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George DINCA*) and Daniel MATEESCU*)

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*) *Bucharest University, Faculty of Mathematics, Str. Academiei 14, 70109 Bucharest, Romania.*

# WELL SET VARIATIONAL PROBLEMS
# AND GENERALIZED SOLUTIONS FOR MULTIVALUED OPERATOR EQUATIONS

George DINCA and Daniel MATEESCU

Let X be a real normed space and X* its dual. For any $x^* \in X^*$ and $x \in X$ we shall write $x^*(x) = \langle x^*, x \rangle$.

If $f: X \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is a functional, then

$$f^*: X^* \longrightarrow \overline{\mathbb{R}} \tag{1}$$
$$(\forall)\, x^* \in X^*,\; f^*(x^*) = \sup_{x \in X} \left\{ \langle x^*, x \rangle - f(x) \right\}$$

is called the conjugate functional of f.

The conjugate functional of f*, that is the functional

$$f^{**}: X^{**} \longrightarrow \overline{\mathbb{R}}$$
$$(\forall)\, x^{**} \in X^{**},\; f^{**}(x^{**}) = \sup_{x^* \in X^*} \left\{ \langle x^{**}, x^* \rangle - f^*(x^*) \right\} \tag{2}$$

is called the biconjugate of f.

PROPOSITION 1 (see for example [1], p.95, proposition 1.8). The functional f* is convex and lower semicontinuous in the weak star topology of X*.

REMARK 1. If X is reflexive, there follows from proposition 1: $f^*$ is convex and lower semicontinuous in the weak topology of $X^*$ and $f^{**}$ is convex and lower semicontinuous in the weak topology of X.

DEFINITION 1. A functional $f:X \longrightarrow (-\infty,+\infty]$ is said to be proper if

$$\mathrm{dom}\, f = \{x \in X / f(x) < +\infty\} \neq \emptyset.$$

PROPOSITION 2 (see for example [1], p.89, proposition 1.5). Let X be a real normed space. A proper convex functional $f:X \longrightarrow (-\infty,+\infty]$ is lower semicontinuous on X if and only if it is lower semicontinuous with respect to the weak topology of X.

From proposition 2 and remark 1 there follows:

PROPOSITION 3. If X is reflexive and $f^*$ is proper then $f^*$ is lower semicontinuous (with respects to the strong topology of $X^*$).

PROPOSITION 4 (see for example [1], p.97, corollary 1.4). A lower semicontinuous convex functional is proper if and only if its conjugate is proper.

PROPOSITION 5 (see for example [1], p.103, proposition 2.1). Let $f:X \longrightarrow (-\infty,+\infty]$ be a convex, proper and lower semicontinuous functional. Then:

$$x^* \in \partial f(x) \iff \chi(x) \in \partial f^*(x^*) \qquad (3)$$

$\chi$ being the embedding injection from X to $X^{**}$. If X is reflexive then identifying X with $X^{**}$, (3) can be written

$$x^* \in \partial f(x) \Longleftrightarrow x \in \partial f^*(x^*) \qquad (3)'$$

Finally, let us observe that from (1) and (2) there follows

$$f(x) + f^*(x^*) \geq \langle x^*, x\rangle \text{ (the Young inequality)}$$

PROPOSITION 6 (see for example [1], p.99, corollary 1.5). If f is convex then

$$f^{**} = \text{cl}f,\ (\text{cl}f)(x) = \liminf_{y\to x} f(x) = \sup_{V\in \mathcal{V}(x)} \inf_{y\in V} f(y) \qquad (4)$$

DEFINITION 2. A functional $f: X \to (-\infty, +\infty]$ is said to be uniformly convex if there is a constant $c > 0$ such that

$$\Delta_{\lambda,u_2,u_1} f = \lambda f(u_2) + (1-\lambda) f(u_1) - f(\lambda u_2 + (1-\lambda)u_1) \geq c\lambda(1-\lambda)\|u_2-u_1\|^2 \qquad (5)$$

for any $u_1, u_2 \in \text{dom } f$ and $\lambda \in (0,1)$.

THEOREM 1. If f is uniformly convex and satisfies the condition $\liminf_{y\to x} f(x) > -\infty$, $(\forall)\ x \in X$, then $f^{**}$ is uniformly convex.

To prove this we first give:

LEMA 1. If $f: X \to (-\infty, +\infty]$ satisfies the condition $\liminf_{y\to x} f(x) > -\infty$ then: for any $u_0 \in \text{dom } f^{**}$ there is $(u_n) \subset \text{dom } f$ such that $u_n \to u_0$ and $f(u_n) \to f^{**}(u_0)$.

Proof. Let

$$d = f^{**}(u_0) = \sup_{U \in \mathcal{V}(u_0)} \inf_{u \in U} f(u) \in \mathbb{R} \tag{6}$$

Consider the sequence of open balls $B_k$ such that

$B_k \supset B_{k+1}$, $\operatorname{diam} B_k \to 0$ as $k \to \infty$, $u_0 \in B_k$, $(\forall)\, k\ \mathbb{N}$.

(for instance $B_k = B(u_0, \frac{1}{k})$).

Let n be a natural number. From (6) there follows:

$$(\exists)\ U_n \in \mathcal{V}(u_0) \text{ such that } d - \frac{1}{n} < \inf_{u \in U_n} f(u) \tag{7}$$

Owing to the way in which the sequence of open balls $B_k$ has been chosen there exists $k_n$ such that

$$B_{k_n} \subset U_n \text{ and } B_i \subset B_{k_n},\ (\forall)\ i \geq k_n \tag{8}$$

From (7) and (8) there follows

$$d - \frac{1}{n} < \inf_{u \in B_{k_n}} f(u)$$

and, all the more

$$d - \frac{1}{n} < f(u),\ (\forall)\ u \in B_{k_n} \tag{9}$$

On the other hand, because $B_{k_n} \in \mathcal{V}(u_0)$, from (6) there follows

$$d + \frac{1}{n} > \inf_{u \in B_{k_n}} f(u)$$

Consequently, there will be $u_{k_n} \in B_{k_n}$ such that

$$f(u_{k_n}) < d + \frac{1}{n} \tag{10}$$

Therefore, for any $n$ there is $U_n \in \mathcal{V}(u_o)$, a natural number $k_n$ and $u_{k_n} \in B_{k_n}$ such that

$$\left.\begin{aligned} & B_i \subset B_{k_n} \subset U_n, \ (\forall)\ i \geq k_n \\ & d - \frac{1}{n} < f(u_{k_n}) < d + \frac{1}{n}. \end{aligned}\right\} \tag{11}$$

(By virtue of (8) we can always assume $k_n < k_{n+1}$ and as $d(u_{k_n}, u_o) \leq \operatorname{diam} B_{k_n} \to 0$, $n \to \infty$, we shall have $u_{k_n} \to u_o$, $n \to \infty$. From (11) we can assert $f(u_{k_n}) \to d$, $n \to \infty$ and the lemma is proved. We can now pass to the proof of theorem 1. Let $u, v \in \operatorname{dom} f^{**}$. As $f^{**}$ is convex we have

$$\lambda u + (1-\lambda) v \in \operatorname{dom} f^{**}, \ (\forall) \quad \lambda \in (0,1).$$

Let $(u_n) \subset \operatorname{dom} f$, $(v_n) \subset \operatorname{dom} f$ such that

$$u_n \to u, \ f(u_n) \to f^{**}(u);$$
$$v_n \to v, \ f(v_n) \to f^{**}(v).$$

Obviously, $\lambda u_n + (1-\lambda) v_n \to \lambda u + (1-\lambda) v$ and, from this,

$$\liminf_{n\to\infty} f(\lambda u_n+(1-\lambda)v_n) \geqslant \liminf_{\zeta\to\lambda u+(1-\lambda)v} f(\zeta) =$$
$$= (\mathrm{cl}f)(\lambda u+(1-\lambda)v) = f^{**}(\lambda u+(1-\lambda)v)$$

Using the uniform convexity of f we can assert:

$$\lambda f(u_n)+(1-\lambda)f(v_n)-c\lambda(1-\lambda)\|u_n-v_n\|^2 \geqslant$$
$$\geqslant f(\lambda u_n+(1-\lambda)v_n)$$

and from where, passing to the inferior limit with $n\to\infty$ we obtain

$$\lambda f^{**}(u)+(1-\lambda)f^{**}(v)-c\lambda(1-\lambda)\|u-v\|^2 \geqslant f^{**}(\lambda u+(1-\lambda)v)$$

which means the uniform convexity of $f^{**}$.

## 2. Well set variational problem

In what follows, X designates a real reflexive Banach space.

Let $\mathcal{D}(P)\subset X$ a dense subspace of X and $P:\mathcal{D}(P)\longrightarrow \mathcal{P}(X^*)$ a mapping from $\mathcal{D}(P)$ into the set of $X^*$'s subsets.

Let us consider yet a functional $G:X\to(-\infty,+\infty]$ so that $\mathrm{dom}\, G=\mathcal{D}(P)$ and finally let $f^*\in X^*$ and

$$\mathcal{F}_{f^*}:X\to(-\infty,+\infty],\ \ \mathcal{F}_{f^*}(v)=G(v)-\langle f^*,v\rangle,\ (\forall)\, v\in X. \quad (12)$$

In [2] we have introduced the concept of well set variational problem defined by the pair $(P, \mathcal{F}_{f^*})$.

DEFINITION 3. We say that the pair $(P, \mathcal{F}_{f^*})$ defines a well set variational problem if for any $f^* \in X^*$,

$$Pu \ni f^* \Longleftrightarrow \mathcal{F}_{f^*}(u) = \min_{v \in \mathcal{D}(P)} \mathcal{F}_{f^*}(v) \quad .$$

The first results connected with this concept were given in [2]. A detailed study with interesting consequences was realized in [3]. Here we mention the following

PROPOSITION 7 ([1]). The pair $(P, \mathcal{F}_{f^*})$ defined a well set variational problem if and only if G is subdifferentiable and $Pu = \partial G(u)$, $(\forall)\ u \in \mathcal{D}(P)$ where $\partial G(u)$ designates the subgradient of G at u.

We further assume that pair $(P, \mathcal{F}_{f^*})$ defines a well set variational problem and that for any $f^* \in X^*$ the following conditions are satisfied:

i) $\mathcal{F}_{f^*}$ is bounded from below;

ii) any minimizing sequence for $\mathcal{F}_{f^*}$ has a limit in X;

iii) all the minimizing sequences have the same limit.

DEFINITION 4 ([5]). We shall called generalized solution of equation $Pu \ni f^*$ (in the sense of Sobolev) and write it as $u_{s,f^*}$, the limit in X of any minimizing sequence for the functional $\mathcal{F}_{f^*}$.

The name of "generalized solution" is justified by

PROPOSITION 8. 1) The classical solution of the equation $Pu \ni f^*$ (if it exists) is a generalized solution.

2) If the generalized solution $u_{s,f^*} \in \mathcal{D}(P)$ then it is a classical one.

Proof. 1) Indeed, if $u_c \in \mathcal{D}(P)$ satisfies $Pu_c \ni f^*$ then $\mathcal{F}_{f^*}(u_c) = \min_{u \in \mathcal{D}(P)} \mathcal{F}_{f^*}(u)$ and it is enough to observe that in this case, a minimizing sequence for $\mathcal{F}_{f^*}$ is the constant sequence $u_n = u_c$, $(\forall)\ n \in \mathbb{N}$.

2) We assume that $u_{s,f^*} \in \mathcal{D}(P)$. Let $(u_n) \subset \mathcal{D}(P)$ be a minimizing sequence of $\mathcal{F}_{f^*}$. Then:

$$\mathcal{F}_{f^*}(u_n) \to \inf_{v \in \mathcal{D}(P)} \mathcal{F}_{f^*}(v),$$

$$u_n \to u_{s,f^*} \in \mathcal{D}(P) \quad \text{(see definition 4)}$$

$$Pu_{s,f^*} = \partial G(u_{s,f^*}) \neq \emptyset.$$

Let $g^* \in \partial G(u_{s,f^*})$. Then:

$$G(u_n) - G(u_{s,f^*}) \geq \langle g^*, u_n - u_{s,f^*} \rangle$$

or

$$\mathcal{F}_{f^*}(u_n) - [G(u_{s,f^*}) - f^*, u_n] \geq \langle g^*, u_n - u_{s,f^*} \rangle$$

Passing to the limit with $n \to \infty$ then follows

$$\inf_{v \in \mathcal{D}(P)} \mathcal{F}_{f^*}(v) - \mathcal{F}_{f^*}(u_{s,f^*}) \geq 0$$

that is $\mathcal{F}_{f^*}(u_{s,f^*}) = \inf_{v \in \mathcal{D}(P)} \mathcal{F}_{f^*}(v)$ from where, as the pair $(P, \mathcal{F}_{f^*})$ defines a well posed variational problem we can assert $Pu_{s,f^*} \ni f^*$.

REMARK 2. Given the uniqueness of the generalized solution, proposition 8 intrinsically containes a proof of the uniqueness of the classical solution (if it exists).

## 3. Extension theorems. Variational characterization of the generalized solution

In what follows we shall offer a variational characterization of the generalized solution for the equation $Pu \ni f^*$.

This characterization is actually contained in the theorem 2.

Let $G^*$ be the conjugate of $G$. Then:

$$\left.\begin{array}{l} G^* \text{ is proper; dom } G^* = X^* \\ (\forall) f^* \in X^*,\ G^*(f^*) = -\inf\limits_{v \in \mathcal{D}(P)} \mathcal{F}_{f^*}(v) \in \mathbb{R} \end{array}\right\} \tag{13}$$

Indeed, $(\forall) f^* \in X^*$,

$$G^*(f^*) = \sup_{v \in X} \left\{ \langle f^*, v \rangle - G(v) \right\} = \sup_{v \in \text{dom}\, G} \left\{ \langle f^*, v \rangle - G(v) \right\} = \sup_{v \in \mathcal{D}(P)} (-\mathcal{F}_{f^*}(v)) = -\inf_{v \in \mathcal{D}(P)} \mathcal{F}_{f^*}(v) \in \mathbb{R}$$

Taking into account (13) as well as propozition 3 we can assert that $G^*$ is convex, proper and lower semicontinuous. From proposition 4 it now follows that $G^{**}$ is proper, convex and lower semicontinuous on $X^{**} = X$.

Finally, from proposition 5 it follows that

$$u \in \partial G^*(f^*) \Longleftrightarrow f^* \in \partial G^{**}(u) \tag{14}$$

THEOREM 2. We have

$$\partial G^{**}(u_{s,f^*}) \ni f^*, \ (\forall)\ f^* \in X^* . \tag{15}$$

Proof. To prove (14) it means to show that (see (14))

$$\partial G^*(f^*) \ni u_{s,f^*}$$

what is

$$(\forall) g^* \in X^*,\ G^*(g^*) - G^*(f^*) \geq \langle u_{s,f^*}, g^* - f^* \rangle \tag{16}$$

Let $(u_n) \subset \mathcal{D}(P) = \mathrm{dom}\ G$ be a minimizing sequence for $\mathcal{F}_{f^*}$. Then

$$u_n \to u_{s,f^*},$$
$$\mathcal{F}_{f^*}(u_n) \to \inf_{v \in \mathcal{D}(P)} \mathcal{F}_{f^*}(v) = -G^*(f^*) \quad \text{(see (13))}$$

According to the Young inequality, we have that

$$G^*(g^*) + G(u_n) \geq \langle g^*, u_n \rangle$$

which can also be written as

$$G^*(g^*) + \mathcal{F}_{f^*}(u_n) \geq \langle u_n, g^* - f^* \rangle \tag{17}$$

Passing to the limit in (17) with $n \to \infty$ follows (16).

We shall now show that, indeed, theorem 2 intrinsically containes a variational characterization of the generalized solution of the equation $Pu \ni f^*$.

Actually, let us consider, for each $f^* \in X^*$ the functional

$$(\forall)\ v \in X\ (= X^{**}),\quad \widetilde{\mathcal{F}}_{f^*}(v) = G^{**}(v) - \langle v, f^* \rangle \tag{18}$$

Let us also consider $\widetilde{P} = \partial G^{**}$.

According to proposition 7, the variational problem for the pair $(P, \mathcal{F}_{f^*})$ is well set.

Then, from proposition 2 it follows that

$$\partial G^{**}(u) = \widetilde{P}u \ni f^* \iff \widetilde{\mathcal{F}}_{f^*}(u) = \min_{v \in \mathcal{D}(P)} \widetilde{\mathcal{F}}_{f^*}(v) \tag{19}$$

In conclusion: if $u_{s,f^*}$ is the generalized solution of the equation $Pu \ni f^*$ then $u_{s,f^*}$ is a classical solution of the equation $\partial G^{**}(u_{s,f^*}) \ni f^*$ and, according to (19) we have

$$\widetilde{\mathcal{F}}_{f^*}(u_{s,f^*}) = \min_{v \in \mathcal{D}(P)} \widetilde{\mathcal{F}}_{f^*}(v),$$

this last equality being the variational characterization we have announced.

Finally, to complete the significance of (14) let us observe that $\partial G^{**}$ is an extension of P.

Indeed, let $u \in \mathcal{D}(P)$ and $f^* \in Pu$. Then, according to proposition 8, $u = u_{s,f^*}$ and according to theorem 2, $f^* \in \partial G^{**}(u)$. Therefore $Pu \subset \partial G^{**}(u)$, $(\forall)\ u \in \mathcal{D}(P)$.

Coupling these remarks with theorem 2 we get:

THEOREM 3. We assume that the pair $(P, \mathcal{F}_{f^*})$ determines a well set variational problem, $\mathcal{F}_{f^*}$ being defined by

$$\mathcal{F}_{f^*}(v) = G(v) - \langle f^*, v \rangle$$

where X is a real reflexive Banach space, $\mathcal{D}(P) \subset X$ is a dense subspace, $P: \mathcal{D}(P) \longrightarrow \mathcal{P}(X^*)$, $G: X \to (-\infty, +\infty]$ with dom $G = \mathcal{D}(P)$ and $f^* \in X^*$.

We also assume that $\mathcal{F}_{f^*}$ have all the properties (i)-(iii) for any $f^* \in X^*$. Then:

1) $\tilde{P} = \partial G^{**}$ is an extension of P;

2) the generalized solution of the equation $\tilde{P}u \ni f^*$ is a classical solution of the equation $Pu \ni f^*$;

3) the generalized solution of the equation $\tilde{P}u \ni f^*$ is a minimizer on $\mathcal{D}(P)$ of the functional $(\forall)\ v \in X \quad (= X^{**})$, $\mathcal{F}_{f^*}(v) = G^{**}(v) - \langle v, f^* \rangle$.

THEOREM 4. If, in addition to the hypotheses of theorem 3, G is uniformly convex then, for any $f^* \in X^*$, $\partial G^{**}(u) \ni f^*$ has a unique solution which is a uniformly continuous function of $f^*$.

(In other words, under the hypotheses of theorem 4, the equation $\tilde{P}u \ni f^*$ has a unique solution which is precisely the generalized solution of the equation $Pu \ni f^*$ and this generalized solution is a uniform continuous function of $f^*$).

Proof. The existence of the solution is given by point 2 of theorem 3.

To obtain the uniqueness of the solution let us notice that, because G is uniformly convex, $G^{**}$ is uniformly convex too (theorem 1).

Because $G^{**}$ is uniformly convex and lower semicontinuous it follows that $\partial G^{***}$ is single valued and uniformly continuous (see [4], theorem 2.2). But (see [1], p.95, proposition 1.8) $G^{***} = (G^*)^{**} = G^*$. Consequently, $\partial G^{***} = \partial G^*$ is single valued and uniformly continuous. Such being the case, from (3)' (written for $G^*$ and $G^{**}$) we obtain:

$$\partial G^*(f^*) = u \Longleftrightarrow f^* \in \partial G^{**}(u) \; . \tag{20}$$

From (20) there follows the uniqueness of the solution of the equation $\partial G^{**}(u) \ni f^*$.

Indeed if $f^* \in \partial G^{**}(u_1)$ and $f^* \in \partial G^{**}(u_2)$ then from (20) it follows that

$$u_1 = u_2 = G^*(f^*).$$

The uniformly continuous dependence of the solution on the right member also follows from (20) considering that $\partial G^*$ is single valued and uniformly continuous.

## 4. APPLICATIONS

We consider a well set variational problem for the pair $(P, \mathcal{F}_{f^*})$ where $\mathcal{F}_{f^*}(u) = G(u) - \langle f^*, u \rangle$ and $G(u) = \phi(u) + \beta(u)$, $(\forall)$ $u \in \text{dom}\, G$.

The variational problem being well set we have (proposition 1).

$$Pu = \partial G(u) = \partial(\phi+\beta)(u)$$

so that the equation $Pu \ni f^*$ becomes

$$\partial(\phi+\beta)(u) \ni f^* \tag{21}$$

We assume that G satisfies the conditions of theorem 4. Then the generalized solution of the equation (21) is the unique classical solution of the equation

$$\partial(\phi+\beta)^{**}(u) \ni f^* \tag{22}$$

and is a uniformly continuous function of $f^*$.

PROPOSITION 9. If $\beta$ is continuous convex with dom $\beta$ = X and $\phi$ is convex and proper then

$$\partial(\phi+\beta) = \partial\phi + \partial\beta \tag{23}$$

$$\partial(\phi+\beta)^{**} = \partial\phi^{**} + \partial\beta \tag{24}$$

Proof. The equality (23) immediately follows from a classical result.

To prove (24) we first show that

$$(\phi+\beta)^{**} = \phi^{**} + \beta \tag{25}$$

that is

$$\mathrm{dom}\,(\phi^{**} + \beta) = \mathrm{dom}\,(\phi + \beta)^{**} \tag{26}$$

and for any $u_o \in \mathrm{dom}\,(\phi^{**} + \beta) = \mathrm{dom}\,(\phi + \ )^{**}$ we have

$$(\phi^{**} + \beta)(u_o) = (\phi + \beta)^{**}(u_o) \tag{27}$$

To prove (26) let $u_o \in \mathrm{dom}\,(\phi^{**} + \beta) = \mathrm{dom}\,\phi^{**}$. Then (see lema 1):

$(\exists)$ $(u_n) \subset \mathrm{dom}\,\phi$ such that $u_n \to u_o$ and $\phi(u_n) \to \phi^{**}(u_o)$.

Hence

$$\phi^{**}(u_o) + \beta(u_o) = \lim_{n\to\infty} \phi(u_n) + \lim_{n\to\infty} \beta(u_n) =$$
$$= \lim_{n\to\infty} (\phi(u_n) + \beta(u_n)) \geq \liminf_{u\to u_o} (\phi(u) + \beta(u)) =$$
$$= (\phi + \beta)^{**}(u_o)$$

therefore $u_o \in \mathrm{dom}\,(\phi + \beta)^{**}$.

In conclusion,

$$\mathrm{dom}\,(\phi^{**} + \beta) \subset \mathrm{dom}\,(\phi + \beta)^{**}$$

and

$$(\forall)\ u \in \mathrm{dom}\,(\phi^{**} + \beta) \Rightarrow \phi^{**}(u) + \beta(u) \geq (\phi + \beta)^{**}(u) \tag{28}$$

Let now $u_o \in \mathrm{dom}\,(\phi+\beta)^{**}$. Like before,

$$(\exists)\ (u_n) \subset \mathrm{dom}\,(\phi+\beta) \text{ so that } u_n \to u_o \text{ and}$$
$$(\phi+\beta)(u_n) \longrightarrow (\phi+\beta)^{**}(u_o).$$

Taking into account that $\beta$ is continuous it follows that

$$\phi(u_n) \longrightarrow (\phi+\beta)^{**}(u_o) - \beta(u_o)$$

therefore

$$(\phi+\beta)^{**}(u_o) - \beta(u_o) = \lim_{n\to\infty} \phi(u_n) \geqslant \liminf_{u\to u_o} \phi(u) =$$
$$= \phi^{**}(u_o)$$

so that $u_o \in \mathrm{dom}\,(\phi^{**}+\beta)$.

As $u_o$ was arbitrarily chosen, it follows that

$$\mathrm{dom}\,(\phi+\beta)^{**} \subset \mathrm{dom}\,(\phi^{**}+\beta)$$

and, $(\forall)\ u \in \mathrm{dom}\,(\phi+\beta)^{**} \Rightarrow (\phi+\beta)^{**}(u) \geqslant \phi^{**}(u) + \beta(u)$ (29)

From (28) and (29) the result is (27).

On the basis of (27) we have

$$\partial(\phi+\beta)^{**} = \partial(\phi^{**}+\beta) = \partial\phi^{**} + \partial\beta$$

the last equality being justified by the fact that $\phi$ and $\beta$ are convex, proper, $\beta$ is continuous and int $(\mathrm{dom}\,\beta) \cap \mathrm{dom}\,\phi^{**} = X \cap \mathrm{dom}\,\phi^{**} = \mathrm{dom}\,\phi^{**} = \emptyset$.

The fact that, under the hypotheses of proposition 9 the equation (22) can be written as $(\partial\phi^{**} + \partial\beta)(u) \ni f^*$ is useful in obtaining some results concerning the regularity of the generalized solution.

REFERENCES


1. Barbu, Viorel and Precupanu Theodor, Convexity and Optimization in Banach Spaces, Second Revised and Extended Edition, Editura Acad.R.S.R. and D.Riedel Publishing Company, 1986.
2. Dincă, G., et Mateescu, D., Problemes variationnels bien posés, Rev.Roum.Math.Pures Appl., 30 (1985), 1, 23-27.
3. Rosca I., Ionescu R., Sofonea M., A variational method for nonlinear operators with applications in mechanics. Preprint INCREST, no.38/1983.
4. Zălinescu, C., On Uniformly Convex Functions, Journ.Math. Analysis and Applications, 95, 2 (1983) p.344-374.
5. Petryshyn W.V., On a class of K-p.d. and non-K.p.d. operators and operator equations, J.Math.Anal.Appl. vol.10 (1965), pp.1-24.