INSTITUTUL
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MATEMATICA

INSTITUTUL NATIONAL
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STIINTIFICA SI TEHNICA

ISSN 0250 3638

HYPERBOLIC SYSTEMS OF QUASILINEAR CONSERVATION LAWS

1. THE RIEMANN PROBLEM

by

Liviu DINU

PREPRINT SERIES IN MATHEMATICS
No. 38/1987

BUCURESTI

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1. THE RIEMANN PROBLEM

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November 1987

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HYPERBOLIC SYSTEMS OF QUASILINEAR CONSERVATION LAWS
1. THE RIEMANN PROBLEM

This work is occasioned by the 30th anniversary of the fundamental paper

Peter D. Lax, "Hyperbolic systems of conservation laws, II"

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1. The nonlinearity hierarchy. Hyperbolicity.

We consider the system ($u \in \mathbb{R}^n$; two independent variables x,t)

$$\frac{\partial u}{\partial t} + A(x,t,u)\frac{\partial u}{\partial x} = B(x,t,u) \qquad (1)$$

with A and B given in a convenient region $\mathcal{R}$ (see §4).

DEFINITION 1. The system (1) is called linear if A does not depend on u and B depends linearly on u (i.e. if the dependence on u is linear), semilinear if A does not depend on u and B depends nonlinearly on u, quasilinear if A depends on u (particularly, if $A=A(u)$).

The NONLINEARITY HIERARCHY:

$$\text{LINEAR SYSTEM} \prec \text{SEMILINEAR S.} \prec \text{QUASILINEAR S.} \prec \text{NONLINEAR S.} \qquad (2)$$

The displacement (from left to right) in the hierarchy (2) is related to the raising of the nonlinearity level.

D2. The semilinear/quasilinear system (1) is called (completely) hyperbolic in the point (x_o, t_o)/respectively (x_o, t_o, u_o) of $\mathcal{R}$ if in this point the eigenvalues λ of the matrix $A(x,t)$/respectively $A(x,t,u)$ are all real (strictly hyperbolic if they are also distinct).

REMARK 1. If the system (1) is strictly hyperbolic then the matrix A is diagonalizable (let P/respectively P^{-1} be the matrix whose columns/rows are right/left eigenvectors of A; then, $P^{-1}AP=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$).

2. Characteristic form of a hyperbolic system.

THEOREM 1. If the strictly hyperbolic system (1) is semilinear then it can be put in a characteristic form.

◀ We define

$$v = P^{-1}u \qquad (3)$$

and put (1) in the form (see R1)

$$\frac{\partial v}{\partial t} + T\frac{\partial v}{\partial x} = C \qquad (4)$$

$$T=P^{-1}AP, \quad C=P^{-1}B+\left(\frac{\partial}{\partial t}P^{-1}+T\frac{\partial}{\partial x}P^{-1}\right)Pv, \qquad (5)$$ ▶

T2 (R. Courant, P. Lax, [10]). If the strictly hyperbolic sistem (1) is quasilinear then it can be put in a characteristic form.

◀ We suppose A^{-1} exists (a nonessential restriction). Since P^{-1} depends on u we cannot use (3) any more and then proceed in two stages.

At first we define $\tilde{u}$ by

$$\frac{\partial}{\partial t} u = P\tilde{u} \tag{6}$$

so that (1) gives (here we need A^{-1})

$$\frac{\partial}{\partial x} u = A^{-1}(B - P\tilde{u}) \tag{7}$$

Then we require $u \in C^2$ and use $\frac{\partial^2}{\partial x \partial t} u = \frac{\partial^2}{\partial t \partial x} u$ to obtain

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial t} + (P^{-1}AP)\frac{\partial \tilde{u}}{\partial x} = {} & P^{-1}A\{-\frac{\partial P}{\partial x} + ([A^{-1}(P\tilde{u}-B)].\mathrm{grad}_u)P\}\tilde{u} + \\ & +P^{-1}\{-\frac{\partial A}{\partial t} + [(P\tilde{u}).\mathrm{grad}_u]A\}A^{-1}(P\tilde{u}-B) + \\ & +P^{-1}\{\frac{\partial B}{\partial t} + [(P\tilde{u}).\mathrm{grad}_u]B\} - \\ & -P^{-1}\{\frac{\partial P}{\partial t} + [(P\tilde{u}).\mathrm{grad}_u]P\}\tilde{u}\end{aligned} \tag{8}$$

Next we consider the vector

$$v = (u_1, \ldots, u_n, \tilde{u}_1, \ldots, \tilde{u}_n)^t \tag{9}$$

(here (^{t}) indicates the transposition). We have (according to (6), (8))

$$\frac{\partial v}{\partial t} + T\frac{\partial v}{\partial x} = C \tag{10}$$

where T is a diagonal matrix whose first n diagonal elements are equal to zero. ▶

R2. The raising of the nonlinearity level (see (2)) has no echo in the formulations T1, T2/ the hierarchy(2) does not filter the property stated by T1, T2.

3. Riemann invariants ([55]). The Riemann form of a hyperbolic system.

R3. For a semilinear system (1) the characteristic form (4) is a Riemann form too. The Riemann invariants (abbreviated RI) $v_1(u), \ldots, v_n(u)$ are given by (3). The invariance (along the characteristics) is allowed to be manifest by the typical case A=constant, B=0 in (1).

R4. For a quasilinear system (1), the characteristic form (10) is not a Riemann form. As $A=A(u)$, $n>2$, a Riemann form can result under only certain restrictions.

Let

$$\overset{1}{R}(u),\ldots,\overset{n}{R}(u);\ \overset{1}{L}(u),\ldots,\overset{n}{L}(u);\ \lambda_1(u),\ldots,\lambda_n(u) \tag{11}$$

be the eigenelements (right eigenvectors, left eigenvectors, eigenvalues) of the matrix $A(u)$. According to the assumed strict hyperbolicity the eigenvectors R/respectively L are independent and the eigenvalues λ are distinct.

Requirement/restriction: the forms

$$\sum_{j=1}^{n} \overset{i}{L}_j(u)\,du_j\ , \qquad 1\le i\le n \tag{12}$$

should be integrable.

Let us suppose, for exemple (see the item 5.1 hereinbelow), that each of the forms (12) has an integrating factor α_i:

$$\alpha_i(u)\overset{i}{L}_j(u)=\frac{\partial}{\partial u_j}v_i(u), \qquad 1\le i\le n \tag{13}$$

We call $v_i(u)$, $1\le i\le n$, Riemann invariants. There are two reasons for it.

First of all, taking (13) into account we obtain succesively

$$\alpha_i\overset{i}{L}\left[\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}\right]=\alpha_i\overset{i}{L}\,B$$

$$\alpha_i\overset{i}{L}\left[\frac{\partial u}{\partial t}+\lambda_i(u)\frac{\partial u}{\partial x}\right]=\alpha_i\overset{i}{L}\,B$$

$$\frac{\partial}{\partial t}v_i+\bar{\lambda}_i(v)\frac{\partial}{\partial x}v_i=\alpha_i\overset{i}{L}\,B,\quad 1\le i\le n \tag{14}$$

where we denote $\bar{\lambda}_i(v)\equiv\lambda_i[u(v)]$. The invariance (along the characteristics) is allowed to be manifest by the typical case $A=A(u)$, $B\equiv 0$.

On the other hand the relation (13) extends the relation (3) in an obvious sense.

As the nonlinearity level rises, the possibility of finding a Riemann form for the system (1) is filtered by the hierarchy (2).

4. A review of the assumptions.

We consider the hodograph space $H=\{u\,|\,u\in\mathbb{R}^n\}$ and the physical plane $E=\{(x,t)\,|\,(x,t)\in\mathbb{R}^2\}$ and study, in the sequel, the initial value problem

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=0,\quad -\infty<x<\infty,\quad t>0 \tag{15}$$

$$u(x,0)=u_0(x),\quad -\infty<x<\infty \tag{16}$$

where the system (15) is strictly hyperbolic and, on an open, bounded, simply connected region $\mathcal{R}\subset H$, we have $A\in C^m(\mathcal{R})$, $m\ge 1$. The points of the hodograph space will (often) be called states.

5. Consequences in the hodograph space of the assumptions considered.

5.1. Existence of the Riemann invariants. [26]

We denote

$$L(u)=(\overset{i}{L}_j(u)) \overset{R1}{=} P^{-1}(u) \tag{17}$$

PROPOSITION 1. If $L\varepsilon C^1(\mathcal{R})$ and in $\mathcal{R}$ the conditions

$$\frac{\partial \overset{i}{L}_j}{\partial u_m} - \frac{\partial \overset{i}{L}_m}{\partial u_j} = 0, \qquad 1\leq i,j,m\leq n \tag{18}$$

are fulfiled, then for every pair (u_o, v_o), $u_o\varepsilon\mathcal{R}$, $v_o\varepsilon\mathbb{R}^n$, the solution $v(u;u_o,v_o)$ of the problem

$$dv-L(u)du=0, \qquad v(u_o)=v_o \tag{19}$$

exists (uniquely) in the whole region $\mathcal{R}$ and $v\varepsilon C^1(\mathcal{R})$.

R5. If the conditions (18) are not fulfilled an analoqous result can be obtained when an integrating factor exists for each of the forms (12) (as n=3 the integrability conditions are written in this case $\overset{i}{L}.\text{rot}_u\overset{i}{L}=0$, $1\leq i\leq 3$).

5.2. Riemann-Lax invariants([36])

Let us consider the autonomous system

$$u' = \overset{i}{R}(u) \tag{20}$$

where $\overset{i}{R}$ is an element of (11), $\overset{i}{R}\varepsilon C^1(\mathcal{R})$.

R6. Since $\mathcal{R}$ does not contain critical points, all the results corresponding to the nonautonomous systems (existence, uniqueness, continuation of the solution) keep valid. As $\overset{i}{R}\varepsilon C^1(\mathcal{R})$, through each point of $\mathcal{R}$ an orbit passes which goes from bord to bord in $\mathcal{R}$.

D3. We say that a nonconstant function $\varphi(u)$, which is C^1 in $\mathcal{R}_o\subset\mathcal{R}$, is a first integral in $\mathcal{R}_o$ for the system (20) if it keeps constant along each orbit included in $\mathcal{R}_o$ of the system (20) (the constant depends on orbit).

D4. We say that a nonconstant function $\varphi(u)$, which is C^1 in $\mathcal{R}_o\subset\mathcal{R}$, is a Riemann-Lax invariant of index i (abbreviated i-RLI) in $\mathcal{R}_o$ if it satisfies in $\mathcal{R}_o$ the equation

$$\overset{i}{R}(u).\text{grad}_u\varphi(u)=0 \tag{21}$$

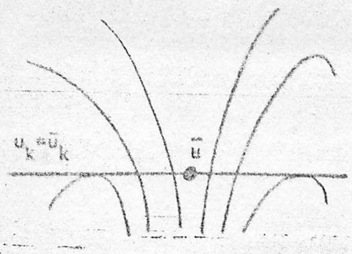
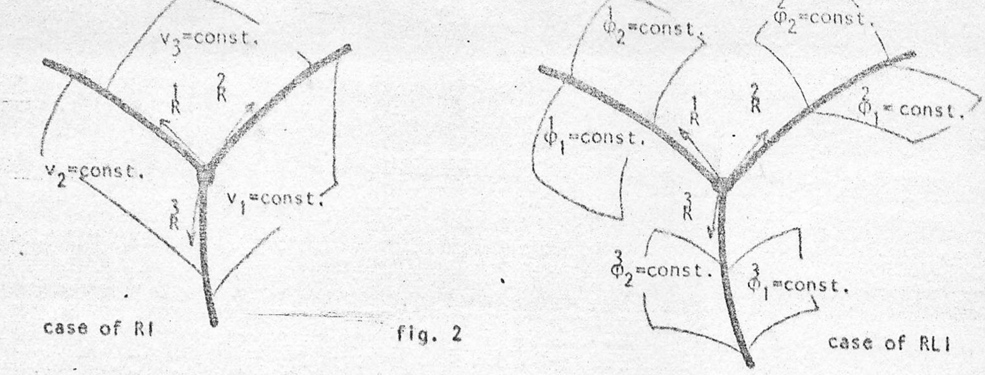


fig. 1



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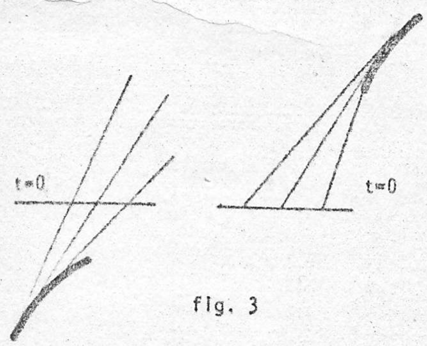

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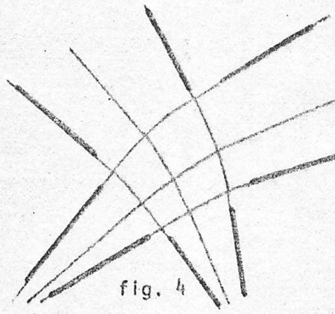

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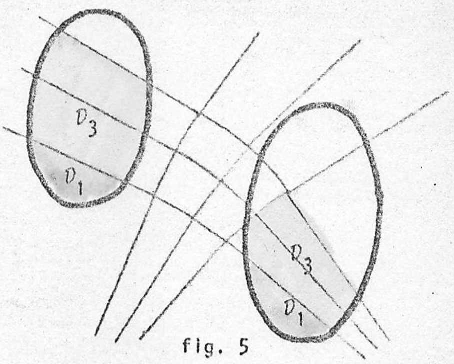


fig. 5

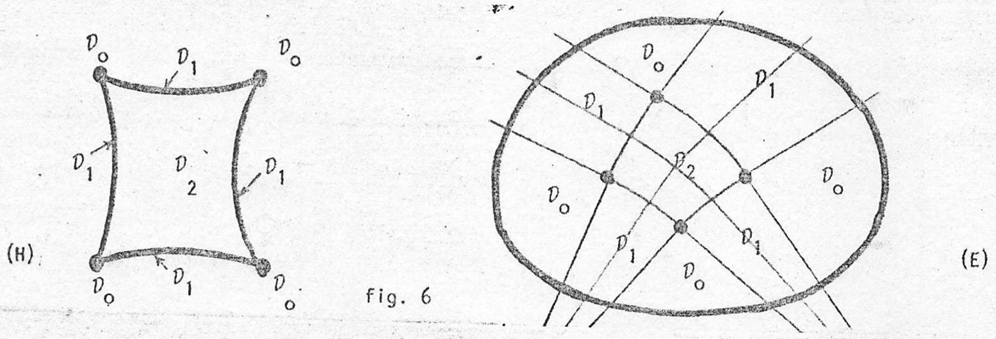


fig. 6

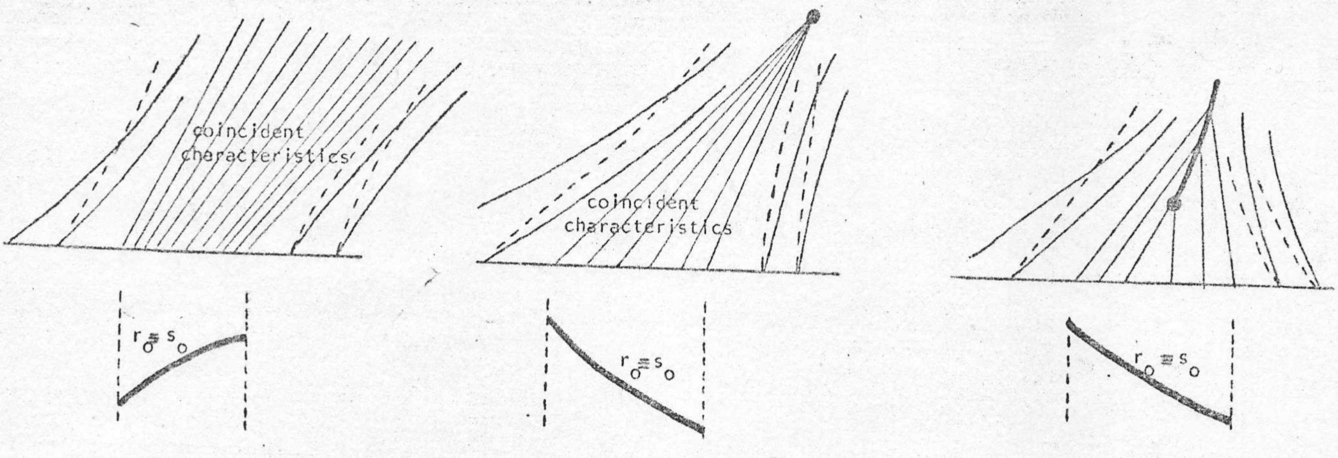


fig. 7

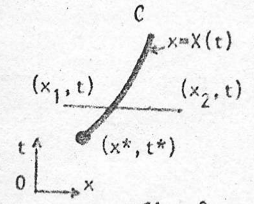


fig. 8

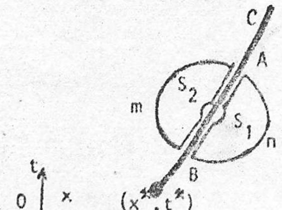


fig. 9

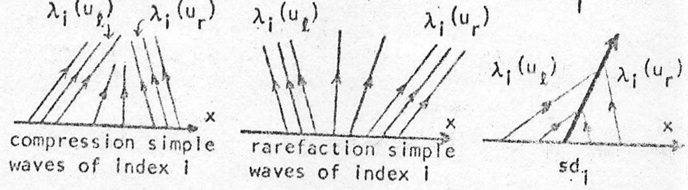
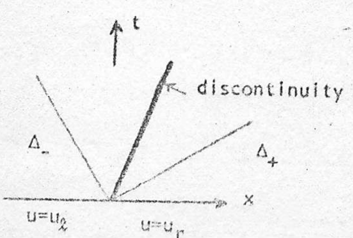
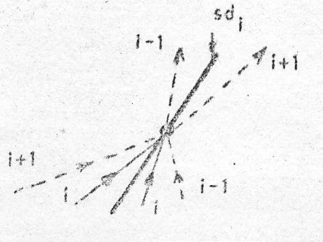
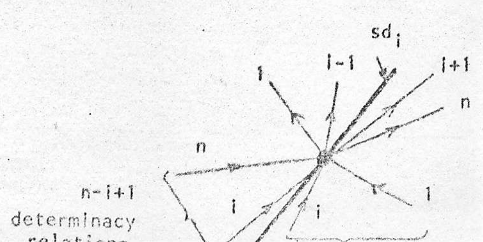


fig. 10







P2. The function $\varphi(u)$ is an i-RLI in $\mathcal{R}_o$ iff it is a first integral in $\mathcal{R}_o$ of the system (20).

Since $\mathcal{R}$ does not contains critical points of the system (20), we have ([2]).

P3. (i) There exist exactly n-1 independent i-RLI, $\overset{i}{\varphi}_1(u),\ldots,\overset{i}{\varphi}_{n-1}(u)$, in a neighbourhood $U(\bar{u})$ of every point $\bar{u}\varepsilon\mathcal{R}$ [1],

(ii) The general solution of (21) can be represented as

$$\varphi=\mathcal{F}[\overset{i}{\varphi}_1(u),\ldots,\overset{i}{\varphi}_{n-1}(u)], \qquad u\varepsilon W\subset U(\bar{u})$$

where $\mathcal{F}$ is an arbitrary C^1 function {defined in a neighbourhood V of the point $[\overset{i}{\varphi}_1(\bar{u}),\ldots$ $\ldots,\overset{i}{\varphi}_{n-1}(\bar{u})]$}.

R7. Since u is not a critical point of the system (20), let $\overset{i}{R}_k(\bar{u})\neq 0$ (fig.1). This circumstance (on which the construction implied in the proof of P3 depends) is generally local: in another point, $\bar{\bar{u}}$, we could have $\overset{i}{R}_j(\bar{\bar{u}})\neq 0\,(k\neq j)$, $\overset{i}{R}_k(\bar{\bar{u}})=0$. Henceforth the local statement of the proposition P3.

Here is an example in which U coincides with $\mathcal{R}$.

EXAMPLE 1. In the adiabatic gasdynamics (in Eulerian coordinates; in the usual notations) the system (15) has the form

$$\begin{aligned}&\frac{\partial\rho}{\partial t}+\frac{\partial m}{\partial x}=0\\&\frac{\partial m}{\partial t}+\frac{\partial}{\partial x}\left(\frac{m^2}{\rho}+p\right)=0\\&\frac{\partial}{\partial t}(\rho S)+\frac{\partial}{\partial x}\left[\frac{m}{\rho}(\rho S)\right]=0\end{aligned}\qquad(22)$$

where

$$p=p(\rho,\,S)$$

Denoting $u=(u_1,\,u_2,\,u_3)=(\rho,\,m,\,\rho S)$ we obtain the following eigenelements of the matrix A in (15) (we choose the length of the eigenvectors R according to $(24)_{1,3}$ and $|R|^2=1$)

[1] We say that the functions $g_1(u),\ldots,g_k(u)$ are independent in a neighbourhood of the point $\bar{u}$ if rank $\|\partial g_i/\partial u_j\|_{u=\bar{u}}=k\leq n$ $(1\leq j\leq n)$.

$$\lambda_1(u)\equiv\frac{m}{\rho}-c(\rho,S)\equiv\frac{u_2}{u_1}-c(u_1,\frac{u_3}{u_1})$$

$$\lambda_2(u)\equiv\frac{m}{\rho}\equiv\frac{u_2}{u_1}$$

$$\lambda_3(u)\equiv\frac{m}{\rho}+c(\rho,\ S)\equiv\frac{u_2}{u_1}+c(u_1,\frac{u_3}{u_1})$$

$$\overset{1}{R}(u)\equiv-(\frac{\partial c}{\partial\rho}+\frac{c}{\rho})^{-1}[1,\frac{m}{\rho}-c,\ S]$$

$$\overset{2}{R}(u)\equiv[(\frac{\partial p}{\partial S})^2+(\frac{m}{\rho}\frac{\partial p}{\partial S})^2+(\rho c^2-S\frac{\partial p}{\partial S})^2]^{1/2}[-\frac{\partial p}{\partial S},\ -\frac{m}{\rho}\frac{\partial p}{\partial S},\ \rho c^2-S\frac{\partial p}{\partial S}] \quad (23)$$

$$\overset{3}{R}(u)\equiv(\frac{\partial c}{\partial\rho}+\frac{c}{\rho})^{-1}[1,\frac{m}{\rho}+c,S]$$

Since we have $\overset{1}{R}_1\neq0$, $\overset{2}{R}_1\neq0$, $\overset{3}{R}_1\neq0$ in the whole $\mathcal{R}$ we can give the RLI in the whole $\mathcal{R}$:

(i) 1 - RLI are

$$\overset{1}{\varphi}_1(u)\equiv\frac{m}{\rho}+\int_{\rho_o}^{\rho}\frac{c(\xi,S)}{\xi}d\xi\equiv\frac{u_2}{u_1}+\int_{u_{10}}^{u_1}\frac{c(\xi,\frac{u_3}{u_1})}{\xi}d\xi$$

$$\overset{1}{\varphi}_2(u)\equiv S\equiv\frac{u_3}{u_1}$$

(ii) 2 - RLI are

$$\overset{2}{\varphi}_1(u)\equiv\frac{m}{\rho}\equiv\frac{u_2}{u_1}$$

$$\overset{2}{\varphi}_2(u)\equiv p(\rho,S)\equiv p(u_1,\frac{u_3}{u_1})$$

(iii) 3-RLI are

$$\overset{3}{\varphi}_1(u)\equiv\frac{m}{\rho}-\int_{\rho_o}^{\rho}\frac{c(\xi,S)}{\xi}d\xi\equiv\frac{u_2}{u_1}-\int_{u_{10}}^{u_1}\frac{c(\xi,\frac{u_3}{u_1})}{\xi}d\xi$$

$$\overset{3}{\varphi}_2(u)\equiv S\equiv\frac{u_3}{u_1}$$

It is easy to see that

$$\overset{1}{R}(u).\text{grad}_u\ \lambda_1(u)=1$$

$$\overset{2}{R}(u).\text{grad}_u\ \lambda_2(u)=0 \quad (24)$$

$$\overset{3}{R}(u).\text{grad}_u\ \lambda_3(u)=1$$

<u>COROLLARY 1</u>

(i) $\overset{i}{R}(u),\text{grad}_u\overset{i}{\phi}_1(u),\ldots\text{grad}_u\overset{i}{\phi}_{n-1}(u)$ are independent in $U(\bar{u})$

(ii) $\overset{k}{L}(u)$, $\text{grad}_u\overset{i}{\phi}_1(u),\ldots,\text{grad}_u\overset{i}{\phi}_{n-1}(u)$, $k\neq i$, are dependent in $U(\bar{u})$.

5.3. The relation between the sets of Riemann and Riemann-Lax invariants.

P4. If there exist RI for the system (15), let $v_1(u),\dots,v_n(u)$ be their expressions, then

$$v_1(u),\dots,v_{i-1}(u),\ v_{i+1}(u),\dots,v_n(u) \qquad (25)$$

are the i-RLI.

◁ According to (13). ▷

In other words, if there exist RI for the system (15) then $v_i(u)$ is an k-RLI, $1\le k\le n$, $k\neq i$.

At this point we ought to revisit the conclusions of P1 and the details of R7, particularly in case n=2.

R8. The RI describe a (local, cf. [63]) characteristic coordinate system in $\mathcal{R}$ (fig. 2). The surface $v_i(u)$=constant is stratified by the lines of index j, $1\le j\le n$, $j\neq i$.

6. Genuine nonlinearity. Linear degeneracy. Convexity. Continuous solution of a strictly hyperbolic system in two independent variables.

6.1. Genuine nonlinearity. Linear degeneracy. Convexity.

D5 (P.Lax [36]) A characteristic field of index i of the strictly hyperbolic system (15) is called genuinely nonlinear if

$$\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)\neq 0, \quad \text{in } \mathcal{R} \qquad (26)$$

and linearly degenerate respectively if

$$\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)\equiv 0 \quad \text{in } \mathcal{R} \qquad (27)$$

D6. (P.Lax [36]). A strictly hyperbolic system (15) is called convex if its characteristic fields are all either genuinely nonlinear or linearly degenerate.

Here are some examples of convex systems.

E2. The system (22), according to (24).

E3. The system

$$\frac{\partial \rho}{\partial t} + \frac{\partial m}{\partial x} = 0$$
$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left(\frac{m^2}{\rho} + p\right)=0, \quad p=p(\rho) \tag{28}$$

of the isentropic gasdynamics in Eulerian coordinates (in the usual notations). Here

$$u=(\rho,m), \quad f(u)=(m, \frac{m^2}{\rho} + p)$$

and

$$\overset{1}{R}(u)=\left(\frac{dc}{d\rho} + \frac{c}{\rho}\right)^{-1}[1,\lambda_1(u)], \overset{2}{R}(u)=-\left(\frac{dc}{d\rho} + \frac{c}{\rho}\right)^{-1}[1,\lambda_2(u)]$$

$$\lambda_1(u)= \frac{m}{\rho} -c, \ \lambda_2(u) = \frac{m}{\rho} + c$$

(we choose the length of eigenvectors R according to (29)).

We have

$$\overset{i}{R}(u).\text{grad}_u\lambda_i(u)=1, \qquad i=1,2 \tag{29}$$

The RI have the expressions

$$v_1(u)\equiv\frac{m}{\rho} - \int \frac{c(\rho)}{\rho}d\rho, \quad v_2(u)\equiv\frac{m}{\rho} + \int\frac{c(\rho)}{\rho}d\rho \tag{30}$$

E4. As n=1 in (15) we put a=f' and then we have $A(u)\equiv a(u)$, $\lambda(u)\equiv a(u)$ in $\mathcal{R}$ so that the requirement (26) can be written [$R(u)f''(u)\neq 0$ in $\mathcal{R}$, i.e.]

$$f''(u)\neq 0 \qquad \text{in } \mathcal{R} \tag{31}$$

(f should be genuinely nonlinear : convex/concave). Under this restriction the length of R can be chosen according to $R(u)f''(u)=1$. On the other hand, the requirement (27) can be transcribed

$$f''(u)\equiv 0 \qquad \text{in } \mathcal{R} \tag{32}$$

(f should depend linearly on u).

6.2. Smooth solution

A smooth solution can be constructed ([10], [15], [17], [27]) from smooth initial data in a convenient neighbourhood of the initial line. This neighbourhood is naturally limited by the presence of singularities (see 6.10).

6.3. Continuous solution. Simple waves solution.

Let us consider, under the assumptions of §4, the autonomous system

$$\frac{dU}{d\alpha} = \Lambda(U)\overset{i}{R}(U) \tag{33}$$

where i is a genuinely nonlinear index, $\Lambda \in C^m(R)$, $m \geq 1$, and $\Lambda \neq 0$ in R. Let $U(\alpha)$ be an orbit of (33) isolated with the condition

$$U(\alpha_o) = U_o$$

We use $U(\alpha)$ to construct the function

$$\beta(\alpha) \equiv \lambda_i[U(\alpha)] \tag{34}$$

(λ_i and $\overset{i}{R}$ are elements of (11)). We have

$$\frac{d\beta}{d\alpha} = \Lambda[U(\alpha)]\{\overset{i}{R}[U(\alpha)].\mathrm{grad}_u \lambda_i[U(\alpha)]\} \neq 0 \tag{35}$$

Next, we take into account the initial value problem

$$\frac{\partial\alpha}{\partial t} + \beta(\alpha)\frac{\partial\alpha}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0 \tag{36}$$

$$\alpha(x,0) = \theta(x), \quad -\infty < x < \infty \tag{37}$$

where θ is a continuous function for which a continuity neighbourhood of $t=0$ exists (example: θ is an increasing or smooth function).

<u>R9.</u> The characteristics of (36), described by

$$\frac{dx}{dt} = \beta[\alpha(x, t)] \tag{38}$$

are straight lines along which α=constant.

It is easy to see, cf.R9, that for $\frac{d\beta}{d\alpha} > 0$/respectively $\frac{d\beta}{d\alpha} < 0$ the solution of (36), (37) can be represented in the implicit form

$$\alpha = \theta(\xi), \quad \xi = x - \beta(\alpha)t \tag{39}$$

Particularly, we consider a smooth θ in (37). The representation (39) can be read in two ways:

$$F_1(\xi, x, t) \equiv \xi - x + \beta[\theta(\xi)]t = 0$$

or

$$F_2(\alpha, x, t) \equiv \alpha - \theta[x - \beta(\alpha)t] = 0 \tag{40}$$

The requirements $\frac{\partial F_1}{\partial \xi} \neq 0$/respectively $\frac{\partial F_2}{\partial \alpha} \neq 0$ imposed to the functions F_1 and F_2 respectively in order to apply the implicit function theorem are both equivalent with the condition

$$1 + \frac{d\beta}{d\alpha}[\theta(\xi)]\frac{d\theta}{d\xi}t \neq 0 \tag{41}$$

Under the restriction (41) we can obtain, explicitly, from $(40)_1$ the correspondence $\xi = \xi(x, t)$ realised by the family of characteristics and from $(40)_2$ the solution $\alpha = \alpha(x,t)$.

Taking $(40)_2$ into account we can calculate

$$\frac{\partial\alpha}{\partial t} = -\frac{\beta(\alpha)\frac{d\theta}{d\xi}}{1+\frac{d\beta}{d\alpha}\frac{d\theta}{d\xi}t}, \quad \frac{\partial\alpha}{\partial x} = \frac{\frac{d\theta}{d\xi}}{1+\frac{d\beta}{d\alpha}\frac{d\theta}{d\xi}t} \tag{42}$$

The derivatives (42) keep bounded in a regularity neighbourhood of $t=0$ (possibly placed on both sides of this axis) determined by (41). According to (41) or (42), given ξ a singularity can appear for

$$t=\bar{t}(\xi)=-\left\{\frac{d\beta}{d\alpha}[\theta(\xi)]\frac{d\theta}{d\xi}\right\}^{-1} \tag{43}$$

As $\frac{d\beta}{d\alpha}\frac{d\theta}{d\xi}<0$ we have $\bar{t}(\xi)>0$ and so a singularity appears in $t>0$ the earliest for $t=t^*=\inf\{\bar{t}(\xi),\xi\in\mathbb{R}\}$. As $\frac{d\beta}{d\alpha}\frac{d\theta}{d\xi}>0$ for each $\xi\in\mathbb{R}$, we find that a regularity neighbourhood strictly contains the halfplane $t>0$ (fig.3,a,b).

The envelope of the one-parameter family of straight lines $(40)_1$ is described by

$$x=\xi-\frac{\beta[\theta(\xi)]}{\frac{d\beta}{d\alpha}[\theta(\xi)]\frac{d\theta}{d\xi}}, \quad t=-\frac{1}{\frac{d\beta}{d\alpha}[\theta(\xi)]\frac{d\theta}{d\xi}} \tag{44}$$

(fig.3,a,b).

Now we use the solution $\alpha(x,t)$ of (36), (37) in order to construct the function

$$u(x,t)=U[\alpha(x,t)] \tag{45}$$

This function is a continuous (weak; see 7.1) solution of (15) (classical if θ is smooth in (37)).

<u>D7 (S.D.Poisson [52]).</u> For the system (15), a nonconstant continuous solution constructed in $\mathcal{D}\subset E$ by the procedure described hereinabove is called <u>a simple waves solution of index i in $\mathcal{D}$.</u> We also say that $\mathcal{D}$ is <u>a simple waves region of index i.</u>

<u>P5.</u> In a simple waves region $\mathcal{D}$ of index i the characteristics of index i of the system (15) are straight lines along which the solution keeps constant.

◀ We have $\beta[\alpha(x,t)]=\lambda_i[u(x,t)]$ in $\mathcal{D}$. We use R9. ▶

As θ is constant along certain intervals of the initial line, the continuous solution constructed by the procedure described hereinbefore consists of constant regions separated by simple waves regions.

The mentioned construction reflects the possibility of a branching through a straight line characteristic and the weak discontinuity character of such a characteristic.

R10. Since (33) and (20) are parallel we can apply the RLI theory in order to characterize a simple waves solution.

(i) Given a point $\bar{u}\in R$, we consider the neighbourhood $U(\bar{u})$ mentioned in P3 (i) and R7. The hodograph of a simple waves solution of index i lays along a line of the field $\overset{i}{R}$. In the limits of the neighbourhood U this hodograph is given, according to P2, by

$$\overset{i}{\varphi}_1(u)=C_1,\dots,\overset{i}{\varphi}_{n-1}(u)=C_{n-1} \qquad (46)$$

(ii) Let us consider next a nonconstant continuous solution u of the system (15) defined in $\mathcal{D}\subset E$ and a conveniently close neighbourhood $\mathcal{D}_o$ of the point $(x_o,t_o)\in\mathcal{D}$ so that $u(\mathcal{D}_o)\subset U(\bar{u})$, $\bar{u}=u(x_o,t_o)$. If

$$\overset{i}{\varphi}_1[u(x,t)]\equiv C_1,\dots,\overset{i}{\varphi}_{n-1}[u(x,t)]\equiv C_{n-1} \text{ in } \mathcal{D}_o \qquad (47)$$

then the RLI theory, the previous remark and the implicit function theorem show that in $\mathcal{D}_o$ the considered solution is a simple waves solution of index i.

6.4. A hierarchy of the smooth solutions.

A smooth simple waves solution is a particular case of rank 1 solution; the statement of P5 shows a particular form of the rank theorem statement ([61], 1.74).

HIERARCHY:

SMOOTH SIMPLE WAVES SOLUTION ≺ RANK 1 SOLUTION ≺ SMOOTH SOLUTION (48)

6.5. The Friedrichs theorems ([18])

LEMMA 1. The boundary of a constant region contained in the domain $\mathcal{D}$ of a continuous solution is a polygonal line whose sides are segments of characteristic straight lines.

◀ Argument 1. For a hyperbolic system the perturbations propagate on characteristics which in a constant region are straight lines. The appearence of a nonconstant neighbouring region is equivalent to a perturbation.

Argument 2. Let C be a noncharacteristic arc of the boundary of a constant region. It appears that the constant solution can be continued outside the constant region, a contradiction. ▶

D6 (K.O.Friedrichs). We say that an open segment (a connected set which does not contain vertices) of the polygonal boundary of a constant region is essentially isolated (fig.4).

T3 (K.O.Friedrichs). Let $\mathcal{D}$ be the domain of a continuous solution u and $\mathcal{D}_1$, $\mathcal{D}_2$ open subsets of $\mathcal{D}$ adjacent along the open arc C. We denote u_1/respectively u_2 the restriction of the solution u to $\mathcal{D}_1$/respectively $\mathcal{D}_2$.

If u_1 is constant, u_2 is smooth and nonconstant and C is essentially isolated then there exists a region $\mathcal{D}_3 \subset \mathcal{D}_2$ adjacent to $\mathcal{D}_1$ along C so that restriction u_3 of u to $\mathcal{D}_3$ is a simple waves solution (fig.5).

◀ Let $\bar{u}=u(\mathcal{D}_1)$ and let i be the index of the characteristic C. According to C1(ii) we have in a neighbourhood U of $\bar{u}$ in H

$$\overset{k}{L}(u)=\sum_{j=1}^{n-1}\Theta_{kj}(u).\mathrm{grad}_u\overset{i}{\varphi}_j(u),\quad 1\le k\le n,\quad k\ne i \qquad (49)$$

By (15) and (49) we obtain in $\mathcal{D}_2$:

$$0=\overset{k}{L}(u)\left[\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}\right]=\overset{k}{L}(u)\left[\frac{\partial u}{\partial t}+\lambda_k(u)\frac{\partial u}{\partial x}\right]=$$
$$=\sum_{j=1}^{n-1}\Theta_{kj}(u)\left[\frac{\partial}{\partial t}\overset{i}{\varphi}_j(u)+\lambda_k(u)\frac{\partial}{\partial x}\overset{i}{\varphi}_j(u)\right],\quad 1\le k\le n,\quad k\ne i \qquad (50)$$

Let us consider in $\mathcal{D}_2$ the initial value problem which consist of the linear system (associated to the solution u)

$$\sum_{j=1}^{n-1}\Theta_{kj}[u(x,t)]\left\{\frac{\partial}{\partial t}\Phi_j(x,t)+\lambda_k[u(x,t)]\frac{\partial}{\partial x}\Phi_j(x,t)\right\}=0,\ 1\le k\le n,\ k\ne i \qquad (51)$$

and the data

$$\Phi_j(x,t)=\text{constant}=\varphi_j(\bar{u}),\quad 1\le j\le n-1,\ \text{along } C \qquad (52)$$

The equation of index k in (51) requires only differentiations in direction $\lambda_k\ne\lambda_i$ so that C is not characteristic for (51). Then the problem (51), (52) has a unique solution in $\mathcal{D}_3\subset\mathcal{D}_2$

$$\Phi_j(x,t)\equiv\text{constant}=\overset{i}{\varphi}_j(u),\quad 1\le j\le n-1 \qquad (53)$$

On the other hand, we have in $\mathcal{D}_3$ [according to (50)] $\Phi_j(x,t)\equiv\overset{i}{\varphi}_j[u(x,t)]$. We use R10 (ii). ▶

Analogous arguments lead to the following theorem (see C3 hereinbelow, pag. 25):

T4. Let $\mathcal{D}_1\subset\mathcal{D}$ be a simple waves region. The region $\mathcal{D}_1$ cannot be adjacent in $\mathcal{D}$ along an essentially isolated segment of straigth line characteristic but to a constant region or a simple waves region of the same index.

6.6. The Friedrichs rank partition

The Friedrichs theorems show how can we characterise the rank of a piecewise smooth solution in its domain $\mathcal{D}$. So, let, in a rank partition, $\mathcal{D}_j$ be a rank j region (j=0,1,2); the Friedrichs theorems state that $\mathcal{D}_2$ and $\mathcal{D}_o$ cannot have in common but isolated points. Thus a description (related to the mentioned context) of the manner in which the rank can change is offered: with one unit through curves, with two units through points. Moreover (see the hierarchy (41)), the Friedrichs theorems show that a rank 1 region adjacent to a rank zero region must be a simple waves region.

A typical example of rank partition is given in fig.6 (we ignore the possible change of the simple waves index in the region $\mathcal{D}_1$ of this figure).

6.7. The hodograph of a simple waves solution

Let i be the index of a genuinely nonlinear field. The following result will be useful to us hereinbelow.

L2. In every point $u\varepsilon\mathcal{R}$ we have

$$\overset{i}{L}(u)\cdot\overset{i}{R}(u)=\frac{\overset{i}{L}(u)\cdot\{[\overset{i}{R}(u).\mathrm{grad}_u]A(u)\}\cdot\overset{i}{R}(u)}{\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)} \tag{54}$$

◄ We use

$$\overset{i}{R}_k(u).\frac{\partial}{\partial u_k}[a_{j\ell}\overset{i}{R}_\ell]=\overset{i}{R}_k\{\frac{\partial a_{j\ell}}{\partial u_k}\overset{i}{R}_\ell+a_{j\ell}\frac{\partial\overset{i}{R}_\ell}{\partial u_k}\},$$

$$\overset{i}{R}_k\frac{\partial a_{j\ell}}{\partial u_k}\overset{i}{R}_\ell=\overset{i}{R}_k\frac{\partial}{\partial u_k}(\lambda_i\overset{i}{R}_j)-\overset{i}{R}_k a_{j\ell}\frac{\partial}{\partial u_k}\overset{i}{R}_\ell=$$

$$=(\overset{i}{R}_k\frac{\partial}{\partial u_k}\lambda_i)\overset{i}{R}_j-(a_{j\ell}-\lambda_i\delta_{j\ell})(\overset{i}{R}_k\frac{\partial}{\partial u_k}\overset{i}{R}_\ell).$$ ►

R11. (P.Lax [36]). (i) Let $A\varepsilon C^{m-1}(\mathcal{R})$, $m\geqslant 3$, in (15). For each $u_\ell\varepsilon\mathcal{R}$ the set of the vectors u_r which can be joined (as states to the right in $\mathcal{R}$) with u_ℓ by a simple waves hodograph of (genuinely nonlinear) index i are laid along a line of the field $\overset{i}{R}$.

Let us introduce, instead of α, a new parameter, denoted ε, cf.

$$\varepsilon=\lambda_i(u)-\lambda_i(u_\ell) \text{ in the points of the orbit } U \tag{55}$$

The relation between the two parametrizations is given, in the points of the mentioned orbit, by

$$\varepsilon=\varepsilon(\alpha)\equiv\lambda_i[U(\alpha)]-\lambda_i(u_\ell)\overset{(34)}{=}\beta(\alpha)-\lambda_i(u_\ell),\quad \varepsilon(\alpha_o)=0 \tag{56}$$

Since i is a genuinely nonlinear index we have, according to (56), (33) and (26),

$$\frac{d\varepsilon}{d\alpha}=\Lambda[U(\alpha)]\{\overset{i}{R}[U(\alpha)].\mathrm{grad}_u\lambda_i[U(\alpha)]\}\neq 0 \tag{57}$$

We can describe an orbit $U[\alpha(\varepsilon)]$ by

$$u = R_i(\varepsilon, u_\ell)$$
$$R_i(0, u_\ell) = u_\ell \tag{58}$$

where the representation (58) is C^{m-1} in a convenient neighbourhood of each point ($\varepsilon=0$, $u_\ell=\bar{u}_\ell$), $\bar{u}_\ell \varepsilon R$. Moreover, u_ℓ cannot be a singular point on the curve (58) and we have

$$\frac{du}{d\varepsilon} = \frac{\overset{i}{R}(u)}{\overset{i}{R}(u)\,.\,\mathrm{grad}_u \lambda_i(u)} \tag{59}$$

$$\frac{d^2u}{d\varepsilon^2} = \frac{[\overset{i}{R}(u)\,.\,\mathrm{grad}_u]\overset{i}{R}(u)}{[\overset{i}{R}(u)\,.\,\mathrm{grad}_u\lambda_i(u)]^2} - \overset{i}{R}(u)\,\frac{\frac{d}{d\varepsilon}[\overset{i}{R}(u)\,.\,\mathrm{grad}_u\lambda_i(u)]}{[\overset{i}{R}(u)\,.\,\mathrm{grad}_u\,\lambda_i(u)]^2} \tag{60}$$

Thus, around each $u_\ell \varepsilon R$ we can find in the hodograph space a convenient neighbourhood whose points can be displaced by a convenient (unique, smooth) mouvement along the lines of the field $\overset{i}{R}$.

(ii) In case of a linearly degenerate field we obtain from (56) $\beta(\alpha) \equiv \lambda_i(u_\ell)$. Then, cf.(27), the envelope given by (44) goes off to the infinity. This circumstance hinders the "fanning out" or the "approach" (described, in the sense of increasing time, by fig.3) of a (linearly degenerate) simple waves region.

(iii) The parameter ε is a magnitude and does not reflect the structure of the simple waves solution in the halfplane $t>0$. This structure is described by the correspondence $\varepsilon=\varepsilon[\alpha(x,t)]$. This correspondence depends, in its turn, on the (given/or determined - see R36) function Θ.

D9. A simple waves region for which $\varepsilon>0$/respectively $\varepsilon<0$ (corresponds, according to (58), to gasdynamic rarefaction/compression and) will be called rarefaction/compression simple waves region.

6.8. Linear degeneracy and weak nonlinearity. The Rozdestvenskii theorem of global well-posedness

D10. The convex system (15) is called weakly nonlinear if for it all the characteristic fields are linearly degenerate.

Let us now suppose, cf.(15), B=0 in (14):

$$\frac{\partial}{\partial t} v_i + \bar{\lambda}_i(v)\frac{\partial}{\partial x} v_i = 0, \quad 1 \le i \le n \tag{61}$$

L3. For a convex system (15) for which the RI exist the requirement (27) is equivalent to the restriction

$$\frac{\partial}{\partial v_k} \bar{\lambda}_k(v) = 0 \quad \text{in } \bar{\mathcal{R}} \tag{62}$$

(without summation; $\bar{\mathcal{R}}$ is the image of $\mathcal{R}$).

◀ Using (13) we find

$$\frac{\partial \lambda_k}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial \bar{\lambda}_k}{\partial v_i} \cdot \frac{\partial v_i}{\partial u_j} = \sum_{i=1}^{n} \alpha_i(u) \overset{i}{L}_j(u) \frac{\partial \bar{\lambda}_k}{\partial v_i}$$

$$\Downarrow$$

$$\overset{k}{R}(u) \cdot \mathrm{grad}_u \lambda_k = \sum_{j=1}^{n} \overset{k}{R}_j(u) \sum_{i=1}^{n} \alpha_i(u) \overset{i}{L}_j(u) \frac{\partial \bar{\lambda}_k}{\partial v_i} =$$

$$= \sum_{i=1}^{n} \alpha_i(u) \Big\{ \sum_{j=1}^{n} \overset{k}{R}_j(u) \overset{i}{L}_j(u) \Big\} \frac{\partial \bar{\lambda}_k}{\partial v_i} =$$

$$= \sum_{i=1}^{n} \alpha_i(u) \delta_{ki} \frac{\partial \bar{\lambda}_k}{\partial v_i} = \alpha_k(u) \frac{\partial \bar{\lambda}_k}{\partial v_k} \quad ▶$$

Therefore, for a weakly nonlinear system for which the RI exist the requirement (62) is fulfilled in (61) for each k, $1 \le k \le n$.

D11. We say that the initial data v_o associated to (61) according to

$$v(x, 0) = v_o(x), \quad x \in \mathbb{R} \tag{63}$$

are mutually disjoint if we can find the intervals (m_k, M_k), $1 \le k \le n$, so that we should have simultaneously

(i) $m_k \le v_{ok}(x) \le M_k$, $\quad x \in \mathbb{R}$

(ii) $M_j < m_{j+1}$

We denote

$$\bar{\varepsilon} = \min_{1\le k\le n-1} (m_{k+1} - M_k) \qquad (64)$$

T5 (B. L. Rozdestvenskii [56],[57]). For a weakly nonlinear system for which the RI exist the initial value problem with smooth disjoint initial data has a (unique) global smooth solution.

◀ All the ingredients of the general proof are present in the analysis of the case n=2. We only consider this case. Cf. (62) we have

$$\bar{\lambda}_1=\bar{\lambda}_1(v_2),\ \bar{\lambda}_2=\bar{\lambda}_2(v_1)$$

We suppose

$$\bar{\lambda}_1'\neq 0,\quad \bar{\lambda}_2'\neq 0 \qquad (65)$$

and put

$$r=\bar{\lambda}_2(v_1),\quad s=\bar{\lambda}_1(v_2) \qquad (66)$$

Then, the problem (61), (63) can be transcribed

$$\frac{\partial r}{\partial t} + s\frac{\partial r}{\partial x} = 0,\quad \frac{\partial s}{\partial t} + r\frac{\partial s}{\partial x} = 0 \qquad (67)$$

$$[r(x,0),s(x,0)]=[r_o(x),s_o(x)],\quad x\in\mathbb{R} \qquad (68)$$

We put

$$z \overset{\text{def}}{=} \frac{\partial r}{\partial t} + r\frac{\partial r}{\partial x} = (r-s)\frac{\partial r}{\partial x} \qquad (69)$$

$$\bar{z} \overset{\text{def}}{=} \frac{\partial s}{\partial t} + s\frac{\partial s}{\partial x} = (s-r)\frac{\partial s}{\partial x} \qquad (70)$$

Applying $(\frac{\partial}{\partial t} + s\frac{\partial}{\partial x})$ and $(\frac{\partial}{\partial t} + r\frac{\partial}{\partial x})$ in (70) and (69) respectively and taking (67) into account we find

$$(\frac{\partial}{\partial t} + s\frac{\partial}{\partial x})z=0,\quad (\frac{\partial}{\partial t} + r\frac{\partial}{\partial x})\bar{z}=0 \qquad (71)$$

A singularity cannot appear - thus limiting the smoothness neighbourhood of the initial line - since in the smoothness neighbourhood we (uniformly) have the following estimates

$$|r|<K,\ |s|<K,\ |r-s|>\bar{\varepsilon}>0$$

$$\left|\frac{\partial r}{\partial x}\right| = \frac{|z|}{|r-s|} < \frac{1}{\bar{\varepsilon}}|z| < 2\frac{KK_1}{\bar{\varepsilon}},\quad \left|\frac{\partial s}{\partial x}\right| < 2\frac{KK_1}{\bar{\varepsilon}}$$

$$K=\max(|m_1|,\ |M_2|)$$

$$K_1=\max\{\sup[|r_o'(x)|,\ x\in R],\sup[|s_o'(x)|,\ x\in R]$$ ▶

C2. A solution of the problem (61), (63) which corresponds to (discontinuous) piecewise smooth initial data can be obtained, globally, as a limit of a sequence of smooth solutions.

◀ The initial data are smoothed and the theorem T5 is applied. ▶

If we "force" the weak nonlinearity allowing

$$\bar{\lambda}_1'(v_2)\equiv 0 \ [\text{yet } \bar{\lambda}_2'(v_1)\neq 0]$$

in (65), then (67) is replaced by (k=constant)

$$\frac{\partial r}{\partial t} + k\frac{\partial r}{\partial x} = 0, \qquad \frac{\partial s}{\partial t} + r\frac{\partial s}{\partial x} = 0,$$

which leads to the linear equation

$$\frac{\partial s}{\partial t} + r_o(x-kt)\frac{\partial s}{\partial x} = 0$$

D12. A convex quasilinear system (15) which is not weakly nonlinear will be called strongly nonlinear.

R12. The definitions D10, D12 characterise the nonlinearity intensity of a strictly hyperbolic system (15).

R13. The following HIERARCHY filters the global existence of a (smooth) solution from smooth data:

$$\text{LINEAR SYSTEM} \prec \text{WEAKLY NONLINEAR S.} \prec \text{STRONGLY NONLINEAR S.} \qquad (72)$$

6.9. The importance of the strict hyperbolicity.

R14. From (67) we obtain $\frac{D(r,s)}{D(x,t)} = (r-s)\frac{\partial r}{\partial x}\frac{\partial s}{\partial x}$.

The disjoint initial data ensure the strict hyperbolicity of the weakly nonlinear system (61). As the initial data are not disjoint the strict hyperbolicity can be compromised and together with it the global existence of the smooth solution as well.

Indeed, let C_s and C_r the families of characteristics of slopes s and r respectively. Using (71) we can see that z=constant on each characteristic of C_s (the constant depends on characteristic) and, similarly, z=constant on each characteristic of C_r. Let $(x^*, 0)$ a point of the initial line in which the data (68) have the same value:

$$r_o(x^*) = s_o(x^*) \qquad (73)$$

and for which

$$[r_o'(x^*)]^2 + [s_o'(x^*)]^2 \neq 0 \qquad (74)$$

We have

$$z(x^*,0)=0, \ \bar{z}(x^*, 0)=0 \qquad (75)$$

so that, cf. (74), the two characteristics - from C_s and C_r respectively - through $(x^*, 0)$ coincide in a convenient neighbourhood of the initial line. From (67) it then appears that their common arc is a straight line segment. This remark can be used in order to supply examples in which, for nondisjoint data, the global existence of the smooth solution is compromised. The simplest example of this kind supposes that the restrictions (73), (74) are fulfilled on a whole interval of the initial line (fig.7) so that in a convenient neighbourhood of this interval the problem (67), (68) corresponding to a weakly nonlinear system degenerates in the problem

$$\frac{\partial r}{\partial t} + r\,\frac{\partial r}{\partial x} = 0, \qquad r(x,0)=r_o(x) \tag{76}$$

which coresponds to a single genuinely nonlinear equation.

According to the item 6.3 the (smooth) data of the problem (76) can be chosen in such a way so as to supply singularities in solution (fig.7).

6,10. Development of a singularity: some estimates.

The manner in which the singularities can be developed in solution is described, in case of a single genuinely nonlinear equation, in the item 6.3. We shall re-make here this description in case of a strictly hyperbolic strongly nonlinear system for which n=2.

Let us consider the Riemann form of the mentioned system

$$\frac{\partial r}{\partial t} + \lambda(r,s)\,\frac{\partial r}{\partial x} = 0, \quad \frac{\partial s}{\partial t} + \mu(r,s)\frac{\partial s}{\partial x} = 0 \tag{77}$$

The strong nonlinearity requirement can be written

$$\frac{\partial \lambda}{\partial r} \neq 0 \text{ and / or } \frac{\partial \mu}{\partial s} \neq 0 \tag{78}$$

while the assumption of the strict hyperbolicity is equivalent to

$$\lambda - \mu \neq 0 \text{ in } \bar{R} \tag{79}$$

The two families of characteristics of the system (77) can be described by the solutions

$$x=X(t,x_o), \quad x=\bar{X}(t,\bar{x}_o) \tag{80}$$

of the problems

$$\frac{d}{dt}X(t,x_o)=\lambda\{r[X(t,x_o),t],s[X(t,x_o),t]\}=$$
$$=\lambda\{r_o(x_o),s[X(t,\ x_o),t]\}\ ,\quad X(0,\ x_o)=x_o \tag{81}$$

$$\frac{d}{dt}\bar{X}(t,\ \bar{x}_o)=\mu\{r[\bar{X}(t,\ \bar{x}_o),t],\ s[\bar{X}(t,\ \bar{x}_o),t]\}=$$
$$=\mu\{r[\bar{X}(t,\ \bar{x}_o),\ t],\ s_o(\bar{x}_o)\},\quad \bar{X}(0,\ \bar{x}_o)=\bar{x}_o \tag{82}$$

Let us put

$$h(r,s)=\int_0^s \frac{\frac{\partial\lambda}{\partial s}(r,\ \xi)\ d\xi}{\lambda(r,\ \xi)-\mu(r,\xi)}\ ,\ \bar{h}(r,s)=\int_0^r \frac{\frac{\partial\mu}{\partial r}(\xi,s)\,d\xi}{\mu(\xi,s)-\lambda(\xi,s)} \tag{83}$$

$$\left.\begin{array}{l} H(t,\ x_o)=\exp\{h[r_o(x_o),\ s(X(t,\ x_o),t)]\} \\ \bar{H}(t,\ \bar{x}_o)=\exp\{\bar{h}[r(\bar{X}(t,\ \bar{x}_o),t),\ s_o(\bar{x}_o)]\} \end{array}\right\} \tag{84}$$

T6 (P. Lax [37]) (i) If in the problem (77), (68) the initial data are nonconstant smooth of compact support or nonconstant periodic then a singularity can appear as there is a $x_o^*\varepsilon R$ so that

$$r_o'(x_o^*)\frac{\partial\lambda}{\partial r} < 0 \text{ and/or } s_o'(x_o^*)\ \frac{\partial\mu}{\partial s} < 0 \tag{85}$$

(ii) Let β, ζ constants so that we have, for each $t>0$, $-\infty<x_o,\bar{x}_o<\infty$, the apriori estimates

$$\left|\frac{1}{H(t,\ x_o)}\ \frac{\partial\lambda}{\partial r}[X(t,\ x_o),x_o]\right|<\beta\quad ,\quad \left|\frac{1}{\bar{H}(t,\ \bar{x}_o)}\ \frac{\partial\mu}{\partial s}[\bar{X}(t,\ \bar{x}_o),\ \bar{x}_o]\right|<\beta$$
$$|r_o'(x_o)H(0,\ x_o)|<\zeta,\ |s_o'(\bar{x}_o)\bar{H}(0,\ \bar{x}_o)|<\zeta \tag{86}$$

Then a singularity can appear the earliest at the time

$$t^*=(\beta\zeta)^{-1}$$

Remark: in the particular cases $r_o(x_o)\not\equiv$constant, $s_o(x_o)\equiv$constant or $r_o(x_o)\equiv$constant, $s_o(x_o)\not\equiv$constant (the system (77) is reduced to an equation of the form (35) and) the statement (i) results from the considerations of 6.3.

◀ Differentiating (81) with respect to x_o we obtain for $(\partial X/\partial x_o)$ the following problem

$$\frac{d}{dt}\left(\frac{\partial X}{\partial x_o}\right)=\frac{\partial\lambda}{\partial r}r_o'(x_o)+\frac{\partial\lambda}{\partial s}\frac{\partial s}{\partial x}\frac{\partial X}{\partial x_o},\quad \frac{\partial X}{\partial x_o}(0,\ x_o)=1 \tag{87}$$

From $(77)_2$ we get

$$\frac{\partial\lambda}{\partial s}\frac{\partial s}{\partial x}=\frac{1}{\lambda-\mu}\frac{\partial\lambda}{\partial s}\frac{d}{dt}s[X(t,\ x_o),t]=\frac{d}{dt}\ln H(t,\ x_o) \tag{88}$$

Furthermore, from (87), (88) it results

$$H\frac{d}{dt}\left(\frac{1}{H}\frac{\partial X}{\partial x_o}\right)=\frac{d}{dt}\left(\frac{\partial X}{\partial x_o}\right)-\frac{\partial X}{\partial x_o}\frac{d}{dt}\ell nH=\frac{\partial\lambda}{\partial r}\,r_o'(x_o)$$

which leads to

$$\frac{\partial X}{\partial x_o}=\frac{H(t,x_o)}{H(0,x_o)}E(t,x_o),\quad E(t,x_o)=1+r_o'(x_o)\int_0^t\frac{\partial\lambda}{\partial r}\frac{H(0,x_o)}{H(\tau,x_o)}d\tau \tag{89}$$

Similarly, for $(\partial\bar{X}/\partial\bar{x}_o)$ we find

$$\frac{\partial\bar{X}}{\partial\bar{x}_o}=\frac{\bar{H}(t,\bar{x}_o)}{\bar{H}(0,\bar{x}_o)}\bar{E}(t,\bar{x}_o),\quad \bar{E}(t,\bar{x}_o)=1+s_o'(x_o)\int_0^t\frac{\partial\mu}{\partial s}\frac{\bar{H}(0,\bar{x}_o)}{\bar{H}(\tau,\bar{x}_o)}d\tau \tag{90}$$

Differentiating in $r[X(t,x_o),t]\equiv r_o(x_o)$ with respect to x_o, we find

$$\frac{\partial}{\partial x}r[X(t,x_o),t]\,\frac{\partial X}{\partial x_o}(t,x_o)=r_o'(x_o) \tag{91}$$

From (91) we can see that a singularity can only appear if for a certain pair x_o^*,t^* we simultaneously have

$$r_o'(x_o^*)\neq 0 \text{ and } \frac{\partial X}{\partial x_o}(t^*,x_o^*)=0 \text{ and/or } s_o'(x_o^*)\neq 0 \text{ and } \frac{\partial\bar{X}}{\partial\bar{x}_o}(t^*,x_o^*)=0 \tag{92}$$

From (89), (90) it results that the requirements (92) can be fulfilled only under restrictions (85).

Finally, from (86), (89), (90) we obtain

$$\min[E(t,x_o),\bar{E}(t,\bar{x}_o)]>1-\beta\zeta t>0 \quad \text{as } t<(\beta\zeta)^{-1}.$$ ▶

<u>R15.</u> The value t^* given by T6(ii) for the systems ought to be compared to the value t^* given in 6.3 for a single equation.

<u>E5.</u> Let us consider, as $p(\rho)\equiv A\rho^\gamma$ (polytropic equation of state), the system (28) of the isentropic gasdynamics

$$\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}+\frac{\gamma-1}{2}c\frac{\partial u}{\partial x}=0 \qquad u=\frac{m}{\rho}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{2}{\gamma-1}c\frac{\partial c}{\partial x}=0$$

We have, according to (30),

$$r=u-\frac{2}{\gamma-1}c,\qquad s=u+\frac{2}{\gamma-1}c$$

$$\lambda(r,s)=u(r,s)-c(r,s)=r(\gamma-\frac{1}{2}) + s(\frac{3}{2}-\gamma)$$

$$\mu(r,s)=u(r,s)+c(r,s)=r(\frac{3}{2}-\gamma)+s(\gamma-\frac{1}{2})$$

$$\frac{\partial\lambda}{\partial s}=\frac{\partial\mu}{\partial r}=\frac{3}{2}-\gamma,\quad \frac{\partial\lambda}{\partial r}=\frac{\partial\mu}{\partial s}=\gamma-\frac{1}{2}>0,\quad \lambda-\mu=2(\gamma-1)(r-s)$$

$$h(r,s)=\frac{3-2\gamma}{4(\gamma-1)}\ln|1-\frac{s}{r}|\ ,\quad \bar{h}(r,s)=\frac{3-2\gamma}{4(\gamma-1)}\ln|1-\frac{r}{s}|=h(s,r)$$

$$H(t,x_o)=\left|1-\frac{s[X(t,x_o),t]}{r_o(x_o)}\right|^{\frac{3-2\gamma}{4(\gamma-1)}},\ \bar{H}(t,\bar{x}_o)=\left|1-\frac{r[\bar{X}(t,\bar{x}_o),t]}{s_o(\bar{x}_o)}\right|^{\frac{3-2\gamma}{4(\gamma-1)}}$$

The requirement (85) can be written

$$r_o'(x_o^*)<0 \text{ and/or } s_o'(x_o^*)<0$$

The condition of strict hyperbolicity ($\lambda\neq\mu$) is fulfilled as $r\neq s$ in $\bar{R}$. Or, this last requirement is satisfied in case of disjoint data.

7. Hyperbolic systems of conservation laws

7.1. Integral/differential/weak form of a (strictly) hyperbolic system of conservation laws

Let us consider first the integral form of a system of conservation laws concerning a (vectorial) entity. This form asserts that in any region V of the space we have

$$\frac{d}{dt}\int_V u\,dV + \int_{\partial V} fn\,dS = 0 \tag{93}$$

where u and f denote respectively the (vectorial) density and the (vectorial) flux of the mentioned entity. Motivating (for example) by gasdynamics we suppose f depends on u alone in a region $R\subset H$.

In case of a single space dimension V is an interval $[x_1, x_2]$ and (93) passes into

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx+f[u(x_2,t)]-f[u(x_1, t)]=0 \tag{94}$$

for each $x_1,x_2\in\mathbb{R}$, $x_1<x_2$, $t>0$.

It is easy to be seen that if a solution of (94) is smooth in a certain region of $t>0$ then in that region it also ("classically") fulfils the system

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \tag{95}$$

The form (95) is said to be a divergence form.

We assume in the following that (see §4)

$$f \in C^m(\mathbb{R}), \quad m>3 \tag{96}$$

and that the matrix

$$A=A(u)=[a_{ij}(u)]_{1\leq i,j\leq n}, \quad a_{ij}=\frac{\partial}{\partial u_j} f_i(u) \tag{97}$$

has real and distinct eigenvalues.

The conclusions of the item 6.10 require, in the attempt to find global solutions of the initial value problem (15), (16), extensions of the concept of solution (for an illuminating discussion see Lax [39], [40], [41]). Here are two ways of accomplishing this. At first,

D13. We regard the (possibly nonsmooth) solutions of the integral form (94) as generalized solutions of the system (95).

Alternatively,

D14. We say that a bounded measurable function $u(x,t)$ is a weak (generalized) solution of (95) if

$$\int_0^\infty \int_{-\infty}^\infty [u \frac{\partial \varphi}{\partial t} + f(u)\frac{\partial \varphi}{\partial x}]dxdt = 0 \tag{98}$$

for any $\varphi \in C_0^1$ with support in $t>0$.

The definition D14 is justified by the fact that if a weak solution is smooth in a certain region of $t>0$ then, as it is easy to be seen, in that region it classically satisfy (95).

Now the definitions D13, D14 can be related to the initial value problem (95), (16). In the sequel a solution of (94), (16) should be regarded as a generalized solution (according to D13) of the problem (95), (16). Alternatively, a bounded measurable function $u(x,t)$ for which

$$\int_0^\infty \int_{-\infty}^\infty [u \frac{\partial \varphi}{\partial t} + f(u) \frac{\partial \varphi}{\partial x}]dxdt + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\,dx=0, \quad (\forall)\varphi \in C_0^1 \tag{99}$$

corresponding to a bounded measurable initial data u_0, should also be regarded as a generalized solution (according to D14) of the initial value problem (95), (16).

If necessary we shall specify (according to D13, D14) the nature of the generalized solution considered.

The opportunity of the two types of extension (considered cf.D13, D14) and the relationship between them is discussed by R19.

C3 (of T3). In the class of continuous functions, the only type of solution which can be connected to a constant state is the simple wave solution.

7.2. Conservativity.

The considerations of the previous item points out the possibility to associate one of the forms (94) or (98) to the form (95). The form (95) occurs in a smooth framework. In order to adapt the consideration 7.1 to the system (15) it is therefore natural to ask ourselves, in a smooth framework, under what conditions this system can be put in divergent form.

D15. The equation

$$\frac{\partial}{\partial t}\varphi(u) + \frac{\partial}{\partial x}\psi(u) = 0 \tag{100}$$

for which $\varphi,\psi\varepsilon C^1(R)$ is called a conservation law associated to the system (15) if it results from (15) for each smooth solution of this system (i.e. in a smooth framework).

In order that a conservation law should result the functions φ, ψ are to satisfy the following restrictions [see (97)]

$$\frac{\partial\psi}{\partial u_i} = \sum_{k=1}^{n} a_{ki}(u)\frac{\partial\varphi}{\partial u_k}, \quad 1\leq i\leq n \tag{101}$$

R16. As n=1 (underdeterminacy) or n=2 the existence of (at least) one pair φ,ψ satisfying (100)/of a conservation law associated to (15) is guaranteed. As n=1, given φ we can determine ψ by integrating $\psi'(u)=a(u)\varphi'(u)$. As n=2, (101) is written

$$\begin{aligned} &\frac{\partial\psi}{\partial u_1} - \left(a_{11}\frac{\partial\varphi}{\partial u_1} + a_{21}\frac{\partial\varphi}{\partial u_2}\right) = 0\\ &\frac{\partial\psi}{\partial u_2} - \left(a_{12}\frac{\partial\varphi}{\partial u_1} + a_{22}\frac{\partial\varphi}{\partial u_2}\right) = 0 \end{aligned} \tag{102}$$

It is easy to be seen that if the system (15) is strictly hyperobolic then the linear system (102) is strictly hyperbolic too. Then a solution of this system results by the method of characteristics.

If n≥3 it is possible that the system (101) should not have but the trivial solution (φ,ψ)=constant. In this case we say that the system (15) is completely nonconservative.

E6(B.L.Rozdestvenskii [56]). The system

$$\frac{\partial u_1}{\partial t} + u_2 \frac{\partial u_1}{\partial x} = 0, \quad \frac{\partial u_2}{\partial t} + u_3 \frac{\partial u_2}{\partial x} = 0, \quad \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} = 0 \qquad (103)$$

is completely nonconservative.

D16. We say that the conservation laws (100) corresponding to the pairs (φ_i, ψ_i), $1 \le i \le \ell$ are independent if the functions $1, \varphi_1, \ldots, \varphi_\ell$ are linearly independent in R.

R17. If for the system (15) there exist n independent conservation laws for which

$$\frac{D(\varphi_1, \ldots, \varphi_n)}{D(u_1, \ldots, u_n)} \neq 0 \quad \text{in } R \qquad (104)$$

then this system can be converted into a divergence form.

So, let

$$\frac{\partial}{\partial t} \varphi_i(u) + \frac{\partial}{\partial x} \psi_i(u) = 0, \qquad 1 \le i \le n \qquad (105)$$

be n independent conservation laws associated to (15). Then on putting $U = \varphi(u)$ and using (104) it results $u = g(U)$, $f(U) = \psi[g(U)]$ so that (105) gets the form

$$\frac{\partial}{\partial t} U + \frac{\partial}{\partial x} f(U) = 0 \qquad (106)$$

8. Discontinous piecewise smooth solution of a convex strictly hyperbolic system of conservation laws.

8.1. The Rankine-Hugoniot jump relations.

D17 (D. Schaeffer [58]). Let u be a solution of (94) or (98) and let $\mathcal{D}$ be a region of its domain as $t \ge 0$. Let C be a smooth curve along which u is discontinuous. We say that the arc $C \cap \mathcal{D}$ is isolated in $\mathcal{D}$ with respect to u if for each point (ξ, τ) of $C \cap \mathcal{D}$ there exists a neighbourhood $V(\xi, \tau) \subset \mathcal{D}$ so that

(i) the solution u is smooth in the regions V_ℓ, V_r of V adjacent along C,

(ii) the limits

$$u_\ell(\xi, \tau) = \lim_{\substack{x \to \xi \\ (x,\tau) \in V_\ell}} u(x, \tau), \qquad u_r(x, \tau) = \lim_{\substack{x \to \xi \\ (x,\tau) \in V_r}} u(x, \tau).$$

exist.

We say that u is a discontinuous piecewise smooth solution in $\mathcal{D}$ if $\mathcal{D}$ is a join of an (at most countable) number of isolated arcs with respect to u and open sets on which u is (continuous, piecewise) smooth.

Let $x=X(t)$ the equation of the arc C. Given $t>0$, we denote

$$u_\ell=u[X(t)-0,t],\ u_r=u[X(t)+0,t],[\![u]\!]=u_r-u_\ell \tag{107}$$

L4. In the points of a discontinuity line C a piecewise smooth solution satisfies the conditions

$$[\![f(u)]\!]=D[\![u]\!]\quad,\quad D=X'(t) \tag{108}$$

◄ The case of the integral form (94) of the system of conservation laws (fig.8). For $t>t^*$ we calculate the rate of change of

$$\int_{x_1}^{x_2}u(x,t)dx=\int_{x_1}^{X(t)-0}u(x,t)dx+\int_{X(t)+0}^{x_2}u(x,t)dx$$

We have

$$\frac{d}{dt}\int_{x_1}^{x_2}u(x,t)dx=\int_{x_1}^{X(t)-0}\frac{\partial u}{\partial t}dx+u[X(t)-0,t]X'(t)+$$

$$+\int_{X(t)+0}^{x_2}\frac{\partial u}{\partial t}dx-u[X(t)+0,t]X'(t)= \tag{109}$$

$$=-[\![u]\!]D+\left(\int_{x_1}^{X(t)-0}+\int_{X(t)+0}^{x_2}\right)\frac{\partial u}{\partial t}dx$$

Since in the adjacent regions u is smooth we can use (95) in order to obtain

$$\frac{d}{dt}\int_{x_1}^{x_2}u(x,t)dx=-D[\![u]\!]-\left(\int_{x_1}^{X(t)-0}+\int_{X(t)+0}^{x_2}\right)\frac{\partial}{\partial x}f(u)=$$

$$=-D[\![u]\!]+[\![f(u)]\!]-\{f[u(x_2,t)]-f[u(x_1,t)]\} \tag{110}$$

The condition (108) then results from (110) and (94).

The case of the weak form (98) (fig.9). Let $S=\text{supp}\,\varphi$ intersects the discontinuity line. We have

$$0=\int_0^\infty\int_{-\infty}^\infty[u\frac{\partial\varphi}{\partial t}+f(u)\frac{\partial\varphi}{\partial x}]dxdt=\int_S[u\frac{\partial\varphi}{\partial t}+f(u)\frac{\partial\varphi}{\partial x}]dxdt=$$

$$=\sum_{i=1}^{2}\int_{S_i}[u\frac{\partial\varphi}{\partial t}+f(u)\frac{\partial\varphi}{\partial x}]dxdt=\sum_{i=1}^{2}\int_{S_i}\{\frac{\partial}{\partial t}(u\varphi)+\frac{\partial}{\partial x}[\varphi f(u)]\}dxdt-$$

$$-\sum_{i=1}^{2}\int_{S_i}[\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}]\varphi dxdt=\int_{B(r)}^{A}\varphi[-udx+f(u)dt]+\int_{A(\ell)}^{B}\varphi[-udx+f(u)dt]=$$

$$=\int_A^B\varphi([\![u]\!]dx-[\![f(u)]\!]dt)=\int_B^A\varphi([\![f(u)]\!]-D[\![u]\!])dt$$

where the indices ℓ and r indicates the left/right side of the discontinuity line. Shrinking supp φ around a given point of the line C we find (108). ▶

We call (108) the Rankine-Hugoniot jump relations.

R18. It is possible that two different systems under divergent form should have the same smooth solutions. Example (as n=1): the forms

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(\frac{1}{2}u^2) = 0 \qquad (111)$$

and

$$\frac{\partial}{\partial t}(\frac{1}{2}u^2) + \frac{\partial}{\partial x}(\frac{1}{3}u^3) = 0 \qquad (112)$$

have distinct discontinuous solutions, since for them (108) gives respectively:

$$D_1 = \frac{[\![\frac{1}{2}u^2]\!]}{[\![u]\!]} = \frac{1}{2}(u_\ell + u_r)$$

$$D_2 = \frac{[\![\frac{1}{3}u^3]\!]}{[\![\frac{1}{2}u^2]\!]} \neq D_1 \quad (\text{as } u_r \neq u_\ell),$$

but they have the same smooth solutions.

In a smooth context the differentiation (111), (112) is of no importance for the equation (of the form (15))

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad (113)$$

The considerations 7.1 are however connected to the divergent form of the system (15). Therefore, in case there are several divergent forms for this system, we have to make it clear to which of these forms these considerations are referred.

On the other hand, we shall notice that the system (103) cannot be put in a divergent form.

R19. (i) Cf. to the considerations of 7.1 and to L4 the integral and the weak form of a system of conservation laws have the same discontinuous piecewise smooth solutions.

(ii) In the horizontal strips of the Glimm scheme the Glimm approximate solution satisfies both the integral and weak form of the system of conservation laws considered. Glimm's arguments are referred to the relationship between this solution and an (exact) solution of the weak form (99). Thus, an equivalence is suggested between the integral and weak approaches, wider than the one stated by the remark (i) (see [39], [40], [41], [12]).

R20. A divergent form, as Glimm's method requires, cannot be associated to the system (103). However, as it is easy to see, this system is weakly nonlinear with RI so that, in this particular case, the problem of a global solution is solved by T5 and C2.

8.2. The Hugoniot curves. The Hugoniot -Lax theorem.

We regard now, given $u_\ell=u[X(t)-0,t]$, $t>0$, the jump conditions (108) as n relations for n+1 unknowns u_r and D. We shall subsequently consider, by eliminating D from (108) [then let $D=D(u_r, u_\ell)$], the projections

$$f(u)-f(u_\ell)=D(u, u_\ell)(u-u_\ell) \tag{114}$$

of these relations in the hodograph space H. Thus we have n-1 independent relations (114) between u and u_ℓ. We think about the problem of finding the set of vectors u which can be associated, as states to the right for the considered discontinuity, by these relations to a given vector u_ℓ (regarded as a state to the left).

T7 (H.Hugoniot [30], P.Lax [36]). Let $f\varepsilon C^m(R)$, $m\geq 3$, in the convex strictly hyperbolic system (95). For each $u_\ell \varepsilon R$ the set of vectors u_r which can be joined in H to u_ℓ as states to the right according to the relations (114) are laid, in a conveniently close neighbourhood of u_ℓ on the join of n (unique) smooth curves, one for each index i, $1\leq i\leq n$. For each of these curves a parametrization can be found so that each of them could be represented by

$$u=S_i(\delta_i, u_\ell), \quad S_i(0, u_\ell)=u_\ell \tag{115}$$

A representation (115) for which the function S_i is C^{m-1} holds in a conveniently small neighbourhood of each point $(\delta_i=0, u_\ell=\bar{u}_\ell)$, $\bar{u}_\ell \varepsilon R$ and for it

(i) u_ℓ cannot be a singular point on the curve (115),

(ii) the tangent in u_ℓ to S_i has the direction $\overset{i}{R}(u_\ell)$

(iii) $$D(0, u_\ell)=\lambda_i(u_\ell) \tag{116}$$

where we put, on each S_i, $D(\delta_i, u_\ell)\equiv D[S_i(\delta_i, u_\ell), u_\ell]$.

◀ (J.Conlon [7]). The conditions (114) can be written

$$[H(u, u_\ell)-D(u, u_\ell)I](u-u_\ell)=0 \tag{117}$$

where

$$H(u, u_\ell)=\int_0^1 A[u_\ell+z(u-u_\ell)]dz \tag{118}$$

As $u\neq u_\ell$ in (117), D is an eigenvalue of $H(u, u_\ell)$. Let i be its index. Then, from (118), it appears that as $u\to u_\ell$ each of the eigenvalues of $H(u, u_\ell)$ tends to an eigenvalue of $A(u_\ell)$. Since the system (95) is strictly hyperbolic it results that, for u conveniently close to u_ℓ, all the eigenvalues of $H(u, u_\ell)$ are real and distinct.

We can put (114) in the form

$$\Omega_k(u, u_\ell)\equiv\overset{k}{\ell}(u, u_\ell)(u-u_\ell)=0, \quad 1\leq k\leq n, \quad k\neq i$$

where $\overset{i}{\ell}$ is the left eigenvector of index i of the matrix H. We have

$$\left[\frac{\partial}{\partial u_j}\Omega_k\right]_{u=u_\ell}=\overset{i}{\ell}_j(u_\ell, u_\ell)=\overset{i}{L}_j(u_\ell)$$

and use the implicit function theorem.

In order to prove (ii) let us differentiate (114) along the curve S_i. We obtain

$$[A(u)-DI]\frac{du}{d\delta}=(u-u_\ell)\frac{dD}{d\delta} \tag{119}$$

As $\delta\to 0$ it results

$$[A(u_\ell)-D(0, u_\ell)I]\frac{du}{d\delta}\Big|_{\delta=0}=0 \tag{120}$$

The possibility $\frac{du}{d\delta}\Big|_{\delta=0}=0$ must be excluded according to the conclusion (i). Thus it appears that

$$\frac{du}{d\delta}\Big|_{\delta=0}=\alpha\overset{i}{R}(u_\ell), \quad \alpha\neq 0 \tag{121}$$

and so we get (116). ▶

<u>D 18.</u> The curve S_i described by (115), given u_ℓ, is called <u>a Hugoniot curve of index i</u>. A point of S_i characterizes <u>a discontinuity of index i</u> in the physical plane E.

<u>E7.</u> In the 1D adiabatic gasdynamics we have $\overset{i}{R}_j\neq 0$, $1\leq i,j\leq 3$, in R (cf. E1). Then for each i, $1\leq i\leq 3$, we have, in a close neighbourhood of u_ℓ (according to T7(ii)),

$[\![u]\!]_j \neq 0$, $1 \leq j \leq 3$, along a discontinuity of index i in E.

R21. The theorem T7 can be restated as follows: around each $u_\ell \varepsilon R$ we can find in the hodograph space H a convenient neighbourhood whose points can be displaced by a convenient (unique, smooth) mouvement along the Hugoniot curves of index i.

8.3. The case of the linearly degenerate fields. Contact discontinuity.

In case of the linearly degenerate fields the theorem T7 has the following remarkable consequence

T8 (P.Lax[36]). A discontinuity corresponding to a linearly degenerate index must propagate, as u_ℓ and u_r are conveniently close to each other along a characteristic.

◀ If k is a linearly degenerate index then (cf.(27)) $\lambda_k(u)$ is a k-RLI and the line through u_ℓ of the field $\overset{k}{R}(u)$ is laid on the surface $\lambda_k(u)$=constant=$\lambda_k(u_\ell)$. So, given u_ℓ, each point $u \neq u_\ell$ of the mentioned line satisfies (108) with $D=\lambda_k(u)=\lambda_k(u_\ell)$ because along this line we have $\frac{d}{d\delta}[f(u)-\lambda_k(u_\ell)u]=0$. It results that the line considered is a Hugoniot line of index k (see also R11(i)). The theorem follows from the uniqueness stated (as u_r and u_ℓ are conveniently close to each other) by T7. ▶

R22. Since $D=\lambda_k(u_\ell)$, given u_ℓ a discontinuity line x=Dt is isolated in E. Since $\frac{dD}{d\delta}$ =0 it appears that along this line a branching takes place (compatible with the fact that the line is laid on a characteristic curve). The possibility of a branching along a discontinuity line is typical of the linear degeneracy. The example E8 presents some details of this branching.

E8. For the system (22) the jump relations (108) have the form

$$[\![\rho(\frac{m}{\rho} - D)]\!] = 0$$

$$[\![m(\frac{m}{\rho} - D) + p]\!] = 0 \qquad (122)$$

$$[\![\rho S(\frac{m}{\rho} - D)]\!] = 0$$

In case of the linearly degenerate field of this system, $\lambda_2(u)$ given by (23) is a 2-RLI and along a line of the field $\overset{2}{R}$ we have

$$D = (\frac{m}{\rho})_\ell = (\frac{m}{\rho})_r \qquad (123)$$

$$[\![\frac{m}{\rho}]\!] = 0 \tag{124}$$

From $(122)_2$ we also obtain

$$[\![p]\!] = 0 \tag{125}$$

On the other hand, the relation $(122)_1$ allows, given u_ℓ (and, thus, given D), an arbitrary value for $[\![\rho]\!]$. This fact together with the remarks (124), (125) shows that the branching is characterized by the values of $[\![\rho]\!]$; particularly $\delta=[\![\rho]\!]$ can be taken as a parameter along of the Hugoniot curve S_2. The remark E7 also reflects, in case of the linearly degenerate field, the dependence of $[\rho,m,\rho S]$ on $[\![\rho]\!]$ as (124), (125) hold.

We motivate by R22 in order to give

D19. A discontinuity corresponding in E (under the assumptions of T8) to a linearly degenerate index is called a contact discontinuity (abbreviated cd).

We have ignored, in the proof of T7, the details of the parametrizations along the Hugoniot curves. In case of a linearly degenerate index such details are now considered.

R23. Let k be a linearly degenerate index. We start with the expressions $\overset{k}{R}(u)$ which define a vector field in $\mathcal{R}$. We next normalize this field by $|\overset{k}{R}|=1$ and determine $\overset{k}{L}$ according to $\overset{k}{L}.\overset{k}{R}=1$. Hence the requirement $|\overset{k}{R}|=1$ is supplemented by the choice (of the expressions $\overset{k}{R}(u)$ we start with, i.e.) of the orientation of the field $\overset{k}{R}$. Then we choose as a parameter ε_k the arc (conveniently oriented; originating with u_ℓ) of the line through u_ℓ of the mentioned field. Along this line we have

$$\frac{du}{d\varepsilon_k} = \overset{k}{R}(u) \tag{126}$$

and $u(0)=u_\ell$ (hence $\alpha=1$ in (121)). From (126) we further obtain

$$\frac{d^2u}{d\varepsilon_k^2} = [\overset{k}{R}(u).\mathrm{grad}_u]\overset{k}{R}(u) \tag{127}$$

8.4. The case of the genuinely nonlinear fields. Shock discontinuity. The Lax admissibility conditions

In case of a genuinely nonlinear index the details of a parametrization along a Hugoniot curve are given by

T9 (P.Lax [36]). Under the assumptions of T7 we can find a parameter ε_i along the curve S_i of genuinely nonlinear index i so that

$$\left.\frac{du}{d\varepsilon_i}\right|_{\varepsilon_i=0}=\frac{\overset{i}{R}(u_\ell)}{[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]_{u=u_\ell}} \tag{128}$$

$$\left.\frac{d^2u}{d\varepsilon_i^2}\right|_{\varepsilon_i=0}=\left.\frac{[\overset{i}{R}(u).\mathrm{grad}_u]\overset{i}{R}(u)}{[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]}\right|_{u=u_\ell}-$$

$$-\overset{i}{R}(u)\left.\frac{\frac{d}{d\varepsilon_i}[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]}{[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]^2}\right|_{u=u_\ell} \tag{129}$$

$$\left.\frac{\alpha D}{\alpha\varepsilon_i}\right|_{\varepsilon_i=0}=\frac{1}{2}>0 \tag{130}$$

$$\mathrm{sign}\ \varepsilon_i=\mathrm{sign}[\lambda_i(u)-\lambda_i(u_\ell)] \tag{131}$$

According to T7 (ii) and (26) it appears, as u_r and u_ℓ are conveniently close to each other, that the curve S_i has no points in common with the surface $\lambda_k(u)$=constant=$\lambda_k(u_\ell)$ but the point u_ℓ.

Hence we can choose

$$\delta=\lambda_i(u)-\lambda_i(u_\ell) \tag{132}$$

as a parameter on S_i (this choice is parallel to (55)). Under the smoothness assumptions of T7 we differentiate along S_i in

$$\delta\equiv\lambda_i[u(\delta)]-\lambda_i(u_\ell)$$

and obtain

$$1\equiv\frac{du}{d\delta}.\mathrm{grad}_u\lambda_i(u) \tag{133}$$

Then, taking the genuinely nonlinear nature of the index into account, we get from (121) and (133)

$$\alpha=[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]^{-1}\Big|_{u=u_\ell} \tag{134}$$

Carrying (134) into (121) we arrive at

$$\left.\frac{du}{d\delta}\right|_{\delta=0}=\frac{\overset{i}{R}(u_\ell)}{[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]_{u=u_\ell}} \tag{135}$$

Next, we differentiate (119) along S_i and find

$$[(\frac{du}{d\delta}.\mathrm{grad}_u)A(u)]\frac{du}{d\delta}+[A(u)-DI]\frac{d^2u}{d\delta^2}=2\frac{dD}{d\delta}\frac{du}{d\delta}+\frac{d^2D}{d\delta^2}(u-u_\ell) \tag{136}$$

In the limit $\delta\to0$ we obtain, cf. (135) and (116),

$$[A(u_\ell)-\lambda_i(u_\ell)I](\frac{d^2u}{d\delta^2})_{\delta=0} =$$

$$= -\frac{\{[\overset{i}{R}(u).\mathrm{grad}_u]A(u)\}_{u=u_\ell}\overset{i}{R}(u_\ell)}{[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]^2_{u=u_\ell}} + \tag{137}$$

$$+2(\frac{dD}{d\delta})_{\delta=0}\frac{\overset{i}{R}(u_\ell)}{[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]_{u=u_\ell}}$$

Then we multiply (137) by $\overset{i}{L}(u_\ell)$ and find, taking (54) into account,

$$(\frac{dD}{d\delta})_{\delta=0} = \frac{1}{2} \tag{138}$$

Hence, we can transcribe (137) by

$$[A(u_\ell)-\lambda_i(u_\ell)I](\frac{d^2u}{d\delta^2})_{\delta=0} = (\frac{du}{d\delta})_{\delta=0} - [(\frac{du}{d\delta}.\mathrm{grad}_u)A(u)]_{\delta=0}(\frac{du}{d\delta})_{\delta=0} \tag{139}$$

Now let us differentiate along S_i in $[A(u)-\lambda_i(u)I]\overset{i}{R}(u)=0$. Thus, in the limit $\delta\to0$, we find

$$[A(u_\ell)-\lambda_i(u_\ell)I](\frac{d}{d\delta}\overset{i}{R})_{\delta=0}=$$

$$=(\frac{du}{d\delta})_{\delta=0}[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]_{u=u_\ell} - \tag{140}$$

$$-[(\frac{du}{d\delta}.\mathrm{grad}_u)A(u)]_{\delta=0}\overset{i}{R}(u_\ell)$$

From (137), (140) it appears that

$$[A(u_\ell)-\lambda_i(u_\ell)I]\{(\frac{d^2u}{d\delta^2})_{\delta=0} - \frac{(\frac{d}{d\delta}\overset{i}{R})_{\delta=0}}{[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]_{u=u_\ell}}\} = 0$$

Then we have

$$(\frac{d^2u}{d\delta^2})_{\delta=0} = \frac{(\frac{d}{d\delta}\overset{i}{R})_{\delta=0}}{[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]_{u=u_\ell}} + \beta\overset{i}{R}(u_\ell) \tag{141}$$

We reparametrize S_i taking $\delta=\delta(\varepsilon_i)$ and requiring

$$\delta(0)=0,\ \delta'(0)=1,\ \delta''(0)=-\beta_1 \tag{142}$$

We obtain

$$\delta=\varepsilon_i-\frac{1}{2}\beta_1\varepsilon_i^2+0(\varepsilon_i^3) \tag{143}$$

We use (143) in order to transcribe (135) and (138) by (128) and (130) respectively. We further use (143) in (141) and find

$$\left(\frac{d^2u}{d\varepsilon_i^2}\right)_{\varepsilon_i} = \frac{\left(\frac{d}{d\varepsilon_i}\overset{i}{R}\right)_{\varepsilon_i=0}}{[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]_{u=u_\ell}} + (\beta-\beta_1)\overset{i}{R}(u_\ell) \tag{144}$$

Then taking

$$\beta_1=\beta-\left(\frac{\frac{d}{d\varepsilon_i}[\overset{i}{R}(u)\cdot\mathrm{grad}_u\lambda_i(u)]}{[\overset{i}{R}(u).\mathrm{grad}_u\lambda_i(u)]^2}\right)_{\varepsilon_i=0} \tag{145}$$

in (142), (143) we obtain (129).

Now let us consider for (143) the form

$$\delta=\varepsilon_i-\frac{1}{2}\beta_1\varepsilon_i^2+\frac{1}{4}a\varepsilon_i^3=\frac{1}{4}\varepsilon_i(a\varepsilon_i^2-2\beta_1\varepsilon_i+4)$$

As $a>\frac{1}{4}\beta_1^2$ it appears that sign $[\lambda_i(u)-\lambda_i(u_\ell)]$=sign δ=sign ε_i. ▶

The parametrizations corresponding to R11,(cf.(55)), R23, T9 (cf.(132), (143), (145)) are taken together in fig.10.

<u>L5.</u> Let i be a genuinely nonlinear index. If $u_r\varepsilon S_i(u_\ell)$, $u_r\neq u_\ell$ and u_r, u_ℓ are conveniently close to each other then D is not an eigenvalue for the matrices $A(u_r)$ or $A(u_\ell)$.

◀ We remark that

$$\lim_{\varepsilon_i\to 0}\frac{d}{d\varepsilon_i}\lambda_i[u(\varepsilon_i)]=\lim_{\varepsilon_i\to 0}[\mathrm{grad}_u\lambda_i]\frac{du}{d\varepsilon_i}\overset{(128)}{=}1 \tag{146}$$

Then we compare (146) with (130) and take into account the strict hyperbolicity of the system (95). ▶

Let us consider two constant regions in $\mathbb{R}_+^2$ (the half plane $t\geq 0$) adjacent along a discontinuity of genuinely nonlinear index. We assume that the values $u=u_\ell$, $u=u_r$ the solution takes in these regions respectively and the velocity D with which the discontinuity propagates satisfy the relations (108).

The constant adjacent regions do not generally reach the initial line t=0. Let us continue the constant regions until they reach the initial line. Then we can isolate a piecewise constant function $u:\mathbb{R}_+^2\to\mathbb{R}^n$

$$u(x,t;u_\ell,u_r)=\begin{cases}u_\ell & ,x\leq Dt\\ u_r\varepsilon S_i(u_\ell) & ,x>Dt\end{cases} \tag{147}$$

and an initial value problem which associates to the system (95) the data

$$u(x,0)=\begin{cases}u_\ell & ,\ x<0\\ u_r\varepsilon S_i(u_\ell) & ,\ x>0\end{cases} \tag{148}$$

<u>D20.</u> We say that the function (147) is an <u>admissible/determined</u> solution of the problem (95), (148) if it can be (uniquely) determined from the data (148)(by the aid of the

method of characteristics). In this case we say that the discontinuity involved in (147) is admissible/determined. Motivating by gasdynamics, an admissible discontinuity is also said to be a shock discontinuity (abbreviated sd).

Let $\Delta_{\pm}$ the domains of determinacy corresponding, for the problem (95), (148), to the intervals $\mathbb{R}_{\pm}$ of the initial line (fig.11). Let us consider for each open interval of the initial line containing the origin its closure in $\mathbb{R}$. We denote by I the set of the closed intervals of the initial line obtained in this way. In order to reach the region $\mathbb{R}^2_+\setminus(\Delta_+ \cup \Delta_-$ the construction used in the method of characteristics must start with data associated to an interval of I. In such a case an ingredient of this construction is the use of the jump relations on the discontinuity. The presence of this ingredient (in fact the presence of a discontinuity) gives a selective feature to the admissibility requirement (cf. T10, T11 hereinbelow).

T10 (P. Lax [36]). Let us consider a discontinuity of genuinely nonlinear index i, corresponding to the points $u_\ell, u_r \varepsilon S_i(u_\ell)$.

We suppose that the assumptions of L5 are valid. The considered discontinuity is admissible iff the conditions (the Lax entropy conditions)

$$(CEL_i)\quad \begin{aligned} &\lambda_i(u_r)<D<\lambda_i(u_\ell)\\ &\lambda_{i-1}(u_\ell)<D<\lambda_{i+1}(u_r)\end{aligned} \tag{149}$$

are fulfilled (fig.12).

◁ Since the system (95) is strictly hyperbolic we have $\lambda_1(u)< \ldots <\lambda_n(u)$ in each of the regions adjacent to the discontinuity. Let j_ℓ and j_r the indices for which in the (left, right) adjacent regions we have respectively

$$\begin{aligned} &\lambda_{j_\ell}(u_\ell)<D<\lambda_{j_\ell+1}(u_\ell)\\ &\lambda_{j_r}(u_r)<D<\lambda_{j_r+1}(u_r)\end{aligned} \tag{150}$$

It is easy to see (fig.13) that (in the sense of increasing time)

(i) a characteristic in the left region approaches the discontinuity iff $\lambda>D$,

(ii) a characteristic in the right region approaches the discontinuity iff $\lambda<D$.

Then it appears that in each point of the discontinuity the left and right regions provide $n-j_\ell$ and j_r determinacy relations[1]. Hence in each point of the discontinuity we have

1) Let j be the index of a characteristic which approaches the discontinuity. The relation $v_j(u)=$ constant (see §3; v_p, $1\le p\le n$, are the RI) along the mentioned characteristic is called a determinacy relation.

$n-j_\ell+j_r$ determinacy relations and n jump relations so that the method of characteristics counts on $2n-j_\ell+j_r$ relations in all. These $2n-j_\ell+j_r$ relations allows us to find in each point of the discontinuity the values of the 2n+1 unknowns u_ℓ, u_r, D. Then the admissibility requirement is equivalent to the condition $2n-j_\ell+j_r=2n+1$ which gives $j_\ell=j_r-1$. An easy re-arrangement of (150) then leads to CEL_{j_r}. Now, it is easy to see, as u_r approaches u_ℓ, that $j_r=i$. ▶

R24. A characteristic of index $j\neq i$/respectively a characteristic of index i is refracted/respectively absorbed through a sd of index i (cf. fig.12).

R25. In case of a linearly degenerate index k the conditions (149) degenerate (cf.T8) in

$$\lambda_k(u_r)=D=\lambda_k(u_\ell) \tag{151}$$

In this case the second condition of the degenerate CEL_k results from (151).

T11 (P.Lax [36]). Let i be a genuinely nonlinear index. In a conveniently close neighbourhood of u_ℓ , on a Hugoniot curve $S_i(u_\ell)$ the fulfilment of CEL_i is equivalent to the requirement $\varepsilon_i<0$.

◀ At first we prove that the fulfilment of CEL_i leads to the requirement $\varepsilon_i<0$. This follows from (131) and $(149)_1$.

Next we prove, conversely, that the requirement $\varepsilon_i<0$ results - for conveniently small values of $|\varepsilon_i|$ - in the fulfilment of CEL_i. We remark that

$$\lim_{\varepsilon_i\to 0} \lambda_{i+1}[u(\varepsilon_i)]=\lambda_{i+1}(u_\ell)>\lambda_i(u_\ell) \tag{152}$$

For conveniently small values of $|\varepsilon_i|$ we find, comparing (130) with (146), that

$$\lambda_i[u(\varepsilon_i)]<D(\varepsilon_i)<\lambda_i(u_\ell) \quad \text{as } \varepsilon_i<0 \tag{153}$$

Now we put (152), (153) together. ▶

R26. In case of a genuinely nonlinear index i the theorem T11 isolates the admissible part ($\varepsilon_i<0$) of a Hugoniot curve.

In the second part of the proof of T11 we imposed upon supplementary restrictions on the negative values of ε_i permitted by T7. In the sequel we shall suppose these restrictions hold.

From (149), (151) it appears that in a close neighbourhood of u_ℓ the (admissible part of the) curves S_j, $1\leq j\leq n$, have no points in common but u_ℓ.

R27. A curve S_k of a linearly degenerate index can be regarded as a hybrid object. Indeed, on such a curve the states u_r and u_ℓ are connected to each other by a jump and, on the other hand, this curve is laid on a line of the field $\overset{k}{R}$.

The strange nature of this object has been already remarked by R22, R23.

R28. In the sequel the vector $\overset{i}{R}$ corresponding to a genuinely nonlinear index i will be normalized according to

$$\overset{i}{R}(u)\cdot\mathrm{grad}_u\lambda_i(u)=1 \tag{154}$$

We shall remark that (154) completely determines the field $\overset{i}{R}$ (see R23 in case of a linearly degenerate index). Then we determine $\overset{i}{L}$ according to $\overset{i}{L}.\overset{i}{R}=1$.

Taking (154) into account we give (59), (60), (128), (129) the form

$$\frac{du}{d\varepsilon_i} = \overset{i}{R}(u), \qquad \varepsilon_i\geq 0 \tag{155}$$

$$\frac{d^2u}{d\varepsilon_i^2} =[\overset{i}{R}(u)\cdot\mathrm{grad}_u]\overset{i}{R}(u), \qquad \varepsilon_i\geq 0 \tag{156}$$

along the rarefaction part of each curve $R_i(u_\ell)$, and

$$\frac{du}{d\varepsilon_i}\bigg|_{\varepsilon_i=0} = \overset{i}{R}(u_\ell) \tag{157}$$

$$\frac{d^2u}{d\varepsilon_i^2}\bigg|_{\varepsilon_i=0} =\{[\overset{i}{R}(u).\mathrm{grad}_u]\overset{i}{R}(u)\}_{u=u_\ell} \tag{158}$$

as $\varepsilon_i\to 0$, $\varepsilon_i<0$ in case of a curve $S_i(u_\ell)$ of genuinely nonlinear index.

Finally we complete this table by adding some details on D:

$$D(0,\ u_\ell)=\lambda(u_\ell),$$

cf.(116), and

$$\frac{dD}{d\varepsilon_i}\bigg|_{\varepsilon_i=0}=\begin{cases}1/2 & \text{in case of a genuinely nonlinear index } i\\ 0 & \text{in case of a linearly degenerate index } i\end{cases}$$

Moreover, in case of a linearly degenerate index k we have

$$\frac{dD}{d\varepsilon_i}\equiv 0 \quad \text{along the } S_k \tag{160}$$

R29. According to the result (130) a branching is not possible in case of a sd (see R22).

8.5. Riemann-Hugoniot local system of coordinates

D21. For each $u_\ell\varepsilon\mathcal{R}$ the curves

$$u=H_i(\varepsilon_i,u_\ell)\underset{=}{\mathrm{def}}\begin{cases}\begin{cases}S_i(\varepsilon_i,u_\ell), S_i(0,\ u_\ell)=R_i(0,\ u_\ell), \varepsilon_i\le 0\\ R_i(\varepsilon_i,u_\ell) \qquad\qquad\qquad\qquad \varepsilon_i\ge 0\end{cases} & \text{for a genuinely nonlinear } i\\ \\ S_i(\varepsilon_i,\ u_\ell) & \text{for a linearly degenerate } i\end{cases} \quad (161)$$

$1\le i\le n$, are said to be <u>the Riemann-Hugoniot curves</u> (abbreviated H_i, $1\le i\le n$, or RH curves).

<u>R30.</u> Since for a genuinely nonlinear i the curves R_i and S_i have, according to (59), (60), (128), (129), a second order contact in each $u_\ell \varepsilon R$, it appears, cf.R11, T9, that, for such an index, in a convenient neighbourhood of each point $(\varepsilon_i=0,\ u_\ell=\bar{u}_\ell)$, $\bar{u}_\ell\varepsilon R$ the representation (161) is C^2 in ε_i, C^{m-1} in u_ℓ and has piecewise continuous third (or higher) derivatives (with a jump at $\varepsilon_i=0$).

<u>R31.</u> The remarks R11(i), R21 can be now adapted: around each $u_\ell\varepsilon R$ we can find in H a convenient neighbourhood whose points can be displaced by a convenient (unique, smooth) mouvement along the RH curves of index i.

Let $\sigma=(k_1,\dots,k_n)$ be a permutation of the indices $1,\dots,n$.

From R30 it appears that in a conveniently close neighbourhood of each point $(\varepsilon=0,\ u_\ell=\bar{u}_\ell)$, $\bar{u}_\ell\varepsilon R$ the function

$$F_\sigma(\varepsilon,\ u_\ell)\equiv H_{k_n}[\varepsilon_{k_n},\ H_{k_{n-1}}[\varepsilon_{k_{n-1}},\ H_{k_{n-2}}[\dots,H_{k_1}[\varepsilon_1,\ u_\ell]\dots]]] \quad (162)$$

is C^2 in ε and C^{m-1} in u_ℓ. This function is related with a construction (denoted K_σ) which succesively considers, starting with u_ℓ, the points $u_1=H_{k_1}(\varepsilon_{k_1},\ u_\ell),\dots,u_{k_n}=H_{k_n}[\varepsilon_{k_n},u_{k_{n-1}}]$. The states $u_1,\dots,u_{k_{n-1}}$ are said to be <u>intermediate.</u>

<u>T12 (P.Lax [36]).</u> Under the assumptions of T7

(i) we can find, around each point $\bar{u}\varepsilon R$ and for every permutation σ of the indices $1,\dots,n$, a neighbourhood $W_\sigma(\bar{u})\subset R$ so that any two arbitrary points u_ℓ, $u_r\varepsilon W_\sigma(\bar{u})$ could be (uniquely) connected to each other, as states to the left/respectively to the right, by a construction K_σ for which

$$\varepsilon_{k_i}=\varepsilon_{k_i}(u_\ell,\ u_r),\quad 1\le i\le n \quad (163)$$

are C^2 in $W_\sigma(\bar{u})\times W_\sigma(\bar{u})$ and we have

$$\varepsilon_{k_i}=0 \qquad \text{as} \quad u_r=u_\ell \tag{164}$$

(ii) for each $\bar{u}_\ell \varepsilon R$/ respectively $\bar{u}_r \varepsilon R$ the functions $\varepsilon_{k_i}(\bar{u}_\ell, u)$, $1\leq i\leq n$/ respectively $\varepsilon_{k_i}(u, \bar{u}_r)$, $1\leq i\leq n$, are piecewise C^{m-1} in u with a jump through the surfaces $\varepsilon_{k_i}(\bar{u}_\ell, u)=0$/respectively $\varepsilon_{k_i}(u, \bar{u}_r)=0$ of genuinely nonlinear index.

◀ We consider for each $\bar{u}\varepsilon R$, cf.(162), the function

$$\Phi_\sigma(u, \varepsilon, u_\ell)\equiv u - F_\sigma(\varepsilon, u_\ell)$$

in a conveniently close neighbourhood of $(u=\bar{u}, \varepsilon=0, u_\ell=\bar{u})$. We have

$$\Phi_\sigma(\bar{u}, 0, \bar{u}) = 0$$

and, according to (126), (155), (157), (162),

$$\Delta=\det\left(\frac{\partial}{\partial\varepsilon_i}\Phi_{\sigma i}\right)\Big|_{(\bar{u},0,\bar{u})}\neq 0$$

Then we use the implicit function theorem. ▶

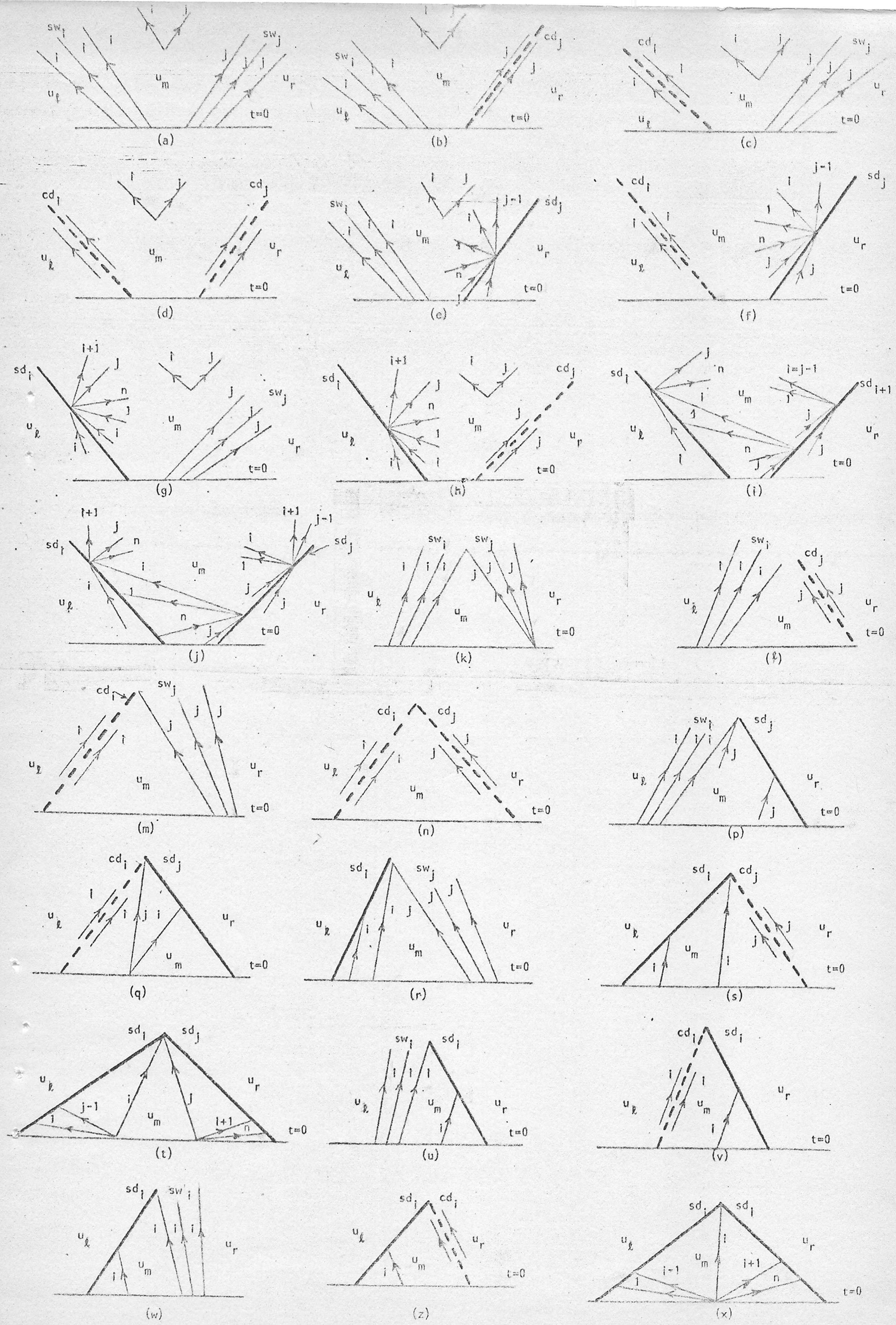
<u>C4.</u> For each point $\bar{u}\varepsilon R$, given a neighbourhood $U(\bar{u})\subset R$ we can find the neighbourhoods $W_\sigma(\bar{u})\subset V_\sigma(\bar{u})\subset U(\bar{u})$ so that any two states of $W_\sigma(\bar{u})$ can be connected to each other by a construction K_σ having its intermediate states in $V_\sigma(\bar{u})$.

<u>D22.</u> Given $u_\ell \varepsilon R$, the n-tuple $[\varepsilon_{k_1}(u_\ell, u_r),\ldots,\varepsilon_{k_n}(u_\ell, u_r)]$ is said to contain <u>the (local) Riemann-Hugoniot coordinates of index σ of the point</u> $u_r \varepsilon R$ <u>with respect to</u> u_ℓ. (abbreviation: <u>σ-RHs</u> for the Riemann-Hugoniot system of coordinates).

The construction K_σ can be carried in E by using four ingredients: constant region, rarefaction simple waves region separating two constant regions, <u>sd</u> separating two constant regions, <u>cd</u> separating two constant regions. In the sequel the last three of these ingredients are called <u>elementary waves</u> (separating two constant regions).

Let us assume that a partition of $\mathbb{R}^2_+$ into three constant regions (corresponding to the states) u_ℓ, u_m, u_r is possible so that the regions u_ℓ, u_m should be the left/respectively right adjacent regions of an elementary wave of index i and, similarly, the regions u_m, u_r should be the left/respectively right adjacent regions of an elementary wave of index j. As t>0 the constant region u_m is bounded by the straight lines $x=P_i t+x_{oi}$ (to the left) and $x=P_j t+x_{oj}$ (to the right), where $x_{oi}\leq x_{oj}$ and

$$P_k=\begin{cases}\lambda_k(u_m) & \text{if the elementary wave of index k is a simple waves region or a } \underline{cd}\\ D_k & \text{if the elementary wave of index k is a } \underline{sd}.\end{cases} \tag{166}$$



(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(ℓ)
(m)
(n)
(p)
(q)
(r)
(s)
(t)
(u)
(v)
(w)
(z)
(x)

D23. The two elementary waves - considered herein above- of indices i and j respectively, for which i labels the left wave, are said to be interactive if $P_i > P_j$ and noninteractive otherwise.

D24. The two elementary waves - considered herein above- are said to be interactive if i>j or if i=j and at least one of them is a sd (noninteractive otherwise).

L6. The definitions D23 and D24 are equivalent.

◀ We exhaustively consider the possible cases.

As i<j, we have (cf.(166))

$P_i=\lambda_i(u_m)<\lambda_j(u_m)=P_j$ (fig.14a,b,c,d) or,

$P_i=\lambda_i(u_m)=\lambda_{j-1}(u_m)<D_j=P_j$ if $j-i=1$ or,

$P_i=\lambda_i(u_m)<\lambda_{j-1}(u_m)<D_j=P_j$ if $j-i>1$ (fig.14 e,f) or,

$P_i=D_i<\lambda_{i+1}(u_m)=\lambda_j(u_m)=P_j$ if $j-i=1$ or,

$P_i=D_i<\lambda_{i+1}(u_m)<\lambda_j(u_m)=P_j$ if $j-i>1$ (fig.14g, h), or,

$\lambda_i(u_m)<P_i=D_i<D_j=P_j<\lambda_{i+1}(u_m)$ if $j-i=1$ (fig.14i)

(we notice that the circumstance $\lambda_i(u_m)<P_j=D_j<D_i=P_i<\lambda_{i+1}(u_m)$ if $j-i=1$ is not possible because - cf.fig.14 t - it leads to $\lambda_i(u_m)>\lambda_{i+1}(u_m)$, a contradiction) or,

$P_i=D_i<\lambda_{i+1}(u_m)<D_j=P_j$ if $j-i\geq 2$ (fig.14j)

As i>j, we have

$P_i=\lambda_i(u_m)>\lambda_j(u_m)=P_j$ (fig.14k,ℓ,m,n) or,

$P_i=\lambda_i(u_m)>\lambda_j(u_m)>D_j=P_j$ (fig.14p,q) or,

$P_i=D_i>\lambda_i(u_m)>\lambda_j(u_m)=P_j$ (fig.14r,s) or,

$P_i=D_i>\lambda_i(u_m)>\lambda_j(u_m)>D_j=P_j$ (fig.14t).

As i=j, we have

$P_i=\lambda_i(u_m)=\lambda_j(u_m)=P_j$ or,

$P_i=\lambda_i(u_m)=\lambda_j(u_m)>D_j=P_j$ (fig.14u,v) or,

$P_i=D_i>\lambda_i(u_m)=\lambda_j(u_m)=P_j$ (fig.14w,z) or,

$P_i=D_i>\lambda_i(u_m)=\lambda_j(u_m)\geq D_j=P_j$ (fig.14x). ▶

D25. A solution in $\mathbb{R}^2_+$ of (95) only consisting of noninteractive elementary waves and constant regions is said to be noninteractive or elementary.

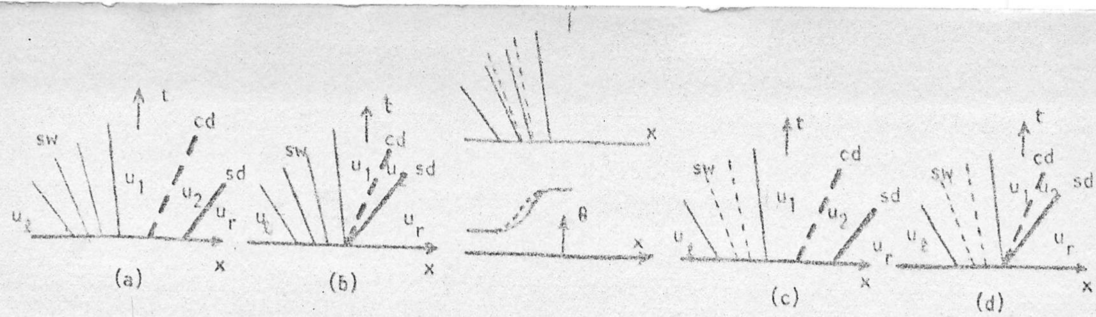


fig. 15

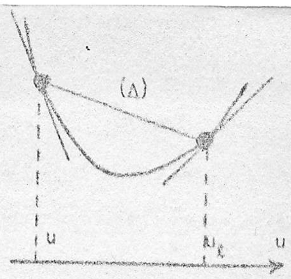


fig. 16

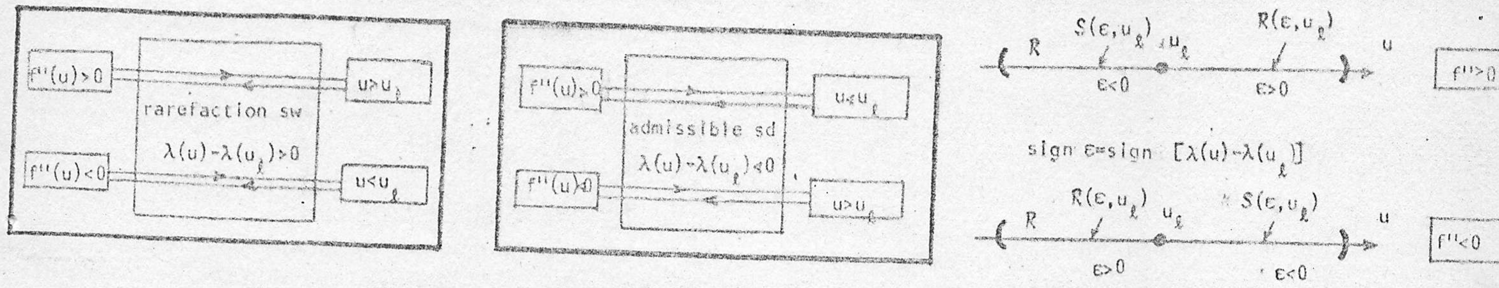


fig. 17

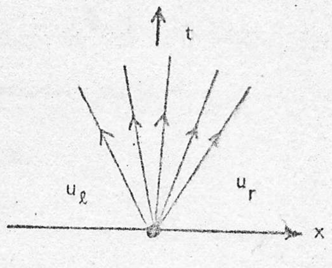


fig. 18

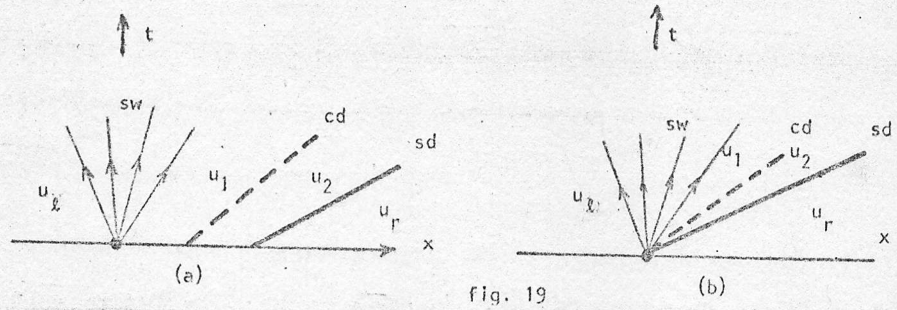


fig. 19

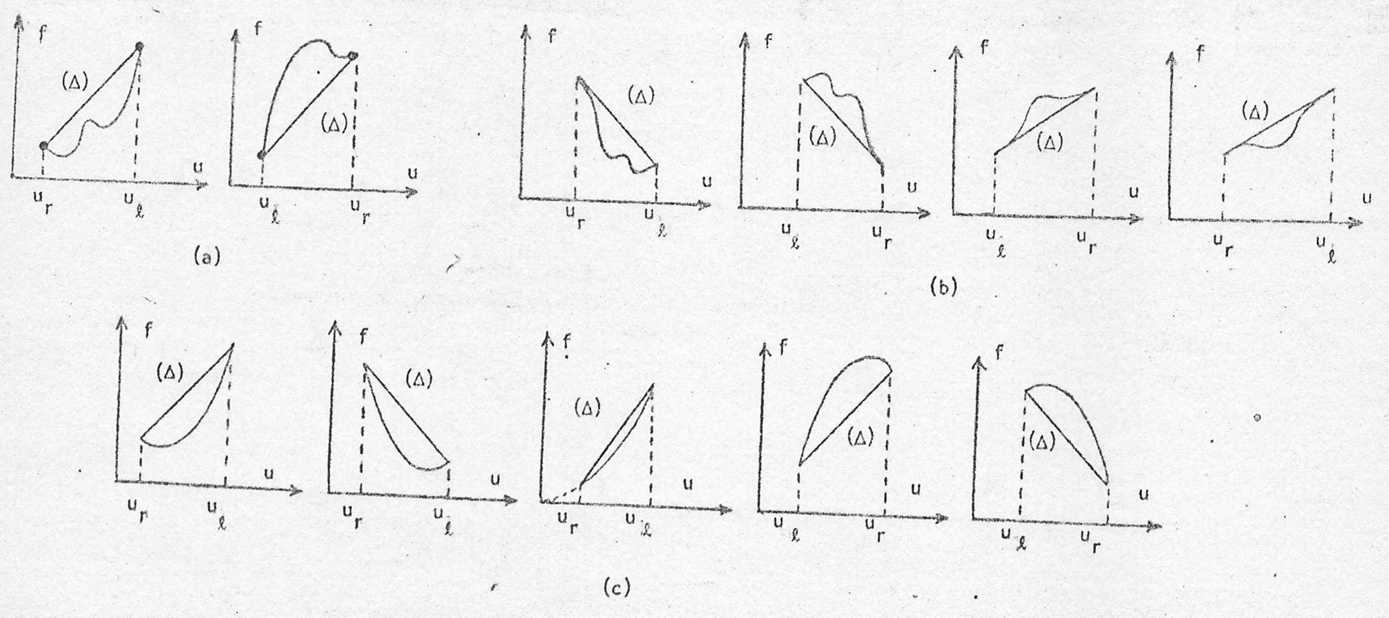
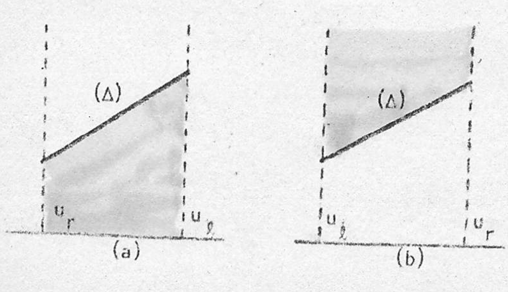


fig. 20



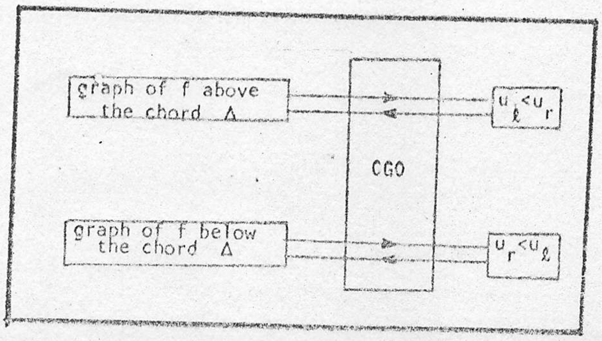


fig. 21

R32. The results of 6.5, 6.6 are reflected by the structure of an elementary solution. The rank of such a solution cannot be greater than one.

The theorem T12 associates to each pair u_ℓ, $u_r \varepsilon W_\sigma(\bar{u})$ a hodograph $\mathcal{H}_\sigma$ consisting of the RH arcs $\mathcal{H}_{k_1},\dots,\mathcal{H}_{k_n}$ of a (unique) construction K_σ. The definition D24 leads to the following significant result.

T13 (P.Lax [36]). The hodograph $\mathcal{H}_\sigma$ can be associated to an elementary solution iff σ is the identical permutation ($k_i=i$, $1\le i\le n$).

R33. As we broaden or narrow the constant regions of an elementary solution the hodograph of this solution keeps unchanged; on the other hand, the finding of a simple waves region in $\mathbb{R}^2_+$ corresponding to a given hodograph depends on the choice of the function θ in (37) and on the width of the interval on which this function is nonconstant (example: the solutions depicted in fig.15 a,b,c,d have the same hodograph)

D26. If the permutation σ is chosen according to T13 then the σ-RHs is called the Riemann-Hugoniot physical coordinate system (abbreviated RHs).

R34. (i) The characteristic coordinate system mentioned in R8 is interactive.

(ii) A compression simple waves region evolves interactively.

R35. In case n=1 we have (cf.E6) for a convex equation:

$$A(u)\equiv a(u),\ \lambda(u)\equiv a(u),\ R(u)\equiv[f''(u)]^{-1} \tag{165}$$

Here the region $\mathcal{R}$ is an open interval of $\mathbb{R}$. The system (59) has the form

$$\frac{du}{d\varepsilon}=[f''(u)]^{-1} \tag{166}$$

which can be transcribed by

$$\frac{d}{d\varepsilon}\lambda(u)=1 \tag{167}$$

Given $u_\ell \varepsilon \mathcal{R}$, as a state to the left (characterized by $\varepsilon=0$), from (167) we find (see(55))

$$\varepsilon=\lambda(u)-\lambda(u_\ell) \tag{168}$$

The jump relation (cf.(108))

$$f(u)-f(u_\ell)=D(u-u_\ell) \tag{169}$$

is graphically analysed in fig.16. Given the point $[u_\ell,\ f(u_\ell)]$ on the graph of f, for each point $u\varepsilon\mathcal{R}$ we can determine D as the slope of the chord Δ which connects this point to the point $[u,f(u)]$ and, on the other hand, to each value $D\ne f'(u_\ell)=\lambda(u_\ell)$ a (unique) point on the mentioned graph can correspond.

$$\lambda(u)=f'(u)<D<\lambda(u_\ell)=f'(u_\ell) \qquad (170)$$

The point u_ℓ divides the region R into two subintervals. Each point of these subintervals can be connected to u_ℓ by a rarefaction simple waves region or a sd[1]: this circumstance determines the nature of the considered subinterval. The nature of each of these subintervals depends on the sign of $f''\neq 0$ in R. This fact is explained in fig.17.

The RH curve through each point $u_\ell \varepsilon R$ can be continued in the whole R.

9. The Riemann problem

9.1. The formulation of the Riemann problem. Selfsimilar solutions. The class $\mathcal{A}$.

A Riemann problem for the system (95) (abbreviated RP) is an initial value problem which associate to this system the data

$$u(x\ ,0)=\begin{cases} u_\ell=\text{constant}, & x<0 \\ u_r=\text{constant}, & x>0 \end{cases} \qquad (171)$$

where the vectors u_ℓ, u_r are arbitrarily prescribed .

R35. The RP is invariant under the transformation

$$\bar{x}=\alpha x, \quad \bar{t}=\alpha t , \quad \alpha>0 \qquad (172)$$

This peculiarity suggeststo look for the RP for a solution of the form

$$u(x,t)\equiv\bar{u}(\tfrac{x}{t}) \qquad (173)$$

D27. A solution of the form (173) is said to be selfsimilar. A selfsimilar simple waves solution is said to be centered (fig.18). The class of the admissible (!) selfsimilar solutions is denoted $\mathcal{A}$.

9.2. The resolution of an arbitrary discontinuity.

The envelope (44) associated (cf.fig.3) to a centered simple waves region consists of a point (= center) only.

A centered simple waves region of index i is structured, according to P5, by a fan of characteristics of index i which radiate from the center. We put $y=\frac{x}{t}$. Then, taking (173) into account, the system (95) can be transcribed $[A(u)-yI]\frac{du}{dy}=0$

R36([36]). The considerations of 6.3 and R11, T13, R33 can be adapted in case of the selfsimilar solutions cf. $y=\lambda_i(u)$, $\varepsilon_i=y-\lambda_i(u_\ell)$. In contrast with the description 6.3 the "initial data" are singular here. With the notations of 6.3 we have $\bar{u}=U[\alpha(x,t)]$ where

1) As n>l only a conveniently close neighbpurhood of u_ℓ can be such a way characterized

$\alpha=\alpha(\frac{x}{t})$, $t>0$. The peculiarity of this case is that the function α (which satisfies the equation (36)) must be determined as a solution of the functional equation

$$\lambda_i\{U[\alpha(y)]\}-y=0, \qquad y\in I\subset\mathbb{R} \tag{174}$$

Then the implicit function theorem shows, using (26), (33), the existence of a unique smooth solution of this equation. Hence a selfsimilar simple waves solution is smooth inside the fan and, generally, only continuous on the characteristic which separate the simple waves region and the adjacent constant region. The system (59) can be transcribed, cf. (154), by $\frac{d\bar{u}}{dy}=\overset{i}{R}(\bar{u})$.

R37. The theorems T12, T13 and the remark R33 show an optimal character of the RP (see fig.19, a variant of fig.15). No element of the construction in $\mathbb{R}_+^2$ is arbitrary now.

We say that a selfsimilar solution of the RP describes the resolution of the arbitrary discontinuity (171) (into elementary waves).

9.3. An example of nonconvexity (O.A.Oleinik [50]).

The peculiarities of a nonconvex graph of f, as n=1 and under usual smoothness asumptions, are presented in fig.20a,b by comparison with a convex graph (fig.20 c).

We denote

$$F(u,v) = \frac{f(u)-f(v)}{u-v}$$

D28. In the nonconvex case n=1 we consider, instead of the admissibility conditions (149), the Oleinik (general) admissibility conditions (abbreviated CGO): a discontinuity which connects the (left, respectively right) states u_ℓ, u_r to each other is said to be admissible if

(i) the points $[u_\ell, f(u_\ell)]$, $[u_r, f(u_r)]$ are consecutive on the graph of f, i.e. the chord Δ connecting these points to each other does not intersect the graph but in these points,

(ii) for each point $[v, f(v)]$, $\min(u_r, u_\ell)<v<\max(u_r, u_\ell)$ one of the (mutually exclusive) restrictions

$$F(u_r, v)<F(u_r, u_\ell)$$
$$F(v, u_\ell)>F(u_r, u_\ell)$$

holds.

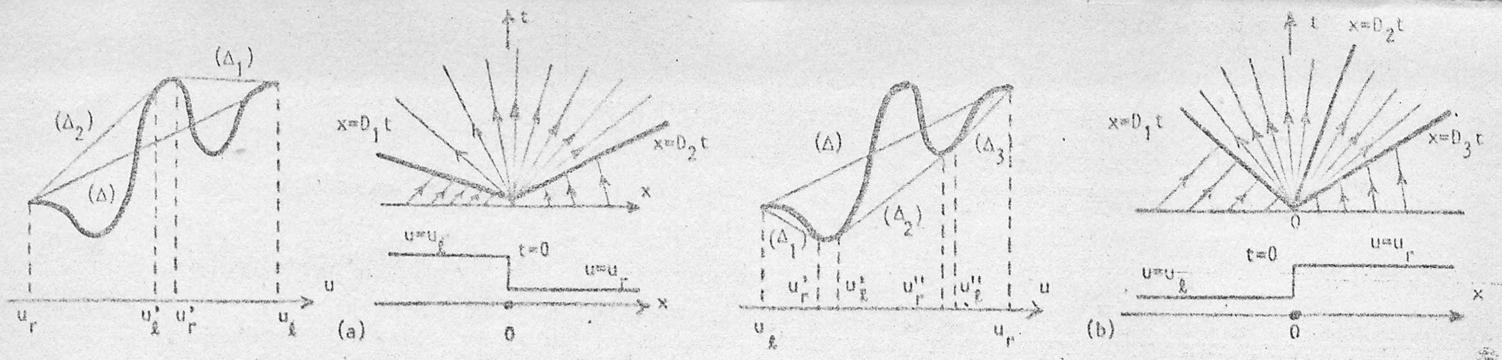


fig. 23

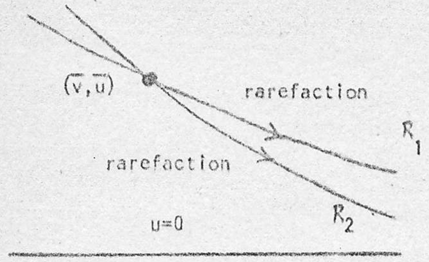


fig. 24

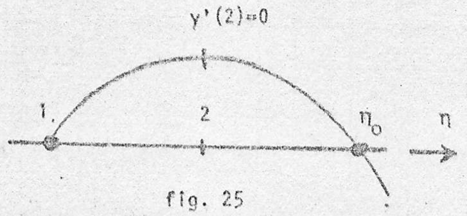


fig. 25

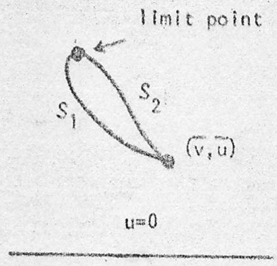


fig. 26

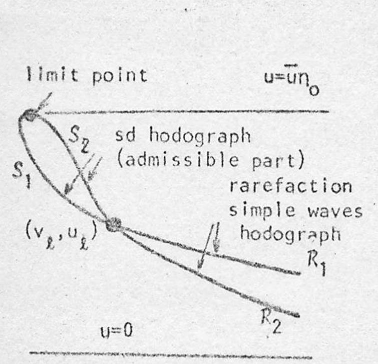


fig. 27

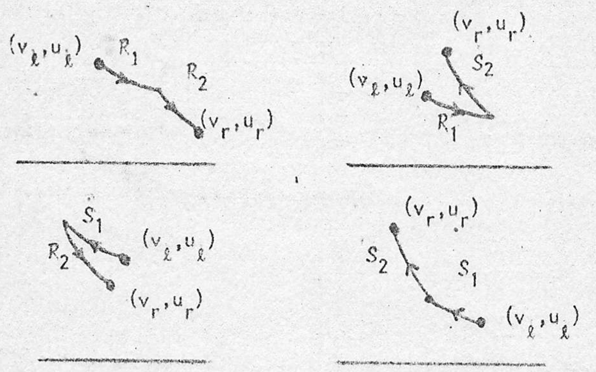


fig. 28

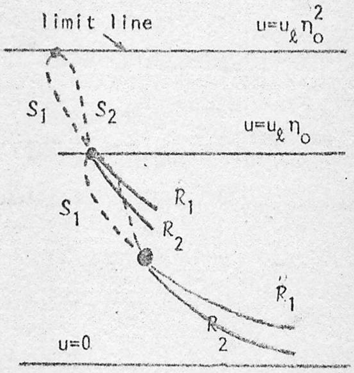


fig. 29

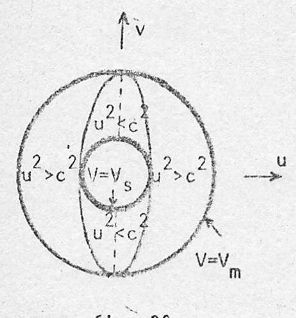


fig. 30

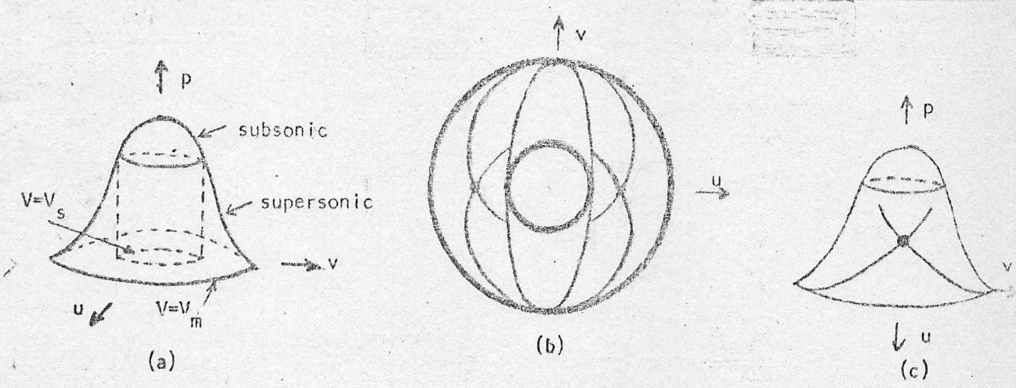


fig. 31

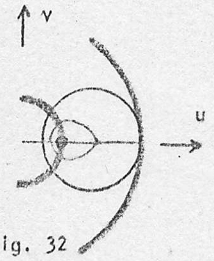


fig. 32

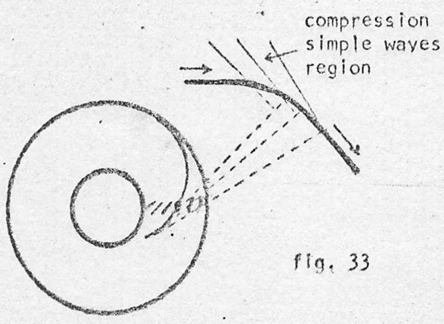


fig. 33

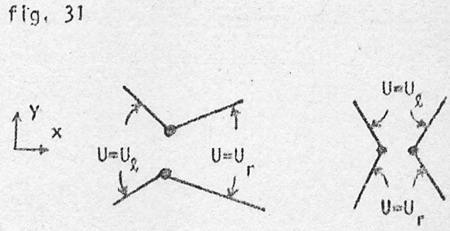
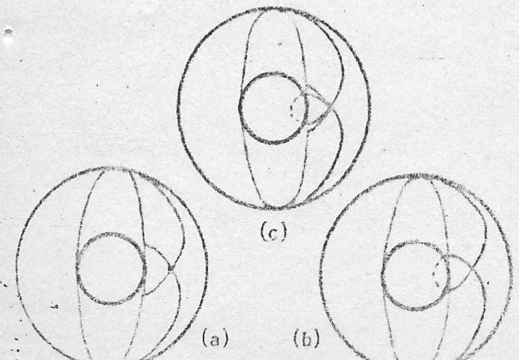


fig. 34



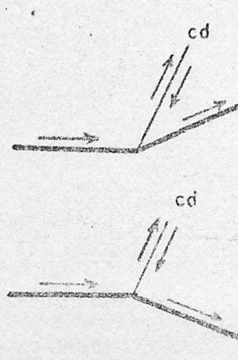


fig. 36

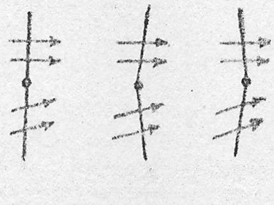


fig. 37


fig. 38

R38. (i) As $u_r<u_\ell$/respectively $u_\ell<u_r$ the requirement CGO (ii) is fulfilled by any point placed in the hachured region of fig.21a/fig.21b. Then it appears that CGO can be reworded by means of the scheme in fig.22.

(ii) As $f''\neq 0$ (the requirement CGO(i) is automatically fulfilled and the requirement CGO(ii) is equivalent to CEL, so that) CGO consist in CEL.

(iii) In case of a nonconvex system the inequalities (170) can be nonstrict (fig.20b; for a gasdynamic motivation (in the theory of detonations) see A.Hanyga [25]).

Let us now consider (given a graph of f) the circumstances $u_r<u_\ell$/respectively $u_\ell<u_r$ depicted in fig.23. The resolution of an arbitrary discontinuity can be obtained in the following way. We consider the concave envelope (fig.23a)/convex envelope (fig.23b) of the graph of f situated under the interval (min (u_ℓ, u_r), max (u_ℓ, u_r)). Each of these evelopes is uniquely determined. For each of the mentioned circumstances only one of these envelopes is admissible in the sense of CGO: the concave envelope in fig.23a, respectively the convex envelope in fig.23b. We explain this fact by means of fig.23a. Let D_1, D_2 the slopes of the chords Δ_1, Δ_2 (tangent to the graph at $u=u_r'$, $u=u_\ell'$ respectively) of the concave envelope. The RP has the solution

$$u(x,t)=\begin{cases} u_\ell & , \quad x<D_1t \\ F(x/t) & , \quad D_1t<x<D_2t \\ u_r & , \quad D_2t\leq x \end{cases}$$

where, cf.(174), the function F satisfies the equation

$$a(F)-s=0, \quad s\varepsilon(D_1,\ D_2) \tag{175}$$

Since (by construction) $f''\neq 0$ as $u\varepsilon(u_\ell',\ u_r')$, the implicit function theorem asserts that a unique solution exists for (175). Thus we can state

T14. In the nonconvex case n=1 a unique solution of the RP exists in $\mathcal{A}$ corresponding to an arbitrary pair $u_\ell,\ u_r\varepsilon R$.

R39. The figure 23 also describe the solution of the RP in $\mathbb{R}^2_+$. For an arbitrarily given $u_\ell\varepsilon R$, if u_r approaches u_ℓ or moves away from u_ℓ respectively the mentioned solution generally changes its structure thus reflecting the nonconvex character of the graph of f (as $f''\neq 0$ the structure of the solution keeps unchanged in each of the circumstances $u_r<u_\ell$ or $u_r>u_\ell$).

9.4. Final remarks

In the sequel global means free from restriction that u_ℓ, u_r should be close to each other.

9.4.1. Notes on the global existence of the Riemann problem solution in the class $\mathcal{A}$

Cf.T14, in the (nonconvex) case n=1 the RP has a global solution in $\mathcal{A}$ for each pair u_ℓ, $u_r \varepsilon R$.

We begin the discussion of the case $n \geq 2$ with an example of global non-existence.

E9 (V.A.Borovikov [4]). Let us consider the RP

$$\begin{cases} \frac{\partial}{\partial t} u + \frac{\partial}{\partial x}(3\log u+v)=0 \\ \frac{\partial}{\partial t} v + \frac{\partial}{\partial x}(\frac{2}{u})=0 \end{cases} \qquad u>0 \tag{176}$$

$$u(x,0),v(x,0) = \begin{cases} (u_\ell, v_\ell), & x<0 \\ (u_r, v_r), & x>0 \end{cases} \qquad u>0,\ u_r>0 \tag{177}$$

(the treatment of the case u<0, $u_\ell<0$, $u_r<0$ is completely analogous; $\log=\log_{10}$).

The system (176) is convex and has the eigenelements

$$\lambda_1=\frac{1}{u},\ \lambda_2=\frac{2}{u},\ \overset{1}{R}=\begin{pmatrix} -\frac{1}{2}u^2 \\ \frac{1}{2}u \end{pmatrix},\ \overset{2}{R}=\begin{pmatrix} -u^2 \\ 2u \end{pmatrix},\ u>0 \tag{178}$$

($\overset{1}{R}$, $\overset{2}{R}$ are normalized by (154)).

We can calculate from (176)

$$\frac{\partial f}{\partial v}\frac{\partial g}{\partial u}=-\frac{2}{u^2}<0 \tag{179}$$

as u>0 or u<0 (here f,g are the components of the flux).

In the hodograph space H the characteristic of index 1 through the point $(\bar{u}, \bar{v})$ is given (by

$$\frac{du}{d\varepsilon}=-\frac{1}{2}u^2,\ \frac{dv}{d\varepsilon}=\frac{1}{2}u,\ (u,v)_{\varepsilon=0}=(\bar{u},\bar{v})$$

which we transcribe) by

$$\frac{dv}{du}=-\frac{1}{u},\ v(\bar{u})=\bar{v} \tag{180}$$

and the characteristic of index 2 through $(\bar{u},\bar{v})$ is described by

$$\frac{dv}{du}=-\frac{2}{u},\quad v(\bar{u})=\bar{v} \tag{181}$$

The equations of the mentioned characteristics are respectively

$$u=\bar{u}\exp[-(v-\bar{v})] \qquad \text{(denoted } R_1)$$

$$u=\bar{u}\exp[-\tfrac{1}{2}(v-\bar{v})] \qquad \text{(denoted } R_2)$$

The rarefaction arcs of these characteristics are* laid in the region $u<\bar{u}$ (fig.24; cf. $\lambda_i(\bar{u})<\lambda_i(u)$, $i=1,2$: see D9).

The Hugoniot curves which connect $(\bar{u},\bar{v})$ (as a left state) to (u,v) are obtained by eliminating D from the jump relations

$$\begin{cases} 3\log u + v - 3\log\bar{u} - \bar{v} = D(u-\bar{u}) \\ \dfrac{2}{u} - \dfrac{2}{\bar{u}} = D(v-\bar{v}) \end{cases} \tag{182}$$

We put

$$\eta=\frac{u}{\bar{u}}, \quad \xi=v-\bar{v} \tag{183}$$

and obtain for the equation of these curves the form

$$\xi^2+3\xi\log\eta+2\,\frac{(\eta-1)^2}{\eta} = 0 \tag{184}$$

It appears that these curves are real if

$$\Delta=9\log^2\eta - 8\,\frac{(\eta-1)^2}{\eta} \geqslant 0 \tag{185}$$

We define

$$y(\eta)=\frac{3}{4}\log\eta-\frac{\eta-1}{\sqrt{2\eta}} \tag{186}$$

and find

$$y(\tfrac{1}{\eta})=-y(\eta) \tag{187}$$

The requirement (185) consists of

$$y(\eta)\leqslant 0 \quad , \quad 0<\eta\leq 1$$

$$y(\eta)\geqslant 0 \quad , \quad 1\leq\eta.$$

This can be transcribed, according to (187), by

$$\begin{matrix} y(\frac{1}{\eta})\geqslant 0 \quad , \quad 0<\eta<1 \\ y(\eta)\geqslant 0 \quad , \quad 1\leq\eta \end{matrix} \tag{188}$$

Motivating by (188) we only consider the circumstance $(188)_2$. We have

$$y'(\eta)=(2\eta)^{-3/2}\,\frac{(2-\eta)(\eta-\frac{1}{2})}{3\sqrt{2\eta}+2(\eta+1)}$$

and

$$y(100)=\frac{3}{2}-\frac{9.9}{\sqrt{2}}<0$$

so that (fig.25) as $\eta>1$ the equation $y(\eta)=0$ has a single root $\eta_o>2$. From (188) it appears that

$$\Delta>0 \text{ iff } 1<\frac{1}{\eta}<\eta_o \text{ or } 1<\eta<\eta_o \text{ i.e. iff } \frac{1}{\eta_o}<\eta<\eta_o$$

In other words, the Hugoniot curves through $(\bar{u}, \bar{v})$, denoted S_1, S_2, are real iff

$$\frac{\bar{u}}{\eta_o}<u<\bar{u}\eta_o \tag{189}$$

According to (183), (184) we have

$$v=\bar{v}-\frac{3}{2}\log\eta\pm\frac{1}{2}\sqrt{\Delta} \text{ along } S_1,\ S_2 \text{ respectively} \tag{190}$$

From (190) it appears that $|v-\bar{v}|$ keeps bounded in case $\frac{1}{\eta_o}<\eta<\eta_o$. The admissible parts of the Hugoniot curves associated to an arbitrarily given point of R are depicted in fig.26; these admissible parts have a second point in common, distinct from $(\bar{v}, \bar{u})$, and they cannot be continued beyond this point. The RH curves associated to an arbitrarily given point of R are depicted in fig.27.

The resolution of the arbitrary discontinuity (177) into a pair of elementary waves consists of four possibilities (sw=simple waves region separating two constant states): (sw_1, sw_2), (sw_1, sd_2), (sd_1, sw_2), (sd_1, sd_2), see fig.28.

From fig.28, fig.29 it appears that given $(u_\ell, v_\ell)\in R$ the RP for which $u_r>u_\ell\eta_o^2$ cannot be solved in $\mathcal{A}$.

In the convex case n=2 it is known that the global existence is guaranteed under the following restrictions on (95):

(R_1) strict hyperbolicity

(R_2) genuine nonlinearity

(R_3) (cf. [63.1]) $\overset{i}{L}(u)\cdot\{[\overset{j}{R}(u)\cdot \text{grad}_u]A(u)\}\cdot\overset{i}{R}(u)<0,\ i\neq j,\ (\forall)\ u\in R$

(we notice that, according to L2 and R28, we have $\overset{i}{L}(u)\cdot\{[\overset{i}{R}(u)\cdot \text{grad}_u]A(u)\}\cdot\overset{i}{R}(u)>0,\ (\forall)\ u\in R$),

$(R_{4,1})$ (cf. [63]) $\frac{\partial f_1}{\partial u_2}\cdot\frac{\partial f_2}{\partial u_1}>0\ ,\quad (\forall)\ u\in R$

$(R_{4,2})$ (cf. [33]) $\overset{i}{L}(u)\cdot\{[\overset{j}{R}(u)\cdot \text{grad}_u]A(u)\}\cdot\overset{i}{R}(u)<0,\ i\neq j,\ (\forall)\ u\in R$

$(R_{4,3})$ (cf. [33]) $\overset{1}{R}(\bar{u})\neq\overset{2}{R}(\bar{\bar{u}}),\quad (\forall)\ \bar{u},\ \bar{\bar{u}}\in R$

(R_5) (cf. [63]) every field line of index 1 intersects every field line of index 2.

The reversed requirement (R_3) and the additional conditions associated with it is, particularly, considered in V.A. Borovikov [4.1].

The restriction $(R_{4,1})$ guarntees that the Hugoniot curves go from bord to bord in R thus avoiding the circumstance isolated, in case of (179), by V.A. Borovikov.

The three requirements (R_4) are parallel to each other. We notice that $(R_{4,2})$ is equivalent to $\overset{i}{R}\cdot \text{grad}_u \lambda_j < 0$, $i \neq j$ and that $(R_{4,3})$ follows from $(R_{4,2})$. Also, we notice that (R_1) follows from $(R_{4,1})$. Examples of systems without property (R_5) are given in [63].

The requirements (R_1), (R_2) guarantee the local existence, cf. 8.2. In order to guarantee the global existence additional restrictions are needed: cf. (R_3)-(R_5).

In order to prove the global existence of a RP solution the requirements (R_2), (R_3), $(R_{4,1})$ are considered in [63]/respectively (R_1), (R_2), (R_3), $(R_{4,3})$, (R_5) are taken into account in [33] 1).

E 10. In case of the system

$$\frac{\partial v}{\partial t} - \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial t} + \frac{\partial \sigma(v)}{\partial x} = 0, \tag{191}$$

we have $\lambda_1 = [-\sigma'(v)]^{1/2} = -\lambda_2$ and suppose [cf. (R_1)] $\sigma' < 0$ and [cf. (R_2)] $\sigma'' \neq 0$.

It is easy to see that (R_3), $(R_{4,1})$ [and also $(R_{4,2})$, whence $(R_{4,3})$] hold.

In [1] the requirement

$$3(\sigma'')^2 < 2\sigma'\sigma''' \quad , \quad v > 0$$

is used, in the context of isentropic gasdynamics, instead of (R_5).

In [42] T.P. Liu considers - in a nonconvex case -

$$(R_2') \qquad \frac{\partial f_1}{\partial u_2} < 0 \; , \quad \frac{\partial f_2}{\partial u_1} < 0$$

$$\frac{\partial f_1}{\partial u_1} \geqslant 0 \; , \quad \frac{\partial f_2}{\partial u_2} < 0$$

instead of (R_2) and takes into account some additional requirements on the flux. We notice that (R_1) and $(R_{4,1})$ follow from $(R'_{2,1})$. In case of the system (191) (R_2') hold and the mentionned additional requirements consist in

1) In [33] the possibility of global existence when $(R_{4,3})$ is replaced by $(R_{4,2})$ is conjectured.

$(R''_{2,1})$ The zeros of σ'' must be isolated

$(R''_{2,2})$ The curve $w=\sigma(v)$ does not have vertical asymptotes and on each compact of the v-axis the number of zeros of σ'' is finite.

$(R''_{2,3})$ The curve $w=\sigma(v)$ does not have horisontal asymptotes.

Also, in [42] the admissibility conditions CEL and CGO are extended respectively in the requirement

(E) $D(u_\ell,u_r)<D(u_\ell,u),\quad u\in S(u_\ell)$

In C.M.Dafermos [11] and V.A.Tupchyev [66], [67], the construction of weak selfsimilar solutions of (95) as a limit (as $\varepsilon \to 0$) of the system

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon^2 t\, \frac{\partial^2 u}{\partial x^2}$$

is discussed as n = 2.

Argument: in case n = 1 A.S.Kalashnikov ([35]) proves that the limit solution does not depend on the presence of t in the dissipation term.

On putting $u = u(y)$, $y = \frac{x}{t}$ in the mentionned system we obtain the following form for the RP:

$$\varepsilon^2 \frac{d^2}{dy^2} u = \frac{d}{dy} f\,[u(y)] - y \frac{d}{dy} u\,, \quad u(-\infty) = u_\ell\,, \quad u(\infty) = u_r\,.$$

On starting with this problem (which corresponds to an autonomous system) an admissibility condition (denoted ET) is isolated in V.A.Tupchyev [66] (and revisited in [67]). In [66] the global existence of an ET - admissible solution to the RP for which in (95) the flux is an entire real analytic function is proved. On the other hand, for a continuously differentiable f in (95) the global existence of an (admissible in the sense of P.Lax [38]) solution to the RP is proved.

In the convex case n=3 an important (though specific) result is isolated by T15 herein below.

The theorems T15, T16 refer to the RP associated to the system of the adiabatic gasdynamics. For this system we assume that the specific internal energy $e=e(\tau,S)$ (S=specific entropy, τ=specific volume) satisfies the usual hypotheses of the ideal gases ($e>0$, $p>0$, $T>0$,

$\frac{\partial p}{\partial \tau}<0$, $\frac{\partial^2 p}{\partial \tau^2}>0$, $\frac{\partial p}{\partial S}>0$ ($p=-\frac{\partial e}{\partial \tau}$ = pressure, $T=\frac{\partial e}{\partial S}$ = temperature) and some reasonable additional requirements concerning its asymptotic properties). We denote $u=(\tau, p, m\tau)$.

T15 (R. Smith [62]). For any data (u_ℓ, u_r) the RP has (at least) a solution in $\mathcal{A}$.

9.4.2. Notes on the global uniqueness of the Riemann problem solution in the class $\mathcal{A}$

The global uniqueness (under convenient admissibility conditions: CEL/CGO/E/ET) is guaranteed in the (nonconvex) case n=1 (cf. T14) and, under restrictions (R_1)-(R_4) (cf. B.Keyfitz and H.Kranzer [33], J.A.Smoller [64]: also see I.M.Gel'fand [20]) or (R'_2), (R''_2) (cf. T.P.Liu [42]) in case n=2. We also notice that the ET-admissible solution constructed by V.A.Tupchyev in [66] is unique.

In the convex case n=3 there is no global uniqueness in $\mathcal{A}$, even in the class of continous solutions, generally (we notice that the solutions in $\mathcal{A}$ are admissible). Examples of global non-uniqueness in the subclass of continuous solutions are due to B.L.Rozdestvenskii, N.N.Yanenko [57] (for a system of the form (15)), V.F.Dyachenko [15] (for a system of the form (95)), V.A.Tupchiyev [68] (for a symmetric hyperbolic system of gradient-type [22]) An example of global uniqueness in $\mathcal{A}$ is presented by T16 hereinbelow.

In [62] the restrictions

MEDIUM	$\frac{\partial}{\partial \tau} p(\tau, e) \leq \frac{p^2}{2e}$	$(\tau, e>0)$
WEAK	$\frac{\partial}{\partial \tau} e(\tau, p) \geq -\frac{p}{2}$	$(\tau, p>0)$

are respectively added to the hypotheses of the ideal gases (see T15). The condition WEAK follows from MEDIUM. Hierarchy:

$$\text{POLYTROPIC} \subset \text{IDEAL} \subset \text{MEDIUM} \subset \text{WEAK} \qquad (192)$$

T16 (R. Smith [62]). The condition MEDIUM (the level MEDIUM in (192)) is necessary and sufficient for the global uniqueness in $\mathcal{A}$ of the RP solution.

In [62] functions e which (violate WEAK/satisfy WEAK but) violate MEDIUM, causing global non-uniqueness in $\mathcal{A}$, are constructed.

In [43], T.P.Liu relaxes the hypotheses, presented hereinbefore, of the ideal gases and proves, under restriction of the (already mentioned) extended admissibility condition proposed in [42], a nonconvex variant of T16.

9.4.3. Types of initial data. Types of solutions.

R40. There are 2^n types of initial data (171) (to each i, $1 \leq i \leq n$, two possibilities correspond: $u_{\ell i} < u_{ri}$ or $u_{\ell i} > u_{ri}$).

If all the characteristic fields of the system (95) are genuinely nonlinear then each index i, $1 \leq i \leq n$, can contribute in the RP solution in two ways: by a simple waves region of index i or by a sd_i. Then there are 2^n types of RP solutions in this case.

We can ask ourselves if a correspondence exists between the set of the 2^n types of initial data and the 2^n types of RP solutions.

In case n=1 (diagonal system) the answer is affirmative in the convex case and negative otherwise (see R39).

If $n > 1$ and there are k linearly degenerate fields then we only have 2^{n-k} distinct types of solutions and the mentioned correspondence does not hold generally.

Nevertheless, for a convex and diagonal system (95) (see for example (61)) having only genuinely nonlinear fields the affirmative answer found in case n=1 keeps valid: there is a correspondence between the signatures of the vectors $u_r - u_\ell$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$.

9.4.4. The importance of the independent variables nature. A reading in the Lax sense of the case of 2D steady adiabatic gasdynamics [14]

Now it is natural to inquire ourselves about the manner in which the Lax theory of the RP, described in the previous paragraphs, is reflected in the mirror of the 2D steady adiabatic gasdynamics context.

An answer to this question shall be presented in the sequel. It integrates, in a reading in the Lax sense, specific methods and motivations due to Prandtl and Busemann.

In particular, this answer shows an optimal character of the case in which one of the independent variables has a temporal nature.

Let us now consider the equations of 2D adiabatic gasdynamics (in usual notations, see [9]; n=4)

$$\begin{aligned}
&\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \\
&\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho u v) = 0 \\
&\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v^2 + p) = 0 \\
&\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}[u(E+p)] + \frac{\partial}{\partial y}[v(E+p)] = 0
\end{aligned} \qquad (193)$$

where we denoted

$$V^2=u^2+v^2, \quad E=\rho(e+\frac{1}{2}V^2), \quad c^2=\gamma\frac{p}{\rho} \tag{194}$$

and, up to an additive constant, we have for a perfect gas

$$e=\frac{1}{\gamma-1}\frac{p}{\rho} \tag{195}$$

We put

$$\begin{aligned} &u=(u_1, u_2, u_3, u_4) \\ &u_1=\rho, \quad u_2=\rho u, \quad u_3=\rho v, \quad u_4=E \end{aligned} \tag{196}$$

and

$$\begin{aligned} &\varphi_1(u)\equiv\rho u\equiv u_2 & &\varphi_2(u)\equiv\rho u^2+p\equiv\frac{u_2^2}{u_1}+p(u) \\ &\varphi_3(u)\equiv\rho uv\equiv\frac{u_2u_3}{u_1} & &\varphi_4(u)\equiv u(E+p)\equiv\frac{u_2}{u_1}[u_4+p(u)] \end{aligned} \tag{197}$$

$$\begin{aligned} &\psi_1(u)\equiv\rho v\equiv u_3 & &\psi_2(u)\equiv\rho uv\equiv\frac{u_2u_3}{u_1} \\ &\psi_3(u)\equiv\rho v^2+p\equiv\frac{u_3^2}{u_1}+p(u) & &\psi_4(u)\equiv v(E+p)\equiv\frac{u_3}{u_1}[u_4+p(u)] \end{aligned} \tag{198}$$

Then, in case of a steady flow, we obtain from (193) a system of the form (105)

$$\frac{\partial}{\partial x}\varphi_i(u)+\frac{\partial}{\partial y}\psi_i(u)=0, \quad 1\leq i\leq 4 \tag{199}$$

The same as in R17 we consider

$$U_i=\varphi_i(u), \quad 1\leq i\leq 4 \tag{200}$$

From (197), (200) we obtain

$$\Delta=\frac{D(\varphi_1,\varphi_2,\varphi_3,\varphi_4)}{D(u_1,u_2,u_3,u_4)}=u^2(u^2-c^2) \tag{201}$$

so that, for u in a region $\mathcal{D}$ where $\Delta\neq 0$ we have

$$u_i=g_i(U), \quad 1\leq i\leq 4 \tag{202}$$

Here are the expressions of the functions g:

$$\left.\begin{aligned} &g_1(U)\equiv\frac{(\gamma+1)U_1^2}{\gamma U_2+[\operatorname{sign}(u^2-c^2)]\{\gamma^2U_2^2-(\gamma^2-1)(2U_1U_4-U_3^2)\}^{1/2}} \\ &g_2(U)\equiv U_1, \quad g_3(U)\equiv\frac{U_3}{U_1}g_1(U), \\ &g_4(U)\equiv\frac{\gamma-1}{\gamma+1}U_2+[\frac{\gamma-1}{\gamma+1}\frac{U_3^2}{U_1^2}+\frac{3-\gamma}{\gamma+1}\frac{U_4}{U_1}]g_1(U) \end{aligned}\right\} \tag{203}$$

where from (197), (200) we have

$$\gamma^2U_2^2-(\gamma^2-1)(2U_1U_4-U_3^2)=\rho^2(u^2-c^2)^2>0$$

R41. We partition the hodograph space U into convexity regions (this term will be motivated later on (see R 43)). In a convexity region $\mathcal{R}$ the following requirements are fulfilled:

$$u\neq 0,\ u^2-c^2\neq 0,\ u^2+v^2-c^2>0 \tag{204}$$

A detailed description of such a region is given in R45 hereinbelow. In the subsequent considerations we suppose U is in a convexity region.

Thus we put

$$f_i(U)=\psi_i[g_1(U),\ g_2(U),\ g_3(U),\ g_4(U)],\quad 1\leq i\leq 4 \tag{205}$$

Taking (200), (202), (205) into account we obtain an analogue of (106)

$$\frac{\partial U}{\partial x}+\frac{\partial f(U)}{\partial y}=0 \tag{206}$$

The jump relations, analogous to (108),

$$[\![f(U)]\!]=D[\![U]\!] \tag{207}$$

can be, alternatively, presented [cf. (197), (198)] in the form

$$\begin{aligned}
&[\![\rho v]\!]=D[\![\rho u]\!]\\
&[\![\rho uv]\!]=D[\![\rho u^2+p]\!]\\
&[\![\rho v^2+p]\!]=D[\![\rho uv]\!]\\
&[\![\rho u(\tfrac{1}{2}V^2+\tfrac{\gamma}{\gamma-1}\tfrac{p}{\rho})]\!]=D[\![\rho v(\tfrac{1}{2}V^2+\tfrac{\gamma}{\gamma-1}\tfrac{p}{\rho})]\!]
\end{aligned} \tag{208}$$

where ρ, u, v, p are given by (202).

We now consider two constant states, let 1,2 be their labels respectively, adjacent to a discontinuity at the points of which (208) holds.

The algebraic study of relations (208) leads to the following results (cf. L7, L8, L9; see [31], [71]).

L7. If

$$\left(\frac{v}{u}\right)_1=\left(\frac{v}{u}\right)_2=D$$

in (208) then

(i) $$[\![p]\!]=0$$

(ii) the relations (208) hold for an arbitrary value of $[\![\rho]\!]$ or $[\![\frac{1}{2}V^2]\!]$.

L8. If

$$\left(\frac{v}{u}\right)_1 = \left(\frac{v}{u}\right)_2 \neq D$$

in (208) then

$$[\![\rho]\!]=0, [\![u]\!]=0, [\![v]\!]=0, [\![p]\!]=0$$

(i.e. there is no jump at all).

L9 (A. Busemann [6])[1]. If

$$\left(\frac{v}{u}\right)_1 \neq \left(\frac{v}{u}\right)_2$$

and

$$v_1=0$$

then the relations (208) are equivalent to

$$
\begin{aligned}
[\![u]\!] &= \frac{2}{\gamma+1}\,\frac{c_1^2}{(u)_1} - \frac{2D^2}{(\gamma+1)(1+D^2)}(u)_1 \\
[\![v]\!]^2 &= \frac{(\gamma+1)(u)_1[\![u]\!]+2[(u)_1^2-c_1^2]}{-(\gamma+1)(u)_1[\![u]\!]+2c_1^2}\,[\![u]\!]^2 \\
[\![p]\!] &= -\rho_1 (u)_1 [\![u]\!] \\
[\![\rho]\!] &= \rho_1 \frac{2(u)_1[\![u]\!]}{(\gamma-1)(u)_1[\![u]\!]-2c_1^2}
\end{aligned}
\tag{209}
$$

L10. If

$$\left(\frac{v}{u}\right)_1 \neq \left(\frac{v}{u}\right)_2$$

in (208) then we have

$$\left[\!\!\left[\frac{1}{2}V^2 + \frac{c^2}{\gamma-1}\right]\!\!\right] = 0$$

We carry (202) into (200) and differentiate the resulting identity in order to obtai[n] the expressions of $(\partial g_i/\partial U_j)$. Then, on using these expressions we calculate

$$
\begin{aligned}
&\frac{\partial f_1}{\partial U_1} = \frac{uv}{\Delta}\left(\frac{\gamma+1}{2}u^2 + \frac{\gamma-1}{2}v^2\right), \quad \frac{\partial f_1}{\partial U_2} = -\gamma\frac{u^2 v}{\Delta}, \\
&\frac{\partial f_1}{\partial U_3} = \frac{u}{\Delta}[(u^2+v^2-c^2)-\gamma v^2], \quad \frac{\partial f_1}{\partial U_4} = (\gamma-1)\frac{uv}{\Delta} \\
&\frac{\partial f_2}{\partial U_1} = 0, \quad \frac{\partial f_2}{\partial U_2} = 0, \quad \frac{\partial f_2}{\partial U_3} = 1, \quad \frac{\partial f_2}{\partial U_4} = 0 \\
&\frac{\partial f_3}{\partial U_1} = (\gamma-1)\frac{u}{\Delta}V^2\left(\frac{1}{2}V^2 + \frac{c^2}{\gamma-1}\right), \quad \frac{\partial f_3}{\partial U_2} = -\frac{u^2}{\Delta}[\gamma V^2-(u^2-c^2)] \\
&\frac{\partial f_3}{\partial U_3} = -2(\gamma-1)\frac{uv}{\Delta}\left[\left(\frac{1}{2}V^2 + \frac{c^2}{\gamma-1}\right) - \frac{u^2}{\gamma-1}\right], \quad \frac{\partial f_3}{\partial U_4} = (\gamma-1)\frac{u}{\Delta}V^2
\end{aligned}
\tag{210}
$$

---

1) Also see [31], [71]

$$\frac{\partial f_4}{\partial U_1}=(\frac{1}{2}V^2+\frac{c^2}{\gamma-1})^2\frac{\partial f_1}{\partial U_4}\ ,\quad \frac{\partial f_4}{\partial U_2}=(\frac{1}{2}V^2+\frac{c^2}{\gamma-1})\frac{\partial f_1}{\partial U_2}$$

(210)

[continued]

$$\frac{\partial f_4}{\partial U_3}=(\frac{1}{2}V^2+\frac{c^2}{\gamma-1})\frac{\partial f_1}{\partial U_3}\ ,\quad \frac{\partial f_4}{\partial U_4}=\frac{\partial f_1}{\partial U_1}$$

We also calculate

$$du=\frac{1}{\rho(u^2-c^2)}[-(\frac{\gamma+1}{2}u^2+\frac{\gamma-1}{2}v^2)dU_1+\gamma u dU_2+(\gamma-1)v dU_3-(\gamma-1)dU_4] \tag{211}$$

$$dv=\frac{1}{\rho u}(-v dU_1+dU_3) \tag{212}$$

$$d\rho=\frac{1}{u(u^2-c^2)}[(\frac{\gamma+3}{2}u^2+\frac{\gamma-1}{2}v^2-c^2)dU_1-\gamma u dU_2-(\gamma-1)v dU_3+(\gamma-1)dU_4] \tag{213}$$

$$dp=(\gamma-1)\frac{u^2}{\Delta}\{u(\frac{1}{2}V^2+\frac{c^2}{\gamma-1})dU_1-[\frac{1}{2}(u^2-v^2)+(\frac{1}{2}V^2+\frac{c^2}{\gamma-1})]dU_2-uv dU_3+u dU_4\} \tag{214}$$

$$d(\frac{1}{2}V^2+\frac{c^2}{\gamma-1})=\frac{1}{\rho u}[-(\frac{1}{2}V^2+\frac{c^2}{\gamma-1})dU_1+dU_4] \tag{215}$$

The matrix

$$A(U)=(\frac{\partial f_i}{\partial U_j}) \tag{216}$$

has the following eigenelements: the eigenvalues

$$\lambda_{1,4}=\frac{uv\pm c(u^2+v^2-c^2)^{1/2}}{u^2-c^2}\ ,\qquad \lambda_2=\lambda_3=\frac{v}{u} \tag{217}$$

and the eigenvectors

$$\overset{1,4}{R}=\Lambda_{1,4}[1,\ \frac{\mp cv+u(u^2+v^2-c^2)^{1/2}}{(u^2+v^2-c^2)^{1/2}}\ ,\ \frac{\pm cu+v(u^2+v^2-c^2)^{1/2}}{(u^2+v^2-c^2)^{1/2}}\ ,\ \frac{1}{2}V^2+\frac{c^2}{\gamma-1}] \tag{218}$$

$$\overset{2}{R}=\Lambda_2[1,2u,2v,(\frac{1}{2}V^2+\frac{c^2}{\gamma-1})+V^2] \tag{219}$$

$$\overset{3}{R}=\Lambda_3[1,u,v,\frac{1}{2}V^2] \tag{220}$$

R42. The expressions (219), (220) correspond to the requirement that

$d\rho=0$ along a line of the field $\overset{2}{R}$

$d(\frac{1}{2}V^2)=0$ along a line of the field $\overset{3}{R}$

The fields of indices k=2,3 are linearly degenerate in a convexity region $\mathcal{R}$

$$\overset{k}{R}(U).\mathrm{grad}_U\lambda_k(U)=0,\ k=2,3,\quad \text{in } \mathcal{R} \tag{221}$$

We normalize the vectors $\overset{1}{R},\overset{4}{R}$ according to

$$\overset{i}{R}(U).\mathrm{grad}_U\lambda_i(U)=1,\ i=1,4,\quad \text{in } \mathcal{R} \tag{222}$$

and the vectors $\overset{2}{R},\overset{3}{R}$ according to $\overset{k}{R}=1$, k=2,3, thus obtaining, particularly,

$$\Lambda_{1,4} = \mp \frac{2}{\gamma+1} \frac{\rho^3}{cV^6}(u^2+v^2-c^2)[\pm cv+u(u^2+v^2-c^2)^{1/2}]^3 \tag{223}$$

We notice that

$$\Lambda_1\Lambda_4 = -\frac{4}{(\gamma+1)^2} \frac{\rho^6}{c^2V^6}(u^2+v^2-c^2)^2(u^2-c^2)^3 \tag{224}$$

Let Δ be the determinant whose rows are $\overset{i}{R}(U)$, $1 \leq i \leq 4$. We have

$$\Delta = -\frac{2}{\gamma-1} \frac{c^3V^2}{(u^2+v^2-c^2)^{1/2}} \Lambda_1\Lambda_2\Lambda_3\Lambda_4 \tag{225}$$

R43. In a convexity region $\mathcal{R}$ the system (206) is convex. In particular, this system is (nonstrictly) hyperbolic and the eigenvectors $R(U)$ are independent.

On carrying (218) or (219), (220) into (211)-(215) and taking R41 into account it appears that

L11. (i) Along a line $\mathcal{R}_k$ of the field $\overset{k}{R}$, k=2,3 we have

$$p(U) = \text{constant} \tag{226}$$

$$\frac{v(U)}{u(U)} = \text{constant} \tag{227}$$

(ii) Along a line $\mathcal{R}_2$ we have

$$\rho(U) = \text{constant}$$

(iii) Along a line $\mathcal{R}_3$ we have

$$\frac{1}{2}V^2(U) = \text{constant}$$

L12. Along a line $\mathcal{R}_i$ of the field $\overset{i}{R}$, i=1,4 we have

$$\frac{1}{2}V^2(U) + \frac{c^2(U)}{\gamma-1} = \text{constant} \tag{228}$$

$$S(U) = \text{constant} \quad (S=\text{entropy}) \tag{229}$$

$$\frac{dv}{du} = -\frac{1}{\lambda_i} \tag{230}$$

R44. The lemma L11 is a variant of L7. The two degrees of freedom found cf. L7(ii) have been associated, cf. R42, to the lines $\mathcal{R}_2$ and $\mathcal{R}_3$ respectively. In this manner the mentioned lines are respectively parametrized by $[\![\rho]\!]$ and $[\![\frac{1}{2}V^2]\!]$ (see E8).

R45. The lemmas L10 and L12 suggest a study of the relation [cf. $c^2 = c^2(p, S)$]

$$C(u,v,p;S,K)=\frac{1}{2}V^2+\frac{c^2}{\gamma-1}-K=0 \tag{231}$$

We shall regard S,K as parameters. Given S,K we denote

$$V_s=(2\frac{\gamma-1}{\gamma+1}K)^{1/2},\quad V_m=(2K)^{1/2},\quad K>0 \tag{232}$$

Given S,K we can describe (231) in space u,v,p as a <u>pressure hill</u> ([71], fig.31 a).

Now, we give to (231) the form

$$V^2-c^2=\frac{\gamma+1}{2}(V^2-V_s^2) \tag{233}$$

Then, it appears that

$$V^2-c^2\geqslant 0 \quad \text{iff } V^2\geqslant V_s^2 \tag{234}$$

On the other hand, (231) can be put in the form

$$V^2-V_m^2=\frac{2}{\gamma-1}c^2\geqslant 0 \tag{235}$$

so that we consider

$$V_s\leqslant V\leqslant V_m \tag{236}$$

We notice that V_s, V_m are only parametrized by K.

Next, the requirement $u^2=c^2$ in (231) leads to the restriction

$$v^2=\frac{\gamma+1}{\gamma-1}(V_s^2-c^2)=\frac{\gamma+1}{\gamma-1}(V_s^2-u^2)$$

i.e., cf. (232),

$$\frac{u^2}{V_s^2}+\frac{v^2}{V_m^2}=1 \tag{237}$$

Thus the images of the convexity regions $\mathcal{R}_K$ associated to (204) result from (236), (237) and are described by fig.30.

<u>R46.</u> Given a point $\tilde{U}$ of a convexity region $\mathcal{R}$ we denote $K=K(\tilde{U})$, $S=S(\tilde{U})$. Cf.L12, K,S keep constant along a field line of index 1 or 4 through $\tilde{U}$ so that the projection in the u,v,p of the image in u,v,p,S of this line is laid on the pressure hill $C_{S,K}$. On substituting $c^2=c^2(u,v;S,K)$ in (230) we obtain the relation

$$F(u,v;\ S,K)=\text{constant} \tag{238}$$

The projection in u,v of the image in u,v,p,S of a field line is an epicycloid (fig.31b; in this figure two epicycloids, from different families, through an arbitrary point (u_o,v_o) of the annulus (236) are depicted). The projection in u,v,p of the mentioned image is depicted in fig.31c. In both of these figures the arrows indicate the way of compression.

$$\overset{1}{\varphi}_1(U)\equiv K(U)\equiv\frac{1}{2}V^2(U)+\frac{c^2(U)}{\gamma-1},\ \overset{1}{\varphi}_2(U)\equiv S(U),\ \overset{1}{\varphi}_3(U)\equiv F_1[u(U),v(U);S(U),K(U)] \quad (239)$$

$$\overset{2}{\varphi}_1(U)\equiv p(U),\ \overset{2}{\varphi}_2(U)\equiv\frac{v(U)}{u(U)},\ \overset{2}{\varphi}_3(U)\equiv\rho(U) \quad (240)$$

$$\overset{3}{\varphi}_1(U)\equiv p(U),\ \overset{3}{\varphi}_2(U)\equiv\frac{v(U)}{u(U)},\ \overset{3}{\varphi}_3(U)\equiv\frac{1}{2}V^2(U) \quad (241)$$

$$\overset{4}{\varphi}_1(U)\equiv K(U),\ \overset{4}{\varphi}_2(U)\equiv S(U),\ \overset{4}{\varphi}_3(U)\equiv F_4[u(U),v(U);S(U),K(U)] \quad (242)$$

We notice that $\overset{2}{\varphi}_3,\overset{3}{\varphi}_3$ correspond to the requirements settled by R42.

At this point it is interesting to revisit comparatively the considerations of 5.2.

R48. In the relation $(209)_2$ the point $((u)_2, v_2)$ describes, given $((u)_1, 0)$, a hypocissoid (Folium of Descartes) called the shock polar (cf.A.Busemann [6]).The admissible part of this polar, isolated by requirement that (an admissible sd must be a compressive discontinuity or, equivalently, that) the velocity must decrease in a transition through a sd, is depicted in fig.32 (which presents the minimal, intermediate and maximal shape of it). Here we have an ad hoc (specific) admissibility criterion instead of CEL. Also this figure makes evidence of the possibility that the Hugoniot curves should go out of the convexity (and hyperbolicity) region.

Now, let us distinguish in $(209)_2$ between the two branches $[\![v]\!]=\pm(\ldots)[\![u]\!]$ and correspondingly denote $(209)_\pm$ the obtained sets of jump relations. On differentiating $(209)_\pm$ in the space U it is easy to verify that (the RH curve H_1/respectively H_4 consist of an arc of Hugoniot curve associated to $(209)_+$/respectively $(209)_-$ and an arc of field line R_1/ respectively R_4, and) the remark R30(i) has an analogue here.

R49. If two states are connected to each other by a sd then in space u,v,p they are laid on different pressure hills corresponding to different values of S but having the same V_s, V_m [cf. L10 and (232)].

R50 (L.Prandtl [53], T.Meyer [48]). The considerations 6.3 and the definition D7 have an analogue here[1].

R51. A well-known orthogonality relation, corresponding to a genuinely nonlinear index, between the set of characteristics in x,y (Mach lines) and the set of epicycloids

[1] Gasdynamic terminology: Prandtl-Meyer flow (instead of simple waves region).

in u,v results from (230). The important (and specific) fact indicated by this relation is that a compression simple waves solution can be selfsimilar (fig. 33).

R52. As is well-known, in a steady adiabatic gasdynamic flow (i) on each streamline we have S=constant and the Bernoulli law $\frac{1}{2}V^2+\frac{c^2}{\gamma-1}=K=$constant holds (the constants depend on streamline generally) and, on the other hand, (ii) if K≡constant, S≡ constant in a certain region $\mathcal{D}$ in x,y then the flow is irrotational in that region: we have $\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0$ in $\mathcal{D}$.

Cf. (217) the streamlines of the flow governed by the system (206) can be characteristics (of index 2 or 3) for this system. Yet, here are an important result concerning the nature of the streamlines:

L13. In a region where K≡constant, S≡constant the streamlines are not characteristics.

◀ Argument 1. To the equations of the isentropic gasdynamics we add the requirement of irrotationality [cf. R52(ii)].

Argument 2. We use the Bernoulli law in order to obtain a system in u,v. For this system the streamlines are not characteristics. ▶

R53. The result L13 shows the importance of the ad hoc (specific) admissibility criterion presented in R48. Also, according to this result we can prescribe data along a streamline (for example along an obstacle).

In the sequel we allow the half-lines x<0, x>0 in (171) to be laid on possibly noncollinear rays (fig.34). Then let, for the RP considered, $K_\ell=K(U_\ell)$, $K_r=K(U_r)$. We separately discuss the cases $K_\ell=K_r=K$ and $K_\ell\neq K_r$.

We notice (according to L10) that K can change through a cd only.

Let us assume $K_\ell\neq K_r$ and K_ℓ, K_r conveniently close to each other. We consider data in $R_{K_\ell}\cap R_{K_r}$ (U_ℓ, U_r are also conveniently close to each other).

R54. Cf. R51 there are two RHs (see D26): RHs_1 (the projection in u,v of its image in u,v,ρ,p is presented in fig.35a) and RHs_2 (the projection in u,v of its image in u,v,ρ,p is presented in fig.35b; these two projections are presented together in fig. 35c). Each of these RHs can serve, depending on the contour in x,y on which the data are prescribed, for the resolution of RP. The RHs_2 has no analogue in the unsteady case.

R55. On taking RHs_1 into account we can see that an analogue of T12 holds. Incidentally, it can be shown that if the construction in the hodograph space can be put in x,y then the ambiguous character of the choice of $\overset{2}{R}$, $\overset{3}{R}$ is not felt by the solution. Thus, the choice R42 is satisfactory.

R56. In the solution of RP K is piecewise constant (with a jump along the cd).

R57. The flow around a dihedral supplies examples of RP without solution in $\mathcal{a}$ (the circumstances presented in fig.36 a,b - for which the "waves" of genuinely nonlinear index are not depicted - are not acceptable). In fact, for each of these circumstances, there is a significant connection between the data U_ℓ, U_r of a resolvable RP.

Here is an example of RP which is always (locally) resolvable in $\mathcal{a}$.

E10. In fig.37 the data are so chosen that the contour C on which they are prescribed is quasitransversal to the velocity directions. An analysis similar to that of fig.33 shows that for such a contour to the solution (in x>0) of the RP only RHs_1 contributes.

The cd can be assimilated to a straight line profile on which $[\![p]\!] = 0$.

R58. The Glimm theorem ([21]) can be adapted (see [12]) to the solution in x>0 corresponding to quasiconstant data prescribed on a quasitransversal contour (in the sense of E10; see fig.38).

R59. The considerations of this item depend on the choice of axes x,y. In particular, the convexity partition (R41) reflects this choice.

We also notice that in L9 the x-axis is in the direction of the velocity.

R60. For the RP solution S and K are piecewise constant.

R61. In the sequel we shall replace the expression (219) by

$$\overset{2}{R}=\Lambda_2\left[\frac{1}{2}\,\frac{V^2+\frac{c^2}{\gamma-1}}{\frac{1}{2}V^2+\frac{c^2}{\gamma-1}}\;,\;u,\;v,\;\frac{1}{2}\left(V^2+\frac{c^2}{\gamma-1}\right)\right] \tag{243}$$

corresponding to the requirement that $d\left(\frac{1}{2}V^2+\frac{c^2}{\gamma-1}\right)=0$ along a line of the field $\overset{2}{R}$. Then we have

$$\Delta = -\frac{2c^5V^2}{(\gamma-1)^2\left(V^2+\frac{c^2}{\gamma-1}\right)(u^2+v^2-c^2)^{1/2}}\,\Lambda_1\Lambda_2\Lambda_3\Lambda_4 \neq 0$$

instead of (225).

R62. In case $K_\ell = K_r = K$ the system (193) can be reduced, according to $\frac{1}{2}V^2 + \frac{c^2}{\gamma-1} \equiv K$, to

$$\begin{aligned}
&\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0\\
&\frac{\partial}{\partial x}\left[\rho\left(\frac{\gamma-1}{\gamma}K+\frac{\gamma+1}{2\gamma}u^2-\frac{\gamma-1}{2\gamma}v^2\right)\right]+\frac{\partial}{\partial y}(\rho uv)=0\\
&\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}\left[\rho\left(\frac{\gamma-1}{\gamma}K+\frac{\gamma+1}{2\gamma}v^2-\frac{\gamma-1}{2\gamma}u^2\right)\right]=0
\end{aligned} \tag{244}$$

Here n=3 and the formulas (196)-(208) have an obvious analogue. The system (244) is parametrized by K. Particularly, we obtain for the present matrix A(U) the eigenelements

$$\lambda_{1,3} = \frac{uv \pm c(u^2+v^2-c^2)^{1/2}}{u^2-c^2}, \quad \lambda_2 = \frac{v}{u} \tag{245}$$

$$\overset{1,3}{R} = \Lambda_{1,3}\left[1, \frac{\mp cv+u(u^2+v^2-c^2)^{1/2}}{(u^2+v^2-c^2)^{1/2}}, \frac{\pm cu+v(u^2+v^2-c^2)^{1/2}}{(u^2+v^2-c^2)^{1/2}}\right] \tag{246}$$

$$\overset{2}{R} = \Lambda_2\left[\frac{1}{2}\,\frac{V^2+\frac{c^2}{\gamma-1}}{\frac{1}{2}V^2+\frac{c^2}{\gamma-1}}, u, v\right] \tag{247}$$

The eigenvector (247) corresponds, according to R61, to (243). Also, the eigenvectors (218) and (246) are respectively related according to (228).

R63. The system (206) is of a mixed type. Other examples of such systems can be found in M.Shearer [60], H.Holden [29]. Particularly, in [60] the case of a system (191) for which the graph of σ is depicted in fig. 39 is considered.

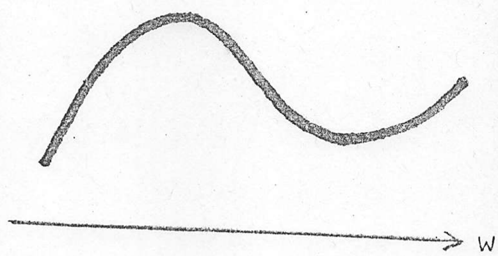


fig. 39

BIBLIOGRAPHY

1. N. Bakhvalov, "On the existence of regular solutions in the large for quasilinear hyperbolic systems", J. Computational Math. and Math. Phys., 10(1970), p.969. (In Russian.)

2. V. Barbu, Ordinary differential equations, Junimea, Iaşi, 1985. (In Romanian .)

3. I. Bohachevsky, "Simple waves and shocks in magnetohydrodynamics", Phys. Fluids, 5(1962), p.1456.

4. V.A. Borovikov, "On the Riemann problem for a system of two quasilinear equations", DAN SSSR, 185(1969), p.19. (In Russian.)

4.1. V.A. Borovikov, "On the Riemann problem for a system of two quasilinear equations", Trudy Moscov. Mat. Obsc., 27 (1972), p.

5.1. M. Burnat, "Riemann invariants", Fluid Dynamics Transactions 4(1969), p.17.

5.2. M. Burnat, "The method of characteristics and Riemann invariants for multidimensional hyperbolic systems", Math.Siberian J., 11(1970), p.279. (In Russian.)

6. A. Busemann, "Verdichtungsstösse in ebenen Gasströmungen" (1930).

7. J. Conlon, "A theorem in ordinary differential equations with an application to hyperbolic conservation laws", Advances in Math., 35(1980), p.1

8. R. Courant, "Cauchy's problem for hyperbolic quasi-linear systems of first order partial differential equations in two independent variables", Comm.Pure Appl. Math., 14(1961), p.257.

9. R. Courant and K.O. Friedrichs, Supersonic flow and shock waves, Interscience, New York, 1948.

10. R. Courant and P.Lax, "On nonlinear partial differential equations with two independent variables", Comm. Pure Appl. Math., 2(1949), p.255.

11.1. C. Dafermos, "Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method", Arch. Rat. Mech. Anal., 52(1973), p.1

11.2. C. Dafermos, "Structure of solutions of the Riemann problem for hyperbolic systems of conservation laws", Arch. Rat. Mech. Anal., 53(1974), p.203.

12.1. L. Dinu, "Hyperbolic systems of quasilinear conservation laws. 2. The Glimm theorem context", INCREST Preprint Series in Math. (to appear).

12.2. L. Dinu, "The Glimm theorem context", Revue Roum. Math. Pures Appl. (to appear).

13. L. Dinu, "The linearized problem in the adiabatic multidimensional gasdynamics. I", Revue Roum. Math. Pures Appl. 32(1987), p.771.

14. L. Dinu and M. Dinu, "A note on the Riemann problem in 2D steady adiabatic gasdynamics", Revue Roum. Math. Pures Appl. 33(1988) (to appear).

15. A. Douglis, "Some existence theorems for hyperbolic systems of partial differential equations in two independent variables", Comm. Pure Appl. Math., 5(1952), p.119.

16. V.F. Dyachenko, "On the conditions of uniqueness of continuous solutions to the Riemann problem for a system of three equations", DAN SSSR, 153(1963), p.1245. (In Russian.)

17. K.O. Friedrichs, "Nonlinear hyperbolic differential equations for functions of two independent variables", Amer. J. Math., 70(1948), p.555.

18. K.O. Friedrichs, Nichtlineare Differenzialgleichungen, Göttingen, 1955.

19. K.O. Friedrichs and H.Kranzer, "Notes on magnetohydrodynamics. VIII Nonlinear waves motion", NYO-6486 Courant Institute, New York, 1958.

20. I.M. Gel'fand, "Problems of the theory of quasilinear equations", Uspehi Mat. Nauk, 14(1959), p.1. (In Russian.)

21. J. Glimm, "Solutions in the large for nonlinear hyperbolic systems of equations", Comm. Pure Appl. Math., 18(1965), p.697.

22. S.K. Godunov, "An interesting class of quasilinear systems", DAN SSSR, 139(1961), p.521. (In Russian.)

23. S.K.Godunov and A.M.Blohin, "Energy integrals in the theory of shock wave stability", in U.Nigul and J.Engelbrecht (editors), Nonlinear deformation waves, Springer Verlag, Berlin-Heidelberg-New York, 1983, p.18.

24. V.V. Gogosov, "Resolution of an arbitrary discontinuity in magnetohydrodynamics", Prikl. Mat. Mech., 25(1961), p.1, (In Russian.)

25. A. Hanyga, "On the solution to the Riemann problem for arbitrary hyperbolic systems of conservation laws", Publ. Inst. Geophys., A-1(98), 1976.

26. Ph. Hartmann, Ordinary differential equations, J.Wiley and Sons, New York - London - Sydney, 1964.

27. Ph. Hartmann and A. Wintner, "On hyperbolic differential equations,"Amer. J.Math., 74(1952), p.855.

28. D. Hoff, "Globally smooth solutions of quasilinear hyperbolic systems in diagonal form", J. Math. Anal. and Appl., 86(1982), p.221.

29. H.Holden, "On the Riemann problem for a prototype of a mixed type conservation law" Comm.Pure Appl.Math., 40 (1987), p.229.

30. H. Hugoniot, "Mémoire sur la propagation du mouvement dans les corps, et spécialement dans les gaz parfaits" (1887, 1889).

31.1. C. Jacob, Introduction mathématique à la mécanique des fluides, Editura Academiei-Gauthier Villars, Bucarest, Paris, 1959.

31.2. N.E. Kochine, Y.A. Kibel' and N.V. Roze, Theoretical hydromechanics vol. 2, fourth edition, Gosizdat, Moscow, 1963. (In Russian).

31.3. L.D. Landau and E. Lifshitz, Mécanique des fluides, Mir, Moscow, 1971.

32. F. John, "Formation of singularities in one-dimensional nonlinear wave propagation", Comm. Pure Appl. Math., 27(1974), p.337.

33. B.Keyfitz and H. Kranzer, "Existence and uniqueness of entropy solutions to the Riemann problem for hyperbolic systems of two nonlinear conservation laws", J. Diff. Eqs., 27(1978), p.444.

34. S. Klainerman and A. Majda, "Formation of singularities for wave equations including the nonlinear vibrating string", Comm. Pure Appl. Math., 33(1980), p.241.

35. A.S. Kalashnikov, "Construction of generalized solutions to first order quasi-linear systems without convexity conditions as limits of solutions to parabolic equations with a small parameter", DAN SSSR, 127(1959), p.27.

36. P. Lax, "Hyperbolic systems of conservation laws. II", Comm. Pure Appl. Math., 10(1957), p.537.

37. P.Lax, "Development of singularities of solutions of nonlinear hyperbolic partial differential equations", J. Math. Phys., 5(1964), p.611.

38. P. Lax, "Shock waves and entropy", in E.A.Zarantonello (editor),Contributions to nonlinear functional analysis, Academic Press, New York, 1971,p.603.

39. P. Lax, "The formation and decay of shock waves", Amer. Math. Monthly, 79(1972), p.227.

40. P. Lax, "Hyperbolic systems of conservation laws and the mathematical theory of shock waves", Regional Conf. Series in Appl. Math., SIAM, 1973.

41. P. Lax, "Shock waves, increase of entropy and loss of information", in S.S. Chern (editor), Seminar of nonlinear partial differential equations, Math. Sci. Research Inst., Publ. 2, Springer Verlag, New York-Berlin-Heidelberg-Tokio, 1984.

42. T.P. Liu, "The Riemann problem for general 2x2 conservation laws", Trans. AMS, 199(1974), p.89.

43. T.P. Liu, "The Riemann problem for general systems of conservation laws", J. Diff. Eqs., 18(1975), p.218.

44. T.P. Liu, "Development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations", J. Diff. Eqs., 33(1979), p.92.

45. T.P. Liu, "Admissible solutions of hyperbolic conservation laws", Memoirs AMS, No.240(1981).

46. Y.M. Lynn, "Magnetohydrodynamic shocks in nonaligned flow", Phys. Fluids, 9(1966), p.314.

47. A. Majda, "The stability of multi-dimensional shock fronts", Memoirs AMS, No.275(1983).

48. T. Meyer, "Uber zweidimensionale Bewegungsvorgange in einem Gas, das mit Überschallgeschwindigkeit strömt" (1908).

49. O.A. Oleinik, "On the uniqueness of the generalized solution of Cauchy problem for a nonlinear system of equations occuring in mechanics", Uspehi Mat. Nauk, 12(1957), p.169.

50. O.A. Oleinik, "On the uniqueness and continuous dependence on data of generalized solutions of the Cauchy problem for a single quasilinear equation", Uspehi Mat. Nauk, 14(1959), p.165. (In Russian.)

51.1. Z. Peradzynski,"On algebraic aspects of the generalized Riemann invariants method", Bull. Acad. Polon. Sci., Ser. Sci. Techn., 18(1970), p.341.

51.2. Z. Peradzynski, "Nonlinear plane k-waves and Riemann invariants", Bull. Acad. Polon. Sci., Ser. Sci. Techn., 19(1971), p.625.

51.3. Z. Peradzynski, "Riemann invariants for the nonplanar k-waves", Bull. Acad. Polon. Sci., Ser. Sci. Techn., 19(1971), p.717.

52. S.D.Poisson, "Mémoire sur la théorie du son" (1808).

53. L.Prandtl, "Neue Untersuchungen über die strömende Bewegung der Gase und Dämpfe" (1907).

54. L. Prandtl and A.Busemann, "Näherungverfahren zur zeichnerischen Ermittlung von ebenen Strömungen mit Uberschallgeschwindigkeit" (1929).

55. G.B. Riemann, "Uber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite" (1859).

56. B.L. Rozdestvenskii, "Discontinuous solutions of hyperbolic systems of quasilinear equations", Uspehi Mat. Nauk, 15(1960), p.59. (In Russian.)

57. B.L. Rozdestvenskii and N.N.Yanenko, Systems of quasilinear equations, Nauka, Moscow, 1978. (In Russian.)

58. D.G.Schaeffer, "A regularity theorem for conservation laws", Adv. in Math., 11(1973), p.368.

59. M. Schatzmann, Introduction à l'analyse des systemes hyperboliques de lois de conservation non linéaires, Preprint INRIA, No.37, 1985.

60. M. Shearer, "The Riemann problem for a class of conservation laws of mixed type", J. Diff. Eqs., 46(1982), p.426.

61. G.E. Shilov, Calculus (functions of several independent variables), Nauka, Moscow, 1972.

62. R. Smith, "The Riemann problem in gas dynamics", Trans. AMS, 249(1979), p.1.

63. J.A. Smoller, "On the solution of the Riemann problem with general step data for an extended class of hyperbolic systems", The Michigan Math. J., 16(1969), p.201.

63.1. J.A. Smoller and J.L. Johnson, "Global solutions for an extended class of hyperbolic systems of conservation laws", Arch. Rat. Mech. Anal., 32(1969), p.169.

64. J.A. Smoller, "A uniqueness theorem for Riemann problems", Arch. Rat. Mech. Anal., 33(1969), p.110.

65. J.A. Smoller (editor), Nonlinear partial differential equations, Contemporary mathematics vol.17, AMS, Providence - Rhode Island, 1983.

66. V.A. Tupchyev, "The Riemann problem for systems of quasilinear equations which do not obey convexity conditions", J. Computational Math. and Math. Phys., 6(1966), p. 527 (In Russian.)

67. V.A. Tupchyev, "On the vanishing viscosity method in the study of the Riemann problem", DAN SSSR, 211(1973), p.55. (In Russian.)

68. V.A. Tupchyev, "On the uniqueness of continuous solution of the Riemann problem for a gradient system", Mat. zametki, 13(1973), p.251. (In Russian.)

69. E. Varley, "Acceleration fronts in viscoelastic materials", Arch. Rat. Mech. Anal, 19(1965), p.215.

70. R. von Mises, "On some topics in the fundamentals of fluid flow theory", Proc. First Natl. Congr.Appl.Mech., Chicago, 1950, p.667.

71. R. von Mises, Mathematical theory of compressible fluid flow, completed by Hilda Geiringer and G.S.S. Ludford, Academic Press, New York, 1958.

72. D. Wagner, "The Riemann problem in two space dimensions for a single conservation law", SIAM J. Math. Anal., 14(1983), p.534.