INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

THE BEST CONSTANTS OF NORMS EQUIVALENCE ON FINITE ELEMENT DISCRETIZATIONS

by

Dumitru ADAM

PREPRINT SERIES IN MATHEMATICS
No. 42/1987

BUCURESTI

THE BEST CONSTANTS OF NORMS EQUIVALENCE
ON FINITE ELEMENT DISCRETIZATIONS

by

Dumitru ADAM

Abstract. We give a way for obtaining on finite element subspaces the best constants of norms equivalence in terms of preconditioning between the Gramians of the basis. As application of this model, we obtain for $H_0^1(\Omega)$, $\Omega \subset R^d$, $d \leq 2$, the best estimations as in classical literature.

1. INTRODUCTION

Many of the practical or theoretical problems concerning the finite element discretizations request a norms equivalence estimation. For this, using the natural framework from [1], formalized in [2], here shortly presented, we give in this paper an practical tool.

Let $\mathcal{H}$ be an real separable Hilbert space for the inner product $\langle .,. \rangle_1$ and let $\{S_{h_i}\}$, $i=1,2,\ldots$, an finite dimension closed subspaces sequence in $\mathcal{H}$, corresponding at the discretization parameters $\{h_i\}$. The main suppositions on this sequence are

(1.1) $$S_{h_i} \subset S_{h_{i+1}}$$

and the following approximation property holds:

(1.2) $$\|u-P_{h_i}u\|_1 := \inf_{v\in S_{h_i}} \|u-v\|_1 \to 0, \quad h_i \to 0$$

for every $u\in H$, equivalently with $i\to\infty$, where P_h is the orthogonal projection operator on H onto S_h. The first is the natural choice of the multilevel discretizations and the second is a caracteristic property of the finite element method.

Now, let S_h be spanned by the linear independent family $\{\phi_h^j; j=1,n_h\}$ in H, and $\mathbb{R}_h$ be the Euclidean real space of same dimension, equipped with the Euclidean inner product $\langle .,.\rangle_h$. Consequently, we use same subscripts for the induced norms and for induced operator norms on S_h and $\mathbb{R}_h$ respectively.

If $\{e_h^j; j=1,n_h\}$ is the canonical basis of $\mathbb{R}_h$, we note by J^h the bijection operator $J^h\in \mathcal{L}(\mathbb{R}_h,S_h)$ defined as follows: $J^h e_h^j=\phi_h^j$, $j=1,n_h$. Thus, for every $\bar{u}_h\in\mathbb{R}_h$, vector of the components $(\bar{u}_h)_i$, the corresponding function in S_h is $u_h:=J^h\bar{u}_h=\sum_{i=1}^{n_h}(\bar{u}_h)_i\phi_h^i$.

Let $J_h\in\mathcal{L}(S_h,\mathbb{R}_h)$ the adjoint operator of J^h, i.e., for every $u_h\in S_h$, $\bar{v}_h\in\mathbb{R}_h$,

$$\langle J_h u_h,\bar{v}_h\rangle_h=\langle u_h,J^h\bar{v}_h\rangle_1$$

With this *usual* notions, we can able to construct our framework. Let $G_h\in B(\mathbb{R}_h)$ be defined by

(1.3) $$G_h:=J_hJ^h=G_h^*>0$$

whose matrix representation in canonical basis is the Gram matrix corresponding at the basis $\{\phi_h^j\}$, denoted by convenience with same simbol G_h, as well any linear operator on $\mathbb{R}_h$. By (1.3) G_h admits a Cholesky factorisation in $\mathbb{R}_h$, (1.4) $$G_h = L_h L_h^* \quad , \quad L_h \in B(\mathbb{R}_h)$$

where L_h has a low-triangular matrix representation. Then, for every $u_h := J^h \bar{u}_h \in S_h$,

(1.5) $$\| u_h \|_1 = \| L_h^* \bar{u}_h \|_h \, .$$

If $\tilde{A}_h \in B(S_h)$, the corresponding Galerkin and preconditioned Galerkin matrices, are the matrix representations of $A_h = J_h \tilde{A}_h J^h$, respectively of $\hat{A}_h = L_h^{-1} A_h L_h^{-*}$, in cannonical basis.

1.1. PROPOSITION. Let $\Lambda_h \in \mathcal{L}(B(S_h), B(\mathbb{R}_h))$ defined by

(1.6) $$\Lambda_h(\tilde{A}_h) = (L_h^{-1} J_h) \tilde{A}_h (J^h L_h^{-*})$$

Then Λ_h is an isomorphic mapping of operator algebras what preserves the spectrum and the norm, i.e.:

(1.7) $$\sigma(\Lambda_h(\tilde{A}_h)) = \sigma(\tilde{A}_h)$$

(1.8) $$\| \Lambda_h(\tilde{A}_h) \|_h = \| \tilde{A}_h \|_1$$

for every $\tilde{A}_h \in B(S_h)$.

Proof. Sketching, from (1.3) and (1.4) it is easy to observe that (1.6) is a similarity relation. This observation and (1.5) are used in [2] for prouving our affirmation. ∎

2. DISCRETE NORMS EQUIVALENCE

Let $\langle \cdot , \cdot \rangle_2$ be a different inner product defined on H. With $J_{h,k}$, k=1,2 we denote the adjoints of J^h corresponding at the ours inner products on H, as above. Then, for every $u_h, v_h \in S_h$ we have:

$$\langle u_h, v_h\rangle_1 = \langle G_{h,1}\bar{u}_h, \bar{v}_h\rangle_h = \langle G_{h,1} J^{-h} u_h, J^{-h} v_h\rangle_h = \tag{2.1}$$
$$= \langle \tilde{G}_h u_h, v_h\rangle_2$$

where $\tilde{G}_h \in B(S_h)$ is the one of the operators of the discrete norms equivalence:

$$\tilde{G}_h := \tilde{G}_{h;1,2} = J^{-1}_{h,2} J_{h,1} \tag{2.2}$$

Now, by (1.6) the spectral equivalent operator of this on $\mathbb{R}_h$ is:

$$\hat{G}_h := \hat{G}_{h;1,2} = \Lambda_{h,2}(\tilde{G}_h) = L^{-1}_{h,2} G_{h,1} L^{-*}_{h,2} \tag{2.3}$$

what is symmetric and positive definite. Then, holds:

2.1. <u>PROPOSITION</u>. The constants of the discrete norms equivalence on S_h are given by the spectral norms of the preconditioned Gram matrices, i.e.

$$C^{(1)}_h \| u_h \|^2_2 \leq \| u_h \|^2_1 \leq C^{(2)}_h \| u_h \|^2_2 , \quad u_h \in S_h \tag{2.4}$$

where

$$C^{(1)}_h = \| \hat{G}_{h;2,1} \|^{-1}_h \tag{2.5}$$

$$C^{(2)}_h = \| \hat{G}_{h;1,2} \|_h \tag{2.6}$$

and this is the best choice of them.

<u>Proof</u>. We prove only the second inequality in (2.4). By (2.1) and (2.2),

$$\| u_h \|^2_1 = \langle \tilde{G}_h u_h, u_h\rangle_2 \leq \| \tilde{G}_h \|_2 \| u_h \|^2_2 = \| \Lambda_{h,2}(\tilde{G}_h) \|_h \| u_h \|^2_2 =$$
$$= \| \hat{G}_{h;1,2} \|_h \| u_h \|^2_2$$

$C_h^{(2)}$ is the best constant because if C is the best, then for $\Psi_h \in S_h$ the eigenfunction corresponding at the largest eigenvalue of $\widetilde{G}_h$, $\lambda_{max}(\widetilde{G}_h) := \|\widetilde{G}_h\|_2$, then,

$$\|\Psi_h\|_1^2 = \langle \widetilde{G}_h \Psi_h, \Psi_h \rangle_2 = \|\widetilde{G}_h\|_2 \|\Psi_h\|_2^2$$

thus $C \geq \|\widetilde{G}_h\|_2$. By a similar way, the first inequality in (2.4) can be proved. ∎

2.1. REMARK. Obviously, if $S_{h_i} \subset S_{h_{i+1}}$, then $C_{h_i}^{(2)} \leq C_{h_{i+1}}^{(2)}$ because for Ψ_{h_i} as above, we have

$$\|\Psi_{h_i}\|_1^2 = \|\widetilde{G}_{h_i}\|_2 \|\Psi_{h_i}\|_2^2 \leq \|\widetilde{G}_{h_{i+1}}\|_2 \|\Psi_{h_i}\|_2^2$$

Hence $C_{h_i}^{(2)} \leq C_{h_{i+1}}^{(2)}$ and analogously, $C_{h_i}^{(1)} \geq C_{h_{i+1}}^{(1)}$. Now, if the sequences $\{C_{h_i}^{(1,2)}\}$ are bounded, then by the approximation property, we can extind at the whole $\mathcal{H}$ the norms equivalence. In this case,

$$C^{(2)} = \lim_{h\to 0} C_h^{(2)} := \lim_{h\to 0} \|\hat{G}_{h;1,2}\|_h = \lim_{h\to 0} \rho(\hat{G}_{h;1,2}) \quad (2.7)$$

This is a way to obtain, as application of our model, the best constants of the norms equivalence on Sobolev space $H_0^1(\Omega)$, what are as in the classical literature of inequalities [5].

3. NORMS EQUIVALENCE ON $H_0^1(\Omega)$.

Let Ω be a bounded domain in R^d, $d \leq 2$. It is knowed the classical result based on the Poincaré-Friedrichs inequality ([6], [7]), that for every $u \in H_0^1(\Omega)$, (3.1) $$\|u\|_1^2 \leq C(\Omega)\,\|u\|_2^2$$

where the Sobolev norms in (3.1) are given by the corresponding inner products, changing the classical notations in our context:

$$\langle u,v\rangle_1 = \langle u,v\rangle_{L^2(\Omega)} + \langle u,v\rangle_2$$

$$\langle u,v\rangle_2 = \sum_{|i|=1} \langle D^i u, D^i v\rangle_{L^2(\Omega)}$$

Moreover, in the following, the subscripts 0,1,2 refere at the norms $L^2(\Omega)$, $\|\cdot\|_1$, $\|\cdot\|_2$-respectively.

In order to obtain our estimations, the choise of the finite element discretization is such that the Gram matrix in $\langle .,.\rangle_2$-inner product be the discrete Laplace matrix corresponding at the finite difference discretization. So, S_h is spanned by a piecewise linear functions family for the uniform grid Ω_h. For d=2 this is obtained by a triangulation where ipotenuses lies on the lines x+y=kh.

In conformity with the previous sections we have:

3.1. PROPOSITION. For $H_o^1((0,1)^d)$, $d\leq 2$ the following inequality holds

(3.2) $$\|u\|_1^2 \leq C_d^{(2)}(\Omega)\cdot\|u\|_2^2$$

where $C_d^{(2)}(\Omega) := C_d^{(2)}$ is the best constant, approximated by

(3.3) $$C_d^{(2)} \leq C_d(\Omega) = 1 + \frac{d+1}{2d}\cdot\frac{1}{\pi^2}$$

Proof. Firstly, let d=1. Then $h(n_h+1)=1$ and with previous notations we have

$$G_{h,1} = G_{h,o} + G_{h,2} \quad ,$$

where

$$G_{h,o}=\frac{h}{6}\begin{bmatrix} 4 & 1 & \ddots & 0 \\ 1 & \ddots & \ddots & 1 \\ 0 & \ddots & 1 & 4 \end{bmatrix}_{n_h x n_h} := \frac{h}{6} G_{h,o}^{(o)}; \quad G_{h,2}=\frac{1}{h}\begin{bmatrix} 2 & -1 & \ddots & 0 \\ -1 & \ddots & \ddots & -1 \\ 0 & \ddots & -1 & 2 \end{bmatrix}_{n_h x n_h} := \frac{1}{h} G_{h,2}^{(o)}$$

By 2.1 Proposition, the best constant $C_h^{(2)}$ is

$$C_h^{(2)} = \| I_h + \frac{h^2}{6} L_{h,2}^{(o)^{-1}} G_{h,o}^{(o)} L_{h,2}^{(o)^{-*}} \|_h = 1 + \frac{h^2}{6} \rho (G_{h,2}^{(o)^{-1}} G_{h,o}^{(o)}) \tag{3.4}$$

Choosing $h_i = 2h_{i+1}$ and $S_{h_i} \subset S_{h_{i+1}}$, we have by 2.1 Remark, that $\{ C_{h_i}^{(2)} \}$ is increasing, and bounded by (3.1), because $C_{h_i}^{(2)}$ are the best. Thus, there exists $C_d^{(2)} = \lim_{h \to 0} C_h^{(2)}$, and this constant is also the best on the whole $H_o^1(\Omega)$ by the following argument. Let $u \in H_o^1(\Omega)$; by the approximation property (1.2), for every $\varepsilon > 0$, there exists $h_{\varepsilon,u}$ such that, if $h < h_{\varepsilon,u}$,

$$\| u - P_h u \|_1 \leq \varepsilon \tag{3.5}$$

Now,

$$\| u \|_1 \leq \| u - P_h u \|_1 + \| P_h u \|_1 \leq \varepsilon + (C_h^{(2)})^{1/2} \| P_h u \|_2$$

$$\leq \varepsilon + (C_d^{(2)})^{1/2} \| P_h u \|_2$$

Because $\left| \| u \|_2 - \| P_h u \|_2 \right| \leq \| u - P_h u \|_2 \leq \| u - P_h u \|_1$ we have $\| P_h u \|_2 \to \| u \|_2$ for $h \to 0$. In the last inequality passing to limit for $h \to 0$, we obtain

$$\| u \|_1 \leq \varepsilon + (C_d^{(2)})^{1/2} \| u \|_2$$

for any ε , i.e. our affirmation (3.2) holds.

Moreover, (3.4) permits a very good upper bound of $C_d^{(2)}$:

$$C_h^{(2)} \leq 1 + \frac{h^2}{6} \| G_{h,2}^{(o)^{-1}} \|_h \cdot \| G_{h,o}^{(o)} \|_h \leq 1 + h^2 / \lambda_{min} (G_{h,2}^{(o)})$$

by Gersghorin theorem applied for $\varrho\,(G_{h,o}^{(o)})$. Because the eigenvalues of $G_{h,2}^{(o)}$ are $\lambda_{h,k}=2(1-\cos\theta_{h,k})$, $\theta_{h,k}=k\pi h\in(0,\pi)$, corresponding of the eigenfunctions $\psi_{h,x}^{(x)}=\sin\,(k\pi x)$, $x\in\Omega_h$, ([8]), we obtain

$$C_h^{(2)} \leq 1+h^2/2(1-\cos\pi h) \ .$$

For $h\to 0$,

$$C_d^{(2)} \leq C_d(\Omega) = 1+\frac{1}{\pi^2} \quad , \ d=1 \quad .$$

For d=2, our choice of finite element subspaces conducts at

$$G_{h,2} = \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} , \qquad G_{h,o} = \frac{h^2}{12} \begin{bmatrix} 1 & 1 & \\ 1 & 12 & 1 \\ & 1 & 1 \end{bmatrix}$$

is the local matrix representations. Because the eigenvalues of $G_{h,2}$ are $\lambda_{h;k,l}=2(2-\cos\theta_{h,k}-\cos\theta_{h,l})$ we obtain in same manner that,

$$C_d^{(2)} \leq C_d(\Omega) = 1+\frac{3}{4} \cdot \frac{1}{\pi^2} \quad , d=2,$$

i.e. (3.3) holds. ∎

3.1. <u>REMARK</u>. If $\Omega\subset R^1$ is an interval of the length α , then $h(n_*+1)=\alpha$ and

$$C_d(\Omega)=1+\frac{\alpha^2}{\pi^2}$$

For $\Omega\subset R^2$ an rectangle of the dimensions $\alpha\times\beta$, we obtain,

$$C_d(\Omega)=1+3\alpha^2\beta^2/2(\alpha^2+\beta^2)\pi^2$$

obviously, the constant of Poincaré-Friedrichs is $C_d^{(2)}-1$ and his evaluation by $C_d(\Omega)-1$ is same as in literature.

Now, we discuss on the approximation property (1.2). By the polinomial basis of finite element choise, we have ([4]) on $H^2(\Omega)\cap H_o^1(\Omega)$ the following property:

$$\inf_{v\in S_h}\|u-v\|_{1,\Omega}\leq Ch\|u\|_{2,\Omega}\quad,\ u\in H^2(\Omega)\cap H_o^1(\Omega)$$

So, choosing $h_{\varepsilon,u}$ such that $Ch\|u\|_{2,\Omega}\leq\varepsilon$, we obtain (3.5). Now, by the density of $H^2(\Omega)\cap H_o^1(\Omega)$ in $H_o^1(\Omega)$ the approximation property can be extinded at the whole $H_o^1(\Omega)$.

Finally, we refere to [3] for preconditioning, to [4], for finite element discretizations on Sobolev spaces and [6], [7] for Poincaré-Friedrichs inequality.

REFERENCES

[1] ADAM, D., "Mesh independence for Galerkin approach using the Cholesky factors of the Gram matrix as preconditioners" - Preprint INCREST 49/1986.

[2] ADAM, D., "On numerical treatment of Galerkin type discretizations" - to appear in Preprint Series.

[3] AXELSSON, O.; GUSTAFSSON,I., "Preconditioning and two-level multigrid methods of arbitrary degree of approximation" - Math. of Comp., vol. 40, no. 161, 1983.

[4] CIARLET, P.G., "The finite element method for elliptic problems" - Studies in Math. and its Appl., vol. 4, North-Holland Publ. Co. 1978.

[5] HARDY, G.H.; LITTLEWOOD, J.E.; POLYA, G., "Inequalities"- 1967.

[6] MERCIER, B., "Topics in finite element solution of elliptic problems" - Springer Verlag, 1979.

[7] NECAS, J., "Les methodes directes en theorie des equations elliptiques" - Ed. de l'Acad. Tchecoslovaque de Sci., Prague 1967.

[8] STUBEN, K.; TROTTENBERG, U., "Multigrid Methods: Fundamental Alg., Model problem analysis, and Applications" - - in "Multigrid Methods", Lecture Notes No. 960.

Department of Mathematics
I N C R E S T
Bd. Pacii 220
79622 Bucharest, Romania.