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Dan POLIŠEVSKI (*)

Summary. We continue in this paper an earlier work [5], by studying the influence of the kinematic viscosity and of the external forces upon the asymptotic behaviour of the Navier-Stokes model of fluid flow through a periodic structure as the characteristic length of the cell tends to zero.

1. Preliminaries

Let Σ^i, $i\in\{1,2,\ldots,6\}$, be the side faces of $Y=[0,1[^3$ and let Γ be a surface of class C^2 included in $\overline{Y}$, which crosses orthogonally the boundary of the cube following some regular curves which are reproduced identically on opposite faces; Γ separates Y in two domains, Y_s (the solid part) and Y_f (the fluid part), with the property that repeating Y by periodicity, the union of all the fluid parts, respectively the solid parts, is connected in $\mathbb{R}^3$ and of class C^2. The origin of the coordinate system is set in a fluid ball; thus all the corners of Y are surrounded by fluid neighbourhoods. We assume also that if Γ attains an edge of $\overline{Y}$, then the normal to Γ in that point is the edge itself.

Let Ω be an open connected bounded set in $\mathbb{R}^3$, locally located on one side of the boundary $\partial\Omega$, a manifold of class C^2, composed of a finite number of connected components. Defining $\varphi : \mathbb{R}^3 \longrightarrow Y$ by

$$\varphi(x_1, x_2, x_3) = (\{x_1\}, \{x_2\}, \{x_3\})$$

where $\{.\}$ denotes the function which associates to any real number its fractional part, we say that a function f defined on $\mathbb{R}^3$ is Y-periodic iff $f = f\circ\varphi$.

Further, for any $\varepsilon \in]0,1[$ we denote

$$\varphi_\varepsilon(x) = \varphi(x/\varepsilon) \quad , \quad x \in \mathbb{R}^3$$
$$\Omega_\varepsilon = \{x \in \Omega \mid \varphi_\varepsilon(x) \in Y_f\} := \text{the fluid part of } \Omega$$
$$\Omega \setminus \overline{\Omega}_\varepsilon = \{x \in \Omega \mid \varphi_\varepsilon(x) \in Y_s\} := \text{the solid part of } \Omega$$
$$\Gamma_\varepsilon = \{x \in \Omega \mid \varphi_\varepsilon(x) \in \Gamma\} := \text{the fluid-solid interface}$$
$$(\partial\Omega)_\varepsilon = \partial\Omega \cap \overline{\Omega}_\varepsilon$$

Let us remark that $\partial\Omega_\varepsilon = (\partial\Omega)_\varepsilon \cup \Gamma_\varepsilon$.

As usual the scalar products and norms in $L^2(\Omega)$ and $H^1_0(\Omega)$ are denoted respectively by $(.,.)$, $|.|$ and $((.,.))$, $\|.\|$; the norm in $L^p(\Omega)$ $(p\neq 2)$ is denoted by $|.|_p$. We agree to use the same notations for the scalar products and norms in $[L^2(\Omega)]^3$, $[H^1_0(\Omega)]^3$ and $[L^p(\Omega)]^3$. To the corresponding notations associated to Ω_ε (instead of Ω) we attach the index ε (for instance the norm in $L^4(\Omega_\varepsilon)$ is denoted by $|.|_{4,\varepsilon}$). Also in this work, Abs(x) stands for the absolute value of the real number x.

Let $\mathcal{V}$ be the space (without topology):

$$\mathcal{V} = \left\{ v \in \mathcal{D}(\Omega) \;\middle|\; \operatorname{div} v = 0 \text{ in } \Omega \right\}.$$

We denote by H and V the closures of $\mathcal{V}$ in $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. To the corresponding concepts associated to Ω_ε (instead of Ω) we attach the index ε.

For any $\varepsilon \in]0,1[$ we consider the Navier-Stokes problem, that is, if the external forces $g_\varepsilon \in H \setminus \{0\}$ and the kinematic viscosity ν_ε are given, we have to find the velocity field u_ε and the pressure p_ε, satisfying in some senses the equations

(1.1) $\operatorname{div} u_\varepsilon = 0$ in Ω_ε

(1.2) $(u_\varepsilon \nabla) u_\varepsilon - \nu_\varepsilon \Delta u_\varepsilon = g_\varepsilon - \nabla p_\varepsilon$ in Ω_ε

and the boundary condition

(1.3) $u_\varepsilon = 0$ on $\partial\Omega_\varepsilon$

The problem (1.1)-(1.3) has a well-known variational formulation:

To find $u_\varepsilon \in V_\varepsilon$ satisfying

(1.4) $\nu_\varepsilon ((u_\varepsilon, v))_\varepsilon + b_\varepsilon (u_\varepsilon, u_\varepsilon, v) = (g_\varepsilon, v)_\varepsilon$ $(\forall)$ $v \in V_\varepsilon$

where b_ε is the trilinear continuous form on V defined by

(1.5) $b_\varepsilon(u,v,w)=\sum_{i,j=1}^{3} \int (u_i \frac{\partial v_j}{\partial x_i} w_j)\,dx$

REMARK 1.1. For any $\varepsilon\in\,]0,1[$, the problem (1.4) has at least one solution (see [9] Ch.II). □

With a proof similar to that of [8] we have the Friedrichs' inequality in Ω_ε :

(1.6) $|v|_\varepsilon \leqslant C_1\varepsilon\|v\|_\varepsilon \qquad (\forall)\ v\in H_0^1(\Omega_\varepsilon)$

where C_1 is independent of ε and v.

We need also the following Sobolev inequality in Ω_ε :

(1.7) $|v|_{4,\varepsilon} \leqslant C_2\,\varepsilon^{1/4}\|v\|_\varepsilon \qquad (\forall)\ v\in H_0^1(\Omega_\varepsilon)$

where C_2 is independent of ε and v.

Proof of (1.7). We consider the classical Sobolev inequality

(1.8) $|u|_6 \leqslant C_0\|u\|\ ,\ (\forall)\, u\in H_0^1(\Omega)$,

where C_0 is independent of u. Then, for any $v\in H_0^1(\Omega_\varepsilon)$, let us choose $u\in H_0^1(\Omega)$ in (1.8) as follows

$$u=\begin{cases} v & \text{in } \Omega_\varepsilon \\ 0 & \text{in } \Omega\setminus\Omega_\varepsilon \end{cases}$$

In a straightforward manner we obtain

(1.9) $|v|_{6,\varepsilon} \leqslant c_0 \|v\|_\varepsilon$.

As $(1/4)/2 + (3/4)/6 = 1/4$, by the Hölder inequality we have

$$|v|_{4,\varepsilon} \leqslant |v|_\varepsilon^{1/4} |v|_{6,\varepsilon}^{3/4} \qquad (1.10)$$

Finally, introducing (1.7) and (1.9) in (1.10) the proof is completed. □

Let $u_\varepsilon \in V_\varepsilon$ be a solution of the problem (1.4); since $b_\varepsilon(u,v,v)=0$ $(\forall)$ $u,v \in V_\varepsilon$, then if we set $v=u_\varepsilon$ in (1.4), we receive

(1.11) $\nu_\varepsilon \|u_\varepsilon\|_\varepsilon^2 \leqslant |g_\varepsilon| \; |u_\varepsilon|_\varepsilon$

and using succesively (1.6) we obtain the following estimations:

$$\|u_\varepsilon\|_\varepsilon \leqslant C\varepsilon |g_\varepsilon| / \nu_\varepsilon \qquad (1.12)$$

$$|u_\varepsilon|_\varepsilon \leqslant C\varepsilon^2 |g_\varepsilon| / \nu_\varepsilon \qquad (1.13)$$

where C denotes constants independent of ε. Now we can prove:

<u>THEOREM 1.1.</u> If the (non-dimensional) Galilean number defined by

$$(1.14) \qquad G_\varepsilon = \varepsilon^{3/2} |g_\varepsilon| / \nu_\varepsilon^2$$

is sufficiently small, then there exists a unique solution of the problem (1.4).

Proof. Let u_1 and u_2 be two possible different solutions of the problem (1.4). If we subtract the equations (1.4) corresponding to u_1 and u_2, and if we denote by $w=u_1-u_2$, then we obtain

$$(1.15) \quad \nu_\varepsilon((w,v))_\varepsilon + b_\varepsilon(u_1,w,v) + b_\varepsilon(w,u_1,v) = 0 \quad (\forall)\ v \in V_\varepsilon .$$

For v=w the relation (1.15) reduces to

$$(1.16) \quad \nu_\varepsilon \|w\|_\varepsilon^2 = -b_\varepsilon(w,u_1,w) \leqslant |w|_{4,\varepsilon}^2 \|u_1\|_\varepsilon$$

Estimating $\|u_1\|_\varepsilon$ by (1.12) and using (1.7), from (1.16) it follows

$$(1.17) \qquad \nu_\varepsilon(1-cG_\varepsilon)\ \|w\|_\varepsilon^2 \leqslant 0$$

with some positive c, independent of ε. For G_ε sufficiently small (1.17) implies $\|w\|_\varepsilon=0$, that is $u_1=u_2$ in V_ε . □

If the homogenization process associated to problem (1.4) is studied, one has to remove the fact that u_ε and p_ε are defined only in Ω_ε. While u_ε can be naturally continued by zero in $\Omega \setminus \Omega_\varepsilon$, the prolongation of p_ε to Ω is not so straight. For the case when ν_ε and g_ε are of ε^0-order, a construction of such a prolongation can be found in [8] and it is done in $L^2(\Omega)$ by transposing some special restriction operator from $H_0^1(\Omega)$

to $H_0^1(\Omega_\varepsilon)$. Unfortunately, it holds only when Y_s is strictly contained into Y, Ω_ε being defined as the domain obtained from Ω by picking out the εY_s parts which do not intersect $\partial\Omega$; thus, from the physical point of view, the flow in [8] is only bidimensional with monophasic border. In [5] we have extended the above mentioned construction to the geometry already presented at the beginning ot this section, which is three-dimensional, with connected phases and biphasic boundary. Still in this case we have succeded in [5] to prove the convergence of the homogenization process, which meant there that, after the prolongation of the solutions, the following convergences hold

(1.18) $u_\varepsilon/\varepsilon^2 \rightharpoonup u$ in $L^2(\Omega)$ weakly

(1.19) $p_\varepsilon \to p$ in $L^{6/5}(\Omega)$ strongly

with the property that u and p satisfy the Darcy problem (see [6] Ch.7).

It is obvious that, for different relative values of the data ν_ε and g_ε with respect to ε, we expect different behaviours of the solutions as ε tends to zero.

2. The restriction operator revisited

As we want now to study a larger range of ν_ε and g_ε, we need more regularity properties for the restriction operator constructed in [5] and therefore we shall reconsider that procedure.

LEMMA 2.1. There exists $f \in \mathcal{L}(W_6^{(1)}(Y), W_6^{(1/2)}(\partial Y_f))$ such that

$$(2.1) \qquad f(u) = 0 \qquad \text{on } \Gamma$$

$$(2.2) \qquad f(u) = u \qquad \text{on } \partial Y_f \text{ if } u=0 \text{ in } Y_s$$

$$(2.3) \qquad \int_{\Sigma_f^i} f(u).n d\sigma = \int_{\Sigma^i} u.n d\sigma, \quad (\forall) \quad i \in \{1,2,\ldots,6\}$$

where $\Sigma_f^i = \Sigma^i \cap \overline{Y}_f$, $\Sigma_s^i = \Sigma^i \cap \overline{Y}_s$ and n denotes the unit outward normal to Σ^i.

Moreover, there exists a constant C such that

$$(2.4) \qquad |f(u)|_{L_\infty(\partial Y_f)} \leqslant C |u|_{L_\infty(\partial Y)} \qquad (\forall) \quad u \in W_6^{(1)}(Y).$$

Proof. This is a slight improvement of Lemma 1 from [5]. The operator f defined there, satisfies (2.1)-(2.3) and also

$$(2.5) \qquad |f(u)|_{W_6^{(1/2)}(\partial Y_f)} \leqslant C |u|_{W_6^{(1)}(Y)}, (\forall) \quad u \in W_6^{(1)}(Y),$$

where C is independent of u. Hence, what is new here is only (2.4).

For this let us remark that if $u \in W_6^{(1)}(Y)$, then $u \in W_6^{(5/6)}(\partial Y)$ and according to the Sobolev imbedding theorems for fractional order spaces (see [1] Ch.VII) it follows $u \in C^0(\partial Y)$. Recalling the definition of f(u), (2.4) is obtained in a straightforward manner. □

LEMMA 2.2. If $u \in W_6^{(1)}(Y)$ then there exists a unique $(v,q) \in H^1(Y_f) \times L^2(Y_f)/\mathbb{R}$, solution of the problem

(2.6) $-\Delta v + \nabla q = -\Delta u$ in Y_f

(2.7) $\mathrm{div}\ v = \mathrm{div}\ u + k(u)$ in Y_f

(2.8) $v = f(u)$ on ∂Y_f (f given by Lemma 2.1)

where, denoting the measure of Y_f by $|Y_f|$, k(u) is given by

(2.9) $$k(u) = \frac{1}{|Y_f|} \int_{Y_s} \mathrm{div}\ u\ dy$$

Moreover, there exists a constant C such that

(2.10) $$|v|_{H^1(Y_f)} \leqslant C |u|_{W_6^{(1)}(Y)}$$

(2.11) $$|v|_{L_\infty(Y_f)}\ C(|u|_{L_\infty(\partial Y)} + \mathrm{Abs}(k(u)))$$

<u>Proof.</u> Everything was proved in [5], except (2.11). For this lest us consider a vector $\zeta \in C^\infty(\partial Y_f)$ such that

(2.12) $$\int_{\partial Y_f} \zeta . n d\sigma = |Y_f|$$

where n is the unit outward normal on Y_f.

Let us consider the system

(2.13) $-\Delta v_\zeta + \nabla q_\zeta = 0$ in Y_f

(2.14) $\mathrm{div}\ v_\zeta = 1$ in Y_f

(2.15) $v_\zeta = \zeta$ on ∂Y_f

This is a classical non-homogeneous Stokes problem. The compatibily condition is satisfied because of (2.12) and according to [2] it follows that there exists a unique $(v_\zeta, q_\zeta) \in H^1(Y_f) \times L^2(Y_f)/\mathbb{R}$, solution of the problem (2.13)-(2.15). Moreover, for any $\alpha > 1$ there exists a positive C (independent of ζ) such that

$$(2.16) \quad |v_\zeta|_{W_\alpha^{(2)}(Y_f)} + |q_\zeta|_{W_\alpha^{(1)}(Y_f)/\mathbb{R}} \leqslant C\,|\zeta|_{W_\alpha^{(2)}(\partial Y_f)}$$

If (v,q) is the solution of the problem (2.6)-(2.8) then it has the form

$$(2.17) \quad v = u + k(u) v_\zeta + \tilde{v}\,, \qquad q = k(u) q_\zeta + \tilde{q}$$

where $(\tilde{v}, \tilde{q})$ is the only solution of the problem

$$(2.18) \quad -\Delta\tilde{v} + \nabla\tilde{q} = 0 \qquad \text{in } Y_f$$

$$(2.19) \quad \operatorname{div} \tilde{v} = 0 \text{ in } \qquad Y_f$$

$$(2.20) \quad \tilde{v} = \Psi(u) := f(u) - u - k(u)\zeta \quad \text{on } \partial Y_f$$

Using properties (2.1)-(2.3) of f one can easily verify the compatibility condition of (2.18)-(2.20). Referring again to [2], we obtain $(\tilde{v}, \tilde{q}) \in H^1(Y_f) \times L^2(Y_f)/\mathbb{R}$.

Obviously, what we need more is an L_∞-estimate for $\tilde{v}$. Taking in account (2.4) we have

$$(2.21) \quad |\Psi(u)|_{L_\infty(\partial Y_f)} \leqslant C(|u|_{L_\infty(\partial Y)} + \operatorname{Abs}(k(u)))$$

For any $i \in \{1,2,3\}$ we denote

$$(2.22) \quad \alpha_i = \sup_{\partial Y_f} \operatorname{Abs}(\Psi_i(u))$$

where $\Psi_i(u)$ are the components of the vector valued function $\Psi(u)$.

Let us define $w=(w_1,w_2,w_3)$ by

$$(2.23) \qquad w_i = \max\left\{\tilde{v}_i - \alpha_i,\ 0\right\}$$

As $\sup_{\partial Y_f} \tilde{v}_i = \sup_{\partial Y_f} \Psi_i(u) \leqslant \alpha_i$, it results

$$(2.24) \qquad w \in H^1_0(\Omega)$$

The domain Y_f is divided in two sets (see [3] Ch. II for inequalities in the sense of H^1),

$$M_i = \left\{ y \in Y_f \mid w_i(y) > 0 \quad \text{in } H^1(Y_f) \right\}$$
$$N_i = \left\{ y \in Y_f \mid w_i(y) = 0 \quad \text{in } H^1(Y_f) \right\},$$

which are determined within a set of measure zero. Moreover, we have

$$(2.25) \qquad \frac{\partial w_i}{\partial y_j} = \begin{cases} \dfrac{\partial \tilde{v}_i}{\partial y_j} & \text{in } M_i \\ 0 & \text{in } N_i \end{cases}$$

Thus, from (2.19) it follows that

$$(2.26) \quad \operatorname{div} w = 0 \quad \text{in } Y_f$$

Now, we take the duality product of (2.18) by w; taking in account (2.24)-(2.26) it yields

$$(2.27)\qquad \sum_{i,j=1}^{3}\int_{Y_f}\frac{\partial \tilde{v}_i}{\partial y_j}\frac{\partial w_i}{\partial y_j}\,dy = \|w\|^2_{H^1_o(Y_f)} = 0$$

and consequently

$$(2.28)\qquad \tilde{v}_i \leqslant \alpha_i \qquad \text{a.e. on } Y_f.$$

Analogously, if we define w, instead of (2.23), by

$$(2.29)\qquad w_i = \min\left\{\tilde{v}_i + \alpha_i,\ 0\right\}$$

we obtain $\tilde{v}_i \geqslant -\alpha_i$ a.e. on Y_f. Hence

$$(2.30)\qquad |\tilde{v}_i|_{L_\infty(Y_f)} \leqslant \sup_{\partial Y_f} \mathrm{Abs}(\Psi_i(u)),\ (\forall)\ i\in\{1,2,3\}.$$

Finally, estimating v via (2.17) and using (2.16), (2.21) and (2.30), one can obtain without difficulties (2.11). □

THEOREM 2.1. For any $\varepsilon>0$ sufficiently small there exists a restriction operator $R_\varepsilon \in \mathcal{L}(\mathring{W}^{(1)}_6(\Omega),\ H^1_o(\Omega_\varepsilon))$ such that

$$(2.31)\qquad u=0 \quad \text{in } \Omega\setminus\Omega_\varepsilon \Rightarrow R_\varepsilon u = u$$

$$(2.32)\qquad \mathrm{div}\ u=0 \quad \text{in } \Omega \Rightarrow \mathrm{div}(R_\varepsilon u)=0.$$

Moreover, for any $u\in\mathring{W}^{(1)}_6(\Omega)$, there exists C independent of ε and u such that

$$(2.33)\qquad \|R_\varepsilon u\|_\varepsilon \leqslant C\varepsilon^{-1}(|u|_6 + \varepsilon|\nabla u|_6)$$

$$(2.34)\qquad |R_\varepsilon u|_{\infty,\varepsilon} \leqslant C(|u|_\infty + \varepsilon^{1/2}|\nabla u|_6)$$

Proof. In fact we have only to prove that the restriction operator defined in [5] satisfy (2.34). Nevertheless we remind here that definition.

First, let us notice that every εY-cube is of the form

$$\varepsilon Y^n = \prod_{i=1}^{3} [\varepsilon n_i, \ \varepsilon n_i + \varepsilon[$$

with $n=(n_1,n_2,n_3)\in\mathbb{Z}^3$ and that the εY-cubes which intersect Ω can be indexed following

$$\mathbb{Z}_\varepsilon = \{ n\in\mathbb{Z}^3 \mid \varepsilon Y^n \cap \Omega \neq \emptyset \}$$

For any $u\in W_6^{(1)}(\Omega)$ we set

$$(2.35) \qquad R_\varepsilon u = 0 \qquad \text{in} \quad \Omega\setminus\Omega_\varepsilon$$

$$(2.36) \qquad R_\varepsilon u = v^n \circ \varphi_\varepsilon \qquad \text{in} \quad \varepsilon Y_f^n$$

where v^n is given by Lemma 2.2 for the datum

$$u_\varepsilon^n = \begin{cases} u(\varepsilon n+\varepsilon(.))\in W_6^{(1)}(Y) & \text{if } \varepsilon Y_f^n \subseteq \Omega \\ u(\varepsilon n+\varepsilon(.)) \text{ continued by zero} & \text{if } \varepsilon Y_f^n \cap \partial\Omega \neq \emptyset. \end{cases}$$

Thus, it follows straightly

$$|R_\varepsilon u|_{\infty,\varepsilon} \leqslant \sup_{n\in\mathbb{Z}_\varepsilon} |v^n \circ \varphi_\varepsilon|_{L_\infty(\varepsilon Y_f^n)} = \sup_{n\in\mathbb{Z}_\varepsilon} |v^n|_{L_\infty(Y_f)}$$

According to (2.11) it yields

$$(2.37) \quad |R_\varepsilon u|_{\infty,\varepsilon} \leqslant C \sup_{n\in\mathbb{Z}_\varepsilon} \left(|u_\varepsilon^n|_{L_\infty(\partial Y)} + \mathrm{Abs}(k(u_\varepsilon^n)) \right)$$

Next, let us evaluate $|u_\varepsilon^n|_{L_\infty(\partial Y)}$ and $\mathrm{Abs}(k(u_\varepsilon^n))$ using the change of variables:

(2.38) $\quad x = \varepsilon n + \varepsilon y, \qquad y \in Y.$

In this way we obtain

$$|u_\varepsilon^n|_{L_\infty(\partial Y)} \leqslant |u|_{L_\infty(\varepsilon Y^n)}$$

$$\int_{Y_s} \operatorname{div} u_\varepsilon^n(y)\,dy = \frac{1}{\varepsilon^2}\int_{\varepsilon Y_s^n} \operatorname{div} u(x)\,dx \leqslant \frac{1}{\varepsilon^2}|\varepsilon Y_s^n|^{5/6}|\nabla u|_{L_6(\varepsilon Y_s^n)}$$

which imply

$$\sup_{n\in \mathbb{Z}_\varepsilon} |u_\varepsilon^n|_{L_\infty(\partial Y)} \leqslant |u|_\infty$$

$$\sup_{n\in \mathbb{Z}_\varepsilon} \mathrm{Abs}(k(u_\varepsilon^n)) \leqslant C\,\varepsilon^{1/2}|\nabla u|_6$$

and the inequality (2.34) is proved via (2.37). □

3. Convergence of the homogenization process for $G_\varepsilon = o(\varepsilon^{-3/2})$

Recalling (1.14), throughout this section we assume the following two hypothesis:

(3.1) $\quad \lim_{\varepsilon\to 0} \varepsilon^{3/2} G_\varepsilon = 0$

(3.2) $\quad (\exists)\ g \in H$ such that $g_\varepsilon / |g_\varepsilon| \to g$ strongly in H.

REMARK 3.1. $|g| = 1$. □

Introducing the hilbertian space

$$(3.3)\quad W=\left\{w\in H^1(Y_f)\,\middle|\, w|_{\Gamma}=0,\ \text{div } w=0,\ w \text{ is Y-periodic}\right\}$$

with the scalar product

$$(3.4)\quad ((u,v))_W = \sum_{i,j=1}^{3}\int_{Y_f}\frac{\partial u_i}{\partial y_j}\frac{\partial v_i}{\partial y_j}\,dy$$

we can formulate for any $k\in\{1,2,3\}$ the so-called local problem:

To find $v^{(k)}\in W$ such that

$$(3.5)\quad ((v^{(k)},w))_W = \int_{Y_f} w_k(y)\,dy,\quad (\forall)\ w\in W$$

where w_k is the k-component of w.

By the Lax-Milgram theorem, there exists a unique $v^{(k)}\in W$, solution of the problem (3.5). Further, one can easily prove that there exists a unique $q^{(k)}\in L^2(Y_f)/\mathbb{R}$ such that

$$(3.6)\quad -\Delta v^{(k)}+\nabla q^{(k)}=e^{(k)}\quad (\text{in the distribution sense in } Y_f)$$

where $e^{(k)}$ is the unit vector of the k-axis.

Moreover, the regularity theorem for the Stokes problem (see [9] Ch.I) implies $v^{(k)}\in H^2(Y_f)$ and $q^{(k)}\in H^1(Y_f)/\mathbb{R}$. Also, as in [6] Ch.7 there is a proof of the Y-periodicity of $q^{(k)}$, we have in conclusion.

$$(3.7)\quad v^{(k)}\in W\cap H^2(Y_f)\ \text{and}\ q^{(k)}\in H^1_{per}(Y_f)/\mathbb{R}$$

Our convergence result is the following:

<u>THEOREM 3.1.</u> If $(u_\varepsilon, p_\varepsilon)$ is a weak solution of the problem (1.1)-(1.3) and if we consider u_ε continued to Ω with value zero out of Ω_ε, then there exists a continuation of p_ε to Ω (denoted with $\tilde{p}_\varepsilon$) such that

(3.8) $\quad \nu_\varepsilon u_\varepsilon / \varepsilon^2 |g_\varepsilon| \longrightarrow u \qquad$ weakly in $L^2(\Omega)$

(3.9) $\quad \tilde{p}_\varepsilon / |g_\varepsilon| \longrightarrow p \qquad$ strongly in $L^{6/5}(\Omega)$

where $(u,p) \in H \times L^{6/5}(\Omega)/\mathbb{R}$ satisfy in the distribution sense in Ω the Darcy equation

(3.10) $\quad u = K(g - \nabla p)$

the homogenized (3x3)- tensor K being defined by

(3.11) $\quad K_{ij} = \int_{Y_f} v_j^{(i)}(y)\,dy \quad (v^{(i)}$ given by (3.5)).

<u>Proof.</u> From (1.13) it results that $\{\nu_\varepsilon u_\varepsilon / \varepsilon^2 |g_\varepsilon|\}_\varepsilon$ is bounded in $L^2(\Omega)$; hence there exists $u \in H$ for which, passing just in case to a subsequence, the convergence (3.8) holds (the fact that the convergence holds on the whole sequence it will be proved by the uniqueness property of the Darcy problem).

Since (1.2) is satisfied in $H^{-1}(\Omega_\varepsilon)$, using R_ε, the operator given by Theorem 2.1, we have for any $v \in \overset{\circ}{W}{}_6^{(1)}(\Omega)$::

(3.12) $\langle \nabla p_\varepsilon , R_\varepsilon v \rangle = -\nu_\varepsilon((u_\varepsilon, R_\varepsilon v))_\varepsilon - b_\varepsilon(u_\varepsilon, u_\varepsilon, R_\varepsilon v) + (g_\varepsilon, R_\varepsilon v)_\varepsilon$

Taking in account (1.6), (1.12) and (1.13) it yields

$$|\langle \nabla p_\varepsilon , R_\varepsilon v\rangle| \leqslant \nu_\varepsilon \|u_\varepsilon\|_\varepsilon \|R_\varepsilon v\|_\varepsilon + |u_\varepsilon|_\varepsilon \|u_\varepsilon\|_\varepsilon |R_\varepsilon v|_{\infty,\varepsilon} +$$

$$+ |g_\varepsilon| \, |R_\varepsilon v|_\varepsilon \leqslant C |g_\varepsilon| (\varepsilon \|R_\varepsilon v\|_\varepsilon + \varepsilon^{3/2} G_\varepsilon |R_\varepsilon v|_{\infty,\varepsilon})$$

Considering also the properties (2.33) and (2.34) of R_ε, we receive

(3.13) $$|\langle \nabla p_\varepsilon , R_\varepsilon v\rangle| \leqslant C |g_\varepsilon| (|v|_6 + \varepsilon |\nabla v|_6 + \varepsilon^{3/2} G_\varepsilon |v|_\infty + \varepsilon^2 G_\varepsilon |\nabla v|_6)$$

where C is independent of ε and v.

Thus we found that the functional

$$F_\varepsilon(.) = \langle \nabla p_\varepsilon , R_\varepsilon(.)\rangle_{\langle H^{-1}, H_0^1\rangle(\Omega_\varepsilon)}$$

is bounded on $\mathring{W}_6^{(1)}(\Omega)$, that is $F_\varepsilon \in W_{6/5}^{(-1)}(\Omega)$. If we continue $v \in \mathring{W}_6^{(1)}(\Omega_\varepsilon)$ with value zero in $\Omega \setminus \Omega_\varepsilon$, from property (2.31) of R_ε it results

(3.14) $$F_\varepsilon \big|_{\Omega_\varepsilon} = \nabla p_\varepsilon$$

Moreover, whenever div v=0, (2.32) implies

$$\langle F_\varepsilon , v\rangle_{\langle W_{6/5}^{(-1)}, \mathring{W}_6^{(1)}\rangle(\Omega)} = 0$$

and hence $(\exists)$ $\tilde{p}_\varepsilon \in \mathcal{D}'(\Omega)$ such that

(3.15) $$\nabla \tilde{p}_\varepsilon = F_\varepsilon \in W_{6/5}^{(-1)}(\Omega)$$

Refering to Corollary 8.12 of [7] it follows

(3.16) $\tilde{p}_\varepsilon \in L^{6/5}(\Omega)/\mathbb{R}$

and comparing this result with (3.14) we see that

(3.17) $\tilde{p}_\varepsilon$ is a continuous of p_ε

Also from (3.13) we obtain

$$\left|\nabla\tilde{p}_\varepsilon\right|_{W^{(-1)}_{6/5}(\Omega)} \leqslant c\left|g_\varepsilon\right|$$

for ε sufficiently small. Consequently, using the inequality

(3.18) $\left|\tilde{p}_\varepsilon\right|_{L^{6/5}(\Omega)/\mathbb{R}} \leqslant c(\Omega)\left|\nabla\tilde{p}_\varepsilon\right|_{W^{(-1)}_{6/5}(\Omega)}$

(see [5] Remark 2 and [4]) we find that the sequence $\left\{\tilde{p}_\varepsilon / |g_\varepsilon|\right\}_\varepsilon$ is bounded in $L^{6/5}(\Omega)/\mathbb{R}$ and therefore there exists $p \in L^{6/5}(\Omega)/\mathbb{R}$ such that on some subsequence

(3.19) $\tilde{p}_\varepsilon / |g_\varepsilon| \rightharpoonup p$ weakly in $L^{6/5}(\Omega)/\mathbb{R}$

(3.20) $\nabla\hat{p}_\varepsilon / |g_\varepsilon| \rightharpoonup \nabla p$ weakly in $W^{(-1)}_{6/5}(\Omega)$

Let us notice that for any $w^\varepsilon \rightharpoonup w$ weakly in $\mathring{W}^{(1)}_6(\Omega)$, via (3.13), we have

$$\begin{aligned}
&\left|\langle\nabla\tilde{p}_\varepsilon/|g_\varepsilon|, w_\varepsilon\rangle - \langle\nabla p, w\rangle\right| \leqslant \\
&\leqslant \left|\langle\nabla\tilde{p}_\varepsilon/|g_\varepsilon|, w_\varepsilon - w\rangle\right| + \left|\langle\nabla\hat{p}_\varepsilon/|g_\varepsilon| - \nabla p, w\rangle\right| \leqslant \\
&\leqslant c\Big[|w_\varepsilon - w|_6 + \varepsilon|\nabla w_\varepsilon - \nabla w|_6 + \varepsilon^{3/2}G_\varepsilon\big(|w_\varepsilon - w|_\infty + \\
&\qquad + \varepsilon^{1/2}|\nabla w_\varepsilon - \nabla w|_6\big)\Big] + \text{(term which} \to 0)
\end{aligned}$$

Taking in account (3.1) and the corresponding compactness theorems we obtain

$$\langle \nabla \tilde{p}_\varepsilon / |g_\varepsilon| , w_\varepsilon \rangle \longrightarrow \langle \nabla p , w \rangle$$

which means

(3.21) $\nabla \tilde{p}_\varepsilon / |g_\varepsilon| \longrightarrow \nabla p$ strongly in $W^{(-1)}_{6/5}(\Omega)$.

Recalling (3.18), from (3.21) we receive (3.9).

Resuming, it remains to prove that (u,p) satisfies (3.10); for this we apply a standard method.

Seting $v_\varepsilon = v^{(i)} \circ \varphi_\varepsilon$ and $q_\varepsilon = q^{(i)} \circ \varphi_\varepsilon$ we write (3.6) in terms of $x = \varepsilon y$

(3.22) $$-\varepsilon^2 \Delta v_\varepsilon + \varepsilon \nabla q_\varepsilon = e^{(i)} .$$

Because $v^{(i)}$ and $q^{(i)}$ are independent of ε, by straight estimations we obtain

(3.23) $$\|v_\varepsilon\|_\varepsilon \leq c\, \varepsilon^{-1} ,\ |v_\varepsilon|_{\infty,\varepsilon} \leq c \ \text{ and } \ |q_\varepsilon|_\varepsilon \leq c$$

where c is independent of ε.

Let $\phi \in \mathcal{D}(\Omega)$; making the duality product of (1.2) and (3.22) by $\phi v_\varepsilon / |g_\varepsilon|$ and respectively $\phi \gamma_\varepsilon u_\varepsilon / \varepsilon^2 |g_\varepsilon|$, by subtraction we get

(3.24) $$\frac{\gamma_\varepsilon}{|g_\varepsilon|} \sum_{k=1}^{3} \left(v_\varepsilon \frac{\partial u_\varepsilon}{\partial x_k} - u_\varepsilon \frac{\partial v_\varepsilon}{\partial x_k} , \frac{\partial \phi}{\partial x_k} \right) + \frac{1}{|g_\varepsilon|} b_\varepsilon (u_\varepsilon , u_\varepsilon , \phi v_\varepsilon) + \frac{\gamma_\varepsilon}{\varepsilon |g_\varepsilon|} (q_\varepsilon , u_\varepsilon \nabla \phi) =$$
$$= (g_\varepsilon / |g_\varepsilon| , \phi v_\varepsilon) + (\hat{p}_\varepsilon / |g_\varepsilon| , v_\varepsilon \nabla \phi) - (\gamma_\varepsilon (u_\varepsilon)_i / \varepsilon^2 |g_\varepsilon| , \phi)$$

where $(u_\varepsilon)_i$ is the i-component of u_ε.

According to (1.12)-(1.13) and (3.23) we have

$$(3.25)\quad \frac{\vartheta_\varepsilon}{|g_\varepsilon|}\left|\left(v_\varepsilon \frac{\partial u_\varepsilon}{\partial x_k} - u_\varepsilon \frac{\partial v_\varepsilon}{\partial x_k}, \frac{\partial \phi}{\partial x_k}\right)\right| \leqslant \frac{c\,\vartheta_\varepsilon}{|g_\varepsilon|}\left(|v_\varepsilon|_\varepsilon \|u_\varepsilon\|_\varepsilon + |u_\varepsilon|_\varepsilon \|v_\varepsilon\|_\varepsilon\right) \leqslant c\,\varepsilon$$

$$(3.26)\quad \frac{1}{|g_\varepsilon|}\left|b_\varepsilon(u_\varepsilon, u_\varepsilon, \phi v_\varepsilon)\right| \leqslant \frac{c}{|g_\varepsilon|}|u_\varepsilon|_\varepsilon \|u_\varepsilon\|_\varepsilon |v_\varepsilon|_{\infty,\varepsilon} \leqslant c\,\varepsilon^{3/2} G_\varepsilon$$

$$(3.27)\quad \frac{\vartheta_\varepsilon}{\varepsilon|g_\varepsilon|}\left|(q_\varepsilon, u_\varepsilon \nabla\phi)\right| \leqslant \frac{c\,\vartheta_\varepsilon}{\varepsilon|g_\varepsilon|}|u_\varepsilon|_\varepsilon |q_\varepsilon|_\varepsilon \leqslant c\,\varepsilon$$

As by the classical lemma on Y-periodic functions we have also

$$(3.28)\quad (v_\varepsilon)_j \rightharpoonup K_{ij} \quad \text{weakly star in } L_\infty(\Omega)$$

where $(v_\varepsilon)_j$ is the j-component of v_ε, then passing (3.24) to the limit we find that u and p satisfy (3.10) in the distribution sense in Ω. □

<u>REMARK 3.2.</u> The tensor K is symmetric and positively defined (see [6] Ch.7). □

4. The macroscopic problem in the transition case $G_\varepsilon = 0(\varepsilon^{-3/2})$

Throughout this section we assume

$$(4.1)\quad (\exists)\, g\in H \text{ such that } g_\varepsilon = \varepsilon^{-3}\vartheta_\varepsilon^2 g \;,$$

from which follows obviously $\varepsilon^{3/2} G_\varepsilon = |g|$.

Reconsidering (1.13), in the present case we obtain that $\{\varepsilon u_\varepsilon / \vartheta_\varepsilon\}_\varepsilon$ is bounded in $L^2(\Omega)$; hence there exists $u\in H$ for which, on some subsequence, it holds

(4.2) $\varepsilon u_\varepsilon / \nu_\varepsilon \longrightarrow u$ weakly in $L^2(\Omega)$.

Also, with the techniques used in the first part of the proof of Theorem 3.1 we can prove that there exists $\tilde{p}_\varepsilon$, a continuation of p_ε to Ω , such that

(4.3) $\varepsilon^3 \tilde{p}_\varepsilon / \nu_\varepsilon^2 \longrightarrow p$ strongly in $L^{6/5}(\Omega)$.

REMARK 4.1. As the energetic method of proving the convergence of the homogenization process seems to fail in this case, we search for asymptotic expansions of u_ε and p_ε as $\varepsilon \to 0$. The heuristic device is to suppose that u_ε and p_ε have two-scale expansions of the form

(4.4) $u_\varepsilon(x) = \varepsilon^{-1}\nu_\varepsilon (u_o(x,y) + \varepsilon u_1(x,y) + \ldots)$

(4.5) $p_\varepsilon(x) = \varepsilon^{-3}\nu_\varepsilon^2 (p_o(x,y) + \varepsilon p_1(x,y) + \ldots)$

where $y = x/\varepsilon$ and the functions $u_k(x,y)$ and $p_k(x,y)$ are Y-periodic in the variable y; their total dependence with respect to x is obtained by the rule

(4.6) $$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}$$

Further on, we consider the "partial" problems obtained by collecting together the terms with the same power of ε in (1.1)-(1.3), as a result of the substitutions (4.4) and (4.5) governed by the rule (4.6).

At the lowest power we receive

(4.7) $$p_o = p_o(x)$$

At the next level we find the so-called local problem:

(4.8) $$\mathrm{div}_y\, u_o=0 \quad \text{in } Y_f$$

(4.9) $$(u_o\nabla_y)u_o-\Delta_y u_o=-\nabla_y p_1+(g-\nabla_x p_o) \quad \text{in } Y_f$$

(4.10) $$u_o=0 \quad \text{on } \Gamma$$

in which x has to be considered as a parameter and $(g-\nabla_x p_o)$ as the given force. Reminding that u_o is Y-periodic in the variable y, it follows that the problem (4.8)-(4.10) is equivalent to the homogeneous Navier-Stokes problem on a torus; hence it has at least one solution (see [9] Ch. II).

If we define the mean value of $w \in W$ by

(4.11) $$\tilde{w} = \int_{Y_f} w(y)\,dy$$

and if we denote by $v(y, g-\nabla_x p_o)$ some solution of the problem (4.8)-(4.10), then the nonlinear version of the Darcean law (3.10) is

(4.12) $$\tilde{u}_o=\tilde{v}(g-\nabla_x p_o) \quad \text{in } \Omega \ . \qquad \square$$

The considerations of Remark 4.1 suggests us the following macroscopic problem.

<u>CONJECTURE</u>. The limits $u \in H$ and $p \in L^{6/5}(\Omega)$ of (4.2) and (4.3) satisfy the equation

(4.13) $$u = \tilde{v} \quad \text{in } \Omega$$

where $v \in W$ is some solution of the problem

(4.14) $$((v,w))_W + b(v,v,w) = (g - \nabla_x p, w), \quad (\forall) w \in W$$

the trilinear continuous form b being defined by

(4.15) $$b(u,v,w) = \sum_{i,j=1}^{3} \int_{Y_f} \left(u_i \frac{\partial v_j}{\partial y_i} w_j\right) dy, \quad (\forall)\ u,v,w \in W. \quad \square$$

References

[1] ADAMS R.A., Sobolev Spaces, Academic Press, New York (1975)

[2] CATTABRIGA L., Su un problema al contorno relativo al sistema di equazioni di Stokes, Rend.Mat.Sem.Univ.Padova, 31, 308-340(1961)

[3] KINDERLEHRER D. and STAMPACCHIA G., An Introduction to Variational Inequalities and Their Applications, Academic Press, New York (1980)

[4] NEČAS, J., Equations aux dérivées partielles, Les Presses de l'Université de Montreal, 103-126(1966)

[5] POLIŠEVSKI D., On the Homogenization of Fluid Flows through Periodic Media, Rend.Sem.Mat.Univers. Politecn. Torino, 44(3), 383-393(1986)

[6] SANCHEZ-PALENCIA E., Non-Homogeneous Media and Vibration Theory, Lecture Notes in Physics, 127, Springer-Verlag, Berlin (1980).

[7] SMITH K.T., Formulas to Represent Functions by Their Derivatives, Math.Ann., 188, 53-77(1970)

[8] TARTAR L., Incompressible Fluid Flow in a Porous Medium-Convergence of the Homogenization Process, Appendix of [6]

[9] TEMAM R., Navier-Stokes Equations - Theory and Numerical Analysis, North-Holland, Amsterdam (1977).

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