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by

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The starting point of this paper is an open problem presented by Ikeda and Watanabe: they prove an almost sure approximation theorem for the solutions of stochastic equations and for their derivatives (with respect to the initial condition) in the case of the poligonal line approximation model and they ask the question if this may be done for more general models. This is done in the present paper.

We consider a square integrable, continuous multidimensional martingale M satisfying the condition $|\langle M\rangle(t) - \langle M\rangle(s)| \leq K|t-s|$ and the equation

$$dX(t,\lambda,\omega) = \alpha(t,\lambda,X) + \varphi(t,\lambda,X)dM(t) + \psi(t,\lambda,X)dt,$$

where λ is an abstract parameter and α, φ, ψ are nonanticipative functions.

A general approximation model M_ε, $\varepsilon > 0$ is defined. M_ε are cadlag processes with finite variation on compact time intervals. The approximating equations will be

$$dX_\varepsilon(t,\lambda,\omega) = \alpha_\varepsilon(t,\lambda,X_\varepsilon) + \varphi_\varepsilon(t,\lambda,X_\varepsilon)\tilde{d}M_\varepsilon(t) + \\ + \psi_\varepsilon(t,\lambda,X_\varepsilon)dt$$

where $\tilde{d}M_\varepsilon(t)$ designes a sort of discrete stochastic integration. The coefficients α_ε, φ_ε and ψ_ε are nonanticipative and converge uniformly to α , respectively to φ and ψ . Conditions for L^p convergence of X_ε to X are derived. In the second section we consider the Markovian case: $\alpha(t,\lambda, X) = \alpha(t,\lambda, X(t))$ and the same form for the other coefficiénts. Here λ is a real multidimensional parameter and infinite differentiably is assumed for the function $(\lambda, x) \to \alpha(t,\lambda, x)$ and for the other coefficients. Then $\int \varphi_\varepsilon \tilde{d}M_\varepsilon$ is replaced by $\int \varphi_\varepsilon dM_\varepsilon - \int g_\varepsilon A_\varepsilon dt$ where dM_ε designes a Stieltjes integral, g_ε depends on M_ε and A_ε on φ_ε . We are now in the classical context. We prove that one may choose a sequence $\varepsilon_n \to 0$ such that

$$\lim_{n\to\infty} \sup_{t\leq T} \sup_{\lambda} |X_{\varepsilon_n}(t,\lambda,\omega) - X(t,\lambda,\omega))| = 0 \quad \text{a.s.}$$

The same convergence is proved for the derivatives of $\lambda \to X(t,\lambda,\omega)$.

INTRODUCTION

Let M be a square integrable continuous multidimensional martingale fulfiling the hypothesis $\langle M\rangle(t) - \langle M\rangle(s) \leq K(t-s)$ for every $0 \leq s \leq t$. We are interested in giving approximation theorems for the solution of the equation

$$\text{(E)} \quad dX(t,\lambda,\omega) = \alpha(t,\lambda, X) + \varphi(t,\lambda, X)dM(t) + \psi(t,\lambda,X)dt,$$

where λ is an abstract parameter and α, φ, ψ are continuous nonanticipative functions fulfiling convenient bound(ed)ness and Lipschitz conditions (see (2.1) - (2.4) below).

In Section 1 we define a general approximation model for $M : M_\varepsilon$, $\varepsilon > 0$ will be cadlag (right continuous with left hand limits) processes with finite variation on compact time intervals fulfiling the following assumptions: $M_\varepsilon(k\varepsilon) = M(k\varepsilon)$ for every $k \in N$ and $M_\varepsilon(t)$ is $F_{k\varepsilon}$ measurable for the $t \leq k\varepsilon$, where $(F_t)_{t\geq 0}$ is the filtration with respect to which M_ε is a martingale.

(Although a large intersection with the general model presented by Ikeda and Watanabe in /1/ exists, the two models are far from being idantical: the first restriction ($M_\varepsilon(k\varepsilon) = M(k\varepsilon)$) constrains us to leave out the important example of the mollifiers and, on the other land, our model is more general in two ways: first of all we deal with cadlag instead of piecewise differentiable approximants and then, no "time omogenity" condition of the type "$M(k\varepsilon+t) = M_\varepsilon(k\varepsilon) + M_\varepsilon(t, \theta_{k\varepsilon})$" is assumed). Except for the two hypotheses above we shall consider the following assumptions:

$$(A_p) \quad E(|V_\varepsilon^k|^p|F_{k\varepsilon})^{1/p} \leqslant e_p\, \varepsilon^{1/2}$$

where V_ε^k is the variation of M_ε on $(k\varepsilon, (k+1)\varepsilon]$ and p is a natural number.

Then we define a sort of "discrete stochastic integral" with respect to M_ε. In order to do this, one defines first a "discrete compensator": for a cadlag process $A(t)$, $t \geqslant 0$ the discrete compensator will be a process $C_\varepsilon(A)$ (which is in fact explicitely defined in (1.8)) such that $A_\varepsilon(k\varepsilon) - C_\varepsilon(A)(k\varepsilon), k \in N$ is a martingale with respect to $(F_{k\varepsilon})_{k\in N}$. Then the discrete stochastic integral will be $\int\varphi\, \tilde{d}M_\varepsilon =: \int\varphi\, dM_\varepsilon - C_\varepsilon(\int\varphi\, dM_\varepsilon)$, where $\int\varphi\, dM_\varepsilon$ is a Stieltjes integral. Except for the symmetry motivation ($\int\varphi dM$ is a martingale and so $\int\varphi\,\tilde{d}M_\varepsilon$ should also be a martingale) the above definition has a calculating reason: the errors of order ε which appear in the calculus involving $\int\varphi\, dM_\varepsilon$ decrease to ε^2 if one replaces this integral by $\int\varphi\,\tilde{d}M_\varepsilon$.

Except for M_ε, $\varepsilon > 0$ we have to use other approximants, $\overline{M}_\varepsilon$, $\varepsilon > 0$. These are equal to M_ε up to $T_\varepsilon = \varepsilon\overline{k}_\varepsilon$ where $\overline{k}_\varepsilon = \min\{k : V_\varepsilon^k > \varepsilon^{3/8}\}$, and equal to zero after T_ε. As $T_\varepsilon \uparrow \infty$ as $\varepsilon \downarrow 0$, $\overline{M}_\varepsilon$ is rather closed to M_ε. The "stochastic integral" $\int\varphi\,\tilde{d}\overline{M}_\varepsilon$ is defined in the same way as with respect to M_ε. For this integral a Burkhölder type inequality is proved

((1.10) below). The proof of this inequality (as many other proofs in the paper) is based on a version of Burkhölder's inequality for discrete time martingales which is presented in the Appendix (Section 4).

In section 2 we deal with the L^p approximation of X. One defines the equations

$$(E_\varepsilon) \quad dX_\varepsilon(t,\lambda,\omega) = \alpha_\varepsilon(t,\lambda,X_\varepsilon) + \varphi_\varepsilon(t,\lambda,X_\varepsilon)\tilde{d}M_\varepsilon + \\ + \psi_\varepsilon(t,\lambda,X_\varepsilon)dt,$$

where α_ε, φ_ε, ψ_ε are nonanticipative, fulfil boundedness and Lipschitz conditions ((2.1) - (2.4) below), and converge uniformly to α respectively φ and ψ (In fact, in both equations (E) and (E_ε) a perturbation β respectively β_ε is useful to be considered (see (2.6), (2.10) and (2.8)). The first result of this section is that, under ($A_{p(p+1)}$), X_ε converge to X in L^p (Theorem 2.1).

The second theorem is a version of the first one: by replacing M_ε by $\overline{M}_\varepsilon$ the same result is derived under weaker assumptions on the coefficients (boundedness is replaced by liniar increments to infinity). The third theorem deals with replacing the compensator $C_\varepsilon(\int \varphi_\varepsilon dM_\varepsilon)$ (which appears in the definition of $\int \varphi_\varepsilon \tilde{d}M_\varepsilon$) by a drift $\int g_\varepsilon A_\varepsilon dt$, where g_ε depends on M_ε and A_ε is calculated starting from φ_ε. With this theorem we come back to the classical approximation context. Both g_ε and A_ε are analogous with the abjects considered by Ikeda and Watanabe in /1/ Cap.VI.7, except one difference: they assume that $\lim_{\varepsilon \to 0} g_\varepsilon = g$ for some function g, which is unnecessary here. This because there, the above mentioned drift does not appear as a compensator (in fact an "asymptotic compensator", being only asymptotically equivalent to $C_\varepsilon(\int \varphi_\varepsilon dM_\varepsilon)$), but as a correction in the limit equation (E). We are also doing this step in Corrolary 2.4. Returning to A_ε we mention that in order to be able to compute it one has to assume that φ_ε is somehow more particular: $\varphi_\varepsilon(t, \lambda, X_\varepsilon) = \varphi_\varepsilon(t, \lambda, X_\varepsilon(t))$ and depends on $x = X_\varepsilon(t)$ in a twice differentiable way. The argument permitting to replace the discrete compensator by a drift is based on Taylor's formula applied to φ_ε in the compensator.

In Section 3 we deal with almost sure convergence. Here the equations considered are Markovian (i.e. $\alpha(t, \lambda, X) = \alpha(t, \lambda, X(t))$ and the same for the other coefficients) and λ is a real multidimensional parameter. We also assume that $(\lambda, x) \to \alpha(t, \lambda, x)$ and the rest of the coefficients are infinitely differentiable. Under this assumption $\lambda \to X_\varepsilon(t, \lambda, \omega)$ is infinitely differentiable and we shall prove that one may find a version of X having the same property. Convergence for the derivatives of X_ε will be discussed too.

The interesting approximating equation here will not be (E_ε) but

$$(\bar{E}_\varepsilon) \qquad X_\varepsilon(t, \lambda, \omega) = \alpha_\varepsilon(t, \lambda, X_\varepsilon(t)) + \varphi_\varepsilon(t, \lambda, X_\varepsilon(t))d\bar{M}_\varepsilon - g_\varepsilon(t)A_\varepsilon(t, \lambda, X_\varepsilon(t))dt + \psi_\varepsilon(t, \lambda, X_\varepsilon(t))dt$$

The main result of the section is Theorem 3.3 which asserts that under the assumption that (A_p) holds for every $p \in N$, X_ε and its derivatives converge in any L^p, $p \in N$ to X and respectively to its derivatives. Then, Sobolev's inequality applied in a classical way ensures that the above convergence holds also under $\sup_\lambda \sup_{t \leq T} |\cdot|$ for every $T > 0$ almost surely. This result is proved by Ikeda and Watanabe in /1/ VI, 7. in the case in which M is the Brownian motion and M_ε is the poligonal line approximation. They set as an open problem whether such results might be obtained for more general approximation models.

1. THE APPROXIMATION MODEL

Let (Ω, F, P) be a probability space with a standard (right continuous and complete) filtration $(F_t)_{t \geq 0}$. Fix $d \in N$ and consider a d-dimensional, continuous, square integrable martingale $M : [0,\infty) X \Omega \to R^d$, $M = (M^1, \ldots, M^d)$. We shall assume that

(1.1) For every $T > 0$ there is a constant c_T such that

$$\langle M^i \rangle (t) - \langle M^i \rangle (s) \leq c_T (t - s)$$

for every $0 \leq s \leq t \leq T$ and $i \leq d$ ($\langle M^i \rangle$ is the compensator of M^{i2}).

Let now fix an $\varepsilon > 0$ and make some general notations. First of all we put $I_\varepsilon^k = (k\varepsilon, (k+1)\varepsilon]$ for $k \in N$. Then, for a function $f : [0, \infty) \to R$ we denote

$$f^*(t) = \sup_{s \leq t} |f(s)|$$

$$\Delta_\varepsilon^k f = f((k+1)\varepsilon) - f(k\varepsilon)$$

$$\bar{\Delta}_\varepsilon^k f = \sup\{|f(t) - f(k\varepsilon)| : k\varepsilon \leq t < (k+1)\varepsilon\}$$

We shall now define the processes approximating M. Let $M_\varepsilon : [0,\infty) x \Omega \to R^d$, $M_\varepsilon = (M_\varepsilon^1, \ldots, M_\varepsilon^d)$ be a d-dimensional cadlag process with finite variation on compact time intervals and such that:

(1.2) i) $M_\varepsilon(k\varepsilon) = M(k\varepsilon)$ for any $k \in N$

ii) $M_\varepsilon(t)$ is $F_{k\varepsilon}$ measurable for any $o \leq t \leq k\varepsilon$

Define than

$$V_\varepsilon^k(\omega) = \max_{i \leq d} \sup\left\{ \sum_{j=0}^{m-1} |M^i(t_{j+1}, \omega) - M^i(t_j, \omega)| : k\varepsilon = \right.$$

$$\left. = t_o < t_1 \ldots < t_m = (k+1)\varepsilon \right\}$$

Clearly V_ε^k is the maximum of the variations of M_ε^i on $[k\varepsilon, (k+1)\varepsilon]$.

The following assumption will be essential through all the paper.

(A_p) For every fixed T 0 there is a constant $e_p = e_p(T)$ such that

$$E(\,|V_\varepsilon^k|^p|F_{k\varepsilon})^{1/p} \leqslant e_p\,\varepsilon^{1/2}, \text{ for every } \varepsilon > 0,\ k \leqslant T/\varepsilon,$$

where $p \in N$. Assuming that (A_2) is fulfiled one may define

$$(1.3)\quad g^{ij}_{\varepsilon,k}(\omega) = \frac{1}{\varepsilon} E(\int_{I_\varepsilon^k} (M_\varepsilon^j(s-) - M_\varepsilon^j(k\varepsilon))dM_\varepsilon^i(s) \| F_{k\varepsilon})(\omega) \text{ and}$$

$$g^{ij}_\varepsilon(t,\omega) = g^{ij}_{\varepsilon,k} \quad \text{for } t \in I_\varepsilon^k.$$

Note that $|g^{ij}_\varepsilon(t)| \leqslant e_2^2 < \infty$ for $t \leqslant T$. The function g^{ij}_ε are an analogous of $s_{ij}(\varepsilon)$ defined in /1/, VI, (7.3). Ikeda and Watanabe assume that $\lim\limits_{\varepsilon\to 0} s_{ij}(\varepsilon)$ exists ((A.7) in the above mentioned work). It turns out that this is not really necessary (see Theorem 2.3 and Corollary 2.4 below).

There are situations in which we are interested in stopping M when V_ε^k becomes large. So we shall define

$$(1.4)\quad \bar{k}_\varepsilon = \min\{k : V_\varepsilon^k > \varepsilon^{3/8}\} \quad \text{and} \quad T_\varepsilon = \varepsilon\,\bar{k}_\varepsilon.$$

Note that T_ε is not a stopping time with respect to the filtration $(F_t)_{t\geqslant 0}$ but it is with respect to $(F_{t+\varepsilon})_{t\geqslant 0}$. Note also that if (A_p) holds for some $p > 8$, then

$$(1.5)\quad \lim_{\varepsilon\to 0} P(T_\varepsilon \leqslant T) = 0 \quad \text{for every } T > 0.$$

Indeed we have:

$$P(T_\varepsilon \leqslant T) = P(\max_{k\leqslant T/\varepsilon} V_\varepsilon^k > \varepsilon^{3/8}) \leqslant \sum_{k\leqslant T/\varepsilon} P(V_\varepsilon^k > \varepsilon^{3/8}) \leqslant$$

$$\leqslant \sum_{k\leqslant T/\varepsilon} \varepsilon^{-3p/8} E(|V_\varepsilon^k|^p) \leqslant T\,\varepsilon^{-(3p/8+1)} e_p^p\,\varepsilon^{p/2} \to 0, \text{as } \varepsilon\to 0.$$

Then we define

$$(1.6)\quad \bar{M}_\varepsilon(t,\omega) = M_\varepsilon(t\wedge T_\varepsilon,\omega)$$

and denote by $\bar{V}_\varepsilon^k$ the maximum of the variations of $\bar{M}_\varepsilon^i$, $i \leqslant d$ on $[k\varepsilon, (k+1)\varepsilon]$. Then $\bar{V}_\varepsilon^k = V_\varepsilon^k$ for $k < \bar{k}_\varepsilon$ and $\bar{V}_\varepsilon^k = 0$ for $k > \bar{k}_\varepsilon$. It is also clear that $\bar{V}_\varepsilon^k \leqslant \varepsilon^{3/8}$.

(cadlag measurable

Let now introduce the norm we shall work with: for a function $\varphi : [0,\infty) \times \Omega \to R$, $p \in N$ and $0 < T, S$

$$(1.7) \qquad \|\varphi\|_{p,T} = E(\sup_{t \leq T} |\varphi(t,\omega)|^{p})^{1/p}$$

$$\|\varphi\|_{p,T,S} = E(\sup_{t \leq S} |\varphi(T+t,\omega) - \varphi(T,\omega)|^{p})^{1/p}$$

Note that $\|\varphi\|_{p,T,S} \leq \|\varphi\|_{p,T} + \|\varphi\|_{p,T,S}$ and, if $\varphi(0) = 0$, then $\|\varphi\|_{p,0,S} = \|\varphi\|_{p,S}$.

We go on and define a "discrete compensator" for $\varphi : [0,\infty) \times \Omega \to R$ cadlag and measurable. Assume that $E(\varphi^{*}(T)) < \infty$ for every $T > 0$ and define

$$(1.8) \quad C_{\varepsilon}(\varphi)(k\varepsilon) =: \sum_{i=0}^{k-1} E(\Delta_{\varepsilon}^{i}\varphi \mid F_{i\varepsilon}), \quad k \in N$$

$$C_{\varepsilon}(\varphi)(t) = C_{\varepsilon}(\varphi)(k\varepsilon) \quad \text{for } k\varepsilon \leq t < (k+1)\varepsilon$$

Then one may define a "discrete stochastic integral" with respect to M_{ε} and $\bar{M}_{\varepsilon}$:

$$(1.9) \quad \int_{0}^{t} \varphi(s,\omega)\,\tilde{d}M_{\varepsilon}(s,\omega) =: \int_{0}^{t} \varphi(s,\omega)\,dM_{\varepsilon}(s,\omega) - - C_{\varepsilon}\left(\int_{0}^{\cdot} \varphi\, dM_{\varepsilon}\right)(t,\omega)$$

$$\int_{0}^{t} \varphi(s,\omega)\,\tilde{d}\bar{M}_{\varepsilon}(s,\omega) =: \int_{0}^{t} \varphi(s,\omega)\,d\bar{M}_{\varepsilon}(s,\omega) - - C_{\varepsilon}\left(\int_{0}^{\cdot} \varphi\, d\bar{M}_{\varepsilon}\right)(t,\omega).$$

We shall now present an analogue of Burkhölder's inequality:

Lema 1.1. Fix an even $p \geq 4$ and assume that $(A_{p(p+1)})$ holds. Then there exists a constant K_p such that (1.10) holds for every $T \geq 0$, $0 \leq S \leq 1$, ε such that $\varepsilon \leq S$ and every $\varphi : [0,\infty) \times \Omega \to R$ such that $E(\varphi^{*}(T)) < \infty$;

$$(1.10) \qquad \left\| \int_{0}^{\cdot} \varphi\, \tilde{d}\bar{M}_{\varepsilon} \right\|_{p,T,S} \leq K_p\, S^{1/p}\, \|\varphi\|_{p,T+S}.$$

Proof. Put $k_\varepsilon = [T/\varepsilon]$ and $k'_\varepsilon = [(T+S)/\varepsilon]$. For $k \leq k'_\varepsilon$ and $0 \leq t < \varepsilon$ such that $k\varepsilon + t \leq T+S$

$$\left|\int_{k\varepsilon}^{k\varepsilon+t} \varphi \, \widetilde{d\bar{M}}_\varepsilon\right| = \left|\int_{k\varepsilon}^{k\varepsilon+t} \varphi \, d\bar{M}_\varepsilon\right| \leq \varepsilon^{3/8} \varphi^*(T+S).$$

Then, as $\varepsilon \leq S$

$$\left\| \int_0^{\cdot} \varphi \, \widetilde{d\bar{M}}_\varepsilon \right\|_{p,T,S} \leq E\Big(\max_{k_\varepsilon < k \leq k'_\varepsilon} \left|\int_{k_\varepsilon}^{k\varepsilon} \varphi \, \widetilde{d\bar{M}}_\varepsilon\right|^p\Big)^{1/p} + 2S^{1/p} \|\varphi\|_{p,T+S}$$

We shall now apply Lemma 4.1. (see Appendix) to the martingale $H_k = \int_0^{k\varepsilon} \varphi \, \widetilde{d\bar{M}}_\varepsilon$, $k \in N$. Fix $2 \leq i \leq p-1$ and $k \leq k'_\varepsilon$. Then by Hölder's inequality and (A_{ip}) hypothesis we have

$$E(|H_{k+1} - H_k|^i | F_{k\varepsilon}) \leq 2^i E(\varphi^*(T+S)^i |V_\varepsilon^k|^i | F_{k\varepsilon}) \leq$$

$$\leq 2^i E(\varphi^*(T+S)^{ip/(p-1)} | F_{k\varepsilon})^{(p-1)/p} E(|V_\varepsilon^k|^{ip} | F_{k\varepsilon})^{1/p} \leq$$

$$\leq 2^i E(\varphi^*(T+S)^{ip/(p-1)} | F_{k\varepsilon})^{(p-1)/p} e_{pi}^i \varepsilon^{i/2}.$$

It follows that

$$\langle H \rangle_i(k'_\varepsilon) - \langle H \rangle_i(k_\varepsilon) =: \sum_{k=k_\varepsilon}^{k'_\varepsilon - 1} E(|H_{k+1} - H_k|^i | F_{k\varepsilon}) \leq$$

$$\leq \varepsilon^{-1} \cdot S \cdot 2^{i+1} e_{pi}^i \varepsilon^{i/2} \max_{k < k'_\varepsilon} E(\varphi^*(T+S)^{ip/(p-1)} | F_{k\varepsilon})^{(p-1)/p}$$

Since $i \geq 2$ one has $S \cdot \varepsilon^{-1} \cdot \varepsilon^{i/2} \leq S$. Then, by Dood's inequality

$$E(|\langle H \rangle_i(k'_\varepsilon + 1) - \langle H \rangle_i(k_\varepsilon)|^{p/i})^{1/p} \leq$$

$$\leq K'_p S^{1/p} E(\max_{k \leq k'_\varepsilon} E(\varphi^*(T+S)^{ip/(p-1)} | F_{k\varepsilon})^{(p-1)/i})^{1/p} \leq$$

$$\leq K'_p S^{1/p} E(\max_{k \leq k'_\varepsilon} E(\varphi^*(T+S)^p | F_{k\varepsilon}))^{1/p} \leq$$

$$\leq K''_p S^{1/p} E(\varphi^*(T+S))^p)^{1/p}.$$

For $i = p$

$$E(|H_{k+1} - H_k|^p | F_{k\varepsilon}) \leq 2^p E(\varphi^*(T+S)^p |\bar{V}_\varepsilon^k|^p | F_{k\varepsilon}) \leq$$

$$\leq 2^p E(\varphi^*(T+S)^p | F_{k\varepsilon}) \, \cancel{\times}\, \varepsilon^{3p/8}$$

Then Dood's inequality yields

$$E(\langle H\rangle_p(k'_\varepsilon+1) - \langle H\rangle_p(k_\varepsilon))^{1/p} \leqslant K''_p\, s^{1/p} E(\varphi^*(T+S)^p)^{1/p}$$

Using now (4.1) the proof of the Lemma is complete.

Q.E.D.

2. L^p APPROXIMATION FOR THE SOLUTIONS OF NON MARKOV EQUATIONS

Consider $W = \{w : [0,\infty) \to R^d : t \to w(t)$ is continuous$\}$ with the σ-algebra $W_t = \sigma(w(s) : s \leqslant t)$ and $\overline{W} = \{\overline{w} : [0,\infty) \to R^d : t \to \overline{w}(t)$ cadlag$\}$ with the σ- algebra $\overline{W}_t = \sigma(\overline{w}(s) : s \leqslant t)$.

Let Γ be an abstract set of parameters and $\varphi_\gamma : [0,\infty)\times\overline{W} \to R$, $\gamma \in \Gamma$.

The following assumptions will be considered:

(2.1) i) $\overline{w} \to \varphi_\gamma(t,\overline{w})$ is $\overline{W}_{t-}$ measurable for every $t \geqslant 0$ and and $t \to \varphi_\gamma(t,\overline{w})$ is cadlag ($\overline{W}_{t-} =: \bigvee_{s<t} \overline{W}_s$).

For every $T > 0$ there exist some constants K_T, K'_T, K''_T, K'''_T and $K^{IV}_T < \infty$ such that for every $\gamma \in \Gamma$, $\overline{w}, \overline{w}' \in \overline{W}$, $\varepsilon > 0$, $k \leqslant T/\varepsilon$, $0 \leqslant t \leqslant T$ and $0 \leqslant s \leqslant \varepsilon$

ii) $|\varphi_\gamma(t,\overline{w})| \leqslant K_T + K'_T \sup_{s<t} |\overline{w}(s)|$

iii) $|\varphi_\gamma(t,\overline{w}) - \varphi_\gamma(t,\overline{w}')| \leqslant K''_T \sup_{s<t} |\overline{w}(s) - \overline{w}'(s)|$

iv) $|\varphi_\gamma(k\varepsilon+s,\overline{w}) - \varphi_\gamma(k\varepsilon,\overline{w})| \leqslant K'''_T\,\varepsilon + K^{IV}_T\, \overline{\Delta}^k_\varepsilon \overline{w}$

(2.2) The hypothesis (2.1) with the supplementary assumption that one may choose $K'_T < 1$, $K''_T < 1$ and $K^{IV}_T < 1$.

(2.3) The hypothesis (2.1) with $K'_T = 0$

(2.4) The hypothesis (2.1) with $K'_T = 0$, $K''_T < 1$ and $K^{IV}_T < 1$.

We shall use the above hypothesis also for functions defined on $[0,\infty) \times W$. The only difference will be that $\varphi_\gamma(t,\cdot)$ will be W_t measurable instead of $\overline{W}_{t-}$ measurable.

Let now consider another abstract set of parameters, Λ, and the process $\beta : [0,\infty) \times \Lambda \times \Omega \to R$, cadlag and such that $\omega \to \beta(t,\lambda,\omega)$ is F_t measurable for every $(t,\lambda) \in [0,\infty) \times \Omega$.

Define:

(2.5) $\|\beta\|^{\Lambda}_{p,T} = \sup_{\lambda\in\Lambda} \|\beta(\cdot,\lambda,\cdot)\|_{p,T}$

$\|\beta\|^{\Lambda}_{p,T,S} = \sup_{\lambda\in\Lambda} \|\beta(\cdot,\lambda,\cdot)\|_{p,T,S}$

for $p \in N$, $T,S \geq 0$.

For a family β_ε, $\varepsilon > 0$ of such processes we shall be interested in the properties

(I_p) $\sup_\varepsilon \|\beta_\varepsilon\|^{\Lambda}_{p,T} < \infty$ for every $T > 0$.

(J_p) There is another family of processes K_ε, $\varepsilon > 0$ fulfiling (I_p) and such that for every even $i \leq p$, $\varepsilon > 0$ and $k \in N$

$$E(|\bar{\Delta}^k_\varepsilon \beta_\varepsilon|^i | F_k)^{1/i} \leq K^*_\varepsilon(k\varepsilon)\varepsilon^{3/8}.$$

We are now able to define the equations we are interested in. Consider some functions $\varphi : [0,\infty) \times \Lambda \times W \to R^d \times R^d$, $\varphi = (\varphi^{ij})_{i,j\leq d}$, $\alpha, \psi : [0,\infty) \times \Lambda \times W \to R^d$, $\alpha = (\alpha^i)_{i\leq d}$, $\psi = (\psi^i)_{i\leq d}$ and a cadlag adapted process $\beta : [0,\infty) \times \Lambda \times \Omega \to$ $\to R^d$, $\beta = (\beta^i)_{i\leq d}$. Consider then the equation

(2.6) $X(t,\lambda,\omega) = \beta(t,\lambda,\omega) + \alpha(t,\lambda,X) +$

$$+ \int_0^t \varphi(s,\lambda,X)dM(s) + \int_0^t \psi(s,\lambda,X)ds,$$

or, componentwise

$$X^i(t,\lambda,\omega) = \beta^i(t,\lambda,\omega) + \alpha^i(t,\lambda,X) + \sum_{j=1}^d \int_0^t \varphi^{ij}(s,\lambda,X)dM^j(s)$$

$$+ \int_0^t \psi^i(s,\lambda,X)ds, \quad i \leq d.$$

(2.7). Remark: Under hypothesis (2.2) for α and (2.1) for φ and ψ the above equation has at most one solution. In the case in which $\alpha = 0$ or $\beta = 0$ it is also known that at least one.

solutions exists. But in fact we are not especially interested in the existence of some solution. So we shall use the expression "Let X be the solution of (2.6)" with the reserve "if such a solution exists".

The same will be true for the approximating equations ((2.10) belor).

(2.8.). Remark: β may be regarded as a perturbation. See for example the way in which Theorem 2.3 follows from Theorem 2.1. To the perturbation in (2.6) there correspond some other perturbations β_ε, $\varepsilon > 0$ in (2.10).

In order to get L^p convergence these perturbations have to fulfil (I_p) and (J_p). This is the meaning of these assumptions: they characterize "good perturbations".

(2.9). Remark: In fact the system (2.6) of equations is more general than it seems to be: one may assume that α, φ and ψ depend not only on X but also on M. That is

$$(2.6') \quad X(t,\lambda,\omega) = \beta(t,\lambda,\omega) + \alpha(t,\lambda, M, X) + \int_0^t \varphi(s,\lambda, M, X)\,dM + \int_0^t \psi(s,\lambda, M, X)\,ds.$$

The system (2.6′) is reductible to a system of type (2.6) by adding the trivial equations $M^i(t) = \int_0^t 1\, dM^i(s)$, $i \leq d$ and taking $Z = (M, X)$ instead of X. In particular all the theorems in the paper may be considered as approximation theorems for the stochastic integrals: one takes $\alpha = \beta = \psi = 0$ and φ independent of X. Then (2.6′) becomes $X(t,\lambda,\omega) = \int_0^t \varphi(s,\lambda, M)\,dM(s)$.

In order to avoid notational complications we restrict ourselves to (2.6).

Let now define the approximating equations. For every $\varepsilon > 0$ consider: $\varphi_\varepsilon : [0,\infty) \times \Lambda \times \bar{W} \to R^d \times R^d$, $\varphi_\varepsilon = (\varphi_\varepsilon^{ij})_{i,j \leq d'}$

$\alpha_\varepsilon, \psi_\varepsilon : [0,\infty) \times \Lambda \times \tilde{W} \to R^d,$

$\alpha_\varepsilon = (\alpha_\varepsilon^i)_{i \leq d}, \quad \psi_\varepsilon = (\psi_\varepsilon^i)_{i \leq d}$ and $\beta_\varepsilon : [0,\infty) \times \Lambda \times \Omega \to R^d$,

$\beta_\varepsilon = (\beta_\varepsilon^i)_{i \leq d}$, the last ones being cadlag and adapted processes.

Consider the equation

$$(2.10) \quad X(t,\lambda,\omega) = \beta_\varepsilon(t,\lambda,\omega) + \alpha_\varepsilon(t,\lambda,X_\varepsilon) + \int_0^t \varphi_\varepsilon(s,\lambda,X_\varepsilon)\tilde{d}M_\varepsilon(s) + \int_0^t \psi_\varepsilon(s,\lambda,X_\varepsilon)ds,$$

or, componentwise

$$X_\varepsilon^i(t,\lambda,\omega) = \beta_\varepsilon^i(t,\lambda,\omega) + \alpha_\varepsilon^i(t,\lambda,X_\varepsilon) + \sum_{j=1}^{d}\int_0^t \varphi_\varepsilon^{ij}(s,\lambda,X_\varepsilon)\cdot$$

$$\cdot\, \tilde{d}M_\varepsilon^j(s) + \int_0^t \psi_\varepsilon^i(s,\lambda,X)ds, \quad i \leq d.$$

Remark:

By a solution of equation (2.10) we shall understand an adapted cadlag process $X_\varepsilon = (X_\varepsilon^1, \ldots, X_\varepsilon^d)$ such that $E(X_\varepsilon^*(T)) < \infty$ for every $T > 0$ and a version of the conditional expectations $E(\int_{I_\varepsilon^k} \varphi_\varepsilon^{ij}(s,\lambda,X_\varepsilon)dM^j(s)|F_{k\varepsilon})$, $1 \leq i,j \leq d$, $k \in N$ such that X_ε verifies (2.10) in which $C_\varepsilon(\int_0^{\cdot} \varphi_\varepsilon^{ij}(s,\lambda,X_\varepsilon)dM_\varepsilon^j(s))$ $1 \leq i,j \leq d$ are defined by means of the above mentioned versions of the conditional expectations ($C_\varepsilon(\int_0^{\cdot}\varphi_\varepsilon^{ij}\, dM_\varepsilon^j)$ is involved in the definition of $\int_0^{\cdot}\varphi_\varepsilon^{ij}\,\tilde{d}M_\varepsilon^j$).

As for the equations (2.6) uniqueness is easily proved and existence of solutions is not especially interesting. If $\alpha_\varepsilon = 0$ or $\beta_\varepsilon = 0$, existance may be proved by using an inductive algorithm on the intervals I_ε^k, $k \in N$. As above, the expression "Let X_ε be a solution of equation (2.10)" will be used under reserve that such a solution exists.

Finally, we have to specify the "distance" between the coefficients of (2.6) and those of (2.10). For $T > 0$ and $p \in N$ $\delta(\varepsilon) = \delta_T^p(\varepsilon)$ and K_T will be positive numbers such that for every $0 \leqslant t \leqslant T$, $w \in W$, $\lambda \in \Lambda$ and $\varepsilon > 0$

(2.11) i) $\max\{|(\alpha_\varepsilon - \alpha)(t,\lambda,w)|,$

$$|(\varphi_\varepsilon - \varphi)(t,\lambda,w)|,\ |(\psi_\varepsilon - \psi)(t,\lambda,w)|\} \leqslant$$

$$\leqslant K_T(\delta(\varepsilon) + \max_{k \leqslant t/\varepsilon} \overline{\Delta}^k_\varepsilon w)$$

ii) $\| \beta_\varepsilon - \beta \|^{\wedge}_{p,T} \leqslant \delta(\varepsilon)$.

We shall now formulate the L^p approximation result in the case of bounded coefficients.

Theorem 2.1. Fix an even $p \geqslant 6$ such that $(A_{p(p+1)})$ holds. Assume that the family φ^{ij}, φ^{ij}_ε, ψ^i, ψ^i_ε, $1 \leqslant i,j \leqslant d$, $0 < \varepsilon$, $\lambda \in \Lambda$ fulfils (2.3) and the family α^i, α^i_ε, $1 \leqslant i \leqslant d$, $0 < \varepsilon$, $\lambda \in \Lambda$ fulfils (2.4).

Assume also that β^i_ε, $1 \leqslant i \leqslant d$, $0 < \varepsilon$ fulfils (J_{p+1}) and that $\lim_{\varepsilon \to 0} \delta^p_T(\varepsilon) = 0$. Then

$$\lim_{\varepsilon \to 0} \| X_\varepsilon - X \|^{\wedge}_{p,T} = 0.$$

Remark (2.12) below and the Borel Cantelli's Lemma ensure that if one chooses a sequence $(\varepsilon_n)_{n \in N}$ such that $\sum_n \varepsilon_n^{1/2p} < \infty$ than, for every fixed $\lambda \in \Lambda$

$$\lim_n \sup_{t \leqslant T} |X_{\varepsilon_n}(t,\lambda,\omega) - X(t,\lambda,\omega)| = 0 \text{ a.s.}$$

But the null set above depends on $\lambda \in \Lambda$. The really interesting result of almost sure convergence will be obtained only in section 3 (Theorem 3.3): there the exceptional set does not depend on λ and the convergence is uniform with respect to $\lambda \in \Lambda$.

Proof. For simplicity we shall use unidimensional notations. Fix $T > 0$. K_i, $i = 1,2, \ldots$ will be constants depending on p and T only. We shall prove that

(2.12) $$\| X_\varepsilon - X \|^{\wedge}_{p,T} \leqslant K_1(\varepsilon^{1/4p} + \delta(\varepsilon))$$

for sufficiently small ε.

(and S, S' are multiples of ε.

Fix $0 \leqslant S'$ and $0 \leqslant S \leqslant 1$ such that $S' + S \leqslant T$ Write

$$(2.13) \quad \| X_\varepsilon - X \|^{\wedge}_{p,S',S} \leqslant \| \beta_\varepsilon - \beta \|^{\wedge}_{p,S',S} + $$

$$+ \| \alpha_\varepsilon - \alpha \|^{\wedge}_{p,S',S} + \| \int_0^{\cdot} \varphi_\varepsilon \tilde{d}M_\varepsilon - \int_0^{\cdot} \varphi \, dM \|^{\wedge}_{p,S',S} +$$

$$+ \| \int_0^{\cdot} \psi_\varepsilon \, ds - \int_0^{\cdot} \psi \, ds \|^{\wedge}_{p,S',S} .$$

Using (2.11) (i) and (2.4) iii) we obtain

$$(2.14) \quad \| \alpha_\varepsilon - \alpha \|^{\wedge}_{p,S',S} \leqslant \| \alpha_\varepsilon - \alpha \|^{\wedge}_{p,S'} + \| \alpha - \alpha_\varepsilon \|^{\wedge}_{p,S'+S} \leqslant$$

$$\leqslant 2\, \delta(\varepsilon) + K_2 \| X_\varepsilon - X \|^{\wedge}_{p,S'} + K_3 \| X_\varepsilon - X \|^{\wedge}_{p,S'+S}$$

with $K_3 < 1$. Then, by (2.11) (i) and (2.3) (ii)

$$(2.15) \quad \| \int_0^{\cdot} \psi_\varepsilon \, ds - \int_0^{\cdot} \psi \, ds \|^{\wedge}_{p,S',S} \leqslant S \| \psi_\varepsilon - \psi \|^{\wedge}_{p,S'+S} \leqslant$$

$$\leqslant K_4 S (\delta(\varepsilon) + \| X_\varepsilon - X \|^{\wedge}_{p,S'+S}).$$

By (2.11) (ii)

$$(2.16) \quad \| \beta_\varepsilon - \beta \|^{\wedge}_{p,S',S} \leqslant 2\delta(\varepsilon).$$

Put $k_\varepsilon = (S'+S)/\varepsilon$, $k'_\varepsilon = S'/\varepsilon$ and write

$$(2.17) \quad \| \int_0^{\cdot} \varphi_\varepsilon \tilde{d}M_\varepsilon - \int_0^{\cdot} \varphi \, dM \|^{\wedge}_{p,S',S} \leqslant$$

$$\leqslant \sup_{\lambda \in \Lambda} E(\max_{k \leqslant T/\varepsilon} | \bar{\Delta}^k_\varepsilon (\int_0^{\cdot} \varphi_\varepsilon \tilde{d}M_\varepsilon) |^p)^{1/p} +$$

$$+ \sup_{\lambda \in \Lambda} E(\max_{k \leqslant T/\varepsilon} | \bar{\Delta}^k_\varepsilon (\int_0^{\cdot} \varphi \, dM) |^p)^{1/p} +$$

$$+ \sup_{\lambda \in \Lambda} E(\max_{k' < k \leqslant k_\varepsilon} | \int_{S'}^{k\varepsilon} \varphi_\varepsilon \tilde{d}M_\varepsilon - \int_{S'}^{k\varepsilon} \varphi \, dM |^p)^{1/p} = J^1_\varepsilon + J^2_\varepsilon + J^3_\varepsilon .$$

Since

$\Delta^k_\varepsilon (\int_0^{\cdot} \varphi_\varepsilon \tilde{d}M_\varepsilon) = \bar{\Delta}^k_\varepsilon (\int_0^{\cdot} \varphi_\varepsilon dM_\varepsilon) \leqslant K_5 V^k_\varepsilon$, one gets

$$J^1_\varepsilon \leqslant K_5 \, E(\max_{k \leqslant T/\varepsilon} | V^k_\varepsilon |^p)^{1/p} \leqslant K_5 E((\sum_{k \leqslant T/\varepsilon} | V^k_\varepsilon |^{p/2})^2)^{1/p}$$

By $(A_{p/2})$, $E(| V^k_\varepsilon |^{p/2} | F_{k\varepsilon}) \leqslant K_6 \, \varepsilon^{p/4}$ and by (A_p)

$E(|V_\varepsilon^k|^p|F_{k\varepsilon}) \leqslant K_7\,\varepsilon^{p/2}$. Then (4.2) yields

$$|J_\varepsilon^1|^{p/2} \leqslant K_8(E((\sum_{k\leqslant T/\varepsilon} E(|V_\varepsilon^k|^{p/2}|F_{k\varepsilon}))^2)^{1/2} +$$

$$+ E(\sum_{k\leqslant T/\varepsilon} E(|V_\varepsilon^k|^p|F_{k\varepsilon}))^{1/2}) \leqslant$$

$$\leqslant K_8((T/\varepsilon)K_6\varepsilon^{p/4} + ((T/\varepsilon)K_7\varepsilon^{p/2})^{1/2}) \leqslant K_9\,\varepsilon^{1/2}.$$

We conclude that

$$J_\varepsilon^1 \leqslant K_{10}\,\varepsilon^{1/p} \tag{2.18}$$

Burkeholder's inequality together with assumptions (1.1) and (2.3) (ii) yields

$$E(|\int_{I_\varepsilon^k}\varphi\, dM|^{p/2}|F_{k\varepsilon}) \leqslant K_{11}E(|\int_{I_\varepsilon^k}\varphi^2 d\langle M\rangle|^{p/4}|F_{k\varepsilon}) \leqslant K_{12}\,\varepsilon^{p/4},$$

and, by the same computation:

$$E(|\int_{I_\varepsilon^k}\varphi\, dM|^p|F_{k\varepsilon}) \leqslant K_{13}\,\varepsilon^{p/2}.$$

Then, the same argument as above yields

$$J_\varepsilon^2 \leqslant K_{14}\,\varepsilon^{1/p}. \tag{2.19}$$

Let us now evaluate J_ε^3. By (2.4) (iv) and (2.3)(*ii*)

$$\overline{\Delta}_\varepsilon^k X \leqslant \overline{\Delta}_\varepsilon^k \beta_\varepsilon + K_{15}\varepsilon + K_{16}\overline{\Delta}_\varepsilon^k X_\varepsilon + K_{17}V_\varepsilon^k,$$

with $K_{16} < 1$. By moving $K_{16}\overline{\Delta}_\varepsilon^k X_\varepsilon$ in the left side of the inequality one gets

$$\overline{\Delta}_\varepsilon^k X \leqslant K_{18}(\varepsilon + \overline{\Delta}_\varepsilon^k \beta_\varepsilon + V_\varepsilon^k).$$

Define now

$$\overline{\varphi}_\varepsilon(t,\lambda,\overline{w}) = \sum_k 1_{[k\varepsilon,(k+1)\varepsilon)}(t)\ \varphi_\varepsilon(k\varepsilon,\lambda,\overline{w})$$

and note that by (2.3)(iv), for every $t\in[k\varepsilon,(k+1)\varepsilon)$ and $\lambda\in\Lambda$

$$|\varphi_\varepsilon - \overline{\varphi}_\varepsilon|(t,\lambda,X_\varepsilon) \leqslant K_{19}(\varepsilon + \overline{\Delta}_\varepsilon^k X_\varepsilon) \leqslant K_{20}(\varepsilon + \overline{\Delta}_\varepsilon^\ell \beta_\varepsilon + V_\varepsilon^k). \tag{2.20}$$

On the other hand, as $M_\varepsilon(k\varepsilon) = M(k\varepsilon)$ and $E(\Delta_\varepsilon^k M\,|F_{k\varepsilon}) = 0$,

$$\int_0^{k\varepsilon} \bar{\varphi}_\varepsilon \,\tilde{d}M_\varepsilon = \int_0^{k\varepsilon} \bar{\varphi}_\varepsilon \,dM_\varepsilon = \int_0^{k\varepsilon} \bar{\varphi}_\varepsilon \,dM .$$

It follows that

$$E(\max_{k \leqslant k_\varepsilon} |\int_{S'}^{k\varepsilon} \varphi_\varepsilon \,\tilde{d}M_\varepsilon - \int_{S'}^{k\varepsilon} \varphi_\varepsilon \,dM|^p)^{1/p} \leqslant$$

$$\leqslant E(\max_{k \leqslant k_\varepsilon} |\int_{S'}^{k\varepsilon} (\bar{\varphi}_\varepsilon - \varphi_\varepsilon)\tilde{d}M_\varepsilon|^p)^{1/p} +$$

$$+ E(\max_{k \leqslant k_\varepsilon} |\int_{S'}^{k\varepsilon} (\bar{\varphi}_\varepsilon - \varphi_\varepsilon)\,dM|^p)^{1/p} = J_\varepsilon^4 + J_\varepsilon^5$$

Burkholder's inequality, (2.20) and (1.1) yield

$$J_\varepsilon^5 \leqslant K_{21}\, E(\max_{k \leqslant k_\varepsilon} (\,\varepsilon + \bar{\Delta}_\varepsilon^k \beta_\varepsilon + V_\varepsilon^k)^p)^{1/p} \leqslant$$

$$\leqslant K_{21}(\,\varepsilon + E(\max_{k \leqslant k_\varepsilon} |\bar{\Delta}_\varepsilon^k \beta_\varepsilon|^p)^{1/p} + E(\max_{k \leqslant k_\varepsilon} |V_\varepsilon^k|^p)^{1/p}).$$

(J_p) and a calculus analogue to the one used to estimate J_ε^1 yield $E(\max_{k \leqslant k_\varepsilon} |\bar{\Delta}_\varepsilon^k \beta_\varepsilon|^p)^{1/p} \leqslant K_{22}\, \varepsilon^{1/2p}$. We conclude that

(2.21) $$J_\varepsilon^5 \leqslant K_{23}\, \varepsilon^{1/4p}$$

To estimate J_ε^4 we shall use Lemma 4.1. for the martingale $H_k = \int_0^{k\varepsilon} (\varphi_\varepsilon - \bar{\varphi}_\varepsilon)\tilde{d}M_\varepsilon$. We write

$$E(|H_{k+1} - H_k|^i | F_{k\varepsilon})^{1/i} \leqslant K_{24}\, E((\varepsilon + \bar{\Delta}_\varepsilon^k \beta_\varepsilon + V_\varepsilon^k)^i (V_\varepsilon^k)^i | F_{k\varepsilon})^{1/i}.$$

For $2 \leqslant i \leqslant p$, Hölder's inequality, (J_{p+1}) and $(A_{p(p+1)})$ yield

$$E(|\bar{\Delta}_\varepsilon^k \beta_\varepsilon V_\varepsilon^k|^i | F_{k\varepsilon})^{1/i} \leqslant E(|\bar{\Delta}_\varepsilon^k \beta_\varepsilon|^{i+1} | F_{k\varepsilon})^{1/(i+1)} E(|V_\varepsilon^k|^{i(i+1)} | F_{k\varepsilon})^{1/i(i+1)} \leqslant$$

$$\leqslant K_\varepsilon^*(k\varepsilon)\, \varepsilon^{3/8}\, e_{i(i+1)}\, \varepsilon^{1/2} \leqslant K_{25}\, K_\varepsilon^*(k\varepsilon)\, \varepsilon^{7/8} ,$$

K_ε being the processes attached to β_ε by (J_{p+1}). Analogous estimations may be done for $\varepsilon V_\varepsilon^k$ and $|V_\varepsilon^k|^2$. Since $i \geqslant 2$, one gets

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$$\langle H \rangle_i (k_\varepsilon) \leqslant \sum_{k=0}^{k_\varepsilon - 1} (K_{26}(K^*_\varepsilon (k\varepsilon) + 1)\, \varepsilon^{7/8})^i \leqslant$$

$$\leqslant K_{27}(K^*_\varepsilon (S' + S) + 1)^i \varepsilon^{3/8}.$$

Now, by using (4.1) and the fact that $\sup_\varepsilon \| K_\varepsilon \|^\wedge_{p,T} < \infty$ we conclude that $J^4_\varepsilon \leqslant K_{28}\, \varepsilon^{3/4p}$. Together with (2.21) this yields

$$J^3_\varepsilon \leqslant K_{29}\, \varepsilon^{1/4p} + \sup_{\lambda \in \Lambda} E(\max_{k'_\varepsilon < k \leqslant k_\varepsilon} | \int_{S'}^{k\varepsilon} (\varphi_\varepsilon - \varphi)\, dM |^p)^{1/p}.$$

By (2.11)(i) and (2.3)(iii)

$$|\varphi_\varepsilon(s, \lambda, X_\varepsilon) - \varphi(s,\lambda, X)| \leqslant K_{30}(\delta(\varepsilon) + \max_{k \leqslant k_\varepsilon} \bar{\Delta}^k_\varepsilon X +$$

$$+ \sup_{t \leqslant S'+S} | X_\varepsilon (t, \lambda, \omega) - X(t, \lambda, \omega) \|),$$

for $s \leqslant S' + S$. Burkholder's inequality, (1.1) and an evaluation of $E(\max_{k \leqslant k_\varepsilon} |\bar{\Delta}^k_\varepsilon X|^p)^{1/p}$ analougos to that used for J^1_ε with $\bar{\Delta}^k_\varepsilon X$ instead of V^k_ε, yields

$$J^3_\varepsilon \leqslant K_{31}(\varepsilon^{1/4p} + S^{1/2}(\delta(\varepsilon) + \| X_\varepsilon - X \|^\wedge_{p,\, S'+S})$$

By using the inequalities we have already proved one gets

$$\| X_\varepsilon - X \|^\wedge_{p,S',S} \leqslant K_{32}(\delta(\varepsilon) + \varepsilon^{1/4p}) + K_{33} \| X_\varepsilon - X \|^\wedge_{p,S'} +$$

$$+ (K_3 + K_4 S + K_{31} S^{1/2}) \| X_\varepsilon - X \|^\wedge_{p,\, S'+S}.$$

As $K_3 < 1$ one may choose S sufficiently small to get $K_3 + K_4 S + K_{31} S^{1/2} < 1$. Then, by writing $\| X_\varepsilon - X \|^\wedge_{p,S'+S} \leqslant \| X_\varepsilon - X \|^\wedge_{p,S'} + \| X_\varepsilon - X \|^\wedge_{p,S',S}$ and moving $(K_3 + K_4 S + K_{31} S^{1/2p}) \cdot \| X_\varepsilon - X \|^\wedge_{p,S',S}$ in the right side of the inequality, one gets

$$\| X_\varepsilon - X \|^\wedge_{p,S',S} \leqslant K_{34}(\delta(\varepsilon) + \varepsilon^{1/4p} + \| X_\varepsilon - X \|^\wedge_{p,S'}).$$

By using the above inequality for $S' = 0, S, 2S, \ldots$ one gets $\| X_\varepsilon - X \|^\wedge_{p,kS,S} \leqslant K_{35}(\delta(\varepsilon) + \varepsilon^{1/4p})$ for every $k < T/S$. Then

$$\| X_\varepsilon - X \|^\wedge_{p,T} \leqslant \sum_{k < T/S} \| X_\varepsilon - X \|^\wedge_{p,kS,S} \leqslant K_{36}(\delta(\varepsilon) + \varepsilon^{1/4p})$$

and so the proof is complete.

Q.E.D.

Now we want to extend the above result to unbounded coefficients having liniar increments at the infinite (hypothesis (2.1)). In order to do this we have to stop M^* at T and consider the equations

$$(2.22)\quad \bar{X}_\varepsilon(t,\lambda,\omega) = \beta_\varepsilon(t,\lambda,\omega) + \alpha_\varepsilon(t,\lambda,\bar{X}_\varepsilon) + \int_0^t \varphi_\varepsilon(s,\lambda,\bar{X}_\varepsilon)\tilde{d}\tilde{M}_\varepsilon(s) + \int_0^t \psi_\varepsilon(s,\lambda,\bar{X}_\varepsilon)ds.$$

Theorem 2.2. Consider an even $p \geqslant 6$ such that $(A_{(p+1)(p+2)})$ holds. Assume that the family φ^{ij}, ψ^i, φ_ε^{ij}, ψ_ε^i, $1 \leqslant i, j \leqslant d$, $0 < \varepsilon$, $\lambda \in \Lambda$ fulfils (2.1) and the family α^i, α_ε^i, $1 \leqslant i \leqslant d$, $0 < \varepsilon$, $\lambda \in \Lambda$ fulfils (2.2). Assume also that β_ε^i, $1 \leqslant i \leqslant d$, $0 < \varepsilon$ fulfils (I_{p+1}) and (J_{p+1}) and $\lim_{\varepsilon\to 0} \delta_T^p(\varepsilon) = 0$. Then

$$\lim_{\varepsilon\to 0} \|\bar{X}_\varepsilon - X\|^\wedge_{p,T} = 0$$

Proof. Put $q = p+1$. We shall prove that

$$(2.23)\qquad \sup_{\varepsilon > 0} \|X\|^\wedge_{q,T} < \infty.$$

For a fixed $N > 0$ let consider a Lipschitz function $\phi_N : R^d \to R^d$ such that $\phi_N(x) = x$ for $|x| \leqslant N$ and $\phi_N(x) = N+1$ for $|x| \geqslant N+1$. Define $\alpha_{N,\varepsilon}(t,\lambda,\bar{w}) = \alpha_\varepsilon(t,\lambda,\bar{w}_N)$ with $\bar{w}_N(s) = \phi_N(\bar{w}(s))$. Then $\alpha_{N,\varepsilon}$ is bounded and equal to α_ε up to the time $T_N(\bar{w}) = \inf\{t > 0 : |\bar{w}(t)| \geqslant N\}$. We define in the same way $\varphi_{N,\varepsilon}$ and $\psi_{N,\varepsilon}$ and consider the equation

$$(2.24)\quad \bar{X}_{N,\varepsilon}(t,\lambda,\omega) = \beta_\varepsilon(t,\lambda,\omega) + \alpha_{N,\varepsilon}(t,\lambda,\bar{X}_{N,\varepsilon}) + \int_0^t \varphi_{N,\varepsilon}(s,\lambda,\bar{X}_{N,\varepsilon})\tilde{d}\tilde{M}_\varepsilon(s) + \int_0^t \psi_{N,\varepsilon}(s,\lambda,\bar{X}_{N,\varepsilon})ds.$$

Fix $t < T_{N,\varepsilon} =: T_N(\bar{X}_{N,\varepsilon}) \wedge T_N(\bar{X}_\varepsilon)$. As $\bar{X}_{N,\varepsilon}(s)$ and $\bar{X}_\varepsilon(s)$, $s \leqslant t$ are the solutions of the same equation, the uniqueness ensures that they are equal. In particular, as α_ε is previsible, it follows that $\alpha_{N,\varepsilon}(T_{N,\varepsilon},\lambda,\bar{X}_{N,\varepsilon}) = \alpha_\varepsilon(T_{N,\varepsilon},\lambda,\bar{X}_\varepsilon)$. The same is true for $\varphi_{N,\varepsilon}$, φ_ε and $\psi_{N,\varepsilon}$, ψ_ε. By using again the equations one gets $\overline{X}_{N,\varepsilon}(T_{N,\varepsilon}) = \overline{X}_{\varepsilon}(T_{N,\varepsilon})$. It follows that $T_N(\overline{X}_{N,\varepsilon}) = T_N(\overline{X}_{\varepsilon}) = T_{N,\varepsilon}$. As $E(\overline{X}^*_{\varepsilon}(T))<\infty$ (definition!) one concludes that $\sup_N T_{N,\varepsilon} > T$ a.s. for every $T > 0$. In particular (2.23) shall be a consequence of

(2.25) $$\sup_{\varepsilon} \| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,T} \leqslant K_1$$

where K_1 (and also K_2, K_3 ... below) is a constant depending on p and T only.

Let us now prove (2.25). Fix $0 \leqslant S'$ and $0 \leqslant S \leqslant 1$ such that $S'+S \leqslant T$. By using (2.1)(ii) for $\varphi_{N,\varepsilon}$ and $\psi_{N,\varepsilon}$ (2.2)(ii) for $\alpha_{N,\varepsilon}$, (I_q) for β_{ε} and (1.10) one gets

$$\| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,S',S} \leqslant K_2 + K_3 \| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,S'} + (K_4 + S^{1/p} K_5) \| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,S'+S}$$

with $K_4 < 1$. One may choose S such that $K_4 + S^{1/p} K_5 < 1$. Write then $\| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,S'+S} \leqslant \| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,S'} + \| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,S',S}$ and move $(K_4 + S^{1/p} K_5) \| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,S',S}$ in the left hand side of the inequality (The fact that $\| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,S',S} < \infty$ is a simple consequence of the boundness of the coefficients of (2.24) , except for which fulfils (I_q)).

One gets

$$\| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,S',S} \leqslant K_6 + K_7 \| \overline{X}_{N,\varepsilon} \|^{\wedge}_{q,S'} .$$

As in the proof of Theorem 2.1, by using the above inequality for $S' = 0, S, 2S, \ldots$ one gets (2.25).

We are now going on to the proof itself. Consider the equations (2.6) and (2.10) stopped at $T_N(X)$ and $T_N(X_{\varepsilon})$ respectively in the same way as (2.22) was stopped in order to get (2.24). Let X_N and $X_{N,\varepsilon}$ respectively denote the solutions of these equations and $\sigma_{N,\varepsilon} = T_{N,\varepsilon} \wedge T_{\varepsilon} \wedge T_N(X)$ (T_{ε} is defined in (1.4)). Note that $\overline{X}_{N,\varepsilon}(t) = \overline{X}_{\varepsilon}(t) = X_{N,\varepsilon}(t)$ and $X_N(t) = X(t)$ for $t < \sigma_{N,\varepsilon}$.

It follows that

$$(2.26)\quad \|\bar{X}_\varepsilon - X\|^{\wedge}_{p,T} \leqslant \sup_{\lambda} E(\sup_{t\leqslant T} |\bar{X}_\varepsilon - X|(t,\lambda,\omega)^p, \sigma_{N,\varepsilon} < T)^{1/p} +$$

$$+ \|X_{N,\varepsilon} - X_N\|^{\wedge}_{p,T}.$$

By (2.23) with $q = 1$ for $\bar{X}_{N,\varepsilon}$ and the same for X, established by an analogous reasoning, based on Burholder's inequality instead of (1.10), it follows that

$$P(\sigma_{N,\varepsilon} < T) \leqslant P(T_{N,\varepsilon} < T) + P(T_N(X) < T) + P(T_\varepsilon < T) \leqslant$$

$$\leqslant K_8 N^{-1} + P(T_\varepsilon < T).$$

Hölder's inequality and (2.23) ensures then that the first term in the right hand side of (2.26) is dominated by

$$(\|\bar{X}_{N,\varepsilon}\|^{\wedge}_{q,T} + \|X\|^{\wedge}_{q,T})(K_8 N^{-1} + P(T_\varepsilon < T))^{1/p(p+1)} \leqslant$$

$$\leqslant K_9 (N^{-1} + P(T_\varepsilon < T))^{1/p(p+1)}$$

By (1.5) $\lim_{\varepsilon\to 0} P(T_\varepsilon < T) = 0$ and by Theorem 2.1. $\lim_{\varepsilon \to 0} \|X_{N,\varepsilon} - X\|^{\wedge}_{p,T} = 0$. Then, for every fixed N

$$\overline{\lim_{\varepsilon}} \|\bar{X}_\varepsilon - X\|^{\wedge}_{p,T} \leqslant K_9 N^{-1/p(p+1)}$$

By letting $N \uparrow \infty$ the proof finishes.

Q.E.D.

Integration with respect to $\widetilde{dM}_\varepsilon$ presents the disadvantage that the "compensator" $C_\varepsilon(\int \varphi\, dM_\varepsilon)$ may generally not be explicitely calculated. If $\varphi_\varepsilon(t, \lambda, \bar{w})$ depends only on $\bar{w}(t-)$, more exactly in a twice differentiable way, then the above "compensator" may be replaced by an "as~~s~~ymptotic compensator". This will be a drift which is as~~s~~ymptotically close to $C_\varepsilon(\int \varphi\, dM_\varepsilon)$. This will be done in Theorem 2.3.

Consider $\varphi, \varphi_\varepsilon : [0,\infty) \times \Lambda \times R^d \to R^d \times R^d$, $\varepsilon > 0$, twice differentiable in $x \in R^d$. We shall denote

$$\partial_k \varphi^{ij}_\varepsilon(t,\lambda,x) =: \frac{\partial}{\partial x_k} \varphi^{ij}_\varepsilon(t,\lambda,x), \quad 1 \leqslant i,j,k \leqslant d.$$

The same notation will be used for any function of this type.

We need the hypothesis

(2.27) i) $\varphi_\varepsilon^{ij}, \partial_k \varphi_\varepsilon^{ij}, \partial_h \partial_k \varphi_\varepsilon^{ij}$, $1 \leq i,j,k,h \leq d$

are bounded uniformly with respect to $\varepsilon > 0, \lambda \in \Lambda$ and $t \in [0,T]$ for every $T > 0$.

ii) $\varphi_\varepsilon^{ij}, \partial_k \varphi_\varepsilon^{ij}$, $1 \leq i,j,k \leq d$ fulfils (2.1)(iv)

Let us then define

$$(2.28) \quad A_\varepsilon^{i,j\ell}(t,\lambda,x) = \sum_{h=1}^{d} \varphi_\varepsilon^{h\ell}(t,\lambda,x)\, \partial_h \varphi_\varepsilon^{ij}(t,\lambda,x),$$

$$1 \leq i,j,\ell \leq d.$$

Consider the equations

$$(2.29) \quad X_\varepsilon^i(t,\lambda,\omega) = \beta_\varepsilon^i(t,\lambda,\omega) + \alpha_\varepsilon^i(t,\lambda,X_\varepsilon) +$$
$$+ \sum_{j=1}^{d} \int_0^t \varphi_\varepsilon^{ij}(s,\lambda,X_\varepsilon(s-,\lambda,\omega))\,dM_\varepsilon^j(s) -$$
$$- \sum_{j,\ell=1}^{d} \int_0^t g_\varepsilon^{j\ell}(s,\omega) A_\varepsilon^{i,j\ell}(s,\lambda,X_\varepsilon(s,\lambda,\omega))\,ds +$$
$$+ \int_0^t \psi_\varepsilon^i(s,\lambda,X_\varepsilon)\,ds, \quad i \leq d,$$

with $g_\varepsilon^{j\ell}$ defined in (1.3).

<u>Theorem 2.3.</u> Consider an even $p \geq 6$ such that $(A_{p(p+1)})$ (with $K_T^{iv}=0$) holds. Assume that φ_ε fulfils (2.27), α_ε fulfils (2.4), ψ_ε fulfils (2.3), β_ε fulfils (2.30) below and $\lim_{\varepsilon \to 0} \delta_T^p(\varepsilon) = 0$. Let X be the solution of (2.6) (in which $\varphi^{ij}(s,\lambda,X)$ are replaced by $\varphi^{ij}(s,\lambda,X(s))$) and X_ε the solution of (2.29). Then

$$\lim_{\varepsilon \to 0} \| X_\varepsilon - X \|_{p,T}^{\wedge} = 0.$$

(2.30) $\bar{\Delta}_\varepsilon^k \beta_\varepsilon \leq K_T \varepsilon$ for every $\varepsilon > 0$ and $k \leq T/\varepsilon$.

<u>Proof.</u> Denote

$$\gamma_\varepsilon^{ij}(t,\lambda,\omega) = C_\varepsilon\Big(\int_0 \varphi_\varepsilon^{ij}(s,\lambda,X_\varepsilon(s-))\,dM_\varepsilon^j(s)\Big)(t) -$$
$$- \sum_{\ell=1}^{d} \int_0^t g_\varepsilon^{j\ell}(s) A_\varepsilon^{i,j\ell}(s,\lambda,X_\varepsilon(s))\,ds.$$

We shall prove that

$$(2.31) \qquad \lim_{\varepsilon \to 0} \| \gamma_\varepsilon^{ij} \|^{\wedge}_{p,T} = 0.$$

Since $g_\varepsilon^{j\ell}$, $A_\varepsilon^{i,j\ell}$, $\varepsilon > 0$ are equally bounded, γ_ε^{ij}, $\varepsilon > 0$ fulfils (J_{p+1}). So, theorem 2.3. will be a consequence of Theorem 2.1. and (2.31).

Let us prove (2.31). K_i, $i = 1,2, \ldots$ will be constants depending on p and T only. We shall denote by $o_\varepsilon = o_\varepsilon(\omega)$, $\varepsilon > 0$ any family of functions such that $\lim_{\varepsilon \to 0} \varepsilon^{-1} E(|o_\varepsilon|^p)^{1/p} = 0$.

Note that

$$(2.32) \qquad \bar{\Delta}_\varepsilon^k x_\varepsilon^i \leqslant K_1(\varepsilon + V_\varepsilon^k), \qquad i \leqslant d.$$

We shall prove that

$$(2.33) \qquad E\left(\int_{I_\varepsilon^k} \varphi_\varepsilon^{ij}(s,\lambda, X_\varepsilon(s-,\lambda,\cdot))\,dM_\varepsilon(s,\cdot) \mid F_{k\varepsilon}\right)(\omega) = o_\varepsilon(\omega) + $$
$$+ \sum_{\ell=1}^{d} g_{\varepsilon,k}^{j,\ell}(\omega)\, A_\varepsilon^{i,j\ell}(k\varepsilon, \lambda, X_\varepsilon(k\varepsilon,\lambda,\omega))\varepsilon.$$

Then by using (2.32), (2.31) appears as a simple consequence of (2.33). In order to prove (2.33) we write the therm in the left hand side as $J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3$ where

$$J_\varepsilon^1 = E\Big(\int_{I_\varepsilon^k} \varphi_\varepsilon^{ij}(s,\lambda, X_\varepsilon(s-)) - \varphi_\varepsilon^{ij}(s,\lambda, X_\varepsilon(k\varepsilon)) -$$
$$- \sum_{h=1}^{d} \partial_h \varphi_\varepsilon^{ij}(s,\lambda, X_\varepsilon(k\varepsilon))(X_\varepsilon^h(s-) - X_\varepsilon^h(k\varepsilon))\,dM_\varepsilon^j(s) \mid F_{k\varepsilon}\Big),$$

$$J_\varepsilon^2 = E\Big(\int_{I_\varepsilon^k} \varphi_\varepsilon^{ij}(s,\lambda, X_\varepsilon(k\varepsilon))\,dM_\varepsilon^j(s) \mid F_{k\varepsilon}\Big),$$

$$J_\varepsilon^3 = \sum_{h=1}^{d} E\Big(\int_{I_\varepsilon^k} \partial_h \varphi_\varepsilon^{ij}(s,\lambda, X_\varepsilon(k\varepsilon))(X_\varepsilon^h(s-) -$$
$$- X_\varepsilon^h(k\varepsilon))\,dM_\varepsilon^j(s) \mid F_{k\varepsilon}\Big).$$

Taylor's formula, hypothesis (2.27) (i) and (2.32) yields $|J_\varepsilon^1| \leqslant E((\varepsilon + V_\varepsilon^k)^2 V_\varepsilon^k \mid F_{k\varepsilon})$. It follows that $J_\varepsilon^1 = \mathcal{O}(\varepsilon)$.

Next, as $E(\Delta_\varepsilon^k M_\varepsilon \mid F_{k\varepsilon}) = 0$,

$$J_\varepsilon^2 = E\Big(\int_{I_\varepsilon^k} (\varphi_\varepsilon^{ij}(s,\lambda,X_\varepsilon(k\varepsilon)) - \varphi_\varepsilon^{ij}(k\varepsilon,\lambda,X_\varepsilon(k\varepsilon))\, dM_\varepsilon^j(s) \mid F_{k\varepsilon}\Big).$$

By (2.27)(ii) and (A_1) one gets $|J_\varepsilon^2| \leqslant K_2\, \varepsilon\, E(V_\varepsilon^k \mid F_{k\varepsilon}) \leqslant K_3\, \varepsilon^{3/2}$.

So J_ε^2 is also an o_ε.

Let us now evaluate J_ε^3. By (2.7) (ii) one deduces

$$J_\varepsilon^3 = o_\varepsilon + \sum_{h=1}^{d} \partial_h \varphi_\varepsilon^{ij}(k\varepsilon,\lambda,X_\varepsilon(k\varepsilon))\, E\Big(\int_{I_\varepsilon^k} (X_\varepsilon^h(s-) - X_\varepsilon^h(k\varepsilon))\, dM_\varepsilon^j(s)) \mid F_{k\varepsilon}\Big).$$

Next, by (2.29)

$$E\Big(\int_{I_\varepsilon^k} (X_\varepsilon^h(s-) - X_\varepsilon^h(k\varepsilon))\, dM_\varepsilon^j(s) \mid F_{k\varepsilon}\Big) = J_\varepsilon^4 + J_\varepsilon^5 + J_\varepsilon^6 + J_\varepsilon^7,$$

with

$$J_\varepsilon^4 = E\Big(\int_{I_\varepsilon^h} (\beta_\varepsilon^h(s-,\lambda,\cdot) - \beta_\varepsilon^h(k\varepsilon,\lambda,\cdot))\, dM_\varepsilon^j(s) \mid F_{k\varepsilon}\Big),$$

$$J_\varepsilon^5 = E\Big(\int_{I_\varepsilon^h} (\alpha_\varepsilon^h(s-,\lambda,X_\varepsilon) - \alpha_\varepsilon^h(k\varepsilon,\lambda,X_\varepsilon))\, dM_\varepsilon^j(s) \mid F_{k\varepsilon}\Big),$$

$$J_\varepsilon^6 = \sum_{\ell=1}^{d} E\Big(\int_{k\varepsilon}^{(k+1)\varepsilon} \int_{k\varepsilon}^{s-} \varphi_\varepsilon^{h\ell}(u,\lambda,X_\varepsilon(u-))\, dM_\varepsilon^\ell(u)\, dM_\varepsilon^j(s) \mid F_{k\varepsilon}\Big),$$

$$J_\varepsilon^7 = E\Big(\int_{k\varepsilon}^{(k+1)\varepsilon} \int_{k\varepsilon}^{s} \Big(-\sum_{r,\ell=1}^{d} g_\varepsilon^{r\ell}(u) A_\varepsilon^{h,r\ell}(u,\lambda,X_\varepsilon(u)) + \psi_\varepsilon^h(u,\lambda,X_\varepsilon)\Big)\, du\, dM_\varepsilon^j(s) \mid F_{k\varepsilon}\Big).$$ with $K_T^{IV} = 0$

By (2,30) J_ε^4 is an o_ε. By (2.4)(iv) it follows that $|J_\varepsilon^5| \leqslant K_3\, \varepsilon\, E(V_\varepsilon^k \mid \mathcal{F}_{k\varepsilon})$ which is an o_ε. Using the boundness of $g_\varepsilon^{r,\ell}$, $A_\varepsilon^{h,r\ell}$ and ψ_ε^h one concludes that J_ε^7 is also an o_ε. Then

$$J_\varepsilon^6 = o_\varepsilon + \sum_{\ell=1}^{d} \varphi_\varepsilon^{h\ell}(k\varepsilon,\lambda,X_\varepsilon(k\varepsilon))\, E\Big(\int_{k\varepsilon}^{(k+1)\varepsilon} \int_{k\varepsilon}^{s-} dM_\varepsilon^\ell(u)\, dM_\varepsilon^j(s) \mid F_{k\varepsilon}\Big) =$$

$$= o_\varepsilon + \sum_{\ell=1}^{d} \varphi_\varepsilon^{h\ell}(k\varepsilon, \lambda, X_\varepsilon(k\varepsilon)) g_{\varepsilon,k}^{j\ell}(\omega)\varepsilon.$$

So we have proved that

$$J_\varepsilon^3 = o_\varepsilon + \sum_{h=1}^{d} \partial_h \varphi_\varepsilon^{ij}(k\varepsilon, \lambda, X_\varepsilon(k\varepsilon)) \sum_{\ell=1}^{d} \varphi_\varepsilon^{h\ell}(k\varepsilon, \lambda, X_\varepsilon(k\varepsilon)) g_\varepsilon^{j\ell}(\omega)\varepsilon =$$

$$= o_\varepsilon + \sum_{\ell=1}^{d} A_\varepsilon^{i,j\ell}(k\varepsilon, \lambda, X_\varepsilon(k\varepsilon)) g_{\varepsilon k}^{j\ell}(\omega)\varepsilon,$$

and so the proof is completed.

Q.E.D.

The convergence problem presented above may appear from another point of view: someone dealing with a stochastic model may be interested in the equations

$$(2.34) \quad X_\varepsilon^i(t,\lambda,\omega) = \beta_\varepsilon^i(t,\lambda,\omega) + \alpha_\varepsilon^i(t,\lambda, X_\varepsilon) + \\ + \sum_{j=1}^{d} \int_0^t \varphi_\varepsilon^{ij}(s,\lambda,X_\varepsilon(s-))\,dM_\varepsilon^j(s) + \\ + \int_0^t \psi_\varepsilon^i(s,\lambda, X_\varepsilon)\,ds, \qquad i \leq d,$$

and wonder if X_ε converges as $\varepsilon \to 0$ and what is the limit. In this case no "compensator" appears, neither exact one, nor as~~s~~ymptotic. It is natural to look for this compensator in the limiting equation. In order to solve this problem we have to make an additional assumption on M_ε, $\varepsilon > 0$:

(B_p) There is a cadlag adapted process $g:[0,\infty) \times \Lambda \times \Omega \to R^d \times R^d$ such that

$$\lim_{\varepsilon \to 0} \| g_\varepsilon^{ij} - \bar{g}^{ij} \|_{p,T}^{\Lambda} = 0 \quad \text{for every } T > 0,\ i,j \leq d.$$

This hypothesis is an analogous of (A.7) in /1/ Cap.VI,7. There, this is a starting assumption and of course it is fulfiled by a large class of examples. We shall also assume that

$$(2.35)\quad \lim_{\varepsilon\to 0} \sup_{\lambda\in\Lambda} \sup_{t\leq T} |\partial_k \varphi_\varepsilon^{ij}(t,\lambda,x) - \partial_k \varphi^{ij}(t,\lambda,x)| = 0$$

for every $T > 0$, $1 \leq i,j,k \leq d$.

Under these supplementary assumptions X_ε will converge to X, the solution of

$$(2.36)\quad X^i(t,\lambda,\omega) = \beta^i(t,\lambda,\omega) + \alpha^i(t,\lambda,X) + \sum_{j=1}^{d}\int_0^t \varphi^{ij}(s,\lambda,X(s))dM^j(s) + \sum_{j,\ell=1}^{d}\int_0^t g^{j\ell}(s)A^{i,j\ell}(s,\lambda,X(s))ds + \int_0^t \psi^i(s,\lambda,X)ds,$$

More precisely

Corollary 2.4. In the context of Theorem 2.3 and the supplementary assumptions (B_p) and (2.35)

$$\lim_{\varepsilon\to 0} \|X - X_\varepsilon\|^{\wedge}_{p,T} = 0$$

where X is the solution of (2.36) and X_ε is the solution of (2.34).

Proof. In order to be able to apply Theorem 2.3, we add and substract

$$\bar\beta^i_\varepsilon(t,\lambda,\omega) = \sum_{j,\ell=1}^{d}\int_0^t (g_\varepsilon^{j\ell}(s)\, A_\varepsilon^{i,j\ell}(s,\lambda,X_\varepsilon(s)) - g^{j\ell}(s)\, A^{i,j\ell}(s,\lambda,X_\varepsilon(s)))ds$$

in the right hand side of (2.34). We get an equation of the same type as (2.29) with the perturbation $\beta'_\varepsilon = \beta_\varepsilon + \bar\beta_\varepsilon$. Note that

$$\|\bar\beta^i_\varepsilon\|^{\wedge}_{p,T} \leq K_1 \max_{j,\ell} \|g_\varepsilon^{j\ell} - g^{j\ell}\|^{\wedge}_{p,T} + K_2 \max_{i,j,\ell}\sup_{t\leq T}\sup_{\lambda,x} |A_\varepsilon^{i.j\ell} - A^{i,j\ell}|(t,\lambda,x).$$

By (B_p) and (2.34), $\lim_{\varepsilon\to 0}\|\bar\beta^i_\varepsilon\|^{\wedge}_{p,T} = 0$ and the proof finishes.

Q.E.D.

3. ALMOST SURE APPROXIMATION FOR THE SOLUTIONS OF MARKOV EQUATIONS

In this section we shall study almost sure convergence. To this end all the set of hypothesis will be strengthened: the parameter λ will be a real multidimensional one, the coefficients $\varphi(t, \lambda, \cdot)$ and $\psi(t, \lambda, \cdot)$ will depend on X by means of $X(t)$ only and α will not depend on X. Indefinite differentiability with respect to λ and x will be assumed. Then both X and X_ε will be indefinitely differentiable in λ. Under the assumption that (A_p) holds for every $p \in N$ we shall prove that the derivatives of any order of X_ε converge, in every L_p, $p \in N$, to the derivatives of X. Then by using Sobolev's inequality in the same way as Ikeda and Watanabe does in Proposition 2.2, Chap.V in /1/, one obtaines almost sure convergence, uniform with respect to $\lambda \in \Lambda$ and t in a compact interval, both for $X_\varepsilon \to X$ and their derivatives of any order.

Let us introduce the new hypothesis. Fix $\bar{d} \in N$ and $\Lambda \subseteq R^{\bar{d}}$ and consider some functions $f, f_\varepsilon : [0,\infty) \times \Lambda \times R^d \to R$, $\varepsilon > 0$. On such a family of functions we shall make the hypothesis:

(3.1) The functions f_ε and f are indefinitely differentiable with respect to $\lambda \in \Lambda$ and $x \in R^d$. For every $T > 0$ there is a constant K_T such that

i) $|f_\varepsilon(t, \lambda, x)| \leqslant K_T$

ii) $|f_\varepsilon(k\varepsilon + s, \lambda, x) - f_\varepsilon(k\varepsilon, \lambda, x)| \leqslant K_T \varepsilon$

for every $\lambda \in \Lambda$, $x \in R^d$, $0 \leqslant t \leqslant T$, $\varepsilon > 0$, $0 \leqslant s < \varepsilon$ and $k \leqslant T/\varepsilon$.

iii) $\lim_{\varepsilon \to 0} \sup_{\lambda \in \Lambda} \sup_{x} \sup_{t \leqslant T} |f_\varepsilon(t, \lambda, x) - f(t, \lambda, x)| = 0.$

The same properties are assumed for the derivatives of any order of f and f_ε (the constant K_T depends on the order of the considered derivative).

Consider now a family of functions $g, g_\varepsilon : [0,\infty) \times \Lambda \to R$, $\varepsilon > 0$. We shall make the hypothesis

(3.2) The family $g, g_\varepsilon, \varepsilon > 0$ fulfils (3.1) in which $x \in R^d$ do not appear.

Finally let us introduce a notation: for a function $h : [0,\infty) \times \Lambda \times \Omega \rightarrow R$ which is indefinitely differentiable with respect to $\lambda \in \Lambda$ and for a multi-index $\gamma = (\gamma_1, \dots, \gamma_m)$, $1 \leq \gamma_i \leq \bar{d}$ we denote

(3.3) $D_\gamma h(t,\lambda,\omega) = \partial_{\gamma_1} \dots \partial_{\gamma_m} h(t,\lambda,\omega)$ with $\partial_k = \dfrac{\partial}{\partial \lambda_k}$.

Considered now some functions $\varphi, \varphi_\varepsilon : [0,\infty) \times \Lambda \times R^d \longrightarrow R^d \times R^d$, $\psi, \psi_\varepsilon : [0,\infty) \times \Lambda \times R^d \longrightarrow R^d$ and $\alpha, \alpha_\varepsilon : [0,\infty) \times \Lambda \longrightarrow R^d$, $\varepsilon > 0$. Assume that (3.1) holds for $\varphi^{ij}, \varphi^{ij}_\varepsilon$, $i,j \leq d$, $\varepsilon > 0$ and $\psi^i, \psi^i_\varepsilon$, $i \leq d$, $\varepsilon > 0$ and (3.2) holds for $\alpha^i, \alpha^i_\varepsilon$, $i \leq d$, $\varepsilon > 0$. The equations we are interested in are

(3.4) $$X^i(t,\lambda,\omega) = \alpha^i(t,\lambda) + \sum_{j=1}^{d} \int_0^t \varphi^{ij}(s,\lambda, X(s,\lambda,\omega))dM^j(s)$$

$$+ \int_0^t \psi^i(s,\lambda, X(s,\lambda,\omega))ds, \quad i \leq d,$$

(3.5) $$X^i_\varepsilon(t,\lambda,\omega) = \alpha^i_\varepsilon(t,\lambda) + \sum_{j=1}^{d} \int_0^t \varphi^{ij}_\varepsilon(s,\lambda,X_\varepsilon(s,\lambda,\omega))dM^j_\varepsilon(s)$$

$$- \sum_{j,\ell=1}^{d} \int_0^t g^{j\ell}_\varepsilon(s,\omega)A^{i,j\ell}_\varepsilon(s,\lambda,X_\varepsilon(s,\lambda,\omega))ds +$$

$$+ \int_0^t \psi^i_\varepsilon(s,\lambda, X_\varepsilon(s,\lambda,\omega))ds, \quad i \leq d,$$

and

$$(3.6)\quad \bar{X}^i_\varepsilon(t,\lambda,\omega) = \alpha^i_\varepsilon(t,\lambda) + \sum_{j=1}^{d} \int_0^t \varphi^{ij}_\varepsilon(s,\lambda,\bar{X}_\varepsilon(s-,\lambda,\omega))\,d\bar{M}^j_\varepsilon(s) - \sum_{j,\ell=1}^{d} \int_0^t g^{j\ell}_\varepsilon(s,\omega)\, A^{i,j\ell}_\varepsilon(s,\lambda,\bar{X}_\varepsilon(s,\lambda,\omega))\,ds + \int_0^t \psi^i_\varepsilon(s,\lambda,\bar{X}_\varepsilon(s,\lambda,\omega))\,ds,\quad i \leqslant d.$$

In order to prove the annonced result we have to give two preliminary lemmas. Consider for every $1 \leqslant i,j,\ell \leqslant d$ and $\varepsilon > 0$ a cadlag adapted process $\beta^i_\varepsilon : [0,\infty) \times \Lambda \times \Omega \to R$ and the functions $\sigma^{ij\ell}_\varepsilon$, $\rho^{ij}_\varepsilon : [0,\infty) \times \Lambda \times R^d \to R$, indefinitely differentiable in $(\lambda, x) \in \Lambda \times R^d$. Consider the equations

$$(3.7)\quad Y^i_\varepsilon(t,\lambda,\omega) = \beta^i_\varepsilon(t,\lambda,\omega) + \sum_{j,\ell=1}^{d} \int_0^t \sigma^{ij\ell}_\varepsilon(s,\lambda,\bar{X}_\varepsilon(s-,\lambda,\omega))\,Y^j_\varepsilon(s-,\lambda,\omega)\,d\bar{M}^\ell_\varepsilon(s) + \sum_{j=1}^{d} \int_0^t \rho^{ij}_\varepsilon(s,\lambda,\bar{X}_\varepsilon(s,\lambda,\omega))\,Y^j_\varepsilon(s,\lambda,\omega)\,ds,\quad i \leqslant d,$$

with $\bar{X}_\varepsilon$ the solution of (3.6).

Consider also the following stronger form of (J_p):

$(\bar{J}_p)$ There is a family of cadlag adapted processes K_ε, $\varepsilon > 0$ fulfiling (I_p) and such that for every $\varepsilon > 0$ and $k \in N$

$$\bar{\Delta}^k_\varepsilon\, \beta_\varepsilon \leqslant K^*_\varepsilon(k\varepsilon)\,(\bar{V}^k_\varepsilon + \varepsilon).$$

<u>Lemma 3.1.</u> Assume that (A_p) holds for every $p \in N$, $\sigma^{ij\ell}_\varepsilon$, ρ^{ij}_ε, $1 \leqslant i,j,\ell \leqslant d$, $0 < \varepsilon$, $\lambda \in \Lambda$ fulfil (3.1) and β^i_ε, $1 \leqslant i \leqslant d$, $\varepsilon > 0$, $\lambda \in \Lambda$ fulfil (I_p) and $(\bar{J}_p)$ for every $p \in N$. Then, if Y^i_ε, $1 \leqslant i \leqslant d$, $\varepsilon > 0$, $\lambda \in \Lambda$ verify (3.7), they fulfil (I_p) and $(\bar{J}_p)$ for every $p \in N$.

<u>Proof.</u> The idea of the proof is the same as for Theorema 2.3.: one uses Taylor's formula. To avoid notational complications we shall

consider the one-dimensional case only; (there are no real extra difficulties in the multidimensional case).

Fix $p \in N$ and $T > 0$. K_i, $i = 1, 2, \dots$ will be constants depending on p and T only. Note that

(3.8) $\bar{\Delta}_\varepsilon^k \bar{X}_\varepsilon \leqslant K_1(\bar{V}_\varepsilon^k + \varepsilon) \leqslant K_2\,\varepsilon^{3/8}$.

For $N > 0$ define $T_N^\varepsilon = \inf\{t > 0 : |Y_\varepsilon(t)| > N\}$ and $Y_{N,\varepsilon}(t,\lambda,\omega) = Y_\varepsilon(t,\lambda,\omega)$ for $t < T_N^\varepsilon$, $Y_{N,\varepsilon}(t,\lambda,\omega) = Y_\varepsilon(T_N^\varepsilon-,\lambda,\omega)$ for $t \geqslant T_N^\varepsilon$.

Then $|Y_{N,\varepsilon}| \leqslant N < \infty$. Write

$$|\bar{\Delta}_\varepsilon^k Y_{N,\varepsilon}| \leqslant \bar{\Delta}_\varepsilon^k \beta_\varepsilon + K_5 Y_{N,\varepsilon}^*((k+1)\varepsilon)(\bar{V}_\varepsilon^k + \varepsilon) \leqslant$$

$$\leqslant K_6(K_\varepsilon^*(k\varepsilon) + Y_{N,\varepsilon}^*(k\varepsilon))(\bar{V}_\varepsilon^k + \varepsilon) + 2\,K_5\varepsilon^{3/8}\bar{\Delta}_\varepsilon^k Y_{N,\varepsilon}$$

where K_ε, $\varepsilon > 0$ are the processes associated with β_ε, $\varepsilon > 0$ in $(\bar{J}_p)$. For small ε, $2\,K_5\,\varepsilon^{3/8} < 1$. So, by passing $2\,K_5\,\varepsilon^{3/8}\bar{\Delta}_\varepsilon^k Y_{N,\varepsilon}$ in the left hand side of the inequality and get

(3.9) $$\bar{\Delta}_\varepsilon^k Y_{N,\varepsilon} \leqslant K_7(K_\varepsilon^*(k\varepsilon) + Y_{N,\varepsilon}^*(k\varepsilon))(\bar{V}_\varepsilon^k + \varepsilon) \leqslant$$

$$\leqslant K_8(K_\varepsilon^*(k\varepsilon) + Y_{N,\varepsilon}^*(k\varepsilon))\,\varepsilon^{3/8}.$$

Letting N tend to infinity one gets the same inequality for $\bar{\Delta}_\varepsilon^k Y_\varepsilon$. So, if we prove (I_p) for Y_ε, $\varepsilon > 0$, then $(\bar{J}_p)$ follows from (3.9).

For $t < T_N^\varepsilon$ one may write

$$Y_{N,\varepsilon}(t) = \beta_\varepsilon(t) + \int_0^t \sigma_\varepsilon(s,\lambda,\bar{X}_\varepsilon(s-))\cdot$$

$$\cdot\, Y_{N,\varepsilon}(s-)\,d\tilde{\bar{M}}_\varepsilon(s) + C_\varepsilon\Big(\int_0^{\cdot} \sigma_\varepsilon Y_{N,\varepsilon}\,d\bar{M}_\varepsilon\Big)(t) +$$

$$+ \int_0^t \rho_\varepsilon(s,\lambda,\bar{X}_\varepsilon(s))\,Y_{N,\varepsilon}(s)\,ds,$$

Fix $0 \leqslant S'$ and $0 \leqslant S \leqslant 1$ such that $S' + S \leqslant T$. By (1.10) and (I_p) for β_ε, $\varepsilon > 0$ one gets

$$(3.10)\quad \| Y_{N,\varepsilon} \|^{\wedge}_{p,S',S} \leq K_9 + K_{10} S^{1/p} \| Y_{N,\varepsilon} \|^{\wedge}_{p,S'+S} +$$

$$+ \| C_\varepsilon (\int_0^{\cdot} \sigma_\varepsilon Y_{N,\varepsilon} d\bar{M}_\varepsilon \|^{\wedge}_{p,S',S} .$$

We wish to evaluate the last term in the right hand side of the inequality. To this end we write

$$\int_{I_\varepsilon^k} \sigma_\varepsilon(s,\lambda,\bar{X}_\varepsilon(s-)) Y_{N,\varepsilon}(s-) d\bar{M}_\varepsilon(s) = \sum_{j=1}^{4} J_\varepsilon^j(k)$$

with

$$J_\varepsilon^1(k) = \int_{I_\varepsilon^k} \Big[\sigma_\varepsilon(s,\lambda,\bar{X}_\varepsilon(s-)) Y_{N,\varepsilon}(s-) -$$

$$- \sigma_\varepsilon(s,\lambda,\bar{X}_\varepsilon(k\varepsilon)) Y_{N,\varepsilon}(k\varepsilon) -$$

$$- \frac{\partial}{\partial x} \sigma_\varepsilon(s,\lambda,\bar{X}_\varepsilon(k\varepsilon)) Y_{N,\varepsilon}(k\varepsilon)(\bar{X}_\varepsilon(s-) - \bar{X}_\varepsilon(k\varepsilon)) -$$

$$- \sigma_\varepsilon(s,\lambda,\bar{X}_\varepsilon(k\varepsilon))(Y_{N,\varepsilon}(s-) - Y_{N,\varepsilon}(k\varepsilon)) \Big] d\bar{M}_\varepsilon(s),$$

$$J_\varepsilon^2(k) = \int_{I_\varepsilon^k} \sigma_\varepsilon(s,\lambda,\bar{X}_\varepsilon(k\varepsilon)) Y_{N,\varepsilon}(k\varepsilon) d\bar{M}_\varepsilon(s),$$

$$J_\varepsilon^3(k) = \int_{I_\varepsilon^k} \frac{\partial}{\partial x} \sigma_\varepsilon(s,\lambda,\bar{X}_\varepsilon(k\varepsilon)) Y_{N,\varepsilon}(k\varepsilon)(\bar{X}_\varepsilon(s-) - \bar{X}_\varepsilon(k\varepsilon)) d\bar{M}_\varepsilon(s),$$

$$J_\varepsilon^4(k) = \int_{I_\varepsilon^k} \sigma_\varepsilon(s,\lambda,\bar{X}_\varepsilon(k\varepsilon))(Y_{N,\varepsilon}(s-) - Y_{N,\varepsilon}(k\varepsilon)) d\bar{M}_\varepsilon(s).$$

By (3.8), (3.9) and Taylor's formula for $f(x,y) = \sigma(s,\lambda,x)y$ one gets

$$(3.11)\quad |J_\varepsilon^1(k)| \leq K_{11}(\bar{\Delta}_\varepsilon^k \bar{X}_\varepsilon \bar{\Delta}_\varepsilon^k Y_{N,\varepsilon} +$$

$$+ (\bar{\Delta}_\varepsilon^k \bar{X}_\varepsilon)^2 Y^*_{N,\varepsilon}((k+1)\varepsilon)) \bar{V}_\varepsilon^k \leq$$

$$\leq K_{12}(K_\varepsilon^*(k\varepsilon) + Y_\varepsilon^*(k\varepsilon)) \varepsilon^{9/8}.$$

Then, let us write $J_\varepsilon^2 = A_\varepsilon + B_\varepsilon$ with

$$A_\varepsilon = \int_{I_\varepsilon^k} (\sigma_\varepsilon(s,\lambda,\bar{X}_\varepsilon(k\varepsilon)) - \sigma_\varepsilon(k\varepsilon,\lambda,\bar{X}_\varepsilon(k\varepsilon))) Y_{N,\varepsilon}(k\varepsilon) d\bar{M}_\varepsilon(s),$$

$$B_\varepsilon = \sigma_\varepsilon(k\varepsilon,\lambda,\bar{X}_\varepsilon(k\varepsilon)) Y_{N,\varepsilon}(k\varepsilon) \Delta_\varepsilon^k \bar{M}_\varepsilon$$

By (3.1) (ii) $|A_\varepsilon| \leqslant K_{13} Y^*_{N,\varepsilon}(k\varepsilon)\, \varepsilon^{11/8}$. As $E(\Delta^k_\varepsilon M \,|F_{k\varepsilon}) = 0$

$$|E(\Delta^k_\varepsilon \bar{M}_\varepsilon |F_{k\varepsilon})| \leq |E(\Delta^k_\varepsilon M_\varepsilon, V^k_\varepsilon \leqslant \varepsilon^{3/8} |F_{k\varepsilon})| =$$

$$= |E(\Delta^k_\varepsilon M_\varepsilon, V^k_\varepsilon > \varepsilon^{3/8} |F_{k\varepsilon})| \leqslant E(|\Delta^k_\varepsilon M_\varepsilon|^2 |F_{k\varepsilon})^{1/2} \cdot$$

$$\cdot\, E(V^k_\varepsilon > \varepsilon^{3/8} |F_{k\varepsilon})^{1/2} \leqslant K_{14} E(|V^k_\varepsilon|^8 > \varepsilon^3 |F_{k\varepsilon})^{1/2} \cdot \varepsilon^{1/2} \leqslant$$

$$\leqslant K_{14}\, \varepsilon^{-1}\, E(|V^k_\varepsilon|^8 |F_{k\varepsilon})^{1/2} \leqslant K_{15}\, \varepsilon\, .$$

Since σ_ε is bounded, it follows that $|E(B_\varepsilon |F_{k\varepsilon})| \leqslant K_{16} Y^*_{N,\varepsilon}(k\varepsilon)\varepsilon$.

We conclude that

$$(3.12) \quad |E(J^2_\varepsilon(k) | F_{k\varepsilon})| \leqslant K_{17}\, Y^*_{N,\varepsilon}(k\varepsilon)\varepsilon.$$

By (3.8) and (3.1)(i)

$$(3.13) \quad |E(J^3_\varepsilon(k) |F_{k\varepsilon}) \leqslant K_{18}\, Y^*_{N,\varepsilon}(k\varepsilon) E(\bar{\Delta}^k_\varepsilon \bar{X}_\varepsilon V^k_\varepsilon |F_{k\varepsilon}) \leqslant$$

$$\leqslant K_{19} Y^*_{N,\varepsilon}(k\varepsilon) E((V^k_\varepsilon + \varepsilon) V^k_\varepsilon |F_{k\varepsilon}) \leqslant K_{20} Y^*_{N,\varepsilon}(k\varepsilon)\varepsilon.$$

By (3.1)(i) and the first inequality in (3.9) one gets

$$(3.14) \quad E(|J^4_n(k)| \,|F_{k\varepsilon}) \leqslant K_{21}\, E(\bar{\Delta}^k_\varepsilon Y_{N,\varepsilon}\, \bar{V}^k_\varepsilon |F_{k\varepsilon}) \leqslant$$

$$\leqslant K_{22} \left(K^*_\varepsilon(k\varepsilon) + Y^*_{N,\varepsilon}(k\varepsilon)\right) E((V^k_\varepsilon + \varepsilon) V^k_\varepsilon |F_{k\varepsilon}) \leqslant$$

$$\leqslant K_{23} \left(K^*_\varepsilon(k\varepsilon) + Y^*_{N,\varepsilon}(k\varepsilon)\right)\varepsilon.$$

Resuming (3.11) - (3.14) and noting that K_ε, $\varepsilon > 0$ fulfils (I_p) we conclude that

$$\| C_\varepsilon (\int_0^{\cdot} \sigma_\varepsilon Y_{N,\varepsilon}\, d\bar{M}_\varepsilon) \|^{\wedge}_{p,S',S} \leqslant S\, K_{24}(1 + \| Y_{N,\varepsilon} \|^{\wedge}_{p,S'+S}).$$

Then, (3.10) yields

$$\| Y_{N,\varepsilon} \|^{\wedge}_{p,S',S} \leqslant K_{25} + K_{26}\, S^{1/p} \| Y_{N,\varepsilon} \|^{\wedge}_{p,S'+S} \leqslant$$

$$\leqslant K_{25} + K_{26} S^{1/p} \| Y_{N,\varepsilon} \|^{\wedge}_{p,S'} + K_{26} S^{1/p} \| Y_{N,\varepsilon} \|^{\wedge}_{p,S',S}\, .$$

By taking S such that $K_{26} S^{1/p} < 1$ *one gets*

$$\| Y_{N,\varepsilon} \|^{\wedge}_{p,S',S} \leqslant K_{27}(1 + \| Y_{N,\varepsilon} \|^{\wedge}_{p,S'}).$$

By using the above inequality for $S' = 0, S, 2S, \ldots$ the proof finishes.

Q.E.D.

Let us now denote

$\mathcal{H}_\Lambda =: \{\beta_\varepsilon : [0,\infty) \times \Lambda \times \Omega \to R,\ \varepsilon > 0 : \beta_\varepsilon,\ \varepsilon > 0$ are cadlag adapted processes fulfiling (I_p) and $(\bar{J}_p)$ for every $p \in N\}$.

Lemma 3.2. $\mathcal{H}_\Lambda$ is closed under summation, product and multiplication by any process fulfiling (3.1). If $(\beta_\varepsilon)_{\varepsilon>0} \in \mathcal{H}_\Lambda$, then $(\int_0^\cdot \beta_\varepsilon d\bar{M}_\varepsilon)_{\varepsilon>0}$, $(\int_0^\cdot \beta_\varepsilon ds)_{\varepsilon>0} \in \mathcal{H}_\Lambda$ ($\bar{M}_\varepsilon$ is one of $\bar{M}_\varepsilon^i$, $i \leq d$)

Proof. The only nontrivial point is $(\int_0^\cdot \beta_\varepsilon d\bar{M}_\varepsilon)_{\varepsilon>0} \in \mathcal{H}_\Lambda$. Fix $p \in N$ and $T > 0$. As above, K_i, $i = 1,2,\ldots$ will be constants depending on p and T only. K_ε, $\varepsilon > 0$ is the family of processes associated with β_ε, $\varepsilon > 0$ by $(\bar{J}_p)$. Note first that

$$(3.15)\quad \bar{\Delta}_\varepsilon^k\Big(\int_0^\cdot \beta_\varepsilon d\bar{M}_\varepsilon\Big) \leq (\beta_\varepsilon^*(k\varepsilon) + \bar{\Delta}_\varepsilon^k \beta_\varepsilon)\bar{V}_\varepsilon^k \leq$$
$$\leq (\beta_\varepsilon^*(k\varepsilon) + K_\varepsilon^*(k\varepsilon)(\bar{V}_\varepsilon^k + \varepsilon))\bar{V}_\varepsilon^k$$

and so $(\bar{J}_p)$ is proved. Then

$$(3.16)\quad \Big\|\int_0^\cdot \beta_\varepsilon d\bar{M}_\varepsilon\Big\|^\wedge_{p,T} \leq \Big\|\int_0^\cdot \beta_\varepsilon d\tilde{M}_\varepsilon\Big\|^\wedge_{p,T} + \Big\|C_\varepsilon\Big(\int_0^\cdot \beta_\varepsilon d\bar{M}_\varepsilon\Big)\Big\|^\wedge_{p,T}.$$

By (1.10) and (I_p) for β_ε, $\varepsilon > 0$, the first term in the right hand side of the above inequality is bounded, uniformly with respect to $\varepsilon > 0$.

Note then

$$\Big|E\Big(\int_{k\varepsilon}^{\ell\varepsilon} \beta_\varepsilon d\bar{M}_\varepsilon \Big| F_{k\varepsilon}\Big)\Big| \leq \beta_\varepsilon^*(k\varepsilon)|E(\Delta_k^\varepsilon \bar{M}_\varepsilon | F_{k\varepsilon})| + E(\bar{\Delta}_\varepsilon^k \beta_\varepsilon V_\varepsilon^k | F_{k\varepsilon}) \leq$$
$$\leq K_2\, \beta_\varepsilon^*(k\varepsilon)\varepsilon + K_\varepsilon^*(k\varepsilon)E((\bar{V}_k^\varepsilon + \varepsilon)\bar{V}_\varepsilon^k | F_{k\varepsilon}) \leq$$
$$\leq K_3(\beta_\varepsilon^*(k\varepsilon) + K_\varepsilon^*(k\varepsilon))\varepsilon.$$

We conclude that $\sup_\varepsilon \|C_\varepsilon(\int_0^\cdot \beta_\varepsilon d\bar{M}_\varepsilon)\|^\wedge_{p,T} < \infty$ and so the proof finishes.

Q.E.D.

A last remarke is necessary: in the sequel we shall differentiate in (3.6) with respect to $\lambda \in \Lambda$ and consider the system verified by $\bar{X}_\varepsilon$ and its derivatives up to a given order k.

This new system of equations may be written in a square form by adding null coefficients. One has also to verify that this new system is "well compensated" in the sense in which (3.6) is. That is, a drift of the form $\sum_{j,\ell} \int_0^{\cdot} g_\varepsilon^{j\ell}(s)\, \tilde{A}_\varepsilon^{i,j\ell}(s, \lambda, Z)ds$ has to appear. Here $\tilde{A}_\varepsilon$ shall be calculated by (2.28) starting with the coefficients of the new system and Z is the vector made of $\overline{X}_\varepsilon$ and its derivatives. A rather long but simple calculation shows that the needed drift appears by differentiating with respect to λ the initial "assymptotic compensator" in (3.6). We leave out this calculation.

We may now state the main result of this section:

Theorem 3.3. Assume that (A_p) is fulfiled for every $p \in N$; $\varphi, \varphi_\varepsilon$, ψ, ψ_ε, $\varepsilon > 0$, $\lambda \in \Lambda$ fulfil (3.1) and $\alpha, \alpha_\varepsilon$, $\varepsilon > 0$, $\lambda \in \Lambda$ fulfil (3.2). Let X, X_ε and $\overline{X}_\varepsilon$ be the solutions of equations (3.4) and respectively (3.5) and (3.6). Then

(i) One may choose a modification of X such that $\lambda \to X(t, \lambda, \omega)$ is indefinitely differentiable for every $t \geq 0$ and $\omega \in \Omega$.

(ii) For every $p \in N$, $T > 0$ and every multi-index γ

$$\lim_{\varepsilon \to 0} \| D_\gamma X - D_\gamma \overline{X}_\varepsilon \|^{\wedge}_{p,T} = 0 .$$

For every sequence $\varepsilon_n \downarrow 0$ one may choose a subsequence which we denote again by ε_n, $n \in N$ such that

(iii) $\lim_{n \to \infty} \sup_{\lambda \in \Lambda} \sup_{t \leq T} |D_\gamma X(t, \lambda, \omega) - D_\gamma \overline{X}_{\varepsilon_n}(t, \lambda, \omega)| = 0$ a.s.

(iv) $\lim_{n \to \infty} \sup_{\lambda \in \Lambda} \sup_{t \leq T} |D_\gamma X(t, \lambda, \omega) - D_\gamma X_{\varepsilon_n}(t, \lambda, \omega)| = 0$ a.s.

for every $T > 0$ and every multi-index γ.

Proof. As the proof is quite analogous with that of Theorem 2.3. in /1/ Cap.V we shall sketch it only. The first step is to prove that for every multi-index γ, $D_\gamma \overline{X}_\varepsilon$, $\varepsilon > 0$ fulfils (I_p) and $(\overline{J}_p)$ for every $p \in N$. This follows by induction on the length of γ by using Lemma 3.1 and Lemma 3.2. Consider then $\overline{X}_\varepsilon$ and all its derivatives up to a given order k. As we remarked above they verify a system of the same form as (3.6) or (2.29) (with $d\overline{M}_\varepsilon$ instead of dM_ε). Unfortunately this system has unbounded coefficients. By using a truncation argument (the same as in Theorem 2.2) based on (I_p), one may reduces the problem to the case in which the coefficients are bounded.

Then, by using Theorema 2.3 (the fact that dM_ε is replaced by $d\overline{M}_\varepsilon$ does not represent real difficulty: see the final part of the proof of Theorema 2.2) one concludes that $D_\gamma \overline{X}_\varepsilon$, $\varepsilon > 0$ is Cauchy under $\|\cdot\|^\wedge_{p,T}$. An argument based on Sobolev's inequality (see Proposition 2.2 Cap.V in /1/) ensures that one may choose a sequence $\varepsilon_n \to 0$ such that $D_\gamma \overline{X}_{\varepsilon_n}$, $n \in N$ is Cauchy under $\sup\limits_{\lambda\in\Lambda} \sup\limits_{t\leq T} |\cdot|$. Then (i), (ii) and (iii) are proved. The last point follows from (iii) and (1.5).

Q.E.D.

A corollary of the same type as Corollary 2.4. may be given:

Corollary 2.4. Under the hypothese of Theorem 3.3., if (2.34) and (B_p) hold for every $p \in N$, the assymptotic compensator may be moved in the limit equation.

4. APPENDIX: A VERSION OF BURKHOLDER'S INEQUALITY FOR DISCRETE MARTINGALES

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(F_k)_{k\in N}$ and $M_k : \Omega \to R$. $k \in N$ a martingale. For a fixed even natural number $p = 2q$ define

$$\langle M \rangle_i (n) = \sum_{k=1}^{n-1} E(|M_{k+1} - M_k|^i | F_k), \qquad 1 \leq i \leq p, \; n \in N$$

Lemma 4.1. For every $p = 2q$, $q \in N$ there is a constant K_p depending on p only such that for every martinjale $(M_k)_{k\in N}$ and avery $n \in N$

$$(4.1)\quad E(\max_{k \leqslant n} M_k^{\,p})^{1/p} \leqslant K_p \max_{1 \leqslant i \leqslant q} E(\langle M \rangle_{2i}(n)^{p/2i})^{1/p}.$$

<u>Proof.</u> Denote

$$N_k = (M_{k+1} - M_k)^2 \quad \text{and} \quad \bar{N}_k = \sum_{i=1}^{k-1} N_i$$

By Burkholder's inequality for discrete martingales

$$E(\max_{k \leqslant n} |M_k|^p) \leqslant K_1\, E(\bar{N}_n^q),$$

with K_1 (as $K_2, K_3, \ldots$ below) constants depending on p only.
Let us now evaluate the term in the right hand side of the above inequality:

$$E(\bar{N}_n^q) = E((\bar{N}_{n-1} + N_{n-1})^q) \leqslant E(\bar{N}_{n-1}^q) + E(N_{n-1}^q) +$$

$$+ K_2 \sum_{k=1}^{q-1} E(\bar{N}_{n-1}^{q-k} N_{n-1}^k) = E(\bar{N}_{n-1}^q) + E(N_{n-1}^q) +$$

$$+ K_2 \sum_{k=1}^{q-1} E(\bar{N}_{n-1}^{q-k} E(N_{n-1}^k | F_{n-1})).$$

We write the same inequality for $E(\bar{N}_{n-1}^q)$ and dominate $\bar{N}_{n-2}^{q-k}$ by $\bar{N}_{n-1}^{q-k}$. One gets

$$E(\bar{N}_{n-1}^q) \leqslant E(\bar{N}_{n-2}^q) + E(N_{n-2}^q) + K_2 \sum_{k=1}^{q-1} E(\bar{N}_{n-1}^{q-k} E(N_{n-2}^k | F_{n-2})).$$

By using inductively this type of inequalities one concludes that

$$E(\bar{N}_n^q) \leqslant \sum_{\ell=0}^{n-1} E(N_\ell^q) + K_2 \sum_{k=1}^{q-1} E(\bar{N}_{n-1}^{q-k}(\sum_{\ell=0}^{n-1} E(N_\ell^k \;| F_\ell))) =$$

$$= E(\langle M \rangle_p^{(n)}) + K_2 \sum_{k=1}^{q-1} E(\bar{N}_{n-1}^{q-k} \langle M \rangle_{2k}(n)) \leqslant$$

$$\leqslant E(\langle M \rangle_p^{(n)}) + K_2 \sum_{k=1}^{q-1} E(\bar{N}_{n-1}^q)^{(q-k)/q}\, E(\langle M \rangle_{2k}(n)^{q/k})^{k/q}.$$

Denote $x = E(\bar{N}_{n-1}^q)$, $\gamma_k = E(\langle M \rangle_{2k}(n)^{p/2k})^{1/p}$ and

$\gamma = \max_{1 \leqslant k \leqslant q} \gamma_k$.

With this notations the above inequality becomes

$$x \leqslant \mu_q^p + K_2 \sum_{k=1}^{q-1} x^{(q-k)/q} \mu_k^{2k} \leqslant K_2 \sum_{k=1}^{q} x^{(q-k)/q} \mu^{2k}.$$

Elementary operations yield $K_2^{-1} \leqslant \sum_{k=1}^{q} (\mu/x^{1/p})^{2k}$. At least one of the terms in the sum has to dominate $K_3 =: (q\,K_2)^{-1}$. Then, for at least one k, $\mu \geqslant K_3^{1/2k}\, x^{1/p} = K_4 x^{1/p}$ and so the proof is complete.

Q.E.D.

A more general form of the above assertion will be useful:

Corollary 4.2. For every even $p \in N$ there is a constant K_p such that for every sequence of integrable random variables $f_k : \Omega \to R$, $k \in N$ such that f_k is F_k measurable

$$(4.2) \quad E(\max_{k \leqslant n} |\sum_{i=1}^{k} f_i|^p)^{1/p} \leqslant E(|\sum_{k=0}^{n-1} E(f_{k+1}|F_k)|^p)^{1/p} +$$

$$+ K_p \max_{1 \leqslant i \leqslant p/2} E((\sum_{k=0}^{n-1} E(|f_{k+1}|^{2i}|F_k))^{p/2i})^{1/p}.$$

Proof. (4.2) is a consequence of Lemma 4.1. applied to the martingale $M_n = \sum_{k=0}^{n-1} (f_{k+1} - E(f_{k+1}|F_k))$.

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