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COMBINATORIAL PROPERTIES OF GROUPS AND SIMPLE

C^*-ALGEBRAS

(preliminary version)

by

Florin BOCA and Viorel NIȚICĂ

Let G be an infinite discrete group and $C_r^*(G)$ be its reduced C^*-algebra. Our interest is to find combinatorial properties for G such that $C_r^*(G)$ is simple with a unique faithful tracial state.

Powers showed that the reduced C^*-algebra of the free group with two generators $C_r^*(F_2)$ is simple with a unique trace ([6]). Then, the class of these groups was extended by Paschke and Salinas ([5]), Akemann and Lee ([1]), Bédos, de la Harpe and Jhabvala ([2], [3]).

In the first part of our work we give a weaker property than that of Powers' group (see [2]), for which $C_r^*(G)$ is simple with unique trace. The groups with this property will be called weak Powers' groups. They have good extension properties.

In the second section we improve the results from [4]. We show that if G is a weak Powers' group, A a unital C^*-algebra, G-simple, with a unique trace, G-invariant for the action $\alpha : G \to \mathrm{Aut}(A)$, and $c : G\times G \to Z_u(A)$ a normalized 2-cocycle, then the reduced cross-product of G and A by c relatively to α, $A \rtimes_{\alpha, c} G$ is a C^*-algebra which is simple with a unique trace. We also prove that if G is an extension of a weak Powers' group by a weak Powers' group, then $C_r^*(G)$ is simple with a unique trace.

We are indebted to Mihai Pimsner for useful discussions and encouragement throughout this work.

§1. COMBINATORICS

Let G be an infinite discrete group. The following definition is from [2].

DEFINITION 1.1. A Powers' group is a group G having the following property: given any non empty finite subset $F\subset G\setminus\{1\}$ and any integer $N\geq 1$, there exist a partition $G=A\amalg B$ and elements $g_1,\ldots,g_N$ in G such that:

(i) $fA\cap A=\emptyset$, for all $f\in F$;

(ii) $g_jB\cap g_kB=\emptyset$, for $j,k=1,\ldots,N$, $j\neq k$.

It is easy to see that the free groups with n generators F_n , $n\in \mathbb{N}\cup\{\infty\}$ are Powers' groups. In [1], [2], [3], [5], [6] are given classes which consist of Powers' groups.

We introduce the following definition:

DEFINITION 1.2. A weak Powers' group is a group G having the following property: given any non empty finite subset $F\subset G\setminus\{1\}$, which is included into a conjugacy class, and any integer $N\geq 1$, there exist a partition $G=A\amalg B$ and elements $g_1,\ldots,g_N$ in G such that:

(i) $fA\cap A=\emptyset$, for all $f\in F$;

(ii) $g_jB\cap g_kB=\emptyset$, for $j,k=1,\ldots,N$, $j\neq k$.

It is clear that any Powers' group is weak Powers' group. It is also clear that in both definitions (i) is true for any $f \in F \cup F^{-1}$.

The weak Powers' groups have, like the Powers' groups, the following elementary properties (see [2], prop. 1):

PROPOSITION 1.3. Let G be a weak Powers' group.

(a) Any conjugacy class is G other than $\{1\}$ is infinite.

(b) The group G is not amenable.

(c) Any subgroup G' of G of finite index is a Powers' group.

<u>Proof</u>. See [2], prop.1.

Hence all observations from [2], section 1, are also true for weak Powers' groups.

We shall denote $\langle f\rangle_M=\{xfx^{-1} \mid x\in M\}$, for a set $M\subset G$ and $f\in G$.

PROPOSITION 1.4. Let G_1 and G_2 be weak Powers' groups. Then the direct product $G=G_1 x G_2$ is a weak Powers' group.

<u>Proof</u>. Let $f=(f_1,f_2)$ be an element of $G\setminus\{1\}$. We may assume $f_1\neq 1$. Let an integer $N\geq 1$ and F be a non empty finite subset of $G\setminus\{1\}$ which is included into the conjugacy class of f . So $F=\langle f\rangle_M$, where M is a finite subset of G.

Set $M_1=pr_1 M$ finite subset of $G_1 \cdot G_1$ is a weak Powers' group, so $G_1=A_1 \amalg B_1$ and there exist $g_1',\dots,g_N'$ G_1 such that:

$$g'A_1 \cap A_1=\emptyset, \quad \text{for } g'\in \langle f_1\rangle_{M_1}$$

$$g_j'B_1 \cap g_k'B_1=\emptyset, \quad \text{for } j,k=1,\dots,N, \quad j\neq k .$$

Set $A=A_1xG_2$, $B=B_1xG_2$, $g_j=(g_j',1)\in G$, $j=1,\ldots,N$. Then $G=A\amalg B$ and (i), (ii) from definition 1.2 follow imediatly. Q.E.D.

This proposition and 2.9 show that if G_1, G_2 are weak Powers' groups, then $C_r^*(G_1xG_2)$ is simple with unique trace. In fact, it is easy to see that C*-tensor product $C_r^*(G_1) \otimes C_r^*(G_2)$ is isomorphic with $C_r^*(G_1xG_2)$. Using this and Cor. 4.21 from [8] it follows that if $C_r^*(G_i)$, $i=1,2$ are simple with unique trace, then $C_r^*(G_1xG_2)$ is simple with unique trace.

De la Harpe asks in [2] if for G_1, G_2 Powers' groups, G_1xG_2 is Powers' group. The answer is affirmative for a large class of Powers' groups. If G_1 and G_2 act by homeomorphism on the Hausdorff topological spaces L_1 and L_2, and the actions are minimal, strongly faithful and strongly hyperbolic (see [2], lemma 4), then G_1xG_2 acts on L_1xL_2 minimal, strongly faithful and strongly hyperbolic, hence G_1xG_2 is Powers' group. In the general case, the answer is unknown.

PROPOSITION 1.5. Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of groups, with G' Powers' group and G" weak Powers' group. Then G is a weak Powers' group.

Proof. G' is identified with a normal subgroup of G, and G" with the quotient group G/G'. We denote the quotient map $G \to G/G'$ by π. Let $\{\gamma_i\}_{i\in I}$ be a complete sistem of representants for G modulo G'.

Let $f\in G\setminus\{1\}$, $M\subset G$ a finite set and an integer $N\geq 1$. The following cases appear:

1) $f \in G' \setminus \{1\}$

G' is a normal subgroup of G so $\langle f \rangle_M \subset G' \setminus \{1\}$. Because G' is a Powers' group, there exist a partition $G' = A_1 \amalg B_1$ and $h_1, \dots, h_N \in G'$ such that:

$$hA_1 \cap A_1 = \emptyset, \quad \text{for } h \in \langle f \rangle_M$$

$$h_j B_1 \cap h_k B_1 = \emptyset, \quad \text{for } j,k = 1, \dots, N, \quad j \neq k.$$

Set $A = \coprod_{i \in I} A_1 \gamma_i$, $B = \coprod_{i \in I} B_1 \gamma_i$. It is easy to verify that $G = A \amalg B$ and

$$hA \cap A = \emptyset, \quad \text{for } h \in \langle f \rangle_M$$

$$h_j B \cap h_k B = \emptyset, \quad \text{for } j,k = 1, \dots, N, \ j \neq k.$$

2) $f \in G \setminus G'$

In this case, there are unique $i_o \in I$, $h \in G'$ such that $f = h\gamma_{i_o}$. Clearly $\gamma_{i_o} \notin G'$. We consider the finite set:

$$\{\pi(\gamma_i)\pi(\gamma_{i_o})\pi(\gamma_i)^{-1} \mid G'\gamma_i \cap M = \emptyset\} = \langle \pi(\gamma_{i_o}) \rangle_{\pi(M)} \subset G/G' \setminus \{\pi(1)\} \ .$$

G/G' is a weak Powers' group, so $G/G' = A' \amalg B'$ and there are $y_j \in G/G'$, $j = 1, \dots, N$ such that:

$$yA' \cap A' = \emptyset, \quad \text{for } y \in \langle \pi(\gamma_{i_o}) \rangle_{\pi(M)}$$

$$y_j B' \cap y_k B', \quad \text{for } j,k = 1, \dots, N, \ j \neq k \ .$$

It is clear that $G = \pi^{-1}(A') \amalg \pi^{-1}(B')$. For $g_j \in \pi^{-1}(y_j)$, $j = 1, \dots, N$ it is easy to verify:

$\gamma A \cap A=\emptyset$, for $\gamma \in \langle f\rangle_M$;

$g_j B \cap g_k B=\emptyset$, for $j,k=1,\dots,N$, $j\neq k$. Q.E.D.

The following proposition appears in [2] and [4]:

PROPOSITION 1.6. Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be an increasing family of Powers' groups (weak Powers' groups). Then $G=\bigcup_{\lambda\in\Lambda} G_\lambda$ is a Powers' group (weak Powers' group).

Proof. We assume G_λ Powers' groups. Let $N\geq 1$ and $F\subset G\setminus\{1\}$ be a finite set. There exist $\lambda_o\in\Lambda$ such that $F\subset G_{\lambda_o}\setminus\{1\}$. So, $G_{\lambda_o}=A_{\lambda_o}\amalg B_{\lambda_o}$ and there are $g_1,\dots,g_N\in G_{\lambda_o}$ which verify:

$fA_{\lambda_o}\cap A_{\lambda_o}=\emptyset$, for $f\in F$

$g_j B_{\lambda_o}\cap g_k B_{\lambda_o}=\emptyset$, for $j,k=1,\dots,N$, $j\neq k$.

Let $\{\gamma_i\}_{i\in I}$ be a right complete sistem of representants of G modulo G_{λ_o}, so $G=\coprod_{i\in I} G_{\lambda_o}\gamma_i$.

Set $A=\coprod_{i\in I} A_{\lambda_o}\gamma_i$, $B=\coprod_{i\in I} B_{\lambda_o}\gamma_i$. Then $G=A\amalg B$ and:

$fA\cap A=\emptyset$, for $f\in F$;

$g_j B\cap g_k B=\emptyset$, for $j,k=1,\dots,N$, $j\neq k$.

The proof is similar for weak Powers' groups. Q.E.D.

We shall consider now some examples of fundamental groups of graphs of groups which are weak Powers' groups (see [7] for definitions and results about graphs and fundamental groups of graphs of groups).

LEMMA 1.7. Let (G,T) be an infinite tree of groups such that G_Q has at least two elements for each $Q \in$ vert T. Let $P \in$ vert T fixed and

$$\ldots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_o = \{P\}$$

the inverse sistem of sets asociated to P ([7], I.2.2). If there are infinite many indices k such that between X_k and X_{k+1} there exists an edge y with $G_y = \{1\}$, then the direct limit $G_T = \varinjlim (G,T)$ is Powers' group.

Proof. It is known from [5] that for G_1, G_2 groups, where G_1 has at least three elements and G_2 at least two elements, then the free product $G_1 * G_2$ is a Powers' group. Hence G_T is the union of an increasing family of Powers' groups. Now, it follows from 1.6 that G_T is Powers group. Q.E.D.

Next lemma is an exercise in [7], I 5.1.

LEMMA 1.8. Let (G,Y) be a non-empty connected graph of groups, and let T be a maximal tree of Y. Let $(\tilde{Y},T)$ be the universal cover of Y relative to T; the graph $\tilde{Y}$ is a tree, on which the group $\pi_1(Y,T)$ acts freely. If $Q \in$ vert Y projects to $P \in$ vert Y, we put $G_Q = G_P$; we define similarly G_y for $y \in$ edge $\tilde{Y}$ as well as $G_y \longrightarrow G_{t(y)}$; the result is a tree of groups $(G,\tilde{Y})$ on which $\pi_1(Y,T)$ acts in a natural way.

Then $\pi_1(G,Y,T)$ is canonically isomorphic to the semidirect product of $\pi_1(Y,T)$ and the group $\pi_1(G,\tilde{Y},\tilde{Y})=\underrightarrow{\lim}(G,\tilde{Y})$.

PROPOSITION 1.9. Let (G,Y) be a non-empty connected graph of groups and let T be a maximal tree of Y. If the group G_Q has at least two elements for every $Q\in$ vert Y, there exists $y\in$ edge Y with $G_y=\{1\}$ and the fundamental group $\pi_1(Y,T)$ has at least two generators, then the fundamental group $\pi_1(G,Y,T)$ is a weak Powers' group.

<u>Proof</u>. From 1.8, $\pi_1(G,Y,T)$ is the semidirect product of $\pi_1(Y,T)$ and $\pi_1(G,\tilde{Y},\tilde{Y})$. The group $\pi_1(Y,T)$ is isomorphic with the free group with n generators ($n\in N\cup\{\infty\}$) and $\pi_1(G,\tilde{Y},\tilde{Y})$ is Powers' group (see 1.7). Hence $\pi_1(G,Y,T)$ is the semidirect product of two Powers' groups. It follows from 1.5 that it is a weak Powers' group. Q.E.D.

§2. <u>FUNCTIONAL ANALYSIS</u>

Let A be a C*-algebra with unit, which acts faithful on $\mathcal{B}(H)$, G be a discrete group and $\alpha:G\to\mathrm{Aut}(A)$ be an action.

We denote by $Z_u(A)$ the unitaries' group from the centre of A.

A (normalised) 2-cocycle on G with values in G-module $Z_u(A)$ is a map $c:G\times G\to Z_u(A)$ which verify:

1) $c(g_1,g_2)c(g_1g_2,g_3)=\alpha_{g_1}(c(g_2,g_3))c(g_1,g_2g_3)$, for $g_1,g_2,g_3\in G$;

2) $c(g,g^{-1})=1$, for $g\in G$;

3) $c(1,g)=c(g,1)=1$, for $g\in G$.

The set of these cocycles is denoted by $Z^2(G,Z_u(A))$.

Let $\pi_\alpha : A \to \mathcal{B}(\ ^2(G,H))$ and $\Lambda_c : G \to \mathcal{U}(l^2(G,H))$, where:

$$(\pi_\alpha(x)\xi)(g)=\alpha_{g^{-1}}(x)\xi(g), \quad \text{for } x\in A,\ \xi\in l^2(G,H),\quad g\in G$$

$$(\Lambda_c(g)\xi)(g_1)=c(g_1^{-1},g)\xi(g^{-1}g_1), \quad \text{for } g,g_1\in G,\ \xi\in l^2(G,H).$$

The C*-algebra generated by $\pi_\alpha(A)$ and $\Lambda_c(G)$ in $\mathcal{B}(l^2(G,H))$ is called the (reduced) cross-product of G and A by c relatively to α and is denoted by $A\rtimes_{\alpha,c}G$.

When $A=\mathbb{C}$ and $c\in Z^2(G,T)$, the (reduced) c-C*-algebra of the group G, denoted $C_r^*(G,c)$ is obtained. If c is a trivial 2-cocycle, it is obtained the reduced C*-algebra of G, denoted $C_r^*(G)$.

Let $u_g=\Lambda_c(g)$. Then u_g are unitaries and

$$u_{g_1}u_{g_2}=c(g_1,g_2)u_{g_1g_2}, \quad \text{for } g_1,g_2\in G \ ;$$

$$u_1=1 \ ;$$

$$u_g^*=u_{g^{-1}} \ ;$$

$$u_g a u_g^*=\alpha_g(a), \quad \text{for } g\in G,\ a\in A.$$

For further details, one can see [9].

If A has a G-invariant trace τ_o, then $A\rtimes_{\alpha,c}G$ has a canonical trace τ :

$$\tau(\sum_g a_g u_g)=\tau_o(a_1).$$

The map $e:B\to A$, $e(\sum_g a_g u_g)=a_1$ is the canonical conditional expectation. The following equality is easy to verify:

$$e(u_g x u_g^*)=\alpha_g(e(x)), \quad \text{for } g\in G, \quad x\in B. \qquad (1)$$

We also have:

$$u_{g_1}\dots u_{g_n}=c_l(g_1,\dots,g_n)\cdot u_{g_1\dots g_n}, \quad \text{for } g_1,\dots,g_n\in G,\ n\geq 2; \qquad (2)$$

$$u_{g_1}\dots u_{g_n}=u_{g_1\dots g_n}\cdot c_r(g_1,\dots,g_n), \quad \text{for } g_1,\dots,g_n\ G,\ n\geq 2, \qquad (3)$$

where:

$$c_l(g_1,\dots,g_n)=c(g_1,g_2)c(g_1g_2,g_3)\dots c(g_1\dots g_{n-1},g_n)\in Z_u(A) \ ;$$

$$c_r(g_1,\dots,g_n)=\alpha_{(g_1\dots g_n)^{-1}}(c_l(g_1,\dots,g_n))\in Z_u(A) \ .$$

LEMMA 2.1. Let $A\subset\mathcal{B}(H)$ be a unital C*-algebra, G be a discrete group, $\alpha : G\rightarrow \mathrm{Aut}(A)$ be an action, $c\in Z^2(G,Z_u(A))$ and D be a subset of G. Then:

$$u_g\cdot P^{l^2(G,H)}_{l^2(D,H)}\cdot u_g^*=P^{l^2(G,H)}_{l^2(gD,H)} \quad , \text{ for } g\in G.$$

Proof. If $G=D\amalg E$, then $l^2(G,H)=l^2(D,H)\oplus l^2(E,H)$. Let $p=P^{l^2(G,H)}_{l^2(D,H)}$ be the projection of $l^2(G,H)$ onot $l^2(D,H)$. We verify that:

$$u_g\cdot p=P^{l^2(G,H)}_{l^2(gD,H)}\cdot u_g \quad .$$

Set $\xi\in l^2(G,H)$, $\xi_1=p\xi$, $\xi_2=(1-p)\xi$.

$$(u_g p\xi)(\gamma)=(u_g\xi_1)(\gamma)=c(\gamma^{-1},g)\xi_1(g^{-1}\gamma) \ ;$$

$$(P^{l^2(G,H)}_{l^2(gD,H)}u_g\xi)(\gamma)=P^{l^2(G,H)}_{l^2(gD,H)}(\gamma\longrightarrow c(\gamma^{-1},g)\xi(g^{-1}\gamma))(\gamma) \ .$$

If $\gamma\in gD$, then $\xi_2(g^{-1}\gamma)=0$, so

$$P^{l^2(G,H)}_{l^2(gD,H)}\,(\gamma\longrightarrow c(\gamma^{-1},g)\xi(g^{-1}\gamma))(\gamma)=c(\gamma^{-1},g)\quad(g^{-1}\gamma)=$$

$$=c(\gamma^{-1},g)\xi_1(g^{-1}\gamma)\ .$$

If $\gamma\notin gD$, then $\xi_1(g^{-1}\gamma)=0$, so

$$P^{l^2(G,H)}_{l^2(gD,H)}\,(\gamma\longrightarrow c(\gamma^{-1},g)\xi(g^{-1}\gamma))(\gamma)=0$$

$$c(\gamma^{-1},g)\xi_1(g^{-1}\gamma)=0$$

Hence $u_g\cdot p=P^{l^2(G,H)}_{l^2(gD,H)}\cdot u_g$. Q.E.D.

LEMMA 2.2. Let A be a unital C*-algebra, G a weak Powers' group, $\alpha:G\longrightarrow \mathrm{Aut}(A)$ an action, $c\in Z^2(G,Z_u(A))$ and $B=A\rtimes_{\alpha,c}G$. Then, for any finite subset $F\subset G\setminus\{1\}$, for any element $x\in B$ of the form $x=\sum_{f\in F}a_f u_f$, $x=x^*$ and for any $\varepsilon>0$, there exist an integer $n\geq 1$, $g_1,\dots,g_n\in G$, $c_1,\dots,c_n\in Z_u(A)$ such that:

$$\left\|\frac{1}{n}\sum_{k=1}^{n}u_{g_k}c_k x c_k^* u_{g_k}^*\right\|\leq\varepsilon\ .$$

Proof. One has:

$$u_g(au_f)u_g^*=\alpha_g(a)u_gu_fu_{g^{-1}}=\alpha_g(a)c(g,f)u_{gf}u_{g^{-1}}=$$

$$=\alpha_g(a)c(g,f)c(gf,g^{-1})u_{gfg^{-1}}=\alpha_g(a)\alpha_g(c(f,g^{-1}))c(g,fg^{-1})u_{gfg^{-1}}= \qquad (4)$$

$$=\alpha_g(ac(f,g^{-1}))c(g,fg^{-1})u_{gfg^{-1}}\ .$$

For any element $x \in B$ of the form $x=\sum_{f\in F} a_f u_f$, $a_f \neq 0$ for $f \in F$, we denote supp $x=F$.

Let $y=au_f+\alpha_{f^{-1}}(a^*)u_f^*$, $f\neq 1$. Then $y=y^*$ and $\tau(y)=0$. Let $M\subset G$ a finite subset which contains r elements and $z=\frac{1}{n}\sum_{g\in M} u_g y u_g^*$. Then (4) implies:

$$F=\text{supp } z=\langle f\rangle_M \cup \langle f^{-1}\rangle_M \ .$$

As F is a finite subset in $G\setminus\{1\}$ and G is a weak Powers' group, there exist $G=D\amalg E$, g_1, g_2, g_3 such that:

(i) $fD\cap D=\emptyset$, for $f\in F$;

(ii) $g_jE\cap g_kE=\emptyset$, for $j,k=1,2,3$, $j\neq k$.

Denote $p=P^{l^2(G,H)}_{l^2(D,H)}$. One has from (i) that $pzp=0$.

Using lemma 2.1 and (ii), we obtain the pairwise orthogonal projections $u_j(1-p)u_j^*$, $j=1,2,3$. By lemma 1 from [4] one has:

$$\left\| \frac{1}{3}\sum_{k=1}^{3} u_{g_k} z u_{g_k}^* \right\| \leq c\,\|z\| \ , \text{ where } c=0,995 \ ,$$

and supp $z'=\langle f\rangle_{g_1M\cup g_2M\cup g_3M} \cup \langle f^{-1}\rangle_{g_1M\cup g_2M\cup g_3M}$, where

$$z'=\frac{1}{3}\sum_{k=1}^{3} u_{g_k} z u_{g_k}^* \ .$$

By this applied several times and by (3) one has that for any $\varepsilon>0$, there exist an integer $r\geq 1$, $g_1,\dots,g_r\in G$, $c_1,\dots,c_r\in Z_u(A)$ such that:

$$\left\| \frac{1}{3} \sum_{j=1}^{r} u_{g_j} c_j y c_j^* u_{g_j}^* \right\| \leq \varepsilon .$$

Now, let x as in lemma. Then, there exist an integer $m \geq 1$ such that $x = x_1 + \ldots + x_m$, where $x_i = a_{f_i} u_{g_i} + \alpha_{f_i^{-1}}(a_{f_i}^*) u_{g_i^{-1}}$, for $i = 1, \ldots, m$.

From the first part of the proof, there exist $g_{11}, \ldots, g_{1n_1} \in G$, $c_{11}, \ldots, c_{1n_1} \in Z_u(A)$ such that:

$$\left\| \frac{1}{n_1} \sum_{j_1=1}^{n_1} u_{g_{1j_1}} c_{1j_1} x_1 c_{1j_1}^* u_{g_{1j_1}}^* \right\| \leq \frac{\varepsilon}{m} \tag{5}$$

Denote $\tilde{x}_2 = \frac{1}{n_1} \sum_{j_1=1}^{n_1} u_{g_{1j_1}} c_{1j_1} x_2 c_{1j_1}^* u_{g_{1j_1}}^*$. As for (5), there exist $g_{21}, \ldots, g_{2n_2} \in G$, $c_{21}, \ldots, c_{2n_2} \in Z_u(A)$ such that:

$$\left\| \frac{1}{n_2} \sum_{j_2=1}^{n_2} u_{g_{2j_2}} c_{2j_2} \tilde{x}_2 c_{2j_2}^* u_{g_{2j_2}}^* \right\| \leq \frac{\varepsilon}{m}$$

By induction, denote $\tilde{x}_{k+1} = \frac{1}{n_k} \sum_{j_k=1}^{n_k} u_{g_{kj_k}} c_{kj_k} x_{k+1} c_{kj_k}^* u_{g_{kj_k}}^*$.

Then, there exist $g_{k+11}, \ldots, g_{k+1n_{k+1}} \in G$, $c_{k+11}, \ldots, c_{k+1n_{k+1}} \in Z_u(A)$ such that:

$$\left\| \frac{1}{n_{k+1}} \sum_{j_{k+1}=1}^{n_{k+1}} u_{g_{k+1j_{k+1}}} c_{k+1j_{k+1}} x_{k+1} c_{k+1j_{k+1}}^* u_{g_{k+1j_{k+1}}}^* \right\| \leq \frac{\varepsilon}{m} ,$$

for $k = 1, \ldots, m-1$.

One has:

$$\left\| \frac{1}{n_1 \ldots n_m} \sum_{j_1=1}^{n_1} \ldots \sum_{j_m=1}^{n_m} u_{g_{mj_m}} c_{mj_m} \ldots u_{g_{1j_1}} c_{1j_1} (x_1 + \ldots + x_m) c^*_{1j_1} u^*_{g_{1j_1}} \ldots c^*_{mj_m} u^*_{g_{mj_m}} \right\| \leq$$

$$\leq \left\| \frac{1}{n_1} \sum_{j_1=1}^{n_1} u_{g_{1j_1}} c_{1j_1} x_1 c^*_{1j_1} u^*_{g_{1j_1}} \right\| + \left\| \frac{1}{n_2} \sum_{j_2=1}^{n_2} u_{g_{2j_2}} c_{2j_2} \tilde{x}_2 c^*_{2j_2} u^*_{g_{2j_2}} \right\| + \ldots +$$

$$+ \left\| \frac{1}{n_m} \sum_{j_m=1}^{n_m} u_{g_{mj_m}} c_{mj_m} x_m c^*_{mj_m} u^*_{g_{mj_m}} \right\| \leq m \cdot \frac{\varepsilon}{m} = \varepsilon .$$

By (2) and by

$$u_g a = \alpha_g(a) u_g \quad , \quad \text{for } g \in G, \ a \in A,$$

one has $n = n_1 \ldots n_m \in N$, $g_1, \ldots, g_n \in G$, $c_1, \ldots, c_n \in Z_u(A)$ such that:

$$\left\| \frac{1}{n} \sum_{k=1}^{n} u_{g_k} c_k x c^*_k u^*_{g_k} \right\| \leq \varepsilon \quad .$$

Q.E.D.

PROPOSITION 2.3. Let A be a unital C*-algebra, G be a weak Powers' group, $\alpha : G \longrightarrow \mathrm{Aut}(A)$ be an action, $c \in Z^2(G, Z_u(A))$ and $B = A \rtimes_{\alpha, c} G$. Then, for any trace τ on B, there exists a G-invariant trace σ on A with $\tau = \sigma e$.

<u>Proof</u>. Let $x = \sum x_g u_g \in B$. Lemma 2.2 implies that the closed convex hull of

$$\{ u_g c_g (x - e(x)) c^*_g u^*_g \mid g \in G, \ c_g \in Z_u(A) \}$$

contains 0. Consequently $\tau(x - e(x)) = 0$, and the assertion follows.

Q.E.D.

COROLLARY 2.4. If G is a weak Powers' group and if there exist a unique G-invariant trace on A, then there exist a unique trace on $A \rtimes_{\alpha,c} G$.

Let A be a unital C*-algebra, $\alpha : G \longrightarrow \mathrm{Aut}(A)$ be an action. A is G-simple if any $\alpha(G)$-invariant closed two-sided ideal in A is either $\{0\}$ or A.

The following assertion is lemma 9 from [4]:

LEMMA 2.5. Assume that A is G-simple. Let $x \in A$ with $x \geq 0$ and $x \neq 0$. There exist $g_1, \ldots, g_n \in G$ and $z_1, \ldots, z_n \in A$ such that:

$$\sum_{j=1}^{n} z_j \alpha_{g_j}(x) z_j^* \geq 1 \ .$$

PROPOSITION 2.6. Let A be a G-simple C*-algebra, $\alpha : G \longrightarrow \mathrm{Aut}(A)$ be an action of a weak Powers' group on A, $c \in Z^2(G, Z_u(A))$. Then $B = A \rtimes_{\alpha,c} G$ is simple.

<u>Proof</u>. Let $I \subset B$ be a two-sided ideal and assume that $x \in I$, $x \neq 0$. One may assume $x \geq 0$, hence $e(x) \geq 0$. By lemma 2.5 there exist $g_1, \ldots, g_n \in G$, $a_1, \ldots, a_n \in A$ such that:

$$\sum_{j=1}^{n} a_j \alpha_{g_j}(e(x)) a_j^* \geq 1$$

Denote $x' = \sum_{j=1}^{n} a_j u_{g_j} x u_{g_j}^* a_j^* \in I$. Then $x' \geq 0$ and one has:

$$e(x') = \sum_{j=1}^{n} e(a_j u_{g_j} x u_{g_j}^* a_j^*) = \sum_{j=1}^{n} a_j e(u_{g_j} x u_{g_j}^*) a_j^* \ .$$

Using (1) it follows that:

$$e(x')=\sum_{j=1}^{n} a_j \alpha_{g_j}(e(x))a_j^* \geq 1.$$

Let $y=\sum_{g\in F} a_g u_g \in B$, with $F\subset G$ finite set, such that $y=y^*$ and $\|x'-y\|\leq \frac{1}{6}$. Then $\|e(x')-e(y)\|\leq \frac{1}{6}$, hence $e(y)\geq e(x')-\frac{1}{6}\geq \frac{5}{6}$.

By lemma 2.2 applied several times, there exist $h_1,\ldots,h_N\in G$, $c_1,\ldots,c_N\in Z_u(A)$ such that:

$$\left\| \frac{1}{N}\sum_{j=1}^{N} u_{h_j} c_j (y-e(y)) c_j^* u_{h_j}^* \right\| \leq \frac{1}{6} \quad .$$

Let $r=\frac{1}{N}\sum_{j=1}^{N} u_{h_j} c_j y c_j^* u_{h_j}^*$ and $r_1=\frac{1}{N}\sum_{j=1}^{N} u_{h_j} e(y) u_{h_j}^* \geq \frac{5}{6}$. It follows that $\|r-r_1\|\leq \frac{1}{6}$, hence $r\geq r_1-\frac{1}{6}\geq \frac{5}{6}-\frac{1}{6}=\frac{2}{3}$.

We denote $z=\frac{1}{N}\sum_{j=1}^{N} u_{h_j} c_j x' c_j^* u_{h_j}^*$. It is clear that $z\in I$, $z\geq 0$ and $\|r-z\|\leq \|x'-y\|\leq \frac{1}{6}$, so $z\geq r-\frac{1}{6}\geq \frac{2}{3}-\frac{1}{6}=\frac{1}{2}$ and z is an invertible element. Q.E.D.

COROLLARY 2.7. If G is a weak Powers' group, A a simple unital C*-algebra, $\alpha: G \longrightarrow \mathrm{Aut}(A)$ an action and $c\in Z^2(G,Z_u(A))$, then $A \rtimes_{\alpha,c} G$ is simple.

COROLLARY 2.8. If G is a weak Powers' group and $c\in Z^2(G,T)$, then $C_r^*(G,c)$ is simple with a unique trace.

COROLLARY 2.9. If G is a weak Powers' group, then $C^*_r(G)$ is simple with a unique trace.

PROPOSITION 2.10. Let G be a discrete group and H a normal subgroup of G. If H and G/H are weak Powers' groups, then $C^*_r(G)$ is simple with a unique trace.

<u>Proof</u>. Let $\varepsilon > 0$ and $Y=Y^* \in \mathbb{C}[G]$ with $\tau(Y)=0$ ($\mathbb{C}[G]$ is the group algebra of G). Then $Y=\sum_{j=1}^{r} \lambda_j U_{g_j} + \bar{\lambda}_j U^*_{g_j}$, where $1 \notin \{g_1,\ldots,g_r\}$. Denote $Y_j=\lambda_j U_{g_j} + \bar{\lambda}_j U^*_{g_j}$, for $j=1,\ldots,r$. One may assume that $g_1,\ldots g_p \in H$ and $g_{p+1},\ldots,g_r \in G\, H$.

H is a weak Powers' group, so by lemma 2.2 there exist $h_1,\ldots,h_n \in H$ such that:

$$\left\| \frac{1}{n} \sum_{k=1}^{n} U_{h_k} \tilde{Y} U^*_{h_k} \right\| \leq \frac{p\varepsilon}{r} , \qquad (6)$$

where $\tilde{Y}=Y_1+\ldots+Y_p$.

Let $\tilde{Y}_p=\frac{1}{n} \sum_{k=1}^{n} U_{h_k} Y_{p+1} U^*_{h_k}$. Then supp $\tilde{Y}_{p+1}$ is included in the conjugacy class of $g_{p+1} \in G \setminus H$, hence supp $\tilde{Y}_{p+1}$ is included in $G \setminus H$. By case 2 from the proof of 1.5 one can see that for any $f \in G \setminus H$, any finite set $M \subset G$ and any integer $N \geq 1$, there exist $G=A \amalg B$ and $g_1,\ldots,g_N \in G$ such that:

$\gamma A \cap A=\emptyset$, for $\gamma \in \langle f \rangle_M$;

$g_j B \cap g_k B=\emptyset$, for $j,k=1,\ldots,N$, $j \neq k$.

As in lemma 2.2 one has $g_{11},\dots,g_{1n_1}\in G$ with:

$$\left\| \frac{1}{n_1} \sum_{k_1=1}^{n_1} U_{g_{1k_1}} \tilde{Y}_{p+1} U^*_{g_{1k_1}} \right\| \leqslant \frac{\varepsilon}{r} . \tag{7}$$

By (6) and (7) one has:

$$\left\| \frac{1}{nn_1} \sum_{i=1}^{n} \sum_{k_1=1}^{n_1} U_{g_{1k_1}} U_{h_i} (\tilde{Y}+Y_{p+1}) U^*_{h_i} U^*_{g_{1k_1}} \right\| \leqslant \left\| \frac{1}{n} \sum_{i=1}^{n} U_{h_i} \tilde{Y} U^*_{h_i} \right\| +$$

$$+ \left\| \frac{1}{n_1} \sum_{k_1=1}^{n_1} U_{g_{1k_1}} \tilde{Y}_{p+1} U^*_{g_{1k_1}} \right\| \leqslant \frac{p\varepsilon}{r} + \frac{\varepsilon}{r} = \frac{(p+1)\varepsilon}{r} .$$

One take now $\tilde{Y}_{p+2} = \frac{1}{nn_1} \sum_{i=1}^{n} \sum_{k_1=1}^{n_1} U_{g_{1k_1}} U_{h_i} Y_{p+2} U^*_{h_i} U^*_{g_{1k_1}}$. By induction, as in lemma 2.2, one has:

$$\left\| \frac{1}{nn_1\dots n_{r-p}} \sum_{i=1}^{n} \sum_{k_1=1}^{n_1} \dots \sum_{k_{r-p}=1}^{n_{r-p}} U_{g_{r-pk_{r-p}}} \dots U_{g_{1k_1}} Y U^*_{g_{1k_1}} \dots U^*_{g_{r-pk_{r-p}}} \right\| \leqslant \varepsilon .$$

Hence, there exist $N=nn_1\dots n_{r-p}$ and $g_1,\dots,g_N\in G$ such that:

$$\left\| \frac{1}{N} \sum_{k=1}^{N} U_{g_k} Y U^*_{g_k} \right\| \leqslant \varepsilon . \tag{8}$$

One can change now $Y\in\mathbb{C}[G]$ to $y\in C^*_r(G)$. The last part is standard (see [2], [4]). Let $I\subset C^*_r(G)$ be a non-zero two-sided ideal in A. Choose $y\neq 0$ in I. One may assume $y\geqslant 0$ and $\tau(y)=1$. By (8) one has:

$$\left\| \frac{1}{N} \sum_{k=1}^{N} U_{g_k} y U^*_{g_k} - 1 \right\| \leqslant \varepsilon .$$

For $\varepsilon<1$, the element $z=\frac{1}{N}\sum_{k=1}^{N} U_{g_k} y U^*_{g_k} \in I$ is invertible, hence $I=C^*_r(G)$.

Let τ' be a trace on $C^*_r(G)$. Then, for ε arbitrarly small, one has $\tau'(y)=1=\tau(y)$, for any $y\in C^*_r(G)$, $y\geq 0$, $\tau(y)=1$. Hence $\tau'=\tau$.

Q.E.D.

Our work suggests the following questions:

1) Does there exist a weak Powers' group which is not Powers' group? The answer is unknown to us even for the semi-direct product of F_2 and F_2.

2) Let G_1 and G_2 discrete groups such that $C^*_r(G_i)$, is simple with unique trace (i=1,2) and $1\to G_1\to G\to G_2\to 1$ be a short exact sequence of groups. Is it true that $C^*_r(G)$ is simple with a unique trace?

If G_1 and G_2 are weak Powers' groups, the answer is affirmative (proposition 2.10).

When G is compatible with the action of G_2 on G_1 (see [9]), then $C^*_r(G)$ is the cross-product of $C^*_r(G_2)$ by G_1 ([9]), hence in this case the answer is affirmative for G_1 weak Powers' group (2.3, 2.6, 2.9).

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