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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS
No. 36/1987

BUCURESTI

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October 1987

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ABSTRACT. An initial and boundary value problem for an elastic-visco-plastic material is considered. An Euler method internal and external approximation techniques are used in order to reduce the continuous problem to a sequence of liniar algebraic systems. The error is estimated over a finite time interval. Supposing that the time step is less then a critical value the error is estimated over an infinite time interval in the viscoelastic case.

0. INTRODUCTION

The following rate-type elastic-visco-plastic constitutive equation:

$$\dot{\sigma}=\mathcal{E}\dot{\varepsilon} + F(\sigma,\varepsilon) \qquad (0.1)$$

is considered. Various results and mechanical interpretations concerning this constitutive law may be found for instance in Freudenthal and Geringer [3], Cristescu-Suliciu [2] and Suliciu [8]. If F depends only on σ equation (0.1) may be reduced to some classical models used in viscoplasticity. Examples of constitutive equations of the form (0.1), involving the full conpling in stress and strain are given for instance in Cristescu--Suliciu [2].

In the paper of Ionescu, Sofonea [6] a quasistatic initial and boundary value problem for this type of materials is considered. Results concerning existence, stability, assymptotic and large time behaviour of the solution are obtained. The goal of the present work is to give error estimates for a numerical approach of this problem.

In the first section some notations are introduced and some preliminary results are recalled. Further on the mechanical problem is stated and the assumptions will be used are given. For the convenience of the reader some results and techniques from Ionescu, Sofonea [6] that will be usefull in this work, are briefly presented.

In section 3, we use an explicite Euler method, internal approximation technique (possibly a finite element one) for the displacement and strain field and an external one for the stress field, in order to reduce the continuous problem to a recursive sequence of linear algebraic systems. The error is estimated (Theorem 3.1) over a finite time interval.

For large time intervals or for a large Lipschitz constant of F (usually at metals) the error estimation obtained in section 3 is not so useful. The following question arise: "How large the time step can be chosen in order to obtain numerical informations about the large time behaviour of the solution?" In order to throw some light on this problem, in section 4, only the viscoelastic case is considered and the function F from (0.1) is supposed to be of the form:

$$F(\sigma,\varepsilon)=-\lambda(\sigma-G(\varepsilon)) \tag{0.2}$$

with $\lambda>0$ and G a strongly monotone function. In this case the error is estimated over an infinite time interval provided that the time step k is restricted to be less then k_o a value which depends on the material constants. It is not established a critical value k_{cr}, for which the numerical solution is divergent for $k>k_{cr}$. But, however, Exemple 1 of section 5 shows that such a critical value exists and hence the restriction on the time step is also a necessar condition in order to have a bound of the error for an infinite time interval. Similar restriction on the time integration step is obtained in Mihailescu-Suliciu, Suliciu [7] where a one-dimensional dinamical problem is considered and the method of characteristics and energetical estimates are used.

Finally three one dimensional numerical examples are given.

1. NOTATIONS AND PRELIMINARIES

Let $\mathcal{S}$ be the set of second order symmetric tensors in R^N (N=1,2,3). We denote by $\cdot$, $|\,|$ the inner product and the euclidian norm in R^N and $\mathcal{S}$. Let $\Omega\subset R^N$ be a bounded domain with a smooth (C^1) boundary $\Gamma=\partial\Omega$ and let Γ_1 be an open subset of Γ with mes $\Gamma_1>0$ and $\Gamma_2=\Gamma-\overline{\Gamma}_1$.

The following Hilbert spaces: $\mathcal{L}=[L^2(\Omega)]_S^{N\times N}$, $L=[L^2(\Omega)]^N$, $\mathcal{H}=[H(\mathrm{div},\Omega)]_S^N$, $H=[H^1(\Omega)]^N$; $H_\Gamma=[H^{1/2}(\Gamma)]^N$ are used and the canonical inner products and norms are denoted by $((\cdot,\cdot),||\cdot||)$, $(((\cdot,\cdot)),|||\cdot|||)$, $((\cdot,\cdot)_\alpha, ||\cdot||_\alpha)$, $((\cdot,\cdot)_H, ||\cdot||_H)$, $((\cdot,\cdot)_\Gamma, ||\cdot||_\Gamma)$ respectively.

Let $V_1=\{u\ H;\ \gamma_o(u)=0 \text{ on } \Gamma_1\}$ where $\gamma_o:H\to H_\Gamma$ is the trace map. The operator $\varepsilon:H\to\mathcal{L}$ given by $\varepsilon=\frac{1}{2}(\nabla+\nabla^T)$ is linear and continuous and since mes $\Gamma_1>0$ the Korn's inequality holds

(1.1) $\quad ||\varepsilon(u)||\geq C||u||_H \quad$ for all $u\in V_1$ *)

If $\sigma\in\mathcal{H}$ then there exists $\gamma_\nu(\sigma)\in H^*_\Gamma$ (the strong dual of H_Γ with the norm denoted by $||\ ||_*$) such that

(1.2) $\quad <\gamma_\nu(\sigma),\gamma_o(v)>=(\sigma,\varepsilon(v))+((\mathrm{div}\ \sigma,v))$

(1.3) $\quad ||\gamma_\nu(\sigma)||_*\leq C||\sigma||_d \quad$ for all $\sigma\in\mathcal{H}$, $v\in H$

By $\sigma\cdot\nu|_{\Gamma_2}$ we shall understand the restriction of $\gamma_\nu(\sigma)$ on $E=\gamma_o(V_1)\subset H_\Gamma$ and the norm in E^* will be denoted by $||\ ||_o$.

As it follows from Geymonat and Suquet [4], $\varepsilon(V_1)$ is the ortogonal complement of $V_2=\{\sigma\in\mathcal{H};\ \mathrm{div}\ \sigma=0;\ \sigma\cdot\nu|_{\Gamma_2}=0\}$ in $\mathcal{L}$. Hence $\mathcal{L}=\varepsilon(V_1)\oplus V_2$ and

(1.4) $\quad (\sigma,\varepsilon(v))=0 \quad$ for all $v\in V_1$, $\sigma\in V_2$

Let $(X,||\ ||_X)$ be one of the above spaces and let denote by $R_+=[0,+\infty)$ and $C^o(R_+,X)=\{z:R_+\to X;\ z$ is continuous$\}$; $C^1(R_+,X)=\{x\in C^o(R_+,X);$ there exists $\dot{z}\in C^o(R_+,X)\}$ where the dot represents the derivative with respect to the time variable. We shall also use the following notation $||z||^\infty_X=\sup_{t\in R_+}||z(t)||_X$ for $z\in C^o(R_+,X)$. Similar to above the spaces $C^o(0,T,X)$ and $C^1(0,T,X)$ can be introduced and the norm in $C^o(0,T,X)$ will be denoted by $||z||_{T,X}=\sup_{t\in[0,T]}||z(\varepsilon)||_X$.

*) Everywhere in this paper $C>0$ will represent a generic constant which depends on Ω, Γ_1 and possibly on some material constants which will be mentioned.

2. PROBLEM STATEMENT. EXISTENCE RESULTS

In this section, after the problem statement and some assumptions, we shall briefly recall some results of Ionescu, Sofonea [6], concerning the existence of the solution, which will be usefull further on.

Let us consider the following mixt problem: find the displacement function $u:R_+ x\Omega\to R^N$ and the stress function $\sigma:R_+ x\Omega\to \mathcal{S}$ such that

(2.1) $\quad \operatorname{div}\sigma(t)+b(t)=0$

(2.2) $\quad \dot{\sigma}(t)=\mathcal{E}\varepsilon(\dot{u}(t))+F(\sigma(t),\varepsilon(u(t)))$ in Ω

(2.3) $\quad u(t)|_{\Gamma_1}=g(t);\quad \sigma(t)\cdot\nu|_{\Gamma_2}=f(t)$ for all $t>0$

(2.4) $\quad u(0)=u_o\ ,\quad \sigma(0)=\sigma_o$ in Ω

where ν is the exterior unit normal at Γ, (2.1) are the Cauchy's equilibrium equations in which $b:R_+ x\Omega\to R^N$ are the given body forces. (2.2) represents a rate-type viscoelastic or viscoplastic constitutive equations, $\mathcal{E}$ is a fourth ordere tensor and $F:\Omega x\mathcal{S}x\mathcal{S}\to\mathcal{S}$ is a constitutive function. u_o and σ_o are the initial data and f,g are the boundary data.

The following assumptions are used.

(2.5)
a) $\mathcal{E}_{ijkl}\in L^\infty(\Omega)$; $|\mathcal{E}(x)\tau|\le Q|\tau|$ for all $x\in\Omega$, $\tau\in\mathcal{S}$, $i,j,k,l=\overline{1,N}$
b) $\mathcal{E}\tau_1\cdot\tau_2=\mathcal{E}\tau_2\cdot\tau_1$ for all $\tau_1,\tau_2\in\mathcal{S}$
c) $\mathcal{E}\tau\cdot\tau\ge d|\tau|^2$, $d>0$ for all $\tau\in\mathcal{S}$

(2.6)
a) $|F(x,\tau_1,\sigma_1)=F(x,\tau_2,\sigma_2)|\le L(|\tau_1-\tau_2|+|\sigma_1-\sigma_2|)$, $L>0$ for all $\tau_1,\tau_2,\sigma_1,\sigma_2\in\mathcal{S}$, $x\in\Omega$
b) $F(x,0,0)=0$

(2.7) $b\in C^1(R_+,L)$, $f\in C^1(R_+,E^*)$, there exists $h\in C^1(R_+,H_\Gamma)$ such that $h(t)=g(t)$ on Γ_1 for all $t\in R_+$

(2.8) a) $u_o\in H$, $\sigma_o\in\mathcal{H}$

b) $\operatorname{div}\sigma_o+b(0)=0$, $\sigma_o\cdot\nu|_{\Gamma_2}=f(0)$, $u_o|_{\Gamma_1}=g(0)$

In order to homogenize the problem (2.1)-(2.4) let us consider $\tilde{u}\in C^1(R_+,H)$ and $\tilde{\sigma}\in C^1(R_+,\mathcal{H})$ the solution of the following linear elastic problem

$$\tilde{\sigma}(t)=\mathcal{E}\varepsilon(\tilde{u}(t)) \tag{2.9}$$

$$\operatorname{div}\tilde{\sigma}(t)+b(t)=0 \qquad \text{in } \Omega \tag{2.10}$$

$$\tilde{u}(t)|_{\Gamma_1}=g(t),\quad \tilde{\sigma}(t)\cdot\nu|_{\Gamma_2}=f(t) \quad \text{for all } t\in R_+ . \tag{2.11}$$

Moreover for all $t\in R_+$, we have

$$\dot{\tilde{\sigma}}(t)=\mathcal{E}\varepsilon(\dot{\tilde{u}}(t)) \tag{2.12}$$

and if $T>0$ and $t_1,t_2\in[0,T]$ we have

$$||\tilde{u}(t_1)-\tilde{u}(t_2)||_H+||\tilde{\sigma}(t_1)-\tilde{\sigma}(t_2)||\le\dot{I}(T)|t_1-t_2| \tag{2.13}$$

where

$$\dot{I}(T)=C(||\dot{b}||_{T,L}+||\dot{f}||_{T,E^*}+||\dot{h}||_{T,H_\Gamma}) \tag{2.14}$$

and C depends only on Ω, Γ_1, d and Q.

Denoting by $\bar{u}_o=u_o-\tilde{u}(0)$, $\bar{\sigma}_o=\sigma_o-\tilde{\sigma}(0)$, $\bar{u}=u-\tilde{u}$, $\bar{\sigma}=\sigma-\tilde{\sigma}$ from (2.1)-(2.4) and (2.9)-(2.12) we obtain:

$$\dot{\bar{\sigma}}(t)=\mathcal{E}\varepsilon(\dot{\bar{u}}(t))+F(\bar{\sigma}(t)+\tilde{\sigma}(t),\varepsilon(\bar{u}(t))+\varepsilon(\tilde{u}(t))) \tag{2.15}$$

$$\operatorname{div}\bar{\sigma}(t)=0 \qquad \text{in } \Omega \tag{2.16}$$

$$\bar{u}(t)|_{\Gamma_1}=0 \qquad \bar{\sigma}(t)\cdot\nu|_{\Gamma_2}=0 \qquad \text{for all } t>0 \tag{2.17}$$

$$(2.18) \qquad \bar{u}(0)=\bar{u}_o \quad , \quad \bar{\sigma}(0)=\bar{\sigma}_o$$

Let consider on V_1 the inner product $(\cdot,\cdot)_a$ given by

$$(2.19) \qquad (u,v)_a=(\mathcal{E}\varepsilon(u),\varepsilon(v)) \qquad u,v\in V_1$$

which generates an equivalent norm denoted by $\|\ \|_a$.

Let $V=V_1 x V_2$ be the product space with the norm denoted by $\|\ \|_V$ generated by the following inner product

$$(2.20) \qquad (x_1,x_2)_V=(v_1,v_2)_a+(\mathcal{E}^{-1}\tau_1,\tau_2)$$

for all $x_i=(v_i,\tau_i)\in V$, $i=1,2$. We shall consider $A:R_+ xV\to V$ a non-linear operator defined as follows

$$(2.21) \qquad (A(t,x_1),x_2)=-(F(\tau_1+\tilde{\sigma}(t),\varepsilon(v_1)+\varepsilon(\tilde{u}(t))),\varepsilon(v_2))+$$
$$+(\mathcal{E}^{-1}F(\tau_1+\tilde{\sigma}(t),\varepsilon(v_1)+\varepsilon(\tilde{u}(t))),\tau_2)$$

for all $x_i=(v_i,\tau_i)\in V$, $i=1,2$, $t\in R_+$

As it follows from Ionescu, Sofonea [6] , $x=(\bar{u},\bar{\sigma})$ is the solution of the problem (2.15)-(2.18) iff $x\in C^1(R_+,V)$ is the solution of the following Cauchy problem:

$$(2.22) \qquad \dot{x}(t)=A(t,x(t)) \qquad t>0$$

$$(2.23) \qquad x(0)=x_o$$

where $x_o=(\bar{u}_o,\bar{\sigma}_o)$. From (2.5)-(2.8) and (2.13) we get

$$(2.24) \qquad \|A(t_1,x_1)-A(t_2,x_2)\|_V\leq CL(\|x_1-x_2\|_V+\dot{I}(T)|t_1-t_2|)$$

for all $x_1,x_2\in V$, $T>0$, $t_1,t_2\in[0,T]$ (the constant C depends only on Ω,Γ_1, Q and d). Hence, there exists a unique solution $x=(\bar{u},\bar{\sigma})\in C^1(R_+,V)$ of (2.22), (2.23) and $u=\bar{u}+\tilde{u}\in C^1(R_+,H)$ $\sigma=\bar{\sigma}+\tilde{\sigma}\in C^1(R_+,\mathcal{H})$ is the unique solution of (2.1)-(2.4).

3. ERROR ESTIMATIONS OVER A FINITE TIME INTERVAL

In this section a numerical aproach of the problem (2.1)-(2.4) is given. Using an explicite Euler's method a recursive sequence of linear elliptic boundary value problems is obtained and the estimation of the error is given. An internal aproximation (we have in mind a finite element aproximation) for the displacement and an external one for the stress lead to a recursive sequence of linear algebraic systems. A final error estimation is obtained.

Let $T>0$, $M\in\mathbb{N}$ and $k=\frac{T}{M}$ be the time step. We consider $V_h\subset V_1$ a finite dimensional subspace of V_1 (constructed for instance using the finite element method), and let $(u_h^n,\sigma_h^n)_{n=\overline{0,M}}$ be the solution of the following recursive algebraic systems:

(3.1) $$u_h^o=\bar{u}_h^o+\tilde{u}(0);\quad \bar{u}_h^o\in V_h;\quad \sigma_h^o\in\mathcal{L}.$$

(3.2) $$\bar{u}_h^{n+1}\in V_h;(\bar{u}_h^{n+1},v_h)_a=(\bar{u}_h^n,v_n)_a-k(F(\sigma_h^n,\varepsilon(u_h^n)),\ \varepsilon(v_h))$$

for all $v_h\in V_h$

(3.3) $$u_h^{n+1}=\bar{u}_h^{n+1}+\tilde{u}((n+1)k)$$

(3.4) $$\sigma_h^{n+1}=\sigma_h^n+\mathcal{E}\varepsilon(u_h^{n+1})-\mathcal{E}\varepsilon(u_h^n)+kF(\sigma_h^n,\varepsilon(u_h^n))$$

The following theorem gives an upper bound of the distance between the exact solution (u,σ) of (2.1)-(2.4) and the approximative one $(u_h^n,\sigma_h^n)_{n=\overline{0,M}}$.

THEOREM 3.1. For all $n=\overline{0,M}$ we have:

(3.5) $$||u(nu)-u_h^n||_H+||\sigma(nk)-\sigma_h^n||\le$$
$$\le C\exp(CLT)\left[k(\mathring{I}(T)+\hat{U}(T)+\mathring{\Sigma}(T))(\exp(CLT)-1)+S(T)+\right.$$
$$\left.+||u_o-u_h^o||+||\sigma_o-\sigma_h^o||\right].$$

where the constant C depends only on Ω, Γ_1, d and Q,

(3.6) $\quad \mathring{U}(T)=||\dot{\bar{u}}||_{T;H} \qquad \mathring{\Sigma}(T)=||\dot{\bar{\sigma}}||_{T,\mathcal{L}}$

(3.7) $\quad S(T)=\sup_{t\in[0,T]}\left(\inf_{v_h\in V_h}||\bar{u}(t)-v_h||\right)$

In order to prove Theorem 3.1 we shall consider the folfowing sequence of linear elliptic boundary value problems:

Find $(u^n,\sigma^n)_{n=\overline{0,M}}$ such that

(3.8) $\quad u^o=u_o \ ; \quad \sigma^o=\sigma_o$

(3.9) $\quad \operatorname{div}\sigma^{n+1}+b((n+1)k)=0$

(3.10) $\quad \sigma^{n+1}-\mathcal{E}\varepsilon(u^{n+1})=\sigma^n-\mathcal{E}\varepsilon(u^n)+kF(\sigma^n,\varepsilon(u^n))$

(3.11) $\quad u^{n+1}|_{\Gamma_1}=g((n+1)k); \qquad \sigma^{n+1}\nu|_{\Gamma_2}=f((n+1)k)$

Using standard arguments from the theory of linear elliptic equations one can get that the problem (3.9)-(3.11) has a unique solution $(u^{n+1},\sigma^{n+1})\in H\times\mathcal{H}$.

<u>LEMMA 3.1</u>. For all $n=\overline{0,M}$ we have:

(3.12) $$||u(nk)-u^n||_H+||\sigma(nk)-\sigma^n||\leq$$
$$\leq kC\left[\mathring{I}(T)+\mathring{U}(T)+\mathring{\Sigma}(T)\right]\cdot\left[\exp(CLT)-1\right]$$

and the constant C depend only on Ω, Γ_1, d and Q.

<u>Proof</u>. Let $\bar{u}^n=u^n-\tilde{u}(nk)$, $\bar{\sigma}^n=\sigma^n-\tilde{\sigma}(nk)$ for all $n=\overline{0,M}$. From (3.9), (3.11) we get $(\bar{u}^n,\bar{\sigma}^n)\in V$ and from (3.10) we can easily deduce that

(3.13) $\quad (\bar{u}^{n+1},v)_a=(\bar{u}^n,v)_a-k(F(\tilde{\sigma}(nk)+\bar{\sigma}^n,\ \varepsilon(\tilde{u}(nk))+\varepsilon(\bar{u}^n)),\ \varepsilon(v))$

for all $v\in V_1$ and

(3.14) $(\mathcal{E}^{-1}\bar{\sigma}^{n+1},\tau)=(\mathcal{E}^{-1}\bar{\sigma}^{n},\tau)+k(\mathcal{E}^{-1}F(\sigma(nk)+\bar{\sigma}^{n},\varepsilon(u(nk))+\varepsilon(\bar{u}^{n})),\tau)$

for all $\tau\in V_2$. Denoting by $y^n=(\bar{u}^n,\bar{\sigma}^n)\in V$, $n=\overline{0,M}$ from (3.13), (3.14), (2.20) and (2.21) we get

(3.15) $y^o=x_o \quad y^{n+1}=y^n+kA(nk,y^n) \quad n=\overline{0,M-1}$

hence $(y^n)_{n=\overline{0,M}}$ is the Euler approximation of the Cauchy problem (2.22), (2.23). As it follows from Henrici [5] p. 26 and (2.24) we have

(3.16) $||x(nk)-y^n||_V\leq k\left[\dot{I}(T)+Z_1\right]\left[\exp(nkCL)-1\right]$

where $Z_1=\sup_{t\in[0,T]}||\dot{x}(t)||_V$.

Having in mind that $x(t)=(\bar{u}(t),\bar{\sigma}(t))$ from (3.16) we can obtain (3.12).

LEMMA 3.2. For all $n=\overline{0,M}$ we have

(3.17) $||u^n-u_h^n||_H+||\sigma^n-\sigma_h^n||\leq C\exp(CLT)\left[D(M)+||u_o-u_h^o||_H+||\sigma_o-\sigma_h^o||\right]$

where C depends only on Ω, Γ_1, d, Q and

(3.18) $D(M)=\sup_{n=\overline{0,M}}\inf_{v_h\in V_h}||\bar{u}^n-v_h||_H$

Proof. Let $(\bar{u}^n, \bar{\sigma}^n)$ as in the proof of Lemma 3.1, and $g^n\in V_1^*$, $g_h^n\in V_h^*$ be given by

(3.19) $g^n(v)=-k(F(\sigma^n,\varepsilon(u^n)),\varepsilon(v))$ for $v\in V_1$

(3.20) $g_h^n(v_h)=-k(F(\sigma_h^n,\varepsilon(u_h^n)),\varepsilon(v_h))$ for $v_h\in V_h$

From (3.2) and (3.13) we get

(3.21) $(\bar{u}^{n+1},v)_a=(\bar{u}^n,v)_a+g^n(v)$ for all $v\in V_1$

(3.22) $$(\bar{u}_h^{n+1},v_h)_a=(\bar{u}_h^n,v_h)_a+g_h^n(v_h) \qquad \text{for all } v_h\in V_h$$

Denoting by $f^n(v)=\sum_{i=0}^{n} g^i(v)+(\bar{u}_o,v)_a$, $f_h^n(v_h)=\sum_{i=0}^{n} g_h^i(v_h)+$ $+(\bar{u}_h^o,v_h)_a$ for all $v\in V_1$, $v_h\in V_h$.

From (3.21) and (3.22) we deduce

(3.23) $$(\bar{u}^{n+1},v)_a=f^n(v) \qquad \text{for all } v\in V_1$$

(3.24) $$(\bar{u}_h^{n+1},v_h)_a=f_h^n(v_h) \qquad \text{for all } v_h\in V_h$$

Having in mind that $|g^i(v_h)-g_h^i(v_h)|\leq$ $\leq CLk(||\sigma^i-\sigma_h^i||+||u^i-u_h^i||_H)||v_h||_H$ and using Strang's lemma (see for instance Ciarlet [1] p. 186) from (3.23), (3.24) we deduce:

(3.25) $$||u^{n+1}-u_h^{n+1}||_H\leq C\Big[\inf_{v_h\in V_h}||u^{n+1}-v_h||_H+$$

$$+kL\sum_{i=0}^{n}(||\sigma^i-\sigma_h^i||+||u^i-u_h^i||)+||\bar{u}_o-\bar{u}_h^o||+||\bar{\sigma}_o-\bar{\sigma}_h^o||$$

If we denote by $a_n=||\bar{u}^n-\bar{u}_h^n||_H=||u^n-u_h^n||_H$, $b_n=||\bar{\sigma}^n-\bar{\sigma}_h^n||=||\sigma^n-\sigma_h^n||$ and $d_n=\sum_{i=0}^{n} a_i+b_i$ for all $n=\overline{0,M}$ from (3.25) we get:

(3.26) $$a_{n+1}\leq C(D(M)+kLd_n+d_o) .$$

If we substitute (3.10) from (3.3) after some algebra we obtain

(3.27) $$b_{n+1}\leq C(a_{n+1}+kLd_n+d_o)$$

From (3.26) and (3.27) we deduce $d_{n+1}\leq d_n(1+CkL)+C(d_o+D(M))$ and recursively we get $d_{n+1}\leq d_o(1+CkL)^n+C(d_o+D(M))\sum_{i=0}^{n}(1+CkL)^i$.

Hence

(3.28) $$d_{n+1} \leq \frac{1}{kL}\left[d_o + D(M)\right](\exp(CLT)-1) + d_o \exp(CLT)$$

for all $n=\overline{0,M-1}$. If we replace (3.28) in (3.26) we get

(3.29) $$a_{n+1} \leq C\left[D(M)+d_o\right]\exp(CLT)$$

and from (3.29), (3.28) and (3.27) we deduce for all $n=\overline{0,M-1}$

(3.30) $$b_{n+1} \leq C\left[D(M)+d_o\right]\exp(CLT)$$

and hence (3.17) holds.

Proof of Theorem 3.1. Having in mind that $D(M) \leq S(T) + \sup_{n=\overline{0,M}} ||u(nk)-u^n||_H$ from (3.12) and (3.17) we deduce (3.5).

4. ERROR ESTIMATIONS OVER AN INFINITE TIME INTERVAL

(viscoelastic case)

We shall study in this section the large time behaviour of the error in a particular case (a viscoelastic one) for which we know from [6] that the system (2.1)-(2.4) is stable. The central result of this section is theorem 4.1 which give an upper bound of the error over an infinite time interval if the time step k is less then k_o which depends on the material constants. It is not established a critical value k_{cr}, the largest k_o for which the statements of theorem 4.1 hold, but however Exemple 1 of the next section sugests us that such a k_{cr} exists.

In this section the constitutive function F of (2.2) is supposed to be of the form:

$$(4.1) \qquad F(\sigma,\varepsilon)=-\lambda\left[\sigma-G(\varepsilon)\right] \qquad \text{for } \sigma,\varepsilon\in\mathcal{S}$$

where $\lambda>0$ and G is a Lipschitz continuous and strongly monotone function i.e.:

$$(4.2) \qquad |G(\tau_1)-G(\tau_2)|\leq L_o|\tau_1-\tau_2| \; ; \quad L_o>0$$

$$(4.3) \qquad (G(\tau_1)-G(\tau_2))\cdot(\tau_1-\tau_2)\geq\alpha|\tau_1-\tau_2|^2 \quad \alpha>0$$

for all $\tau_1,\tau_2\in\mathcal{S}$. Let us remark that $L=\lambda\max(1,L_o)$ and for large λ the Lipschitz constant L from (2.6) is large.

We shall also assume that

$$(4.4) \qquad \begin{aligned} &I=||f||_o^{\infty}+|||b|||^{\infty}+||h||^{\infty}<+\infty \\ &\overset{e}{I}=||f||_o^{\infty}+|||b|||^{\infty}+||h||_{\Gamma}^{\infty}<+\infty \end{aligned}$$

and from [6] theorem 4.2, and (4.4) we can deduce

$$\mathring{U}=\|\mathring{\bar{u}}\|_a^\infty<+\infty \qquad \mathring{\Sigma}=\|\mathring{\bar{\sigma}}\|^\infty<+\infty \tag{4.5}$$

Let $k>0$ be the time step and let us consider the following recursive algebric systems slightly different from (3.1)-(3.4):

$$u_h^o=\bar{u}_h^o+\tilde{u}(0);\quad \bar{u}_h^o\in V_h;\quad \sigma_h^o\in\mathcal{L}. \tag{4.6}$$

$$\bar{u}_h^{n+1}\in V_h;\quad (\bar{u}_h^{n+1},v_h)_a=(\bar{u}_h^n,v_h)_a+\lambda k\big[<f(nk),\gamma_o(v_h)>+ \tag{4.7}$$

$$+((b(nk),v_h))-(G(\varepsilon(u_h^n)),\varepsilon(v_h))\big] \qquad \text{for all } v_h\in V_h$$

$$u_h^{n+1}=\bar{u}_h^{n+1}+\tilde{u}((n+1)k) \tag{4.8}$$

$$\sigma_h^{n+1}=(1-\lambda k)\sigma_h^n+\mathcal{E}\varepsilon(u_h^{n+1}-u_h^n)+\lambda kG(\varepsilon(u_h^n)) \qquad \text{for } n\in\mathbb{N} \tag{4.9}$$

REMARK 4.1. The sequence $(u_h^n)_{n\in\mathbb{N}}$ can be computed from (4.6)-(4.8) without any computation performed on the sequence $(\sigma_h^n)_{n\in\mathbb{N}}$.

THEOREM 4.1. Let $k_o=\min(\frac{1}{2\lambda},\frac{d^2\alpha}{2\lambda L_o^2 Q})$. If $0<k\le k_o$ then

$$\begin{aligned} &q_1=\lambda L_o^2 Q/(d^2\alpha)(1-\exp(-\lambda k\alpha/Q))+\exp(-\lambda k\alpha/Q)<1\ , \\ &q_2=\lambda k(1-\exp(-\lambda k))+\exp(-\lambda k)<1;\qquad q=\max(q_1,q_2)<1 \end{aligned} \tag{4.10}$$

and for all $n\in\mathbb{N}$ we have:

$$\|u(nk)-u_h^n\|_a\le q_1^n\|u_o-u_h^o\|_Q+kCL_o/\alpha(\mathring{I}+\mathring{U})+C(L_o/\alpha S+\mathring{S}/(\lambda L_o)) \tag{4.11}$$

$$\|\sigma(nk)-\sigma_h^n\|\le q_2^n\|\sigma_o-\sigma_h^o\|+CL_o(1-\exp(-\lambda k))nq^n\|u_o-u_h^o\|_a+ \tag{4.12}$$

$$+kC\Big[\mathring{\Sigma}+\frac{L_o}{\alpha}(L_o+\alpha)(\mathring{U}+\mathring{I})\Big]+C(\frac{L_o^2}{\alpha}S+\frac{\mathring{S}}{\lambda})$$

where C depends only on Ω, Γ_1, d, Q and S, S are given by:

$$(4.13)\quad \begin{aligned} S &= \sup_{t\in R_+} \left(\inf_{v_h\in V_h} \|\bar{u}(t)-v_h\|_a \right) \\ \overset{\circ}{S} &= \sup_{t\in R_+} \left(\inf_{v_h\in V_h} \|\overset{\circ}{\bar{u}}(t)-v_H\|_a \right) \quad . \end{aligned}$$

REMARK 4.2. As it follows from (4.11) and (4.13) the error of the initial displacement $\|u_o-u_h^o\|_a$ and stress $\|\sigma_o-\sigma_h^o\|$ is vanishing in the estimation of the error at time nk for n large. This a concequence of the asymptotic stability of the system (2.1)-(2.4) (see theorem 4.2 of [6]).

In the proof of theorem 4.1 the following three abstract lemma will be useful.

LEMMA 4.1. Let $(X,(\cdot,\cdot),\|\ \|)$ a Hilbert space, $A:R_+ xX\to X$ a nonlinear operator and $x\in C^1(R_+,X)$ the solution of the Cauchy problem (2.22), (2.23). Suppose that there exist $L_1,L_2,c>0$ such that

$$(4.14)\qquad \|A(t_1,x_1)-A(t_2,x_2)\|\le L_1\|x_1-x_2\|+L_2\,|t_1-t_2|$$

$$(4.15)\qquad (A(t,x_1)-A(t,x_2),x_1-x_2)\le -c\|x_1-x_2\|^2$$

for all $x_1,x_2\in X$, $t,t_1,t_2\in R_+$. Let $k>0$ be the time step, $B:k\mathbb{N}xX\to X$ and $(Z_n)_{n\in\mathbb{N}}$ a sequence such that

$$(4.16)\qquad \|A(nk,y)-B(nk,y)\|\le Z_n$$

for all $n\in\mathbb{N}$, $y\in X$. Let $(y_n)_{n\in\mathbb{N}}$ be defined as follows:

$$(4.17)\qquad y^o\in X,\quad y^{n+1}=y^n+k\,B(nk,y^n)\qquad n\in\mathbb{N}.$$

If $k<k_o=c/L_1^2$ then $q=kL_1^2(1-\exp(-kc))/c+\exp(-kc)<1$ and for all $n\in\mathbb{N}$ we have:

$$(4.18)\qquad ||x(nk)-y^n||\leq q^n||x_o-y^o||+kL_2/(c-kL_1^2)+$$

$$+(1-\exp(-ck))/c\sum_{i=0}^{n-1}(1+c/L_1)Z_i+kL_1||x(ik)||\,q^{n-1-i}$$

$$(4.19)\qquad ||\dot{x}(nk)-(y^{n+1}-y^n)/k||\leq L_1||x(nk)-y^n||+Z_n\ .$$

Proof. Let $n\in\mathbb{N}$ be fixed. For all $t\in[nk,nk+k]$ we denote by $z(t)=y^n+(y^{n+1}-y^n)(t-nk)/k$ and we remark that:

$$(4.20)\qquad z(nk)=y^n\qquad \dot{z}(t)=B(nk,y^n)\qquad t\in[nk,nk+k]$$

If we denote by $\theta(s)=||x(t)-z(t)||^2$, $t=nk+s$ from (2.22), (2.23), (4.14)-(4.16) we get

$$\tfrac{1}{2}\dot{\theta}(s)\leq(A(t,x(t))-B(nk,y^n),\ x(t)-z(t))=$$

$$=(A(t,x(t))-A(t,z(t)),x(t)-z(t))+(A(t,z(t))-A(nk,y^n),x(t)-z(t))+$$

$$+(A(nk,y^n)-B(nk,y^n),x(t)-z(t))\leq-c\theta(s)+$$

$$+\sqrt{\theta(s)}\left[sL_2+sL_1||y^{n+1}-y^n||/k+Z_n\right]$$

hence we have

$$\dot{\theta}(s)\leq-2c\theta(s)+\sqrt{\theta(s)}\,(kL_2+L_1||y^{n+1}-y^n||+Z_n)\qquad s\in[0,k]$$

and using Lemma 4.1 from [6] we obtain

$$(4.21)\qquad ||x(nk+k)-y^{n+1}||\leq\exp(-ck)\,||x(nk)-y^n||+$$

$$+(1-\exp(-ck))\left[L_1||y^{n+1}-y^n||+kL_2+Z_n\right]/c$$

From (4.14)-(4.17) we can easily deduce:

(4.22) $$\| y^{n+1}-y^n\| \leq k\left[L_1\|x(nk)-y^n\|+Z_n+\|\dot{x}(nk)\|\right]$$

If we replace (4.22) in (4.21) we get

(4.23) $$\|x(nk+k)-y^{n+1}\| \leq q\|x(nk)-y^n\|+$$

$$+(1-\exp(-kc))\left[kL_2+Z_n(1+kL_1)+kL_1\|\dot{x}(nk)\|\right]/c$$

and recursively we obtain (4.18). We also have

$$\|\dot{x}(nk)-(y^{n+1}-y^n)/k\| \leq \|A(nk,x(nk))-A(nk,y^n)\|+$$

$$+\|A(nk,y^n)-B(nk,y^n)\| \leq L_1\|x(nk)-y^n\|+Z_n \ .$$

REMARK 4.3. A larger k_o for which similar inequalities with (4.18), (4.19) hold for $k<k_o$ can be obtained if ck_o is the smallest positive solution of the equation $\exp(-x)+$ $+L_1^2(x-\exp(-x))-1=0$.

LEMMA 4.2. Let X, A and x like in Lemma 4.1, and $Y\subset X$ a closed subspace. We denote by $P:X\to Y$ the projector map on Y and let $B:R_+ xX\to Y$ given by $B(t,z)=PA(t,z)$ for $t\in R_+$, $z\in X$. If $y\in C^1(R_+,Y)$ is the solution of the following Cauchy problem

(4.24) $$y(0)=y_o\in Y \qquad \dot{y}(t)=B(t,y(t)) \qquad t>0$$

then we have:

(4.25) $$\|x(t)-y(t)\| \leq \|x_o-y_o\|e^{-ct}+DL_1/c+\sqrt{\dot{D}D/c}$$

(4.26) $$\|\dot{x}(t)-\dot{y}(t)\| \leq L_1\|x(t)-y(t)\|+\dot{D}$$

for all $t\in R_+$ where:

$$(4.27)\qquad D=\sup_{t\in R_+}\inf_{z\in Y}||x(t)-z|| \qquad \mathring{D}=\sup_{t\in R_+}\inf_{z\in Y}||\dot{x}(t)-z||$$

Proof. If we denote by $\theta(t)=||x(t)-y(t)||^2$ from (4.24), (2.22), (2.23) we get $\frac{1}{2}\dot{\theta}(t)=(A(t,x(t))-B(t,y(t)),x(t)-y(t))$.

For all $z\in Y$ we have $(A(t,y(t))-B(t,y(t)),\ x(t)-y(t))=$ $=(A(t,y(t))-B(t,y(t)),x(t)-z)\leq||x(t)-z||\inf_{v\in Y}||A(t,y(t))-v||\leq$ $\leq||x(t)-z||(\mathring{D}+L_1\sqrt{\theta(t)})$ and hence we deduce

$$(4.28)\qquad (A(t,y(t))-B(t,y(t)),x(t)-y(t))\leq D(\mathring{D}+L_1\sqrt{\theta(t)})$$

Having in mind that $\frac{1}{2}\dot{\theta}(t)=(A(t,x(t))-A(t,y(t)),x(t)-y(t))+$ $+(A(t,y(t))-B(t,y(t)),x(t)-y(t))$ from (4.28), (4.15) we obtain $\frac{1}{2}\dot{\theta}(t)\leq-c\theta(t)+L_1D\sqrt{\theta(t)}+D\mathring{D}$ and using Lemma 4.2 from [6] we deduce (4.25).

If we remark that $||\dot{x}(t)-\dot{y}(t)||\leq||A(t,x(t))-B(t,x(t))||+$ $+||B(t,x(t))-B(t,y(t))||$ we can easily get (4.26).

LEMMA 4.3. Let X, Y, A, B and x like in Lemma 4.2 and let $(y_n)_{n\in N}$ be given by (4.17) with $y^o\in Y$. If $0<k<k_o=c/L_1^2$ then $q=kL_1^2(1-\exp(-ck))/c+\exp(-ck)<1$ and for all $n\in N$ we have

$$(4.29)\qquad ||x(nk)-y^n||\leq q^n||x_o-y^o||+k(L_2+L_1\mathring{Z})/(c-L_1^2k)+$$
$$+2L_1(D+c\mathring{D}/L_1^2)/(c-L_1^2k)$$

$$(4.30)\qquad ||\dot{x}(nk)-(y^{n+1}-y^n)/k||\leq L_1||x(nk)-y^n||+\mathring{D}$$

where D, $\mathring{D}$ are given by (4.27) and $\mathring{Z}=\sup_{t\in R_+}||\dot{x}(t)||$.

Proof. Let $y\in C^1(R_+,Y)$ the solution of (4.24) with $y_o=y^o\in Y$. From Lemma 4.2 we get (4.25) and (4.26) and hence for t=ik we have:

$$(4.31)\qquad ||\dot{y}(ik)||\leq L_1||x_o-y^o||e^{-cki}+DL_1^2/c+\dot{D}+L_1\sqrt{\dot{D}D/c+\dot{Z}},\ i\in\mathbb{N}.$$

If we use now Lemma 4.1 for X=Y, A=B we deduce

$$(4.32)\qquad ||y(nk)-y^n||\leq kL_2/(c-kL_1^2)+kL_1(1-\exp(-ck))/c\sum_{i=1}^{n-1}||\dot{y}(ik)||q^{n-1-i}$$

$$(4.33)\qquad ||\dot{y}(nk)-(y^{n+1}-y^n)/k||\leq L_1||y(nk)-y^n||$$

If we replace (4.31) in (4.32) after some algebra we obtain

$$(4.34)\qquad ||y(nk)-y^n||\leq(q^n-e^{-ckn})||x_o-y^o||+$$

$$+k/(c-L_1^2k)\left[L_2+L_1(DL_1^2/c+\dot{D}+L_1\sqrt{\dot{D}D/c+\dot{Z}})\right]$$

Using now (4.25) with t=nk and (4.33) we get (4.29).

Having in mind that $||\dot{x}(nk)-(y^{n+1}-y^n)/k||\leq$ $\leq||A(nk,x(nk))-B(nk,x(nk))||+||B(nk,x(nk))-B(nk,y^n)||$ we easily deduce (4.30).

Proof of Theorem 4.1. Let $\bar{u}\in C^1(R_+,V_1)$, $\bar{\sigma}\in C^1(R_+,V_2)$ the solution of (2.15)-(2.18). If we multiply (2.14) by $\varepsilon(v)$ and we use (4.1) after integrating over Ω we get

$$(4.35)\qquad (\dot{\bar{u}}(t),v)_a=\lambda\langle f(t),\gamma_o(v)\rangle+\lambda((b(t),v))-$$

$$-\lambda(G(\varepsilon(\dot{\bar{u}}(t))+\varepsilon(\tilde{u}(t))),\ \varepsilon(v))\ \text{for all } v\in V_1\ .$$

Let $J:R_+xV_1\to V_1$ given by

$$(4.36) \qquad (J(t,v),w)_a=\lambda<f(t),\ \gamma_o(w)>+\lambda((b(t),w))-$$

$$-\lambda(G(\varepsilon(v)+\varepsilon(\tilde{u}(t))),\varepsilon(w)) \quad \text{for all } v,w\in V_1$$

From (4.35) and (4.36) we deduce that $\bar{u}$ is the solution of the following Cauchy problem

$$(4.37) \qquad \bar{u}(0)=\bar{u}_o \qquad \dot{\bar{u}}(t)=J(t,\bar{u}(t)) \qquad t>0$$

Using (4.2), (4.3) and (2.5) we get:

$$(4.38) \qquad (J(t,v_1)-J(t,v_2),v_1-v_2)_a\leq -\frac{\lambda\alpha}{Q}||v_1-v_2||_a^2$$

$$(4.39) \qquad ||J(t_1,v_1)-J(t,v_2)||_a\leq\frac{\lambda L_o}{d}||v_1-v_2||_a+\lambda L_o C\hat{I}|t_1-t_2|$$

for all $v_1,v_2\in V_1$, $t,t_1,t_2\in R_+$.

Let $J_h:R_+xV_1\to V_h$, $J_h=P_{V_h}J$ and let us remark that $(\bar{u}_h^n)_{n\in N}$ is the solution of the following recursive system

$$(4.40) \qquad \bar{u}_h^o\in V_h \ , \quad \bar{u}_h^{n+1}=\bar{u}_h^n+kJ_h(nk,\bar{u}_h^n)$$

If we use now Lemma 4.3 with $X=V_1$, $Y=V_h$, $A=J$, $B=J_h$, $L_1=\lambda L_o/d$, $L_2=\lambda LCI$, $c=\lambda L_o/Q$ we deduce that for $0<k<\dfrac{\alpha d^2}{2\lambda L^2 Q}$ we have $q_1=\dfrac{\lambda L_o^2 Qk}{d^2\alpha}(1-\exp(-\alpha\lambda k/Q))+\exp(-\alpha\lambda k/Q)<1$ and (4.11) holds and also we have:

$$(4.41) \qquad ||\dot{\bar{u}}(nk)-(\bar{u}_h^{n+1}-\bar{u}_h^n)_k||a\leq\lambda L_o C||\bar{u}(nk)-\bar{u}_h^n||_a+\hat{S} \ ,$$

Let $W:R_+x\mathcal{L}\to\mathcal{L}$ given by

$$W(t,\tau)=-\lambda\tau+\mathcal{E}\varepsilon(\dot{\bar{u}}(t))-\lambda\tilde{\sigma}(t)+\lambda G(\varepsilon(u(t))$$

for all $\tau\in\mathcal{L}$. Having in mind that $\mathcal{E}\varepsilon(\dot{\bar{u}}(t))-\lambda\tilde{\sigma}(t)+\lambda G(\varepsilon(u(t))=$ $=\dot{\bar{\sigma}}(t)+\lambda\bar{\sigma}(t)\in V_2$ we deduce that $W(t,\tau)\in V_2$ if $\tau\in V_2$ and hence $\bar{\sigma}$ is the solution of the following Cauchy problem:

(4.42) $\quad \bar{\sigma}(0)=\bar{\sigma}_o \in V_2 \qquad \dot{\bar{\sigma}}(t)=W(t,\bar{\sigma}(t)) \qquad t>0$.

If we denote by $W_o(nk,\cdot):\mathcal{L}\to\mathcal{L}$ for $n\in N$ the following operator

(4.43) $\quad W_o(nk,\tau)=-\lambda\tau+\mathcal{E}\varepsilon(\bar{u}_h^{n+1}-\bar{u}_h^n)/k-\lambda\tilde{\sigma}(nk)+\lambda G(\varepsilon(u_h^n))$

for $\tau\in\mathcal{L}$ and $\bar{\sigma}_h^n=\sigma_h^n-\tilde{\sigma}(nk)$ from (4.43), (4.9) we obtain:

(4.44) $\quad \bar{\sigma}_h^{n+1}=\bar{\sigma}_h^n+W_o(nk,\bar{\sigma}_h^n)\cdot k \quad , n\in N$

From (4.43), (4.41) and (4.11) we get

(4.45) $\quad ||W(nk,\tau)-W_o(nk,\tau)||\leq C(\lambda L_o||\bar{u}(nk)-\bar{u}_h^n||_a+\dot{S})$

We can also easily deduce:

(4.46) $\quad (W(t,\tau_1)-W(t,\tau_2),\tau_1-\tau_2)\leq-\lambda||\tau_1-\tau_2||^2$

(4.47) $\quad ||W(t_1,\tau_1)-W(t_2,\tau_2)||\leq\lambda||\tau_1-\tau_2||+\lambda L_o C(\dot{U}+\dot{I})|t_1-t_2|$

for all $\tau_1,\tau_2\in\mathcal{L}$, $t,t_1,t_2\in R_+$.

Using now Lemma 4.1 with $X=\mathcal{L}$, $A=W$, $B=W_o$, $L_1=\lambda$, $L_2=\lambda L_o C(\dot{U}+\dot{I})$, $Z_n=C(\lambda L_o||u(nk)-u_h^n||_a+\dot{S})$ and (4.11) after some algebra we get (4.12).

5. NUMERICAL EXAMPLES

In order to ilustrate the numerical method previously presented we shall give some one-dimensional numerical examples.

In this section $\Omega=(0,1)\subset\mathbb{R}$ and the following initial and boundary value problem (which is an one dimensional version of (2.1)-(2.4)) is considered.

(5.1) $\dot{\sigma}(t,x)=Q\dot{\varepsilon}(t,x) \quad +F(\sigma(t,x),\varepsilon(t,x))$

(5.2) $\varepsilon(t,x)=\frac{\partial u}{\partial x}(t,x) \quad$ for $x\in(0,1)$

(5.3) $\frac{\partial\sigma}{\partial x}(t,x)+b(t,x)=0$

(5.4) $u(0,t)=0$

(5.5') $u(1,t)=q(t)$ or

(5.5") $\sigma(1,t)=r(t) \quad$ for $t>0$

(5.6) $u(0,x)=u_o(x) \quad \sigma(0,x)=\sigma_o(x) \quad$ for $x\in(0,1)$

EXAMPLE 1. Let us consider the linear viscoelastic case $F(\sigma,\varepsilon)=-\lambda(\sigma-b\varepsilon)$ with homogeneous initial data $u_o(x)=\varepsilon_o x$, $\sigma_o(x)=\sigma_o$ and $b(t,x)=0$, $r(t)=\sigma_o$. In this case one can easily integrate (5.1)-(5.6) to obtain the solution $\sigma(t,x)=\sigma_o$, $u(t,x)=\varepsilon(t)x$ and $\varepsilon(t)=\varepsilon_o\exp(-\lambda bt/a)+(1-\exp(-\lambda bta))\sigma_o/b$.

Since V_h is the finite element space constructed with polynomial functions of degree greater or equal to 1 and the problem is homogeneous, we get that $u_h^n=u_k^n$ and $\sigma_h^n=\sigma_k^n=\sigma_o$. Let $E(n,k)$ be the relative strain error

(5.7) $E(n,k)=|\varepsilon(u_k^n)-\varepsilon(nk)|/|\varepsilon(nk)|$

at iteration n for the time step k.

Some numerical evaluations of E(n,k) are presented in figures 1 and 2 for a=20., λ=10., σ_o=40. and b=10.

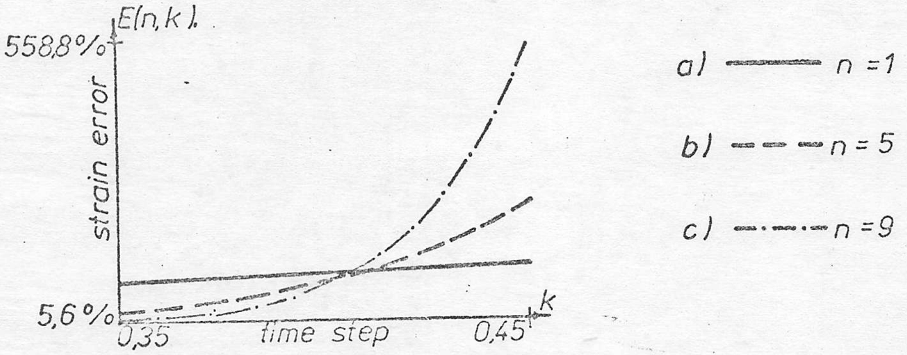

Figure 1

The strain relative error in example 1 at iteration 1 in a) at iteration 5 in b) and at iteration 9 in c).

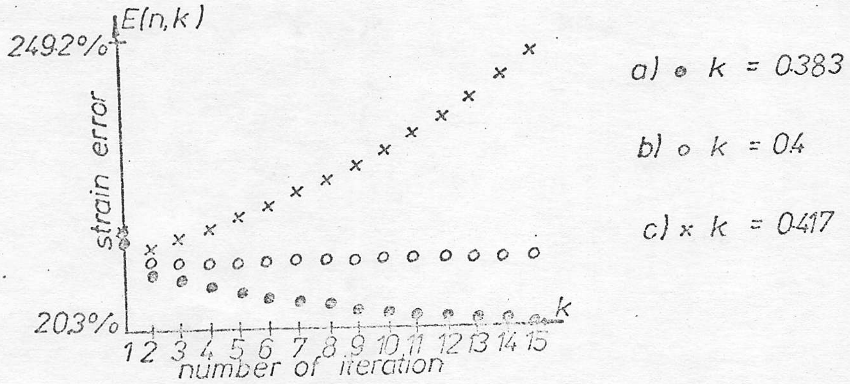

Figure 2

The strain relative error in example 1 for the time step k=0.383 in a) k=0.4 in b) and k=0.417 in c).

In figure 1 one can remark that at iteration 1 the error is linear with respect to the time step k but at iteration 9 the error seems to have an exponential behaviour.

In figure 2 we see that for k=0.383 the error is decreasing when the number n of iterations is increasing, but for k=0.4 the error is almost constant for any n and for k=0.417 the error is quickly increasing. This example suggest that there exists a critical time step k_{cr} (in our case $k_{cr} \simeq 0.4$) such that the error is bounded iff $k<k_{cr}$.

<u>EXAMPLE 2</u>. In this example we shall consider a nonlinear viscoelastic case $F(\sigma,\varepsilon)=-\lambda(\sigma-G(\varepsilon))$ with G a non-monotone function:

$$G(\varepsilon)=\begin{cases} 10\varepsilon & \text{for } \varepsilon\leqq 2 \\ -5\varepsilon+30 & \text{for } 2<\varepsilon<4 \\ 10\varepsilon-30 & \text{for } \varepsilon\geqq 4 \end{cases} \tag{5.8}$$

which is ploted in figure 3.

Let $b(t,x)=0$, $r(t)=\sigma_o(x)=15.$, $a=20$, and $\lambda=10$. We can easily see that $\sigma(t,x)=\sigma_o$ and $\varepsilon(t,x)$ is the solution of

$$\dot{\varepsilon}(t,x)=\lambda(\sigma_o-G(\varepsilon(t,x))/a \tag{5.9}$$

$$\varepsilon(0,x)=\varepsilon_o(x)=\frac{du_o}{dx}(x) \tag{5.10}$$

We remark that $\varepsilon_1(t)=1.5$, $\varepsilon_2(t)=3.$ and $\varepsilon_3(t)=4.5$ are constant solutions for (5.9). The constant solutions ε_1 and ε_3 are asymptotically stable having their domains of atractivity in the set of initial homogeneous strains, $A_1=(-\infty,3)$ and $A_3=(3,+\infty)$ respectively.

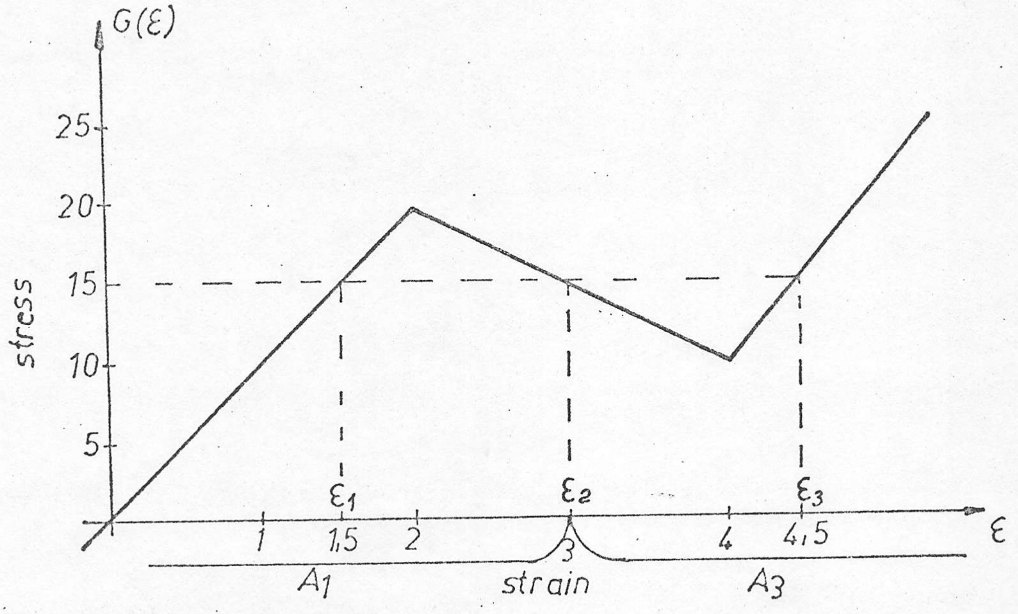

Figure 3
The grafic representation of the function G
from (5.8) example 2

The solution ε_2 is not stable. This is the kind of one dimensional example that Ionescu and Sofonea had in their mind in Remark 4.4 of [6] .

In this example we choose $\varepsilon_o(x)=3.+0.075(x-0.5)$ in order to have $\varepsilon_o(x)\in A_1$ for $x<0.5$, $\varepsilon_o(x)\in A_3$ for $x>0.5$ and ε_o to be "very close" to the unstable solution ε_2. The space V_h is the finite element space constructed with polynomial functions of degree 3 (i.e. $V_h\subset C^1(\bar{\Omega})$), $\Omega=(0,1)$ was divided into 50 finite elements and the time step k=0.05. In figure 4 the computed solution $\varepsilon(u_h^n)$ is ploted and the results agree with the teoretical expectations previously presented.

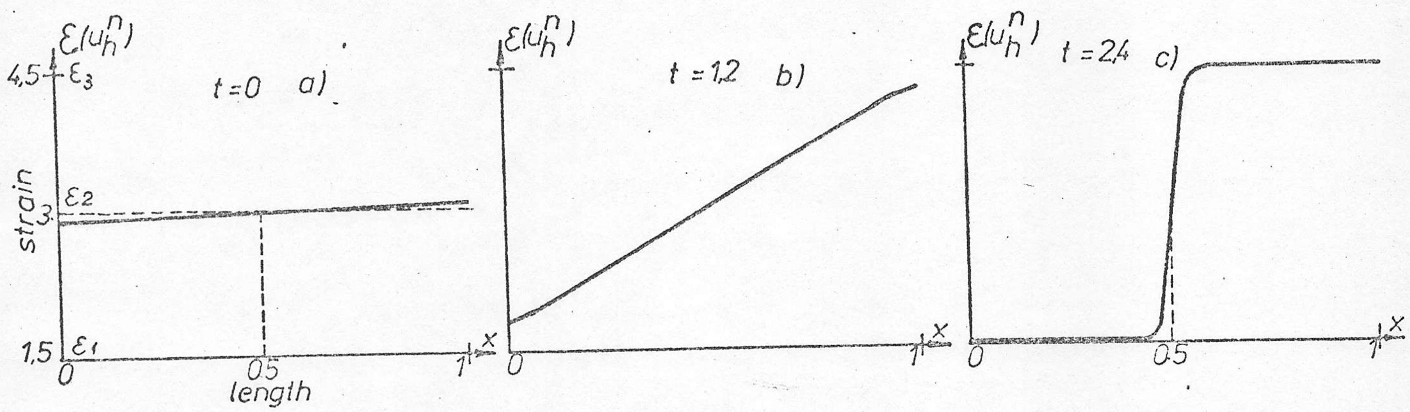

Figure 4

The computed strain field $\varepsilon(u_k^n(x))$ from example 2. The initial field (n=0, t=0) in a) at iteration 24 (t=1.2) in b) and at iteration 48 (t=2.4) in c).

EXAMPLE 3. Let us consider the elastic-viscoplastic case $F(\sigma,\varepsilon) = -\frac{1}{2\mu}(\sigma - P_K(\sigma))$ where P_K is the projection map on the plasticity convex $K=[-1,1]$. Let $q(t)=0$, $u_o(x)=0$, $\sigma_o(x)=0$, $\Omega=1.$ and

$$(5.11) \qquad b(t,x) = \begin{cases} 2tx & \text{for } x<0.5 \\ -2t(1-x) & \text{for } x>0.5 \end{cases}$$

The elastic perfectly plastic version of this example was considered by Suquet [9] in order to show that in the velocity field discontinuities are generated and hence the solution belongs to BD(Ω) (the space of bounded deformation functions).

As it follows from Suquet [9] the solution of the elasto-visco-plastic problem considered here approaches the solution of elastic perfectly plastic problem (considered by Suquet) for small viscosity coefficient μ. In order to obtain in our case similar properties of the solution (described by Suquet) we choose a small $\mu=0.005$. In this way one can consider the elasto--visco-plastic case as a penalized elastic perfectly plastic problem.

Let us remark that for $0<t\leq 6$ the solution of (5.1)-(5.6) is an elastic one:

$$\sigma(t,x)=\begin{cases} t(1/12-x^2) & \text{for } x<1/2 \\ t(1/12-(x-1)^2) & \text{for } x>1/2 \end{cases} \tag{5.12}$$

for $t\leqq 6$

$$u(t,x)=\begin{cases} tx(1/4-x^2)/3 & \text{for } x<1/2 \\ t(x-1)(1/4-(x-1)^2)/3 & \text{for } x>1/2 \end{cases} \tag{5.13}$$

For $t=6$, $\sigma(t,1/2)=-1$ and hence the point $x=1/2$ is plastified.

The space V_h is constructed as in example 2 and the time step $k=0.01$.

In figure 5 the computed stress field σ_h^n is ploted. One can see that at $t=6$ the point $x=1/2$ is plastified, and the stress field remains continuous for $t>6$.

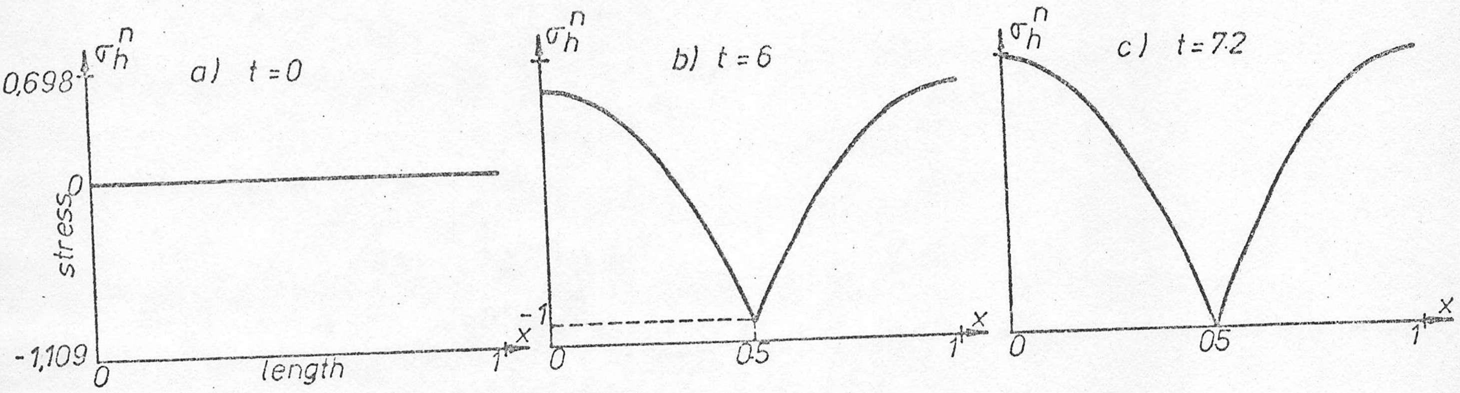

Figure 5

The computed stress field σ_h^n from example 3. The initial strain field in a) (t=0), at t=6 in b) and at t=7.2 in c), (t=nk).

In figures 6 and 7 the computed displacement field u_h^n and velocity field $v_h^n=(u_h^{n+1}-u_h^n)/k$ are ploted. In this figures one can see that for t>6 a "discontinuity" appears in the displacements and velocity fields at x=1/2 which is developing in time.

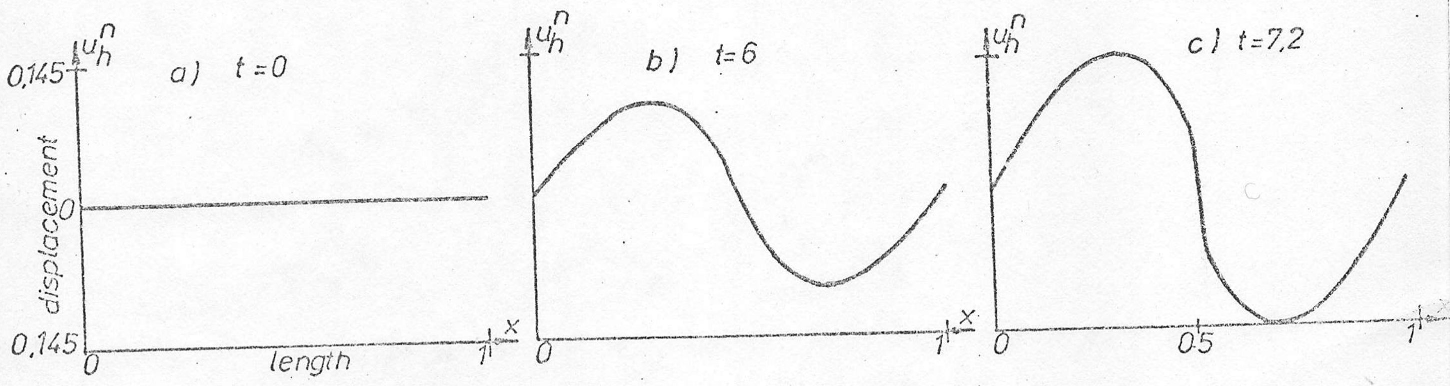

Figure 6

The computed displacement field u_h^n from example 3. The initial displacement field (t=0) in a), at t=6 in b)

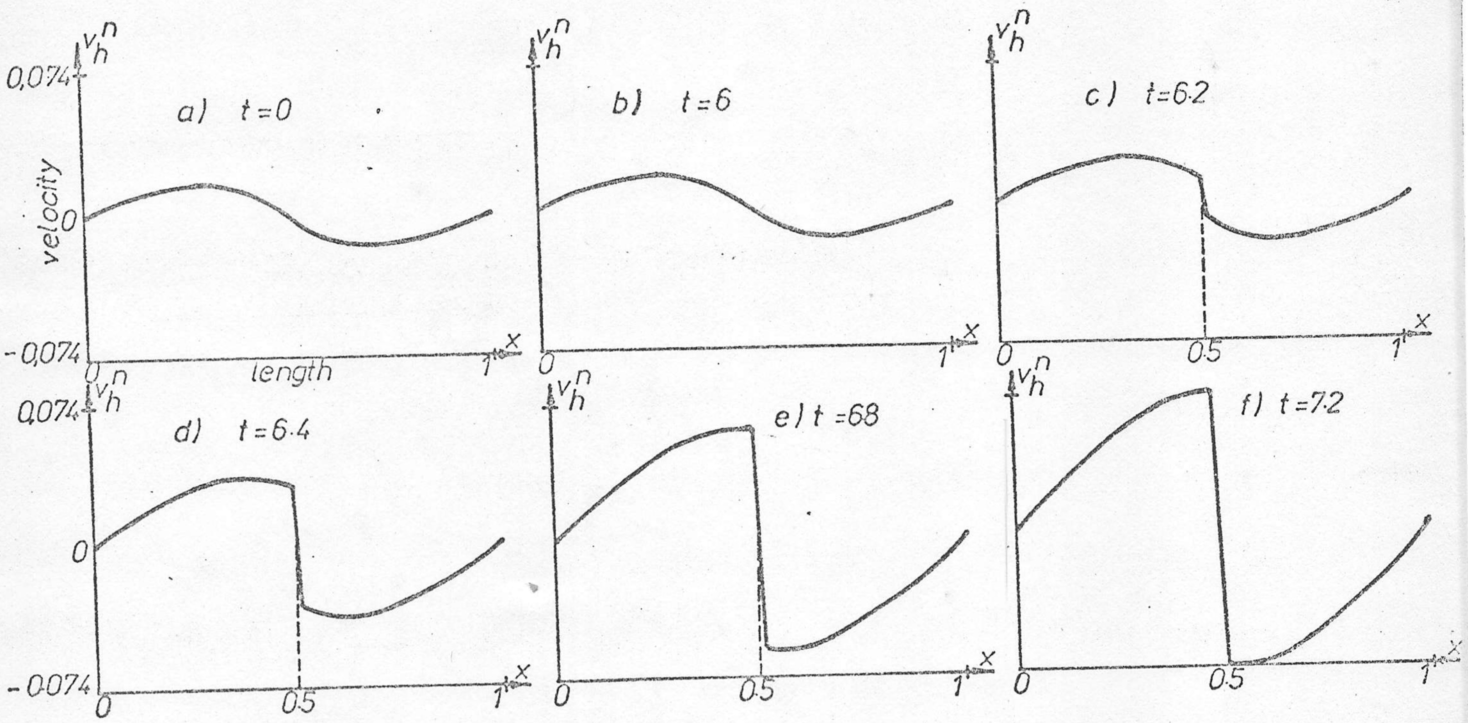

Figure 7

The computed velocity field v_h^n from example 3. The initial velocity field (t=0) in a) at t=6 in b), at t=6.2 in c), at t=6.4 in d), at t=6.8 in e) and at t=7.2 in f), t=nk.

In figure 8 the computed strain field $\varepsilon(u_h^n)$ is ploted, and the remark that the "discontinuity" point x=1/2 the strain is quickly increasing.

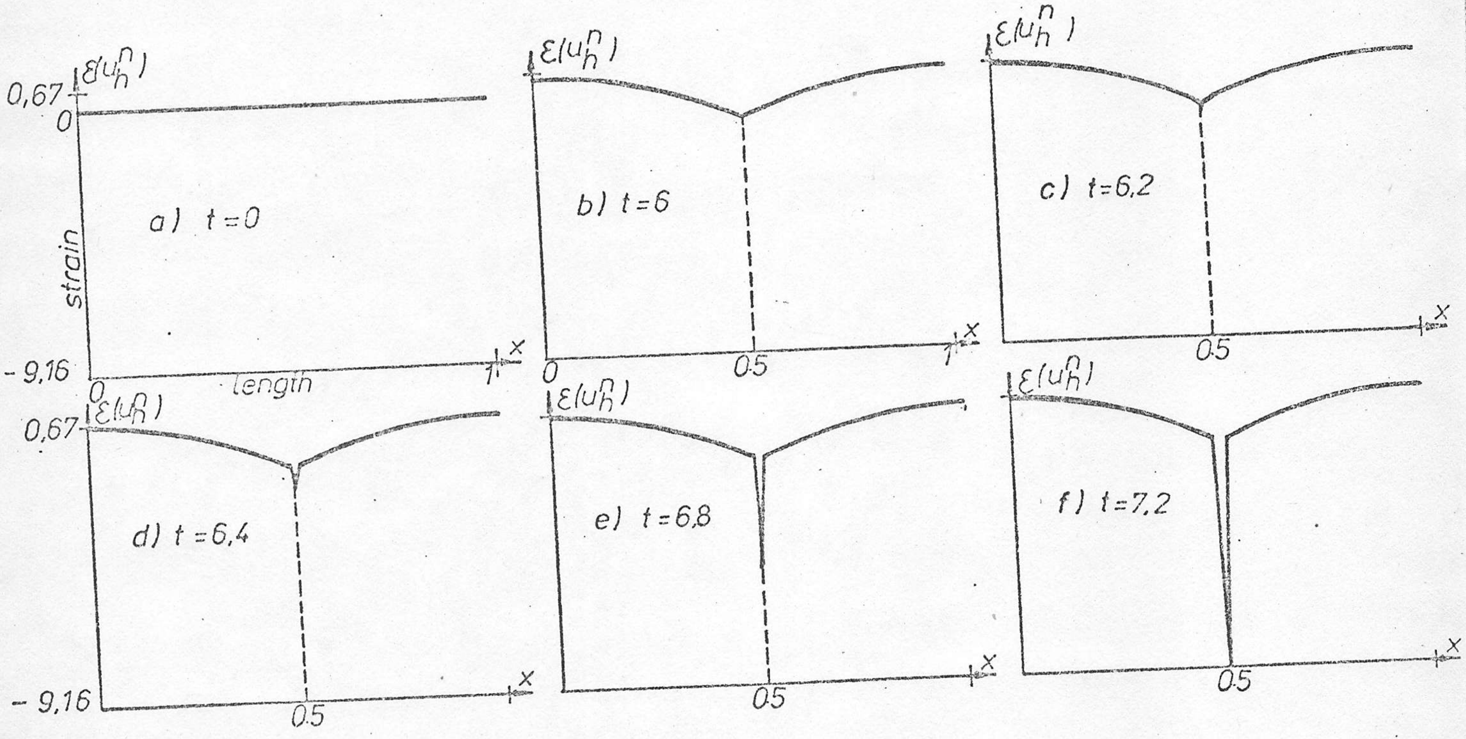

Figure 8
The computed strain field $\mathcal{E}(u_h^n)$ from example 3. The initial strain field (t=0) in a), at t=6 in b), at t=6.2 in c), at t=6.4 in d), at t=6.8 in e) and at t=7.2 in f), t=nk.

AKNOWLEDGEMENT. I thank dr. I. Suliciu for the encouragement to undertake this work, for this usefull sugestions during the research as well as for his remarks which improved the clarity of the text.

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