (1985), no.1,103-128.

[E/L] Effros, E., Lance, L.C., Tensor product of operator algebras, Adv.in Math. 25(1973), 157-176.

[EL] Elliot, G., On approximately finite-dimensional von Neumann algebras, Math. Scand. 39(1976), 91-101.

[GL] Glimm, J., A Stone-Weierstrass theorem for C*-algebras, Ann.Math. 72(1960), 216-244.

[HA1] Haagerup, U., All nuclear C*-algebras are amenable, Invent. Math. 74(1983), 305-319.

[HA2] --, Injectivity and decomposition of completely bounded maps, Operator algebras and their connections with topology and ergodic theory (Busteni, 1983), 170-222, Lecture Notes in Math. 1132, Springer, Berlin etc.1985.

[KA] Kadison, R.V., A generalized Schwarz inequality and algebraic invariants for operator algebras, Ann.Math. 56(1952), 494-503.

[KI1] Kirchberg, E., C*-nuclearity implies CPAP, Math. Nachr. 76(1977), 203-212.

[KI2] --, Positive maps and C*-nuclear algebras, Proc. Intern. Conference on Operator Algebras, Ideals and their Applications in Theoretical Physics, Leipzig, September 12-20, 1977, pp. 347-328, Teubner Texte, Leipzig, 1978.

[KI3] --, The Fubini theorem for exact C*-algebras, J. Operator Theory 10(1983), 3-8.

[KY] Kye,Seung-Hyeok, On the Fubini products of C*-algebras, Thesis work, Seoul National University, October 1974.

[LA1] Lance, L.C., On nuclear C*-algebras, J.Funct.Anal. 12 (1973), 157-176.

[LA2] --, Tensor products and nuclear C*-algebras, Operator algebras and applications, Part 1 (Kingston,Ont.,1980), pp.379-400, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., 1982.

[LI] Lindenstrauss, J., Extensions of compact operators, Mem.Amer.Math.Soc. 48(1964).

[L/T] Lindenstrauss, J., Tzafriri, Classical Banach spaces I, II, Ergebnisse der Math. 92 and 97, Springer, Berlin New York, 1977/1979.

[PAU] Paulsen, V.I., Completely bounded maps on C*-algebras and invariant operator ranges, Proc.AMS,86(1982),91-96.

[SCHA] Schaefer, H.H., Topological Vector Spaces, Macmillan, London, New York 1966.

[TAK] Takesaki, M., Theory of operator algebras I, Springer, New York Inc. 1979.

[WA1] Wassermann, S., The slice map problem for C*-algebras, Proc. London Math. Soc., 32(1976), 537-559.

[WA2] --, Injective W*-algebras, Math.Proc. Camb. Phil. Soc., 82(1977), 39-47.

[WI1] Wittstock, G., Ein operatorwertiger Hahn-Banach Satz, J.Funct.Anal., 40(1981), 127-150.

[WI2] --, Extensions of completely bounded C*-module homomorphisms, Operator algebras and group representations, Vol.II (Neptun 1980), 238-250, Monographs Stud.Math.,18, Pitman,Boston/Mass.-London, 1984.

[YO] Youngson, M.A., Completely contractive projections on C*-algebras, Quart.J.Math. Oxford Ser. (2)34(1983), 507-511.

Autor's adress:
Eberhard Kirchberg
Otto-Buchwitz-Str. 63
1140 Berlin