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by

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0. INTRODUCTION. THE ABSTRACT SETTING

Coarse moduli spaces fail to exist for various basic objects such as finite dimensional algebras, function fields, affine varieties, singularities, linear algebraic groups. The moral of this paper is that for large classes of such objects one can construct "birational moduli spaces"; roughly speaking by a "birational moduli space" we mean a set M together with a family $(k(x))_{x\in M}$ of fields k(x) satisfying formally the basic properties of the residue fields at the (nonnecessary closed) points of a coarse moduli space. Of course birational moduli spaces are very rough moduli spaces since they carry no topological structure; in particular they do not reflect degeneration phenomena. Nevertheless in many cases they are the best global moduli spaces one can expect to dispose of.

The ideology of "birational moduli" originates in the work of Matsusaka and Shimura; Matsusaka essentially constructed [Ma] (see also [Sh] [Koiz]) the fields k(x) for polarized nonsingular complete varieties. In this paper we construct (by an entirely different method) the fields k(x) for several other types of objects namely for: finite dimensional algebras, function fields of general type, affine varieties of nonnegative Kodaira dimension, algebraisable singularities, linear algebraic groups.

Our method has an interest in itself because it relates "birational moduli" to the nonabelian cohomology of certain nonprofinite groups.

In what follows we introduce our concepts, state our main result and discuss the strategy of proving it.

(0.1) Throughout this paper we fix a ground field k of characteristic $p\geq 0$. By a field we will always mean a field extension of k; any field homomorphism (in particular any field automorphism) will be over k. Denote by $\mathcal{K}$ the category of fields.

(0.2) By a birational space we will mean a set X such that for each $x\in X$ we are given a field $k(x)$ finitely generated over k. Any such X induces a functor $h_X:\mathcal{K}\longrightarrow \mathrm{Ens}$

$$h_X(K)=\{(x,u):x\in X,\quad u\in \mathrm{Hom}_{\mathcal{K}}(k(x),K)\}$$

Two birational spaces X and Y are called isomorphic if there is a bijection $f:X\longrightarrow Y$ such that $k(x)\simeq k(f(x))$ for all $x\in X$.

(0.3) A functor $\mathcal{M}:\mathcal{K}\longrightarrow \mathrm{Ens}$ is called pseudo-representable if there exist a birational space M and a functorial homomorphism $\mathcal{M}\longrightarrow h_M$ such that the map $\mathcal{M}(K)\longrightarrow h_M(K)$ is an isomorphism for any algebraically closed field K. Such an M is said to quasi-represent $\mathcal{M}$ and it is an easy exercise to check that in characteristic zero M is uniquely determined up to isomorphism.

(0.4) Let $\mathcal{C}$ be a fibred category over $\mathcal{K}$; by this we mean here that for any $K\in \mathrm{Ob}(\mathcal{K})$ we are given a category $\mathcal{C}_K$ of "objects over K", for any $u \in \mathrm{Hom}_{\mathcal{K}}(K,K')$ we are given a "base change" functor $\mathcal{C}_u : \mathcal{C}_K \longrightarrow \mathcal{C}_{K'}$, and for any pair $(u,v) \in \mathrm{Hom}(K,K') \times \mathrm{Hom}(K',K")$ we are given a functorial isomorphism $\mathcal{C}_{u,v} : \mathcal{C}_v \circ \mathcal{C}_u \longrightarrow \mathcal{C}_{vu}$, all these data being subject to certain compatibility axioms for which we send to Grothendieck's exposition [Gr]. Given $\mathcal{C}$ one can define the "moduli functor" $\mathcal{M} : \mathcal{K} \longrightarrow \mathrm{Ens}$ by $\mathcal{M}(K)=\mathrm{Ob}(\mathcal{C}_K)/\mathrm{iso}$ (=set of isomorphism classes of objects in $\mathcal{C}_K$). If a birational space M quasi-represents the moduli functor $\mathcal{M}$ we say that M is a birational moduli space for $\mathcal{C}$ (so in characteristic zero birational moduli spaces, if they exist, they are unique up to isomorphisms).

The present paper is devoted to the problem of constructing birational moduli spaces for various special $\mathcal{C}$'s.

(0.5) An example (trivial from our viewpoint) when birational moduli spaces exist is the following: let $\mathcal{C}$ be the restriction of some fibred category $\widetilde{\mathcal{C}}$ over the category of k-schemes; if there is a coarse moduli space C for $\widetilde{\mathcal{C}}$ then a birational moduli space for $\mathcal{C}$ can be obtained by just taking the underlying set of (nonnecessary closed) points of C together with the residue fields at these points. Of course our main interest is to construct birational moduli spaces in cases when coarse moduli spaces fail to exist (or at least are not known to exist).

Here is our main result:

(0.6) THEOREM. Birational moduli spaces exist if chark(k)=0 and $\mathcal{C}$ is one of the following:

a) $\mathcal{C}_K$=category of finite dimensional K-algebras,

b) $\mathcal{C}_K$=category of finitely generated regular field extensions of K of general type,

c) $\mathcal{C}_K$=category of affine K-varieties of nonnegative Kodaira dimension,

d) $\mathcal{C}_K$=category of algebraisable formal K-algebras ($k=\mathbb{C}$),

e) $\mathcal{C}_K$=category of linear algebraic K-groups.

For the precise definitions of the above five fibred categories appearing in the statement above we send to Sections 3-5. In particular a formal K-algebra means a local complete noetherian K-algebra A with residue field K; such an A is called algebraisable if $K_a \hat{\otimes}_K A$ is the completion of some finitely generated K_a-algebra at some maximal ideal (K_a=algebraic closure of K). Moreover the Kodaira dimension of an affine K-variety U is by definition the Kodaira dimension of the K_a-variety $(U \otimes_K K_a)_{reg}$ [Sa].

To prove Theorem (0.6) we first reduce it via formal arguments to certain statements about "fields of definition" and "fields of moduli" (see the definitions (0.7) and Theorem (0.11) below). Then we investigate the relationship between the two types of fields above in the special cases a)-e); this will be done in Sections 3-5 by using ideas from [Bu_1] and a technical result on killing nonabelian cocycles (cf. Sections 1-2). Our killing procedure is in some sense analogue to the one used by Kolchin [Kol] p. 394 to construct Picard-Vessiot extensions associated to a given linear differential equation; Kolchin's derivations are replaced here by automorphisms. Note that most of the material in Section 1-4 holds in arbitrary characteristic.

(0.7) In the rest of this Section we introduce fields of definition and fields of moduli for any object $A \in Ob(\mathcal{C}_K)$ ($\mathcal{C}$ a fibred category over $\mathcal{K}$, K an algebraically closed field) and discuss the reduction of Theorem (0.6) to a problem concerning these fields. So we make the following definitions (which are inspired from [Ma], [Sh], [Koiz], [AMP]):

1) A subfield K_o of K is called a field of definition for A if $A \simeq \mathcal{C}_j(A^\circ)$ for some $A^\circ \in Ob(\mathcal{C}_{K_o})$ (where j denotes the inclusion $K_o \subset K$ and $\simeq$ means "isomorphism in $\mathcal{C}_K$"). Call $D(A,\mathcal{C})$ the set of all subfields of K which are fields of definition for A.

2) Define the group $\Sigma(A,\mathcal{C}) = \{\sigma \in Aut(K);\ A \simeq A^\sigma\}$ where $A^\sigma = \mathcal{C}_{\sigma^{-1}}(A)$. A subfield K_o of K is called a field of moduli for A if $\Sigma(A,\mathcal{C}) = Aut(K/K_o)$. Call $M(A,\mathcal{C})$ the set of all subfields of K which are fields of moduli for A.

(0.8) Some general easy remarks are in order. Suppose K is an algebraically closed field and $A \in Ob(\mathcal{C}_K)$. Then the following hold:

1) $D(A,\mathcal{C})$, $\Sigma(A,\mathcal{C})$, $M(A,\mathcal{C})$ depend only on the isomorphism class of A in $\mathcal{C}_K$ i.e. only on the image of A in $\mathcal{M}(K)$.

2) If $K_o \in D(A,\mathcal{C})$ then $Aut(K/K_o) \subset \Sigma(A,\mathcal{C})$; in particular if K_o is perfect then $K^{\Sigma(A,\mathcal{C})} \subset K_o$ (here for any group Γ acting on a field E we denote by E^Γ the field of Γ-invariant elements of E).

3) Let K_A be the intersection of all members of $D(A,\mathcal{C})$. It may happen that $K_A \notin D(A,\mathcal{C})$ even in quite reasonable cases (for instance if $\mathcal{C}_K$=category of smooth projective curves over K in characteristic zero, see [Sh]). In particular it may happen that $K^{\Sigma(A,\mathcal{C})} \notin D(A,\mathcal{C})$. However what one should expect in reasonable cases is that $(K^{\Sigma(A,\mathcal{C})})_a \in D(A,\mathcal{C})$ (where for any subfield E of

K we agree to denote by E_a the algebraic closure of E in K); compare with Theorem (0.10) below.

4) If $M(A,\mathcal{C})\neq\emptyset$ then $K^{\Sigma(A,\mathcal{C})}\in M(A,\mathcal{C})$ and moreover $M(A,\mathcal{C})$ consists precisely of those subfields of K whose perfect closure in K equals $K^{\Sigma(A,\mathcal{C})}$. Furthermore if char(k)=0 it is easy to check that the extension $K^{\Sigma(A,\mathcal{C})}\subset K_A$ is normal algebraic provided K is not the algebraic closure of K_A. Of course this does not imply apriori that there is a member of $D(A,\mathcal{C})$ algebraic over $K^{\Sigma(A,\mathcal{C})}$. Our main concern will be in fact to prove that this happens in various special cases. Note that an interesting problem is to decide whether $K^{\Sigma(A,\mathcal{C})}=K_A$ (see [Koiz]); we will not discuss this problem here.

5) A remark which will play a key role later is the following. Suppose there is a finite extension K_o of $K^{\Sigma(A,\mathcal{C})}$ contained in K with $K_o\in D(A,\mathcal{C})$. Then $K^{\Sigma(A,\mathcal{C})}\in M(A,\mathcal{C})$. This can be seen as follows: we may suppose $K_o/K^{\Sigma(A,\mathcal{C})}$ is normal. By 2) we have

$$Aut(K/K_o)\subset\Sigma(A,\mathcal{C})\subset Aut(K/K^{\Sigma(A,\mathcal{C})})$$

Upon letting H to be the image of $\Sigma(A,\mathcal{C})$ under the projection $Aut(K/K^{\Sigma(A,\mathcal{C})})\longrightarrow Aut(K_o/K^{\Sigma(A,\mathcal{C})})$ we have by usual Galois theory that $H=Aut(K_o/(K_o)^H)$ hence $\Sigma(A,\mathcal{C})=Aut(K/K_o)^H)$ and we are done by 4).

6) There is a remarkable exact sequence in our general situation namely

$$1\longrightarrow Aut_K(A)\longrightarrow G(A,\mathcal{C})\longrightarrow\Sigma(A,\mathcal{C})\longrightarrow 1$$

where $Aut_K(A)$ is the group of automorphisms of A as an object in

$\mathcal{C}_K$ and $G(A,\mathcal{C})$ is the group described as follows. Its elements are pairs $s=(\sigma,v)$ with $\sigma\in\Sigma(A,\mathcal{C})$ and $v:A\longrightarrow A^{\sigma}$ an isomorphism in $\mathcal{C}_K$; the multiplication is defined by the rule

$$(\sigma,v)(\tau,w)=(\sigma\tau,c_{\sigma,\tau}\circ v^{\tau}\circ w)$$

where $v^{\tau}=\mathcal{C}_{\tau^{-1}}(v)\in \mathrm{Hom}(A^{\tau},(A^{\sigma})^{\tau})$ and $c_{\sigma,\tau}=\mathcal{C}_{\sigma^{-1},\tau^{-1}}(A)\in$ $\in \mathrm{Hom}((A^{\sigma})^{\tau},A^{\sigma\tau})$. The projection $G(A,\mathcal{C})\longrightarrow\Sigma(A,\mathcal{C})$ is of course given by $(\sigma,v)\longmapsto\sigma$.

As an example, if $\mathcal{C}_K$=category of associateive unitary K-algebras (and if for any A $\mathrm{Ob}(\mathcal{C}_K)$ we view K as a subset of A) then $G(A,\mathcal{C})$ is precisely the group of all ring automorphisms s of A such that $s(K)=K$.

Note that in general $G(A,\mathcal{C})$ acts on K via $\Sigma(A,\mathcal{C})$ and clearly $K^{G(A,\mathcal{C})}=K^{\Sigma(A,\mathcal{C})}$.

A key point in our approach will be to kill cocycles of $G(A,\mathcal{C})$ with values in general linear groups $GL_n(K)$.

7) Finally let's explain the relation between our setting here and Weil's Galois descent [W]; we won't need this remark later on. Let K_o be a subfield of K. If $K_o\in D(A,\mathcal{C})$, then one can find by 2) a group homomorphism $s:\mathrm{Aut}(K/K_o)\longrightarrow G(A,\mathcal{C})$ which, composed with the projection $\pi:G(A,\mathcal{C})\longrightarrow\Sigma(A,\mathcal{C})$ yelds the natural inclusion $\mathrm{Aut}(K/K_o)\subset\Sigma(A,\mathcal{C})$ such an s will be called a section of π over K_o. Conversely, if such a section s for π over K_o exists one can ask whether $K_o\in D(A,\mathcal{C})$; upon letting $s(\sigma)=(\sigma,s_{\sigma})$ for $\sigma\in \mathrm{Aut}(K/K_o)$ with $s_{\sigma}:A\to A^{\sigma}$ we see that we have

$$s_{\sigma\tau}=c_{\sigma,\tau}\circ s_{\sigma}^{\tau}\circ s_{\tau}$$

for all σ, τ i.e. that the family $(s_\sigma)_\sigma$ satisfies a condition analogue to Weil's "cocycle condition" [W]. So if we assume in addition that K/K_o is a finite (algebraic) extension, Weil's Galois descent will yeld for "reasonable special $\mathcal{C}$'s" (for instance for $\mathcal{C}$ as in Theorem (0.6)) that $K_o \in D(A,\mathcal{C})$. Weil's method does not apply however to the case when K/K_o is transcendental.

(0.9) In case $\mathcal{C}_K$=category of polarized nonsingular complete K-varieties, the relationship between fields of definition and fields of moduli was investigated in detail in [Ma] [Sh] [Koiz]. Their approach was via Chow coordinates (see also the remark at (0.12) below).

In case $\mathcal{C}_K$=category of formal K-algebras the problem of understanding the relationship between the two types of fields above was left open in [AMP] p. 192.

Our main results in Sections 3-5 (cf. (3.9), (4.6), (5.5)) show in particular that:

(0.10) THEOREM. In all the situations a)-e) from Theorem (0.6) we have $(K^{\Sigma(A,\mathcal{C})})_a \in D(A,\mathcal{C})$ and $K^{\Sigma(A,\mathcal{C})} \in M(A,\mathcal{C})$ for all algebraically closed field K and all object $A \in Ob(\mathcal{C}_K)$.

On the other hand we have the following quasi-representability criterion:

(0.11) THEOREM. Assume char(k)=0 and let $\mathcal{C}$ be a fibred category over $\mathcal{K}$ satisfying the following conditions ($\mathcal{M}$ denotes in what follows the moduli functor associated to $\mathcal{C}$):

1) For any algebraically closed field K and any $A \in Ob(\mathcal{C}_K)$ there is a member of $D(A,\mathcal{C})$ which is finitely generated over k.

2) For any field K we have $\mathcal{M}(K) = \varinjlim \mathcal{M}(E)$ where E runs through the ordered set of all subfields of K which are countably generated over k.

3) For any extension $K \subset K'$ of algebraically closed fields the map $\mathcal{M}(K) \longrightarrow \mathcal{M}(K')$ is injective.

Then the following conditions are equivalent:

α) $\mathcal{M}$ is quasi-representable,

β) For any algebraically closed field K of infinite transcendence degree over k and for any $A \in Ob(\mathcal{C}_K)$ we have $(K^{\Sigma(A,\mathcal{C})})_a \in D(A,\mathcal{C})$.

Moreover if conditions α), β) hold and if K is algebraically closed, M is a birational moduli space for $\mathcal{C}$, $A \in Ob(\mathcal{C}_K)$ and $(x_A, u_A) \in h_M(K)$ corresponds to A under the bijection $\mathcal{M}(K) \simeq h_M(K)$ (where $x_A \in M$, $u_A : k(x_A) \longrightarrow K$) then we have $u_A(k(x_A)) = K^{\Sigma(A,\mathcal{C})} \in M(A,\mathcal{C})$

(0.12) Now conditions 1), 2), 3) in Theorem (0.11) are easily seen to be satisfied in cases a)-e) from Theorem (0.6). So Theorem (0.6) follows immediately from Theorems (0.10) and (0.11). Note also that if axiom 1) in Theorem (0.11) holds then condition

$(K^{\Sigma(A,\mathcal{C})})_a \in D(A,\mathcal{C})$

implies that there exists a finite extension K_o of $K^{\Sigma(A,\mathcal{C})}$ belonging to $D(A,\mathcal{C})$ which already implies by remark 5) in (0.8) that $K^{\Sigma(A,\mathcal{C})} \in M(A,\mathcal{C})$. So the hard part in (0.10) is to prove that $(K^{\Sigma(A,\mathcal{C})})_a \in D(A,\mathcal{C})$ and this will be our main concern in Sections 1-5. Note that one could try to prove this fact in a "geometric way" as follows:

Each object $A \in Ob(\mathcal{C}_K)$ can be viewed as a "family" over Spec(K). Then one can try to replace Spec(K) by an algebraic k-variety Spec(S) ($K_o = K^{\Sigma(A,\mathcal{C})}$, $K_o \subset S \subset K$) on which $\Sigma(A,\mathcal{C})$ acts by birational automorphisms and then try to (prove and) use representability of the functor of isomorphisms between "objects in $\widetilde{\mathcal{C}}_S$" where $\widetilde{\mathcal{C}}$ is an "extension" of $\mathcal{C}$ to the category of

K_o-schemes. There are some serious difficulties with this approach. First, although one can always find in each of the cases a)-e) a field of definition K_1 for A which is finitely generated over K_o it is not at all clear that one can find such a K_1 which in addition is stable under $\Sigma(A,\mathcal{C})$. Secondly even if such a stable K_1 was found (and Spec(S) is a model of K_1/K_o) it may happen that the isomorphisms between "objects in $\tilde{\mathcal{C}}_S$" form an infinite dimensional object over S (as it is the case for formal algebras) or a finite dimensional object with infinitely many components (as it may occur in the case of linear algebraic groups).

Note also that one could try to prove that $(K^{\Sigma(A,\mathcal{C})})_a \in$ $\in D(A,\mathcal{C})$ by extending the method of Matsusaka-Shimura. There are difficulties also with this approach. Indeed, in their method it is essential that the moduli functor be of the form

$$\mathcal{M}(K) = \coprod H_i(K)/R_i(K)$$

with H_i certain quasi-projective k-schemes and $R_i \subset H_i \times H_i$ certain "algebraic equivalence relations". Although our $\mathcal{M}$ has this form in cases a) and b) from Theorem (0.6) one does not expect $\mathcal{M}$ to take this form in cases c), d), e).

Note finally that our cohomological method yelds more than quasi-representability of $\mathcal{M}$ (e.g. it yelds the "splitting" assertions in Theorems (3.3), (4.3), (5.12) which we did not explain in our Introduction but have an interest in themselves).

(0.13) We close this section by sketching the proof of Theorem (0.11). Although the proof is purely formal it is somewhat tricky. Implication $\alpha) \Rightarrow \beta)$ is routine. In proving $\beta) \Longrightarrow \alpha)$

it is convenient to make the following definition: $\mathcal{M}$ is said to be quasi-representable on a subcategory $\mathcal{K}_o$ of $\mathcal{K}$ if there is a birational space M and a functorial homomorphism

$$\mathcal{M}|\mathcal{K}_o \longrightarrow h_M|\mathcal{K}_o$$

which induces isomorphisms on the algebraically closed fields of $\mathcal{K}_o$. We say that M quasi-represents $\mathcal{M}$ on $\mathcal{K}_o$. Then we proceed in three steps.

Step 1. Let Ω be an algebraically closed field of infinite transcendence degree over k. Denote by $\mathcal{K}_\Omega$ the subcategory of $\mathcal{K}$ whose objects are the subfields of Ω and whose morphisms are those field homomorphisms between subfields of Ω which can be lifted to automorphisms of Ω; clearly $\mathcal{K}_\Omega$ is not a full subcategory of $\mathcal{K}$. Then one proves that $\mathcal{M}$ is quasi-represented on $\mathcal{K}_\Omega$ by $M = \mathcal{M}(\Omega)/Aut(\Omega)$ (where $Aut(\Omega)$ acts on $\mathcal{M}(\Omega)$ in the natural way and the fields k(x) are defined as follows: one takes an arbitrary section $s: M \longrightarrow \mathcal{M}(\Omega)$ of the projection $\mathcal{M}(\Omega) \longrightarrow M$ and puts $k(x) = \Omega^{\Sigma(s(x),\mathcal{C})}$).

Step 2. Let $\mathcal{K}_\omega$ be the full subcategory of $\mathcal{K}$ whose objects are the countably generated field extensions of k. If Ω is any algebraically closed field of uncountable transcendence degree over k then one can construct a functor $\mathcal{K}_\omega \rightarrow \mathcal{K}_\Omega$ sending each field $K \in Ob(\mathcal{K}_\omega)$ into a subfield $\tilde{K}$ of Ω isomorphic to K and each morphisms $K \longrightarrow E$ in $\mathcal{K}_\omega$ into a morphism $\tilde{K} \rightarrow \tilde{E}$ in $\mathcal{K}_\Omega$ compatible with the isomorphisms $K \simeq \tilde{K}$, $E \simeq \tilde{E}$. Then one checks that if M quasi-represents $\mathcal{M}$ on $\mathcal{K}_\Omega$ then M also quasi-represents $\mathcal{M}$ on $\mathcal{K}_\omega$.

Step 3. If $\varphi : \mathcal{M}|_{\mathcal{K}_\omega} \longrightarrow h_M|\mathcal{K}_\omega$ is a functorial homomorphism making M quasi-represent $\mathcal{M}$ on $\mathcal{K}_\omega$ then φ can be extended to a natural functorial homomorphism $\mathcal{M} \longrightarrow h_M$ by taking direct limits (use property 2) in (0.11)). Note that our construction was noncanonical at several points (the choice of the section $s.M \longrightarrow \mathcal{M}(\Omega)$ the choice of the functor $\mathcal{K}_\omega \to \mathcal{K}_\Omega$). However the quasi-representing object is, as we already remarked, unique up to isomorphism.

1. KILLING NONABELIAN COCYCLES

Let G be a topological group; unlike in [KL] and [Ser] we will not assume here (and this will be important in what follows) that G is finite or profinite. For the results stated in the Introduction, discrete topologies on our groups G would suffice; however to get a satisfactory picture of the situation it is convenient to take nondiscrete topologies into account too and so we will.

By a G-field (respectively G-group, G-ring) we will understand a field (respectively a group, a ring) X together with a G-action on X by field (respectively group, ring) automorphisms; such a G-field (respectively G-group, G-ring) will be called discrete if the action map $G \times X \longrightarrow X$ is continuous where X is given the discrete topology (equivalently if for any $x \in X$ the isotropy group of x is an open subgroup of G); we will also say that G acts continuously on X. If K is a discrete G-field and L is a linear algebraic K^G-group then L(K), the group of K-points of L, has a natural structure of discrete G-group.

Now if Γ is a discrete G-group define the set $Z^1(G, \Gamma)$ of continuous 1-cocycles as the set of all continuous maps $f:G \to \Gamma$ satisfying $f(st)=f(s)s(f(t))$ for all $s,t \in G$; in this definition continuity of f can be replaced by the equivalent condition that $f^{-1}(1)$ is an open subgroup of G. A continuous cocycle f will be called a coboundary if there exists $x \in \Gamma$ such that $f(s)=x^{-1}sx$ for all $s \in G$.

By an extension of discrete G-fields we mean a field extension E/K of discrete G-fields such that the G-actions on K and E agree; such an extension will be called constrained if the extension E^G/K^G is algebraic.

One more definition: a subgroup G_1 of G is called cofinite if there exists a sequence of subgroups $G_1 \subset G_2 \subset \ldots \subset G_m = G$ such that G_i is normal and of finite index in G_{i+1} for $1 \leq i \leq m-1$; Clearly the extension K^{G_1}/K^G is then necessarily finite algebraic.

Our main result in this section is the following:

(1.1) THEOREM. Let G be a topological group, K a discrete G-field, L a linear algebraic K^G-group and $f \in Z^1(G,L(K))$ a continuous cocycle. Then there exists an open cofinite subgroup G_1 of G and a finitely generated constrained extension of G_1-fields K_1/K such that the image of f via the natural map

$$Z^1(G,L(K)) \longrightarrow Z^1(G_1,L(K_1))$$

is a coboundary.

Proof. Embed L into GL_N for some N and suppose L is defined in $k[X]_d$ by an ideal I, where $X=(X_{ij})$ and $d=\det(X)$. Due to the formula $f(st)=f(s)s(f(t))$ for $s,t \in G$, there is a unique G-action on $K[X]$ which agrees with our G-action on K and such that $sX_{ij}=\sum X_{ip}(f(s))_{pj}$ where $f(s) \in L(K)$ is viewed as an element in

$GL_N(K)$. Since $f^{-1}(1)$ is open, $K[X]$ is a discrete G-ring. Since $sd=\det(f(s))d$ the action above continuously extends to a G-action on $K[X]_d$; clearly $J=IK[X]_d$ is globally invariant under G. Let S be the set of all ideals J' in $K[X]_d$ satisfying the following properties:

1) J' contains J and

2) J' is G'-invariant for some open cofinite subgroup G' in G.

By noetherianity S has a maximal member; call it J_1 and call G_1 the corresponding open cofinite subgroup from condition 2). We claim J_1 is a prime ideal. Indeed let $M=\{P_1,\dots,P_m\}$ be the set of all prime ideals in $K[X]_d$ which are minimal over J_1. Clearly G_1 acts on M and put $G_2=\mathrm{Ker}(G_1 \longrightarrow \mathrm{Aut}(M))$ which will still be open and cofinite in G. Since P_1 is G_2-invariant, we must have $P_1 \in S$ hence by maximality of J_1 we have $P_1=J_1$ which proves our claim. We let now K_1 be the quotient field of $R_1=K[X]_d/J_1$ and $x \in L(K_1)$ be the K_1-point of L corresponding to the map $k[X]_d/I \longrightarrow K_1$. Clearly G_1 acts continuously on K_1 and $f(s)=x^{-1}sx$ for all $s \in G_1$ so the cocycle $G_1 \longrightarrow G \xrightarrow{f} L(K) \longrightarrow L(K_1)$ is a coboundary. We are left to prove that $K_1^{G_1}/K^G$ is algebraic. It is sufficient to check that any element $a \in K_1^{G_1}$ is algebraic over K; indeed if $a^n+b_1a^{n-1}+\dots+b_n=0$ with $b_i \in K$ is an equation of minimal degree satisfied by a then for any $s \in G_1$ we will have $(b_1-sb_1)a^{n-1}+\dots$ $\dots+(b_n-sb_n)=0$ and hence by minimality, $sb_i=b_i$ for all i and $s \in G_1$. In other words a turns out to be algebraic over K^{G_1}.

Assume there exists $a \in K_1^{G_1}$ transcendental over K and look for a contradiction. By Chevalley's constructibility theorem there exists $g \in K[a]$, $g \neq 0$ such that the image of the map $\mathrm{Spec}(R_1[a]) \longrightarrow$ $\longrightarrow \mathrm{Spec}(K[a])$ contains $\mathrm{Spec}(K[a]_g)$ (where $R_1[a]$ denotes of course the R_1-subalgebra of K_1 generated by a and analogously for $K[a]$). We claim there exist an open cofinite group G_2 in G and a G_2-invariant prime ideal $P \neq 0$ in $K[a]$ not containing g. If the field K^{G_1} is infinite we may simply take $G_2 = G_1$ and $P = (a-c)K[a]$ where $c \in K^{G_1}$, $g(c) \neq 0$. To prove the claim in general note that there is at least one polynomial $h \in K^{G_1}[a]$ none of whose prime factors $h_1, \dots, h_m$ in $K[a]$ divides g. Clearly G_1 acts continuously on $K[a]$ and also on the finite set of ideals $F = \{h_1K[a], \dots, h_mK[a]\}$. Then the claim follows by taking $G_2 = \mathrm{Ker}(G_1 \longrightarrow \mathrm{Aut}(F))$. With P at hand consider the set $E = \{Q_1, \dots, Q_s\}$ of minimal primes in the fibre of the map $\mathrm{Spec}(R_1[a]) \longrightarrow \mathrm{Spec}(K[a])$ at P; clearly G_2 acts continuously on $R_1[a]$ and also acts on E. Then if we let $G_3 = \mathrm{Ker}(G_2 \longrightarrow \mathrm{Aut}(E))$ we get that $Q = Q_1$ is G_3-invariant, hence so will be $Q \cap R_1$, hence so will be the inverse image of $Q \cap R_1$ in $K[X]_d$ which we call J_3. Now $Q \neq 0$ hence $Q \cap R_1 \neq 0$ (because R_1 and $R_1[a]$ have the same quotient field) so J_3 strictly contains J_1. Since G_3 is cofinite in G, this contradicts the maximality of J_1 and our theorem is proved.

(1.2) In view of Theorem (1.1) a special role will be played by topological groups G having the property

(*) G has no open normal subgroups of finite index except G itself.

Clearly if G satisfies (*) then G has no open cofinite subgroups except G itself hence $G_1 = G$ in Theorem (1.1). Let us discuss a remarkable example of a group with property (*). For any field extension K/K_o give $\mathrm{Aut}(K/K_o)$ the unique structure of topological group for which the identity has a fundamental system of neighbourhoods consisting of all subgroups of the form $\mathrm{Aut}(K/E)$ with

$K_o \subset E \subset K$, E finitely generated over K_o , see [R]; this topology will be called the natural topology on $Aut(K/K_o)$.

(1.3) LEMMA. Let K/K_o be an extension of algebraically closed fields with K of infinite transcendence degree over K_o. Then $Aut(K/K_o)$ with its natural topology has property (*).

Proof. Let H be an open normal subgroup of finite index N in $G=Aut(K/K_o)$ and let E be a finitely generated extension of K_o in K such that $Aut(K/E) \subset H$. We claim that for any $\sigma \in G$ one can find an automorphism $\tau \in G$ such that $\sigma^{-1}\tau^N \in Aut(K/E)$; this will imply that G=H. To prove our claim let EF denote the compositum of E and $F=\sigma E$ in K and let $\varphi : Q(EF \otimes F \otimes \ldots \otimes F) \to K$ be an embedding extending the inclusion $EF \subset K$ (here the tensor products are over K_o , the number of factors is N and Q denotes "taking the quotient field"). Put $F_2=\varphi(1 \otimes F \otimes 1 \otimes \ldots \otimes 1), \ldots,$ $F_N=\varphi(1 \otimes \ldots \otimes 1 \otimes F)$. Now let $\tau_o : EF_2 \ldots F_N \simeq FF_2 \ldots F_N$ be the isomorphism induced via φ by the isomorphism $E \otimes F \otimes \ldots \otimes F \to$ $\to F \otimes F \otimes \ldots \otimes F$ given by

$$x_1 \otimes x_2 \otimes \ldots \otimes x_N \longmapsto x_2 \otimes \ldots \otimes x_N \otimes \sigma x_1$$

Then for any $x \in E$ we have $\tau_o^k x \in F_{N-k+1}$ $(1 \leq k \leq N,\ F_1=F)$ and $\tau_o^N x = \sigma x$. We conclude by letting $\tau \in G$ to be any extension of τ_o.

2. K[G]-MODULES

(2.1) Let G be a topological group and K a discrete G-field. Denote by K[G] the skew group K-algebra of G; recall that as a K-linear space, K[G] has a basis consisting of the elements of G while the multiplication is defined by the formula

$(c_1s_1)(c_2s_2)=(c_1s_1(c_2))(s_1s_2)$ for all $c_1,c_2 \in K$ and $s_1,s_2 \in G$. We will be interested here in the category of $K[G]$-modules (note that the (G,K)-spaces from [KL] are $K[G]$-modules while the converse is not true since we do not assume that the map $G \to \mathrm{Aut}(K)$ defining the G-action on K is injective). The basic relation in a $K[G]$-module M is $s(cx)=(sc)(sx)$ for all $s \in G$, $c \in K$, $x \in M$. When we say a $K[G]$-module is finite dimensional we mean it has finite dimension as a K-linear space. A $K[G]$-module is called discrete if it is so as a G-group. The field K itself is a discrete $K[G]$-module in a natural way.

Now for any $K[G]$-module M put $M^G=\{x \in M;\ sx=x \text{ for all } s \in G\}$; M^G is a K^G-linear space and we have a natural injective K-linear map

$$K \otimes_{K^G} (M^G) \longrightarrow M, \qquad c \otimes x \longmapsto cx$$

We will often identify $K \otimes_{K^G} (M^G)$ with the image of the above map. If this map is also surjective we say that M is a split $K[G]$-module; clearly M is split if and only if it has a K-basis contained in M^G (see also [KL] for related discussion; however the main results in [KL] involve only finite groups G so they are not sufficient for our purpose).

Note that any split $K[G]$-module is a discrete $K[G]$-module. If K_1/K is an extension of G-fields and if M is a (discrete) $K[G]$-module then $K_1 \otimes_K M$ has a natural structure of (discrete) $K_1[G]$-module defined by $s(c \otimes x)=sc \otimes sx$ for all $s \in G$, $c \in K_1$, $x \in M$.

(2.2) A useful remark is that if M is a split $K[G]$-module. G_1 is an open subgroup of G and K_1/K is an extension of discrete G_1-fields then the following hold:

1) $K_1 \otimes_K M$ is a split $K_1[G_1]$-module and

2) the natural map

$$f: (K_1^{G_1}) \otimes_{K^G} (M^G) \longrightarrow (K_1 \otimes_K M)^{G_1}$$

is an isomorphism.

The first assertion is clear since $K_1 \otimes_K M$ has a K_1-basis consisting of G-invariant elements in M. To prove the second assertion it is sufficent to check that f becomes an isomorphism after tensorisation with K_1 over $K_1^{G_1}$. But after tensorisation both the source and the target of f naturally identify with $K_1 \otimes_K M$ and we are done.

Now according with a general principle [KL] our result (1.1) on "killing cocycles" leads to "existence of invariant bases":

(2.3) COROLLARY. Let K be a discrete G-field and M a discrete $K[G]$-module of finite dimension. Then there exist an open cofinite subgroup G_1 of G and a finitely generated constrained extension K_1/K of discrete G_1-fields such that $K_1 \otimes_K M$ is a split $K_1[G_1]$-module.

Proof. It is similar to an argument from [KL]; for convenience we repeat the argument below. Let $x_1, \ldots, x_m$ be a K-basis of M and write $sx_i = \sum a_{ij}(s)x_j$ with $s \in G$, $a(s) = (a_{ij}(s)) \in GL_m(K)$. The map $f: G \longrightarrow GL_m(K)$, $f(s) = (a(s))^{-1}$ is a continuous 1-cocycle, hence by (1.1) one can find a cofinite G_1, a finitely generated constrained extension K_1/K of G_1-fields and a matrix $b = (b_{ij}) \in GL_m(K_1)$ such that $f(s) = b^{-1}sb$ for all $s \in G_1$. Put

$y_i = \sum b_{ij}x_j$ for $1 \leq i \leq m$. Upon letting x and y to be the column vectors with entries $x_1,\ldots,x_m$ and $y_1,\ldots,y_m$ respectively we have for all $s\in G_1$:

$$sy=(sb)(sx)=(bf(s))(a(s)x)=bx=y$$

hence $y_i\in(K_1\otimes_K M)^{G_1}$ and we are done.

3. LOCALLY FINITE K[G]-ALGEBRAS

(3.1) By a K-algebra we will mean here either an associative unitary (not necessarily commutative !) K-algebra or a Lie K-algebra. By a locally finite structure on a K-algebra A we will mean a sequence $(A_n)_{n\geq 0}$ of finite dimensional K-linear subspaces of A such that $A=\sum A_n$ (the sum need not be direct !) and $1\in A_o$ (if there is a unit 1 in A). By a locally finite K-algebra we mean a K-algebra A together with a locally finite structure (A_n) on it. The locally finite K-algebras form a category: a morphism between two locally finite K-algebras $(A,(A_n))$ and $(B,(B_n))$ is by definition a K-algebra map $f:A\longrightarrow B$ such that $f(A_n)\subset B_n$ for all $n\geq 0$.

Here are some standard examples of locally finite K-algebras. Any graded K-algebra A with finite dimensional homogenous pieces has a natural structure of locally finite K-algebra (put A_n=piece of degree n). Any finite dimensional K-algebra A is a locally finite K-algebra in a standard way (we put $A_n=A$ for all n). Given an affine K-variety Spec(A) there is no "canonical" locally finite structure on A; however if we are given a smooth compactification X of U=Spec(A) then associated to it there is a canonical locally finite structure on A defined by $A_n=$ $=H^o(X,\mathcal{O}_X(nD))$ where D is the reduced divisor whose support equals $X\setminus U$ (here by a compactification of a K-variety U we mean a complete K-variety containing U as a Zariski open set). A quite general class of finitely generated commutative K-algebras A possessing a "canonical" locally finite structure will be described in (3.8): it is the class of those A for which Spec(A) is a K-variety with non negative Kodaira dimension.

For any field K denote by $\mathcal{L}_K$ the category of finitely generated locally finite K-algebras; for any field homomorphism $K \longrightarrow K'$ define the base change functor $\mathcal{L}_K \to \mathcal{L}_{K'}$ by $(A,(A_n)) \longmapsto (K' \otimes_K A,(K' \otimes_K A_n))$. We have defined a fibred category $\mathcal{L}$ over the category of fields. Clearly (A_n) define a gradation if and only if $(K' \otimes_K A_n)$ define a gradation.

(3.2) Following [NW] p.952 if G is a topological group and K is a discrete G-field then by a $K[G]$-algebra we mean a K-algebra A which is also a $K[G]$-module such that the multiplication map $A \otimes_K A \longrightarrow A$ and the unit $K \longrightarrow A$ (if there is any) are $K[G]$-module maps (where $A \otimes_K A$ is a $K[G]$- module via $s(a_1 \otimes a_2)=sa_1 \otimes sa_2$ for $s\in G$, $a_1,a_2 \in A$). By a locally finite $K[G]$-algebra we will mean a locally finite K-algebra A which is also a $K[G]$-algebra such that A_n is a $K[G]$-submodule of A for all $n \geq 0$. A (locally finite) $K[G]$-algebra is called discrete if it is so as a $K[G]$-module (i.e. as a G-group).

Following [NW] p.957 we say that the (locally finite) $K[G]$-algebra A is split if there is an isomorphism of (locally finite) K-algebras $A \simeq K \otimes_{K^G} (A^o)$ for some (locally finite) K^G-algebra A^o such that for the induced $K[G]$-algebra structure on $K \otimes_{K^G} (A^o)$ we have $s(c \otimes x)=sc \otimes x$ for all $s \in G$, $c \in K$, $x \in A^o$. Clearly any split (locally finite) $K[G]$-algebra is a discrete $K[G]$-algebra.

The main result of this section is:

(3.3) THEOREM. Let A be a finitely generated locally finite discrete $K[G]$-algebra.

1) If K is algebraically closed then $(K^G)_a \in D(A,\mathcal{L})$.

2) If G has property (*) then there exists a constrained extension $\tilde{K}/K$ of discrete G-fields such that $\tilde{K} \otimes_K A$ is a split locally finite $\tilde{K}[G]$-algebra.

Proof. If (A_n) is the locally finite structure on A then A_n are finite dimensional $K[G]$-submodules of A. Using (2.3) we may construct inductively a sequence $G \supset G_0 \supset G_1 \supset \ldots$ of open cofinite subgroups of G and a sequence $K \subset K_0 \subset K_1 \subset \ldots$ of field extensions of K such that for all $n \geq 0$ the following conditions are satisfied:

a) K_n is a discrete G_n-field (put $k_n = K_n^{G_n}$).

b) K_n/K_{n-1} is an extension of discrete G_n-fields (where $K_{-1} = K$).

c) k_n/k_{n-1} is an algebraic extension (where $k_{-1} = K^G$).

d) $K_n \otimes_K A_n$ is a split $K_n[G_n]$-module (call it B_n and put $C_n = B_n^{G_n}$).

Moreover define $\tilde{K} = \bigcup K_n$, $\tilde{k} = \bigcup k_n$, $A_n^o = \tilde{k} \otimes_{k_n} C_n$ and $A^o = \bigcup (K_n \otimes_K A)^{G_n}$. Then A^o is easily seen to be a sub-$\tilde{k}$-algebra of $\tilde{K} \otimes_K A$. We claim that the natural map $\tilde{K} \otimes_{\tilde{k}} A^o \longrightarrow \tilde{K} \otimes_K A$ is an isomorphisms of $\tilde{K}$-algebras (which we shall think of from now on as the identity). Indeed our map is surjective because $\tilde{K} \otimes_K A = \sum \tilde{K} \otimes_K A_n = \sum \tilde{K} \otimes_{K_n} B_n$ and any element of B_n is a K_n-linear combination of elements from C_n hence it is a $\tilde{K}$-linear combination of elements from A^o. To prove that our map is injective, let

$x_1,\ldots,x_p$ be $\tilde{k}$-linearly independent elements from A^o and let's check that they remain $\tilde{K}$-linearly independent as elements of $\tilde{K}\otimes_K A$. If $\sum a_i x_i=0$ with $a_i\in\tilde{K}$ then there exists an integer $n\geq 0$ such that $a_1,\ldots,a_p\in K_n, x_1,\ldots,x_p\in(K_n\otimes_K A)^{G_n}$. Since the map

$$K_n\otimes_{k_n}(K_n\otimes_K A)^{G_n}\longrightarrow K_n\otimes_K A$$

is injective and $x_1,\ldots,x_p$ are k_n-linearly independent they will remain K_n-linearly independent hence $a_1=\ldots=a_p=0$ and our claim is proved.

Next note that

$$\tilde{K}\otimes_{\tilde{k}}(A_n^o)=\tilde{K}\otimes_{K_n}(K_n\otimes_{k_n}C_n)=\tilde{K}\otimes_K A_n$$

hence the natural maps $A_n^o\longrightarrow A^o$ are injective and $A^o=\sum A_n^o$ so $(A^o,(A_n^o))$ is a finitely generated locally finite $\tilde{k}$-algebra. Now assertion 2) in our theorem follows because if G has property (*) then $G_n=G$ for all $n\geq 0$ hence $\tilde{k}=\tilde{K}^G$ and $A^o=(\tilde{K}\otimes_K A)^G$. To prove assertion 1) note that since A is finitely generated there is a finitely generated K-subalgebra R of $\tilde{K}$ such that $R\otimes_K(K\otimes_{\tilde{k}}A^o)=R\otimes_K A$. Then for all $n\geq 0$ we have

$$R\otimes_K(K\otimes_{\tilde{k}}A_n^o)\subset(R\otimes_K A)\cap(\tilde{K}\otimes_K A_n)=R\otimes_K A_n$$

By symmetry the converse inclusion also holds so we have $R\otimes_K A_n=R\otimes_K(K\otimes_{\tilde{k}}A_n^o)$ for all $n\geq 0$. We conclude by reducing the equality $R\otimes_K(K\otimes_{\tilde{k}}A^o)=R\otimes_K A$ modulo any maximal ideal of R.

(3.4) COROLLARY. Let K be algebraically closed and $A\in Ob(\mathcal{L}_K)$. Then A has a natural structure of locally finite $K[G]$-algebra

$(G=G(A,\mathcal{L}))$, in particular $(K^{\Sigma(A,\mathcal{L})})_a \in D(A,\mathcal{L})$. If in addition A is either commutative or finite dimensional then $K^{\Sigma(A,\mathcal{L})} \in M(A,\mathcal{L})$. Furthermore if tr.deg. $K/k=\infty$ and if $s: \Gamma=\mathrm{Aut}(K/K_o) \to G(A,\mathcal{L})$ is a section of $G(A;\mathcal{L}) \longrightarrow \Sigma(A,\mathcal{L})$ over $K_o=(K^{\Sigma(A,\mathcal{L})})_a$ (cf. 7) in (0.8)) making A a discrete $K[\Gamma]$-algebra then A splits over some constrained extension of K (here $\Gamma=\mathrm{Aut}(K/K_o)$ is viewed with its natural topology cf. (1.2)).

Proof. Recall that the elements of $G=G(A,\mathcal{L})$ are pairs $s=(\sigma,v)$ where $\sigma\in\Sigma(A,\mathcal{L})$ and $v:A \longrightarrow A^{\sigma}$ is an isomorphism in $\mathcal{L}_K$. Then K becomes a discrete G-field and A becomes a locally finite $K[G]$-algebra by letting $sa=p_{\sigma}(v(a))$ for all $s=(\sigma,v)\in G$ and $a\in A$ where $p_{\sigma}=\sigma\otimes 1_A: A^{\sigma}\longrightarrow A$. We conclude by (3.3) and remark 5) in (0.8).

Note that sections s as in the statement of (3.4) always exist by 2) in (0.8).

(3.5) We will give a "birational" application of (3.4). For any field K denote by $\mathcal{R}_K$ the category of finitely generated regular field extensions of K; if $K \longrightarrow K'$ is any field homomorphism define the base change functor $\mathcal{R}_K \to \mathcal{R}_{K'}$ by $F \mapsto F'$ where F' is the quotient field of the integral domain $K'\otimes_K F$. We have defined a fibred category $\mathcal{R}$ over the category of fields. Note that for any $F\in \mathrm{Ob}(\mathcal{R}_K)$, $\Sigma(F,\mathcal{R})$ identifies with the group $\{\sigma\in \mathrm{Aut}(K);$ there exists $\tilde{\sigma}\in \mathrm{Aut}(F)$ such that $\tilde{\sigma}/_K=\sigma\}$. The fields of definition relative to $\mathcal{R}$ are quite significant in algebraic geometry: they can be interpreted as "birational fields of definition" for algebraic varieties. Schimura's counterexamples [Sh] show that $K^{\Sigma(F,\mathcal{R})}$ need not be a field of definition for F (even if K=complex field, tr. deg. ${}_K F=1$). Here is what our method yelds:

(3.6) COROLLARY. Let K be algebraically closed and $F \in \mathrm{Ob}(\mathcal{R}_K)$ a regular field extension of K of general type. Then $({}_K\Sigma(F,\mathcal{R}))_a \in D(F,\mathcal{R})$ and ${}_K\Sigma(F,\mathcal{R}) \in M(F,\mathcal{R})$.

In the above statement by F/K being of general type we mean (in arbitrary characteristic) that there is a non-singular projective model V of F/K and an integer $n \geq 1$ such that the n-canonical rational map $f_n: V \dashrightarrow \mathbb{P}^N$ is birational onto its image.

Proof. Let $K[R_n]$ be the K-subalgebra of the canonical ring $R=\bigoplus_{m\geq 0} R_m$, $(R_m = H^0(V, \omega_{V/K}^{\otimes m})$, $\omega_{V/K}$=canonical bundle on V) generated by R_n. Then $K[R_n]$ has a natural structure of finitely generated locally finite K-algebra induced from the gradation. Moreover we have $\Sigma(K[R_n],\mathcal{L}) = \Sigma(F,\mathcal{R})$ and $D(K[R_n],\mathcal{L}) = D(F,\mathcal{R})$ so we may conclude by (3.4) and remark 5) in (0.8).

(3.7) We will give now an "affine" application of (3.4). For any field K denote by $\mathcal{A}_K$ the category of finitely generated geometrically integral commutative K-algebras (which is of course anti-isomorphic to the category of affine K-varieties): the categories $\mathcal{A}_K$ together with the obvious base change functors yeld a fibred category $\mathcal{A}$ over the category of fields. Clearly once again we have an identification between $\Sigma(A,\mathcal{A})$ and the group $\{\sigma \in \mathrm{Aut}(K)$; there exists $\tilde{\sigma} \in \mathrm{Aut}(A)$ such that $\tilde{\sigma}_{/K} = \sigma\}$. Here Aut(A) denotes the group of ring automorphisms of A.

(3.8) COROLLARY. Let K be algebraically closed of characteristic zero and U=Spec(A) an affine K-variety. Suppose U has non-negative Kodaira dimension. Then $(K^{\Sigma(A,\mathcal{A})})_a \in D(A,\mathcal{A})$ and $K^{\Sigma(A,\mathcal{A})} \in M(A,\mathcal{A})$.

In the above statement by the Kodaira dimension of an affine K-variety U we understand the Kodaira dimension of the noncomplete manifold U_{reg}(=smooth part of U) in Sakai's sense [Sa].

Proof. By the proof of (3.4) it is sufficient to prove that A has a locally finite structure (A_n) such that for any $\sigma\in\Sigma(A,\mathcal{A})$ and any K-isomorphism $v:A\longrightarrow A^\sigma$ we have $v(A_n)=A_n$ for all $n\geq 0$ (here the upper σ will always mean "applying the functor $K^\sigma\otimes_K$?" where K^σ is K itself viewed as a K-algebra via the isomorphism $\sigma^{-1}:K\longrightarrow K$).

We perform the following construction. Let U_{reg} be the smooth locus of U and let X be a smooth projective compactification of U_{reg} such that $X\setminus U_{reg}$ is the support of a reduced divisor D on X with normal crossings. Let $\omega_{X/K}$ be the canonical sheaf on X and choose an integer $m\geq 1$ such that $\mathcal{S}_m(X,D):=H^0(X,\omega_{X/K}^{\otimes m}((m-1)D))\neq 0$.

Put $A_n=\{f\in A;\ f\cdot\beta_1\otimes\beta_2\otimes\cdots\otimes\beta_n\in\mathcal{S}_{nm}(X,D)$ for all $\beta_1,\dots,\beta_n\in\mathcal{S}_m(X,D)\}$,

where we view A as a subspace of the field K(X) of rational functions on X and we view $\mathcal{S}_p(X,D)$ $(p\geq 1)$ as subspaces of the stalk of $\omega_{X/K}^{\otimes p}$ at the generic point of X. Clearly $A=\bigcup A_n$, $1\in A_o$ and $\dim_K A_n<\infty$ for all $n\geq 0$. Suppose now $v:A\longrightarrow A^\sigma$ is a K-isomorphism; we get an induced K-isomorphism $\varphi:U^\sigma\longrightarrow U$. Clearly $(U_{reg})^\sigma=(U^\sigma)_{reg}$ hence $\varphi((U_{reg})^\sigma)=U_{reg}$.

We will still denote by $\varphi : X^{\sigma} \dashrightarrow X$ the rational map induced by φ. By [Bo][Sa] we get induced K-isomorphisms:

$$\varphi^{*} : \mathcal{S}_p(X,D) \longrightarrow \mathcal{S}_p(X^{\sigma},D^{\sigma}) = \mathcal{S}_p(X,D)^{\sigma} \quad \text{for } p \geq 1 .$$

hence v induces K-isomorphisms $A_n \longrightarrow A_n^{\sigma}$ which closes our proof.

(3.9) We close this section by making the following remark. Let $\mathcal{D}$, $\mathcal{C}$ be fibred categories over $\mathcal{K}$; we say that $\mathcal{D}$ is a regular fibred subcategory of $\mathcal{C}$ if for all $K \in Ob(\mathcal{K})$, $\mathcal{D}_K$ is a full subcategory of $\mathcal{C}_K$, the base change functors $\mathcal{D}_K \longrightarrow \mathcal{D}_{K'}$, are induced by the corresponding functors $\mathcal{C}_K \longrightarrow \mathcal{C}_{K'}$, and moreover the following conditions hold:

1) if $A \in Ob(\mathcal{D}_K)$, $B \in Ob(\mathcal{C}_K)$ and $A \simeq B$ in $\mathcal{C}_K$ then $B \in Ob(\mathcal{D}_K)$

2) suppose $K_o \subset K$ is a field extension , $A^o \in Ob(\mathcal{C}_{K_o})$ and $A \in Ob(\mathcal{C}_K)$ is deduced by base change from A^o via $K_o \subset K$; then $A^o \in Ob(\mathcal{D}_{K_o})$ if and only if $A \in Ob(\mathcal{D}_K)$.

Note that if K is algebraically closed and $A \in Ob(\mathcal{D}_K)$ then $D(A,\mathcal{D}) = D(A,\mathcal{C})$, $\Sigma(A,\mathcal{D}) = \Sigma(A,\mathcal{C})$, $M(A,\mathcal{D}) = M(A,\mathcal{C})$.

Define the fibred subcategories $\mathcal{L}^{fin}$, $\mathcal{R}^{gen}$, $\mathcal{A}^{+}$ as follows: for each field K let $\mathcal{L}^{fin}_K$, $\mathcal{R}^{gen}_K$, $\mathcal{A}^{+}_K$ be the full subcategories of $\mathcal{L}_K$, $\mathcal{R}_K$, $\mathcal{A}_K$ whose objects are the finite dimensional algebras (viewed as locally finite in the standard way), the fields F in $\mathcal{R}_K$ such that the extension $K_a \otimes_K F/K_a$ is of general type (K_a=algebraic closure of K) and respectively the

K-algebras A in $\mathcal{A}_K$ such that the K_a-variety Spec$(K_a \otimes_K A)$ has nonegative Kodaira dimension.

$\mathcal{L}^{fin}$ is a regular fibred subcategory of $\mathcal{L}$ and in characteristic zero so are the subcategories $\mathcal{R}^{gen}, \mathcal{A}^+$ of $\mathcal{R}, \mathcal{A}$ respectively. With this remark in mind our Corollaries (3.4), (3.6), (3.8) imply that Theorem (0.10) from the Introduction holds in the cases a), b), c) (which correspond to $\mathcal{C} = \mathcal{L}^{fin}, \mathcal{R}^{gen}, \mathcal{A}^+$ respectively).

4. FORMAL K[G]-ALGEBRAS

(4.1) Recall that by a formal K-algebra we mean a local noetherian complete K-algebra with residue field K. Denote by $\mathcal{F}_K$ the category of formal K-algebras; for any field homomorphism $K \to K'$ define the base change functor $\mathcal{F}_K \to \mathcal{F}_{K'}$ by $A \mapsto K' \hat{\otimes}_K A$ (where $\hat{\otimes}$ is the completed tensor product). We have defined a fibred category $\mathcal{F}$ over the category of fields. Exactly as in (3.5) and (3.7) for any $A \in Ob(\mathcal{F}_K)$, $\Sigma(A, \mathcal{F})$ identifies with the group of all isomorphisms of K which can be lifted to ring automorphisms of A.

(4.2) Let K be a discrete G-field (G a topological group).

By a formal K[G]-algebra we mean a formal K-algebra which is also a K[G]-algebra. A is a called a continuous formal K[G]-algebra if for all $n \geq 1$ the K[G]-algebras A/M^n are discrete where $M = M(A)$ is the maximal ideal of A. The definition of a split formal K[G]-algebra is analogue to the definition of a split K[G]-algebra in (3.2): one has to replace $\otimes$ by $\hat{\otimes}$. Any split formal K[G]-algebra obviously is a continuous formal K[G]-algebra.

(4.3) THEOREM. Let A be a continuous formal $K[G]$-algebra.

1) If K is algebraically closed then $(K^G)_a \in D(A,\mathcal{F})$.

2) If G has property (*) then there exists a constrained extension $\tilde{K}/K$ of discrete G-fields such that $\tilde{K}\otimes_K A$ is a split formal $\tilde{K}[G]$-algebra.

Proof. For all $n \geq 0$, $A_n = A/M^n$ is a finite dimensional discrete $K[G]$-algebra. Using (2.3) we may construct once again sequences $(G_n)_{n\geq 0}$ and $(K_n)_{n\geq 0}$ satisfying properties a)-d) in the proof of (3.3). Let $k_n, B_n, C_n, \tilde{k}, \tilde{K}, A_n^o$ be defined by the same formulae as in the proof of (3.3). Then, exactly as in (3.3) A_n^o is a $\tilde{k}$-subalgebra of $\tilde{K}\otimes_K A_n$ and we have $\tilde{K}\otimes_{\tilde{k}}(A_n^o)=\tilde{K}\otimes_K A_n$ for all $n\geq 1$. Since the natural map $f_n: B_{n+1}\to K_{n+1}\otimes_{K_n} B_n$ is a map of $K_{n+1}[G_{n+1}]$-modules we get by (2.2) that

$$f_n(C_{n+1}) \subset (K_{n+1}\otimes_{K_n} B_n)^{G_{n+1}} = k_{n+1}\otimes_{k_n} C_n$$

Consequently the maps $\tilde{K}\otimes_K A_{n+1}\to \tilde{K}\otimes_K A_n$ send A_{n+1}^o onto A_n^o. We claim that with these data one can construct a formal $\tilde{k}$-algebra A^o and a $\tilde{K}$-isomorphism $f:\tilde{K}\hat{\otimes}_{\tilde{k}} A^o \to \tilde{K}\hat{\otimes}_K A$; moreover if G has property (*) we claim that we can choose f such that the G-action induced via f on $\tilde{K}\otimes_{\tilde{k}} A^o$ is given by $s(c\hat{\otimes} x) = sc\hat{\otimes} x$ for $s\in G$, $c\in K$, $x\in A^o$.

First note that our claim closes the proof of Theorem (4.3). This is clear for statement 2) in (4.3). To prove statement 1) we have to "specialize" the $\tilde{K}$-isomorphism $\tilde{K}\hat{\otimes}_K A \simeq \tilde{K}\hat{\otimes}_K (K\hat{\otimes}_{\tilde{k}} A^o)$; this is possible due to Seidenberg's criterion of analytic equivalence, see [Se].

Now the claim above can be proved using an argument from

[Bu$_1$], Chapter II, Section 5; we reproduce it here for convenience.

For any field F put $F_N = F[[X_1,\dots,X_N]]$ the power series F-algebra in N indeterminates. If N is the embedding dimension of A one can find surjective maps $p_n : \tilde{k}_N \to A_n^o$ which agree with the projections $A_{n+1}^o \longrightarrow A_n^o$. Upon letting $J_n = \mathrm{Ker}(p_n)$ we have $\tilde{K}$-isomorphisms $\tilde{K}_N / J_n\tilde{K}_N \simeq \tilde{K} \otimes_K A_n$ which are compatible with the projections obtained by "passing from n+1 to n", hence we have an injective map

$$\tilde{K}_N / \bigcap_n (J_n\tilde{K}_N) \longrightarrow \varprojlim_n (\tilde{K}_N / J_n\tilde{K}_N) \simeq \varprojlim_n (\tilde{K} \otimes_K A_n) = \tilde{K} \hat{\otimes}_K A$$

This map is also surjective because it is so when composed with the map $\tilde{K} \hat{\otimes}_K A \to \tilde{K} \otimes_K A_2$. Put $J_o = \bigcap_{n\geq 1} J_n$; we shall be done if we prove that $\bigcap_{n\geq 1} (J_n\tilde{K}_N) = J_o\tilde{K}_N$ because if it so we conclude by putting $A^o = \tilde{k}_N / J_o$. Upon letting $I_n = J_n / J_o \subset C = \tilde{k}_N / J_o$ and $B = \tilde{K}_N / J_o\tilde{K}_N$ we are reduced to proving that for any extension $C \subset B$ of local noetherian rings with C complete and for any sequence of ideals $(I_n)_{n\geq 1}$ in C with $\bigcap_{n\geq 1} I_n = 0$ we have $\bigcap_{n\geq 1} (I_n B) = 0$. The last statement can be proved as follows: by [Na] p.103 there is a function $m : \mathbb{N} \longrightarrow \mathbb{N}$ such that $I_{m(n)} \subset (M(C))^n$ for all n, hence $\bigcap_{n\geq 1} (I_{m(n)}B) \subset \subset \bigcap_{n\geq 1} (M(B))^n = 0$. This closes the proof of (4.3).

(4.4) Define the fibred subcategory $\mathfrak{F}^{alg}$ of $\mathfrak{F}$ by putting for any field K, $\mathfrak{F}_K^{alg}$ = full subcategory of $\mathfrak{F}_K$ whose objects are the algebraisable formal K-algebras; recall that $A \in \mathrm{Ob}(\mathfrak{F}_K)$ is called algebraisable if $K_a \hat{\otimes}_K A$ is the completion of a finitely generated K_a-algebra at some maximal ideal; in particular if K is algebraically closed then any $A\in Ob(\mathcal{F}_K^{alg})$ has a field of definition $K_o\in D(A,\mathcal{F})$ finitely generated over k. It is an easy exercise to check that if k is uncountable then $\mathcal{F}^{alg}$ is a regular fibred subcategory of $\mathcal{F}$. We have the following:

(4.5) COROLLARY. If K is algebraically closed and A is a formal K-algebra then $K_o=(K^{\Sigma(A,\mathcal{F})})_a\in D(A,\mathcal{F})$. If in addition k is uncountable and A is algebraisable then $K^{\Sigma(A,\mathcal{F})}\in M(A,\mathcal{F})$. Finally if tr.deg.$K/k=\infty$ and if $s:\Gamma=Aut(K/K_o)\to G(A,\mathcal{F})$ is a section of $G(A,\mathcal{F})\to\Sigma(A,\mathcal{F})$ over K_o making A a continuous formal $K[\Gamma]$-algebra then A splits over a constrained extension of K.

Proof. First assertion follows from (4.3) exactly as in (3.4). To prove the second assertion write $A\simeq K\hat{\otimes}_{K_o}(A^o)$. Since $\mathcal{F}^{alg}$ is regular in $\mathcal{F}$, A^o is still algebraisable, in particular $D(A^o,\mathcal{F})$ contains a field E finitely generated over k. Clearly the compositum $EK^{\Sigma(A,\mathcal{F})}$ in K is a finite extension of $K^{\Sigma(A,\mathcal{F})}$ and belongs to $D(A,\mathcal{F})$ so we may conclude by remark 5) in (0.8). The third assertion also follows from (4.3). Note that exactly as in (3.4) sections s as in the statement of (4.5) always exist by 2) in (0.8).

(4.6) Due to the regularity of $\mathcal{F}^{alg}$ in $\mathcal{F}$, Corollary (4.5) shows that Theorem (0.10) from the Introduction holds in case d) (which corresponds to $\mathcal{C}=\mathcal{F}^{alg}$).

5. HOPF K[G]-ALGEBRAS

(5.1) Hopf algebra terminology will be freely borrowed from [Sw][HSw]. Denote by $\mathcal{H}_K$ the category of finitely generated commutative Hopf K-algebras; for any field homomorphism $K \longrightarrow K'$ define the base change functor $\mathcal{H}_K \to \mathcal{H}_{K'}$ by $A \longmapsto K' \otimes_K A$. We have defined a fibred category over the category of fields; call it $\mathcal{H}$.

As well known $\mathcal{H}_K$ is anti-equivalent to the category of linear algebraic K-groups; if $A \in Ob(\mathcal{H}_K)$ the corresponding linear algebraic K-group will be L=Spec(A) with the multiplication induced by the comultiplication of A. We will often identify A and L above if there is no danger of confusion. Moreover if K is algebraically closed we will sometimes use the letter L to denote also the group L(K) of K-points of L.

(5.2) Following [NW] p.952, by a Hopf K[G]-algebra we mean a Hopf K-algebra A which is also a K[G]-algebra such that the comultiplication $\Delta : A \longrightarrow A \otimes_K A$ and counit $\varepsilon : A \longrightarrow K$ are K[G]-module maps.

(5.3) THEOREM. Let K be an algebraically closed G-field of characteristic zero and A a Hopf K[G]-algebra which is commutative and finitely generated. Then $(K^G)_a \in D(A, \mathcal{H})$.

The above theorem will be deduced from the following:

(5.4) THEOREM. Let K be algebraically closed of characteristic zero $K_o \subset K$ an algebraically closed subfield and L a linear algebraic K-group with unipotent radical U. Then $K_o \in D(L, \mathcal{H})$ if and only if $K_o \in D(\mathrm{Lie}(U), \mathcal{L})$ where Lie(U) is the Lie algebra of U and is viewed in a canonical way as a locally finite K-algebra (cf. (3.1)).

Theorems (5.3), (5.4), Corollary (3.4) and remark 4) in (0.3) yeld:

(5.5) COROLLARY. If L and U are as in Theorem (5.4) then $(K^{\Sigma(L,\mathcal{H})})_a \in D(L,\mathcal{H})$ and $K^{\Sigma(L,\mathcal{H})} \in M(L,\mathcal{H})$. Moreover we have $(K^{\Sigma(L,\mathcal{H})})_a = (K^{\Sigma(Lie(U),\mathcal{L})})_a$.

(5.6) Proof of Theorem (5.4). If $L=L^o \otimes_{K_o} K$ with L^o a linear algebraic K_o-group then $U=U^o \otimes_{K_o} K$ where U^o is the unipotent radical of U hence K_o is a field of definition for U, in particular for Lie(U). Conversely, if K_o is a field of definition for Lie(U) then so it will be for U because U is isomorphic as an affine variety with the spectrum of the symmetric algebra on Lie(U), the isomorphism being given by "exp" while the multiplication on U is defined by the Campbell-Hausdorff formula which involves only rational coefficients [Ho] p.228. So we may write $U \simeq U^o \otimes_{K_o} K$ for some unipotent K_o-group U^o. Now by [Ho] p.117 L is a semidirect product of U with some linearly reductive subgroup $P \subset L$. P is then reductive and in particular $P=P^o \otimes_{K_o} K$ for some reductive K_o-group P^o [De]. By [Ho] p.218 the group Aut(U) of algebraic group automorphisms of U is an algebraic K-group; moreover we must have $Aut(U)=Aut(U^o) \otimes_{K_o} K$ as one can see from the discussion at [Ho] p.217. Furthermore the group homomorphism $\rho: P \longrightarrow Aut(U)$ defined by $\rho(p)u=p^{-1}up$ ($p \in P$, $u \in U$) is also algebraic. We claim there is a K-point σ of Aut(U) and a morphism of algebraic K_o-groups $\rho^o: P^o \longrightarrow Aut(U^o)$ such that $\rho^o \otimes 1_K = Inn_\sigma \circ \rho$ where $Inn_\sigma \in Aut(Aut(U))$ is defined by $Inn_\sigma(\tau)=\sigma^{-1} \circ \tau \circ \sigma$. Indeed since P is linearly reductive, by [DGa] p.194 we have in particular $H^1(P, Lie(Aut(U)))=0$ (with

P acting on Lie(Aut(U)) via ρ and the adjoint representation of Aut(U)). By [DGr] p.116 the above cohomology group identifies with the space of "first order deformations" of ρ modulo the "first order deformations arrising from infinitesimal inner automorphisms of Aut(U)". Now the existence of ρ^o and σ follows for instance from [Bu_2], Theorem 2.11 plus an obvious specialisation argument. With ρ^o and σ at hand we may define an isomorphism of algebraic K-groups

$$\varphi : L = U \rtimes_{\rho} P \longrightarrow U \rtimes_{r} P$$

by the formula $\varphi(u,p) = (\sigma^{-1}(u), p)$ where $U \rtimes_{\rho} P$ is set theoretically $U \times P$ with multiplication given by $(u_1,p_1)(u_2,p_2) = ((\rho(p_2)u_1)u_2, p_1p_2)$ and $U \rtimes_r P$ is defined similarily with $r = \rho^o \otimes 1_K$ instead of ρ. But $U \rtimes_r P = (U^o \rtimes_{\rho^o} P^o) \otimes_{K_o} K$ and Theorem (5.4) is proved.

(5.7) Proof of Theorem (5.3). A is the coordinate Hopf algebra of an algebraic K-group L. Let U be the unipotent radical of L and J the defining prime ideal of U in A. We claim that $s(J)=J$ for all $s \in G$. Indeed upon letting σ to be the image of s in Aut(K) it is sufficient to prove that the natural map $p_\sigma : L^\sigma \longrightarrow L$ given in some matrix representation by $(x_{ij}) \longmapsto (\sigma x_{ij})$ carries the unipotent radical of L^σ onto the unipotent radical of L (here of course $L^\sigma = \mathrm{Spec}(A^\sigma)$). But this follows from the fact that the map p_σ is an abstract group isomorphism (of course not an algebraic K-group isomorphism !), it takes Zariski closed sets into Zariski closed sets and takes unipotent matrices into unipotent matrices, so our claim follows. We deduce that the coordinate Hopf algebra A/J of U is a Hopf $K[G]$-algebra. We need the following:

(5.8) LEMMA. Let $\mathcal{G}$ be a linear algebraic K-group and H its coordinate Hopf K-algebra. Suppose H has a structure of Hopf $K[G]$-algebra. Then Lie($\mathcal{G}$) has a (naturally induced) structure of $K[G]$-algebra.

Proof. Recall from [Sw][HSw] that $H^*=\mathrm{Hom}_K(H,K)$ has a K-algebra structure with multiplication given by convolution (for $f_1,f_2\in H^*$, $x\in H$ we have $(f_1 * f_2)(x)=\sum f_1(x_{(1)})f_2(x_{(2)})$ where we used the "sigma notation" $\Delta x=\sum x_{(1)} \otimes x_{(2)}$). On the other hand H^* has a natural structure of $K[G]$-module defined by $(sf)(x)=$ $=s(f(s^{-1}x))$ for $s\in G$, $f\in H^*$, $x\in H$. Using the fact that Δ and ε are $K[G]$-module maps it is straightforward to check that with the above $K[G]$-module structure, H^* is in fact a $K[G]$-algebra. Let $(H^*)^{\mathrm{Lie}}$ be the K-algebra whose underlying K-linear space is H^* and whose bracket is defined by $[f,g]=f * g-g * f$. Then clearly $(H^*)^{\mathrm{Lie}}$ is still a $K[G]$-algebra. Now recall from [HSw] p.219 that the K-linear space

$$\mathrm{Der}_K^{\varepsilon}(H,K)=\{f\in H^*;\ f(xy)=f(x)\varepsilon(y)+\varepsilon(x)f(y) \quad \text{for all } x,y\in H\}$$

is a Lie K-subalgebra of $(H^*)^{\mathrm{Lie}}$ and is isomorphic to Lie($\mathcal{G}$). On the other hand $\mathrm{Der}_K^{\varepsilon}(H,K)$ is a $K[G]$-submodule of H^* as one can see immediately by using the fact that ε is a $K[G]$-module map. Consequently $\mathrm{Der}_K^{\varepsilon}(H,K)$ has a structure of $K[G]$-subalgebra of $(H^*)^{\mathrm{Lie}}$ and we are done.

(5.9) Returning to the proof of (5.3) and recalling our notations from (5.7) we get by Lemma (5.8) that Lie(U) is a K[G]-algebra. By Theorem (3.3) $(K^G)_a \in D(\mathrm{Lie}(U), \mathcal{L})$ hence by Theorem (5.4) $(K^G)_a \in D(L, \mathcal{H})$ which closes our proof.

(5.10) It worths noting that, as well known [De] reductive groups "don't have moduli" in the sense that they are defined over the algebraic closure of the prime field. This is not the case with unipotent groups [GOH] so $K^{\Sigma(L,\mathcal{H})}$ in (5.5) may be transcendental over k.

(5.11) We close by explaining how one can obtain a splitting result for linear algebraic groups similar th splitting assertions in Theorems (3.3) and (4.3). First make the usual definitions: a Hopf K[G]-algebra is called discrete if it is so as a K[G]-module: it is called split if there is a Hopf K-algebra isomorphism $A \simeq K \otimes_{K^G} (A^o)$ for some Hopf K^G-algebra A^o such that the induced K[G]-algebra structure on $K \otimes_{K^G} (A^o)$ is given by $s(c \otimes x) = sc \otimes x$ for all $s \in G$, $c \in K$, $x \in A^o$. Once again "split" implies "discrete".

The topological group G will be said to have property (**) if for any open normal subgroup H of G the quotient group G/H is divisible. Clearly property (**) implies property (*) from (1.2). On the other hand note that the proof of Lemma (1.3) shows that the topological group $\mathrm{Aut}(K/K_o)$ (K_o, K algebraically closed, tr. deg. $K/K_o = \infty$) has property (**) as well.

(5.12) THEOREM. Let G be a topological group with property (**), let K be an algebraically closed discrete G-field of characteristic zero and A a discrete Hopf K[G]-algebra which is commutative and finitely generated. Then there is a constrained extension

$\tilde{K}/K$ of discrete G-fields such that $\tilde{K} \otimes_K A$ is a split Hopf $\tilde{K}[G]$--algebra.

Proof. Since G has property (*), K^G must be algebraically closed; we denote it by K_o. By (5.3) we have $A \simeq K \otimes_{K_o} (A^o)$ for some Hopf K_o-algebra A^o; $K \otimes_{K_o} (A^o)$ will inherit from A a structure of discrete Hopf $K[G]$-algebra. Let L^o be the linear algebraic K_o-group corresponding to A^o. By [BS] the functor of automorphisms of L^o is representable (on the category of reduced K_o-schemes) by a locally algebraic group scheme $\mathcal{G}$ over K_o which is an extension of an arithmetic group Λ by a linear algebraic K_o-group $\mathcal{G}_1$. We may construct a continuous cocycle $f \in Z^1(G, \mathcal{G}(K))$ as follows: for any $s \in G$ define $f(s) \in \mathcal{G}(K)$ to be the K-automorphisms of $L = L^o \otimes_{K_o} K$ obtained by composing the k-automorphism $a \longmapsto sa$ of $K \otimes_{K_o} (A^o)$ with the k-automorphism $c \otimes x \longmapsto \sigma^{-1} c \otimes x$ of the same algebra. We have an exact sequence of pointed sets (cf. [BS]):

$$H^1(G, \mathcal{G}_1(K)) \to H^1(G, \mathcal{G}(K)) \to H^1(G, \Lambda)$$

with G acting trivially on Λ. Now it is easy to check that there are no nontrivial homomorphisms from a divisible group to $GL_n(\mathbb{Z})$. This immediately implies that there are no nontrivial continuous homomorphisms from a topological group with property (**) into an arithmetic group, in particular $H^1(G, \Lambda) = 1$ so the class of f lifts to some element f_1 in $H^1(G, \mathcal{G}_1(K))$. By Theorem (1.1) one can find a constrained extension $\tilde{K}/K$ of discrete G-fields such that the image of f_1 via the map $H^1(G, \mathcal{G}_1(K)) \to H^1(G, \mathcal{G}_1(\tilde{K}))$ is 1. This implies that f is mapped to 1 via $H^1(G, \mathcal{G}(K)) \to H^1(G, \mathcal{G}(\tilde{K}))$. Finally this implies by standard arguments that $\tilde{K} \otimes_K A$ is a split

Hopf $\tilde{K}[G]$-algebra and Theorem (5.12) is proved.

(5.13) COROLLARY. If L and U are as in (5.4) and if tr.deg. $K/k=\infty$ and $s: \Gamma = \mathrm{Aut}(K/K_o) \longrightarrow G(L,\mathcal{H})$ is a section of $G(L,\mathcal{H}) \longrightarrow \Sigma(L,\mathcal{H})$ over $K_o = (K^{\Sigma(L,\mathcal{H})})_a$ making the coordinate Hopf algebra A of L a discrete Hopf $K[\Gamma]$- algebra then A splits over a constrained extension of K (again Γ is viewed with its natural topology).

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