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by

Tiberiu CONSTANTINESCU

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Tiberiu CONSTANTINESCU*)

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*) *Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.*

PREDICTION OF NONSTATIONARY DISCRETE PROCESSES

T. Constantinescu

I INTRODUCTION

The present paper is motivated in part by some questions in [24]. Roughly speaking, the problem of explicit structure of the Kolmogorov decomposition of a positive definite kernel on the set of integers is taken into account. Such a description was realized in the paper [6] on the basis of the so-called Schur analysis of the given kernel. The starting point of this formalism is a classical paper of I.Schur [18], and then, it was developped in many ways in papers of M.G.Krein (for instance,[14],[1]) , in the contractive intertwining dilations theory([2] and other papers quoted there) or, in a more direct connection with nonstationary processes in [17],[16],[8], and so on.

The content of the paper is the following. In Section 2 we briefly describe the Schur analysis and the form of the Kolmogorov decomposition based on it as presented in [6] . Moreover, we discuss some standard material on Wold decomposition.

In Section 3, we associate with a positive definite kernel on the set of integers, its maximal outer factor as an adaptation to the nonstationary case of the classical method of Lawdenslager [15].

The next section contains some elements of prediction theory-a computation of the prediction-error operator and some elements of filtering nonstationary processes. In the last section we discuss a general result of embedding a nonstationary process into a stationary one and here, some connections with papers as [7],[8] are imposed.

II PRELIMINARIES

Throughout this paper $\mathbb{Z}$ will denote the integers; for two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, $\mathcal{L}(\mathcal{H},\mathcal{H}')$ will denote the set of the linear bounded operators from $\mathcal{H}$ into $\mathcal{H}'$, and most of the notation for Hilbert space operators will be taken from [20]. Thus, for a contraction $T\in\mathcal{L}(\mathcal{H},\mathcal{H}')$ (i.e., $\|T\|\leq 1$), $D_T=(I-T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T=\overline{D_T(\mathcal{H})}$ are the defect operator, respectively the defect space of T. The unitary operator

$$J(T):\mathcal{H}\oplus\mathcal{D}_{T^*}\longrightarrow\mathcal{H}'\oplus\mathcal{D}_T$$

$$J(T)=\begin{bmatrix} T & D_{T^*}\\ D_T & -T^*\end{bmatrix}$$

is the elementary rotation of T. We will consider $O_{\mathcal{H}}(I_{\mathcal{H}})$ as being the zero (identity) operator on the corresponding space and for a closed subspace $\mathcal{L}$ of $\mathcal{H}$, $P^{\mathcal{H}}_{\mathcal{L}}$ will denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{L}$.

A. Positive-definite kernels on $\mathbb{Z}$

We are concerned here with the following object: for a family of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$, there is given an application $\mathcal{T}$ on $\mathbb{Z}\times\mathbb{Z}$ such that $\mathcal{T}(i,j)=T_{ij}\in\mathcal{L}(\mathcal{H}_j,\mathcal{H}_i)$ and the operators

$$M_{ij}(\mathcal{T})=M_{ij}:\bigoplus_{k=i}^{j}\mathcal{H}_k\longrightarrow\bigoplus_{k=i}^{j}\mathcal{H}_k$$
$$M_{ij}=(T_{mn})_{i\leq m,n\leq j} \tag{2.1}$$

for $i,j\in\mathbb{Z}$, $i\leq j$ are positive. We will suppose, without loss of generality, that $T_{ii}=I$ for each $i\in\mathbb{Z}$.

The objective of the Schur analysis is to give a description of $\mathcal{T}$ by means of a family of contractions $\mathcal{G}=\{G_{ij}\}_{i,j\in\mathbb{Z},i\leq j}$, where $G_{ii}=0$ for $i\in\mathbb{Z}$ and for $i,j\in\mathbb{Z}$, $i<j$, $G_{ij}:\mathcal{D}_{G_{i+1,j}}\longrightarrow\mathcal{D}_{G^*_{i,j-1}}$.

Actually, we have an explicit correspondence([6]):

$$T_{i,i+1}=G_{i,i+1} \tag{2.2}$$

for $i\in\mathbb{Z}$, and for $i,j\in\mathbb{Z}$, $j>i+1$,

$$T_{ij}=R_{i,j-1}U_{i+1,j-1}C_{i+1,j}+D_{G^*_{i,i+1}}\dots D_{G^*_{i,j-1}}G_{ij}D_{G_{i+1,j}}\dots D_{G_{j-1,j}} \tag{2.3}$$

where R_{ij}, U_{ij} and C_{ij} are given as follows: for a fixed $i\in\mathbb{Z}$, the family $\{G_{ik}\}_{i<k}$ defines a row contraction

$$R_i:\bigoplus_{k\geq i+1}\mathcal{D}_{G_{i+1,k}}\longrightarrow\mathcal{L}^*_i$$
$$R_i=(G_{i,i+1},D_{G^*_{i,i+1}}G_{i,i+2},\dots) \tag{2.4}$$

and when $j>i$, R_{ij} is the restriction of R_i to $\bigoplus_{k=i+1}^{j}\mathcal{D}_{G_{i+1,k}}$.

By duality are also defined the column contractions C_j and C_{ij}.

Finally, the unitary operators U_{ij}, sometimes called generalized rotations, are given by

$$U_{ii}=I \tag{2.5}$$

for $i\in\mathbb{Z}$ and for $j>i$,

$$U_{ij}:\bigoplus_{k=-j}^{-i}\mathcal{D}_{G^*_{-k,j}}\longrightarrow\bigoplus_{k=i}^{j}\mathcal{D}_{G_{ik}}$$
$$U_{ij}=J_j(G_{i,i+1})J_j(G_{i,i+2})\dots J_j(G_{ij})(U_{i+1,j}\oplus I_{\mathcal{D}_{G^*_{ij}}}) \tag{2.6}$$

where the subscript j at $J(G_{i,i+k})$ means that the elementary rotation of $G_{i,i+k}$ was extended with identity on corresponding spaces.

B. Wold decompositions

Consider a family of Hilbert spaces $\{\mathcal{E}_n\}_{N\leq n\leq M+1}$ with $N\in\mathbb{Z}\cup\{-\infty\}$, $M\in\mathbb{Z}\cup\{\infty\}$, $N<M$ and a family of isometries $V_n\in\mathcal{L}(\mathcal{E}_{n+1},\mathcal{E}_n)$. For $N<n<M+1$ we define the spaces

$$\mathcal{L}_n=\mathcal{E}_n\ominus V_n\mathcal{E}_{n+1} \tag{2.7}$$

and we discern two cases:

-for $M\in\mathbb{Z}$, we also define $\mathcal{L}_{M+1}=\mathcal{E}_{M+1}$ and for $N<n\leq M$,

$$\mathcal{E}_n=\mathcal{L}_n\oplus\bigoplus_{k=1}^{M+1-n}V_n\dots V_{n+k-1}\mathcal{L}_{n+k} \tag{2.8}$$

leading to the following "translation model": we define the unitary operators

$$\Phi_n : \mathcal{E}_n \longrightarrow \bigoplus_{k=0}^{M+1-n} \mathcal{L}_{n+k}$$

$$\Phi_n(e_n, V_n e_{n+1}, \ldots, V_n \ldots V_M e_{M+1}) = (e_n, e_{n+1}, \ldots, e_{M+1}) \tag{2.9}$$

and it follows immediately that $\Phi_n V_n = S_{+,n} \Phi_{n+1}$, where

$$S_{+,n} : \bigoplus_{k=0}^{M-n} \mathcal{L}_{n+k+1} \longrightarrow \bigoplus_{k=0}^{M+1-n} \mathcal{L}_{n+k}$$

$$S_{+,n}(e_{n+1}, \ldots, e_{M+1}) = (0, e_{n+1}, \ldots, e_{M+1}) \tag{2.10}$$

-when $M=\infty$, we get for every $n>N$ and $p \geqslant 0$ that

$$\mathcal{E}_n = \bigoplus_{k=0}^{p} V_n \ldots V_{n+k-1} \mathcal{L}_{n+k} \oplus V_n \ldots V_{n+p} \mathcal{E}_{n+p+1} \tag{2.11}$$

and it is necessary to take into account the residual spaces

$$\mathcal{R}_n = \bigcap_{p=0}^{\infty} V_n \ldots V_{n+p} \mathcal{E}_{n+p+1} \tag{2.12}$$

such that

$$\mathcal{E}_n = \bigoplus_{k=0}^{\infty} V_n \ldots V_{n+k-1} \mathcal{L}_{n+k} \oplus \mathcal{R}_n \tag{2.13}$$

and with respect to these decompositions,

$$V_n = V_n^{(1)} \oplus V_n^{(2)} \tag{2.14}$$

where $V_n^{(2)} : \mathcal{R}_{n+1} \longrightarrow \mathcal{R}_n$ are unitary operators and $V_n^{(1)}$ have "translation models" as (2.10).

C. Kolmogorov decomposition.

The Kolmogorov decomposition produces an operatorial model for a process associated with a given covariance kernel (see for instance [20]). When the kernel is defined on $\mathbb{Z}$ (i.e. an object $\mathcal{T}$ as in (2.1)), the family $\mathcal{G}$ associated to it by (2.2) and (2.3), can be used for an explicit description of the Kolmogorov decomposition of $\mathcal{T}$ ([6]). This construction goes through the following steps: for a contraction of the type of R_1 , i.e.

$$T=(T_1, D_{T_1^*}T_2, \ldots)$$

with T_1 a contraction in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}')$ and T_k contractions in $\mathcal{L}(\mathcal{H}_k, \mathcal{D}_{T_{k-1}^*})$, we give an adequate identification of the defect spaces of T. Define for $k \geqslant 1$ the operators

$$D_k(T): \bigoplus_{j=1}^{k} \mathcal{H}_j (= \mathcal{H}^{[k]}) \longrightarrow \bigoplus_{j=1}^{k} \mathcal{D}_{T_j}$$

$$D_k(T)=\begin{bmatrix} D_{T_1} & -T_1^*T_2 & \dots & -T_1^*D_{T_2^*}\dots D_{T_{k-1}^*}T_k \\ 0 & D_{T_2} & \dots & -T_2^*D_{T_3^*}\dots D_{T_{k-1}^*}T_k \\ \vdots & & \dots & \\ 0 & 0 & \dots & D_{T_k} \end{bmatrix} \tag{2.15}$$

and

$$D_\infty(T): \mathcal{H} \longrightarrow \bigoplus_{j=1}^{\infty} \mathcal{D}_{T_j} = \mathcal{D}(T)$$

$$D_\infty(T) = \operatorname*{s-lim}_{k\to\infty} D_k(T) P^{\mathcal{H}}_{\mathcal{H}^{[K]}} \tag{2.16}$$

where $\mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}_k$ and $\operatorname*{s-lim}_{k\to\infty}$ means the strong operatorial limit.

Now, the operator

$$\alpha(T): \mathcal{D}_T \longrightarrow \mathcal{D}(T)$$

$$\alpha(T)D_T = D_\infty(T) \tag{2.17}$$

is a unitary one. Further on, we define the operator

$$H_\infty(T) = \mathcal{H}' \longrightarrow \mathcal{H}'$$

$$H_\infty(T) = (\operatorname*{s-lim}_{k\to\infty} D_{T_1^*}\dots D^2_{T_k^*}\dots D_{T_1^*})^{\frac{1}{2}} \tag{2.18}$$

and the operator

$$\mathcal{B}(T): \mathcal{D}_{T^*} \longrightarrow \overline{H_\infty(T)(\mathcal{H}')} = \mathcal{D}_*(T)$$

$$\mathcal{B}(T)D_{T^*} = H_\infty(T) \tag{2.19}$$

is also a unitary one. The next step is to consider

$$W(T): \mathcal{D}_*(T) \oplus \mathcal{H} \longrightarrow \mathcal{H}' \oplus \mathcal{D}(T)$$

$$W(T) = \begin{bmatrix} I & 0 \\ 0 & \alpha(T) \end{bmatrix} J(T) \begin{bmatrix} 0 & I \\ \mathcal{B}(T) & 0 \end{bmatrix} \tag{2.20}$$

and to take into consideration this operator for the row contractions R_i given by (2.4)

Consequently, we define the spaces

$$\mathcal{K}_i = \bigoplus_{j=-\infty}^{i-1} \mathcal{D}_*(R_j) \oplus \mathcal{H}_i \oplus \mathcal{D}(R_i) \qquad (2.21)$$

and the unitary operators

$$W_i : \mathcal{K}_{i+1} \longrightarrow \mathcal{K}_i$$
$$W_i = I \oplus W(R_i) \qquad (2.22)$$

with respect obvious decompositions of the spaces. Finally , the Kolmogorov decomposition of $\mathcal{T}$ is given by $\mathcal{V}(\mathcal{T}) = \mathcal{V} = \{V(n)\}_{n\in \mathbb{Z}}$,

$$V(n) : \mathcal{H}_n \longrightarrow \mathcal{K}_o$$

$$V(n) = \begin{cases} W^*_{-1} W^*_{-2} \dots W^*_n / \mathcal{H}_n & n<0 \\ (P^{\mathcal{K}_0}_{\mathcal{L}_0})^* & n=0 \\ W_o W_1 \dots W_{n-1}/\mathcal{H}_n & n>0 \end{cases} \qquad (2.23)$$

in the sense that $T_{ij} = V(i)^* V(j)$ for $i,j \in \mathbb{Z}$ and $\mathcal{K}_o = \bigvee_{n\in\mathbb{Z}} V(n)\mathcal{H}_n$.

Moreover, the last minimality condition yields a natural unicity.

III THE MAXIMAL OUTER FACTOR

In this section we describe the maximal outer factor of a positive definite kernel $\mathcal{T}$ on $\mathbb{Z}$ in terms of the associated parameter $\mathcal{G}$. The method follows the classical idea of Lawdenslager of using the Wold decompositions. General results in stationary case were obtained along this line in [20] for the bounded case and in [19] for the general case (of a semispectral measure on the unit circle).

Actually, a strong connection there exists between these two cases. Thus, for F such a semispectral measure, i.e. a positive linear application on the set $C(\mathbb{T})$ of continuous functions on the unit circle $\mathbb{T}$ with values in $\mathcal{L}(\mathcal{H})$ (we can suppose $F(1)=I$) , we define in a standard manner the analytic function $g(z) = F((e^{it}+z)/(e^{it}-z))$ which has also the property that $\mathrm{Re}\, g \geq 0$ on the unit disc $\mathbb{D}$. Then , we define $h(z)=(g(z)-I)(g(z)+I)^{-1}$, this function being analytic and contractive in $\mathbb{D}$ and for $z\in\mathbb{D}$,

$$F((1-|z|^2)/|e^{it}-z|^2)=(I-h(\bar{z}))^{-1}(I-h(\bar{z})h^*(\bar{z}))(I-h(\bar{z}))^{-1}.$$

This relation gives us an idea on the nature of the maximal outer factor of F (see [19] for details). Now we pass on to the nonstationary case. We fix $\mathcal{T}$ and its $\mathcal{G}$ and introduce a new object: for a family of Hilbert spaces $\{\mathcal{L}_n\}_{n\in\mathbb{Z}}$, we take into account application $\mathcal{F}$ on $\mathbb{Z}\times\mathbb{Z}$ such that $\mathcal{F}(i,j)=F_{ij}\in\mathcal{L}(\mathcal{L}_j,\mathcal{H}_i)$, $F_{ij}=0$ for $j>i$ and $\mathrm{col}_i\mathcal{F}$ are contractions for every i in $\mathbb{Z}$ (by $\mathrm{col}_i\mathcal{F}$ we denote the i-th column of $\mathcal{F}$). An $\mathcal{F}$ will be called outer if $\bigvee_{i\geq k}(\mathrm{col}_i\mathcal{F})\mathcal{H}_i=\bigoplus_{i\geq k}\mathcal{L}_i$ for $k\in\mathbb{Z}$. Starting with such an application $\mathcal{F}$ we define the positive definite kernel $\mathcal{T}_{\mathcal{F}}$ on $\mathbb{Z}\times\mathbb{Z}$ by

$$\mathcal{T}_{\mathcal{F}}(i,j)=(\mathrm{col}_j\mathcal{F})^*\mathrm{col}_i\mathcal{F}\in\mathcal{L}(\mathcal{H}_j,\mathcal{H}_i). \tag{3.1}$$

We also need the standard order for positive definite kernels, which means that for $\mathcal{T}_1$ and $\mathcal{T}_2$ we have $\mathcal{T}_1\leq\mathcal{T}_2$ if $\mathcal{T}_2-\mathcal{T}_1$ is a positive definite kernel. After these preliminaries we can state the main result of the section.

<u>3.1 THEOREM</u> For every positive definite kernel $\mathcal{T}$ there exist a family of Hilbert spaces $\{\mathcal{L}^o_n\}_{n\in\mathbb{Z}}$ and $\mathcal{F}^o$ outer such that:

(1) $\mathcal{T}_{\mathcal{F}^o}\leq\mathcal{T}$

(2) for any other family of Hilbert spaces $\{\mathcal{L}_n\}_{n\in\mathbb{Z}}$ and $\mathcal{F}$ such that $\mathcal{T}_{\mathcal{F}}\leq\mathcal{T}$, we have $\mathcal{T}_{\mathcal{F}}\leq\mathcal{T}_{\mathcal{F}^o}$.

<u>PROOF</u> We will follow the method of Lawdenslager [15] and the proof will appear as an adaptation of the one of Proposition 4.2 V [20] or Theorem 4 [19].

For the unitary operators $\{W_n\}_{n\in\mathbb{Z}}$ given by (2.24), we define

$$\mathcal{K}^+_i=\mathcal{H}_i\oplus\mathcal{D}(R_i) \tag{3.2}$$

and

$$\begin{gathered} W^+_i:\mathcal{K}^+_{i+1}\longrightarrow\mathcal{K}^+_i \\ W^+_i=W_i/\mathcal{K}^+_i \end{gathered} \tag{3.3}$$

are isometries for $i\in\mathbb{Z}$. We now use the Wold decomposition for this family of isometries; define

$$\mathcal{L}_i^o = \mathcal{K}_i \ominus W_i^+ \mathcal{K}_{i+1} = W_i(\ldots \oplus 0 \oplus \mathcal{D}_*(R_i) \oplus 0_{\mathcal{H}_{i+1}} \oplus \ldots) \qquad (3.4)$$

and

$$F_{ij}^o = \begin{cases} P^{\mathcal{K}_j}_{W_{j-1}^* \ldots W_i^* \mathcal{L}_i^o} / \mathcal{H}_j & i < j \\ P^{\mathcal{K}_i}_{\mathcal{L}_i^o} / \mathcal{H}_i & i = j \\ P^{\mathcal{K}_j}_{W_j \ldots W_{i-1}} \mathcal{L}_i^o / \mathcal{H}_j & i > j . \end{cases} \qquad (3.5)$$

We proceed with the properties of $\mathcal{F}^o$. First of all, having in mind the structure of the spaces $\mathcal{K}_i$, the action of W_i and the second equality in (3.4), we immediately get that $F_{ij}^o = 0$ for $j > i$.

The contractivity of $\mathrm{col}_i \mathcal{F}^o$ is also obvious and let us prove that $\mathcal{F}^o$ is outer. As $F_{ij}^o = 0$ for $j > i$, we have only to show that $P^{\mathcal{K}_i}_{\mathcal{L}_i^o} / \mathcal{H}_i = \mathcal{L}_i^o$, and take $e_i \in \mathcal{L}_i^o$ such that $(e_i, P^{\mathcal{K}_i}_{\mathcal{L}_i^o} h_i) = 0$ for $h_i \in \mathcal{H}_i$.

Using again the second equality in (3.4), we obtain that $e_i = 0$ and $\mathcal{F}^o$ is outer. Then, for $h_i \in \mathcal{H}_i$, $i = o, \ldots, n$, $k \in \mathbb{Z}$,

$$\sum_{i,j=o}^{n} (\mathcal{T}_{\mathcal{F}^o}(i+k, j+k) h_i, h_j) = \sum_{i,j=o}^{n} ((\mathrm{col}_{j+k} \mathcal{F}^o)^* (\mathrm{col}_{i+k} \mathcal{F}^o) h_i, h_j) =$$

$$= \Big\| \sum_{i=o}^{n} \mathrm{col}_{i+k} \mathcal{F}^o h_i \Big\|^2 \leq \Big\| \sum_{i=o}^{n} W_k \ldots W_{k+i-1} h_i \Big\|^2 = \sum_{i,j=o}^{n} (T_{i+k, j+k} h_i, h_j)$$

and this proves (1).

Consider $\mathcal{F}$ such that $\mathcal{T}_{\mathcal{F}} \leq \mathcal{T}$ and the application

$$X_i \Big(\sum_{k=o}^{n} V(i+k) h_k \Big) = \sum_{k=o}^{n} \mathrm{col}_{i+k} \mathcal{F} h_k \qquad , i \in \mathbb{Z}.$$

Hence

$$\Big\| X_i \Big(\sum_{k=o}^{n} V(i+k) h_k \Big) \Big\|^2 = \Big\| \sum_{k=o}^{n} \mathrm{col}_{i+k} \mathcal{F} h_k \Big\|^2 = \sum_{k,j=o}^{n} (\mathcal{T}_{\mathcal{F}}(i+k, i+j) h_i, h_j) \leq$$

$$\leq \sum_{k,j=o}^{n} (T_{i+k, i+j} h_i, h_j) = \Big\| \sum_{k=o}^{n} V(i+k) h_k \Big\|^2$$

X_i extends to a contraction between $\mathcal{K}_i^+$ and $\bigoplus_{k \geq o} \mathcal{L}_{k+i}$.

Moreover,

$$X_i \mathcal{R}_i = X_i \bigcap_{n \geq o} W_i \ldots W_{n+i} \mathcal{K}^+_{n+i+1} \subset \bigcap_{n \geq o} \mathrm{col}_{n+i} \mathcal{F} \Big(\bigoplus_{k \geq o} \mathcal{L}_{i+k} \Big) = 0$$

consequently,

$$\sum_{k,j=0}^{n} (\mathcal{T}_{\mathcal{F}}(i+k,i+j)h_i,h_j)=\Big\|\sum_{k=0}^{n} \mathrm{col}_{k+i}\mathcal{F}h_k\Big\|^2=$$

$$=\Big\|X_i\sum_{k=0}^{n} V(i+k)h_k\Big\|^2=\Big\|X_i\sum_{k=0}^{n} \mathrm{col}_{k+i}\mathcal{F}^o h_k\Big\|^2\leq$$

$$\leq\Big\|\sum_{k=0}^{n} \mathrm{col}_{k+i}\mathcal{F}^o h_k\Big\|^2=\sum_{k,j=0}^{n} (\mathcal{T}_{\mathcal{F}^o}(i+k,i+j)h_i,h_j)$$

for $h_k\in\mathcal{H}_k$ and this means that $\mathcal{T}_{\mathcal{F}}\leq\mathcal{T}_{\mathcal{F}^o}$. ∎

3.2 REMARKS

(i) From the proof we also have the condition for equality in (1), that is $\mathcal{R}_i=0$ for every $i\in\mathbb{Z}$.

(ii) As in the stationary case we have an obvious "unicity" of $\mathcal{F}^o$. For this reason we will call it the maximal outer factor of $\mathcal{T}$.

(iii) We have explaned at the level of the parameter $\mathcal{G}$ the interplay given by the Wold decomposition between the so-called Cholesky factorization and the triangular structure of the Kolmogorov decomposition. ∎

IV NONSTATIONARY PREDICTION

It is well known in the stationary case that the maximal outer factor plays a fundamental role in prediction -[11],[23],[10],[20],[19] and so on . It is expected that similar results hold for nonstationary processes. Actually, we already used the correspondence $\mathcal{T}\longleftrightarrow\mathcal{G}$ in some aspects of nonstationary prediction in [3] and a nonstationary Levinson algorithm is given in [21] - a scalar version appeared in [17]. Consequently, we are concerned here with a few facts suggested by some questions in [24].

A. Prediction-error operators

The prediction-error operators are

$$\Delta_i(\mathcal{T})=I-P^{\mathcal{K}_i}_{\mathcal{H}_i}P_{V_i\mathcal{K}^+_{i+1}}P^{\mathcal{K}_i}_{\mathcal{H}_i} \tag{4.1}$$

and , in the stationary case, two classical formulas are given for computing them(actually, in this case there is only one such an operator). The most known is the Szegö formula ([11],[23],[19]) expressing $\Delta(\mathcal{T})$ in terms of the maximal outer factor of $\mathcal{T}$.

Another formula (known as Verblunsky formula) express $\Delta(\mathcal{T})$ directly in terms of the parameter $\mathcal{G}$ - see [9] for the scalar case and [1] for the operatorial variant. Similar results hold in nonstationary case.

<u>4.1 PROPOSITION</u> $\Delta_i(\mathcal{T})=H^2_{\mathcal{L}}(R_i)=(F^o_{ii})^* F^o_{ii}$.

<u>PROOF</u> Notice that

$$\Delta_i(\mathcal{T})= I - P^{\mathcal{K}_i}_{\mathcal{H}_i} W_i P^{\mathcal{K}_{i+1}}_{\mathcal{K}^+_{i+1}} W_i^* P^{\mathcal{K}_i}_{\mathcal{H}_i} =$$

$$=P^{\mathcal{K}_i}_{\mathcal{H}_i} W_i (I - P^{\mathcal{K}_{i+1}}_{\mathcal{K}^+_{i+1}}) W_i^* P^{\mathcal{K}_i}_{\mathcal{H}_i}$$

and the first equality follows immediately from (3.2) and (2.22).

The second one follows from (3.5). ■

<u>B. Orthogonal polynomials</u>

We briefly present some elements of orthogonal polynomials (which represent in fact prediction filters for the underlying process) especially for pointing out that in the context of the analysis based on the correspondence $\mathcal{T} \longleftrightarrow \mathcal{G}$, dimension assumptions (i.e. if $\dim \mathcal{H}_i$ are finite or not) can be avoided- see 9 ii in [24].

For other developments on orthogonal polynomials in the scalar and matricial case- as a consequence of many applications- see for instance [8] ,[16],[24].

Fix $i \in \mathbb{Z}$ and define the left orthogonal polynomials

$$p_{i,n}(z)=a^i_{nn} z^n + a^i_{n,n-1} z^{n-1} + \dots + a^i_{no} \tag{4.2}$$

and the right orthogonal polynomials

$$r_{i,n}(z)=b^i_{nn} z^n + b^i_{n,n-1} z^{n-1} + \dots + b^i_{no} \; . \tag{4.3}$$

Their coefficients appear as columns and rows in the inverses of the Cholesky factorization (i.e. the maximal outer factor) of $\tilde{M}_{i,i+n}$ $(\tilde{M}_{ij}=(T^*_{mn})_{i \le m,n \le j})$ and $M_{i,i+n}$ (we suppose the defects $D_{G_{ij}}$ invertible, this yealding the invertibility of $\tilde{M}_{i,i+n}$ and $M_{i,i+n}$).

Define also the polynomials $P_{i,n}$ and $R_{i,n}$ by "dividing" $p_{i,n}$ and $r_{i,n}$ by their highest coefficients, so that,

$$P_{i,n}=(D^{-1}_{G_{i+n-1,i+n}}\dots D^{-2}_{G_{i,i+n}}\dots D^{-1}_{G_{i+n-1,i+n}})^{\frac{1}{2}}p_{i,n} \tag{4.4}$$

and

$$R_{i,n}=r_{i,n}\cdot(D^{-1}_{G^*_{i,i+1}}\dots D^{-2}_{G^*_{i,i+n}}\dots D^{-1}_{G^*_{i,i+1}})^{\frac{1}{2}}. \tag{4.5}$$

One more notation is also necessary; thus, $P^T_{i,n}(z)=z^n\overline{P}_{i,n}(z^{-1})$ where the coefficients of $\overline{P}_{i,n}$ are obtained as the adjoints of the coefficients of $P_{i,n}$ where each G_{mk} is replaced with the adjoint of its symmetric about the second diagonal (in the triangular matrix formed by the parameters $\{G_{mk}/i\leq m\leq i+n-1, i+1\leq k\leq i+n\}$) and in a similar way we get $R^T_{i,n}$. By direct computation using for instance (1.13) in [6] or (21), (21') in [21] we verify the reccurence formulas:

$$\begin{aligned}&P_{i,n+1}(z)=zP_{i+1,n}(z)-\\ &-D_{G_{i+n,i+n+1}}\dots D_{G_{i+1,i+n+1}}G^*_{i,i+n+1}D^{-1}_{G^*_{i,i+n}}\dots D^{-1}_{G^*_{i,i+1}}R^T_{i,n}(z)\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}&R_{i,n+1}(z)=zR_{i,n}(z)-\\ &-P^T_{i+1,n}(z)D^{-1}_{G_{i+n,i+n+1}}\dots D^{-1}_{G_{i+1,i+n+1}}G^*_{i,i+n+1}D_{G^*_{i,i+n}}\dots D_{G^*_{i,i+1}}\end{aligned} \tag{4.7}$$

which represent operatorial variants for results in [17]. We end with a remark about Christoffel-Darboux formulas. Thus, by direct computation based on (4.6) and (4.7) we check that

$$\begin{aligned}&(r^T_{i,n+1}(z))^*r^T_{i,n+1}(w)-p^*_{i,n+1}(z)p_{i,n+1}(w)=\\ &=(r^T_{i,n}(z))^*r^T_{i,n}(w)-\overline{z}wp^*_{i+1,n}(z)p_{i+1,n}(w)\end{aligned} \tag{4.8}$$

and it is clear that defining

$$k_{i,n}(z,w)=\sum_{k=0}^{n}p^*_{i,k}(z)p_{i,k}(w)$$

we get

$$K_{i,n}(z,w)=(r^T_{i,n}(z))^* r^T_{i,n}(w)+\bar{z}wK_{i+1,n}(z,w)=$$
$$=(r^T_{i,n+1}(z))^* r^T_{i,n+1}(w)-p^*_{i,n+1}(z)p_{i,n+1}(w)+\bar{z}wK_{i+1,n+1}(z,w). \tag{4.9}$$

Of course, when the stationarity is assumed, $K_{i,n}=K_{i+1,n+1}$ and (4.9) is the Christoffel-Darboux formula. In the general case, the relation can be iterated in order to obtain an expression of $K_{i,n}$ in terms of $r_{i,n}$, this being another form of the duality between the left and the right orthogonal polynomials. $K_{i,n}$ lost its meaning as a reproducing kernel for a certain Hilbert space of functions, but instead some triangular objects are imposed. These questions deserve further investigation, but as the way is rather transparent , we stop now here.

C. State-space generators

In [24] it is proposed an approach to the moment problem by using state-space generators.That is, for a positive matrix $M_{ij}=(T_{mn})_{i\leq m,n\leq j}$ there are determined matrices A and B such that

$$T_{mn}=B^* A^{*m}A^nB .$$

In the paper [12] treating the stationary case, it was remarked the connection between formulation and dilation theory. But , the connection goes further, such that in the nonstationary case, the construction in IIC. gives an explicite description of the state-space generator, extendingthe results for the stationary case in [24] and [12] , and thus answering in a certain sense the question 9 i in [24] .

V STATIONARY EMBEDDING

In this section we show that a nonstationary process can be always embedded into a stationary process. Of course, such a result can be derived from general dilation theory, but we will point out its meaning at the level of the parameter $\mathcal{G}$.

More precisely, let be given a positive definite kernel $\mathcal{T}$ and its parameter $\mathcal{G}$. First of all, remark that such a kernel is Toeplitz (i.e. the corresponding process is stationary) if and only if the parameters G_{ij} satisfy $G_{ij}=G_{i+k,j+k}$ for $k\geq 1$. Further on, $\mathcal{T}$ is again arbitrary and we define $\tilde{\mathcal{K}}=\bigoplus_{n\in\mathbb{Z}}\mathcal{H}_n$, $\tilde{G}_0=0_{\tilde{\mathcal{K}}}$ and for $n\geq 1$,

$$\tilde{G}_n:\mathcal{D}_{\tilde{G}_{n-1}}\left(=\bigoplus_{k\in\mathbb{Z}}\mathcal{D}_{G_{k,k+n-1}}\right)\longrightarrow\mathcal{D}_{\tilde{G}^*_{n-1}}\left(=\bigoplus_{k\in\mathbb{Z}}\mathcal{D}_{G^*_{k,k+n-1}}\right)$$

$$(\tilde{G}_n)_{ij}=\begin{cases}G_{ij} & ,\ j=i+n,\ i\in\mathbb{Z}\\ 0 & \text{in rest}\ .\end{cases} \tag{5.1}$$

The sequence $\{\tilde{G}_n\}_{n=0}^{\infty}$ generates by (2.2) and (2.3) the coefficients of a Toeplitz kernel $\tilde{\mathcal{T}}$ and let $\tilde{W}\in\mathcal{L}(\tilde{\mathcal{K}})$ be the Naimark dilation of $\tilde{\mathcal{T}}$, i.e. the unitary operator obtained in (2.24).

<u>5.1 THEOREM</u> A nonstationary process $\mathcal{V}=\mathcal{V}(\mathcal{T})$ can be embedded into the stationary process generated by $\tilde{W}$, i.e.

$$V(n)=P^{\tilde{\mathcal{K}}}_{\mathcal{K}_0}\tilde{W}^n/\mathcal{H}_n$$

(we do not write the natural embeddings of $\mathcal{H}_n$ and $\mathcal{K}_0$ into $\tilde{\mathcal{K}}$).

<u>PROOF</u> The theorem can be easily read out on the associated transmission-line models in the manner of [12]. A formal proof can be given as follows: a short look at the elements involved in $\tilde{W}$ shows that all of them "split correspondingly" with regard to the elements of the family $\{W_n\}_{n\in\mathbb{Z}}$. So that, after reordering, $\tilde{W}$ is an appropriate direct sum of W_n and now taking (2.23) into account, we get the necessary equality. ■

<u>5.2 REMARKS</u>

(i) When a displacement structure (in the sense of [13]) is pointed out , another embedding can be taken into account ([7]). Let us restrict to a simple case (analized in details in [24]) , when we have a positive definite matrix of the form

$$M_n=T_n-S_nS_n^*$$

where T_n is Toeplitz and S_n is strictly lower triagular Toeplitz (i.e. it is zero on and above the main diagonal) and n is the order of the matrices. As M_n appears as a Schur complement, we can take

$$\underline{M}_n=P_n\begin{bmatrix} T_n & S_n \\ S_n^* & I \end{bmatrix}P_n$$

where P_n is an appropriate row interleaving matrix, in order to have the (Cholesky) factorization

$$\underline{M}_n = P_n\begin{bmatrix} F_n^* & 0 \\ S_n^* & I \end{bmatrix}\begin{bmatrix} F_n & S_n \\ 0 & I \end{bmatrix}P_n$$

where $M_n=F_n^* F_n$ is the Cholesky factorization of M_n. In this way, the components (but not the elements m_{ij}) of M_n were embedded in a larger block Toeplitz matrix such that the Cholesky factor (i.e. the maximal outer factor) of M_n was embedded into the Cholesky factor of $\underline{M}_n$. For the embedding (5.1) , we have that also M_n is contained in $\tilde{M}_n$, but the price is the growing of the size of $\tilde{M}_n$; for $\underline{M}_n$ its size is 2nx2n and for $\tilde{M}_n$ it is n^2xn^2.

Actually, this can be viewed even when (5.1) is applied to a Toeplitz kernel: in the passing from general Kolmogorov decomposition to Naimark dilation, it is also involved a "cancellation" process.

(2) The embedding obtained in [7] was further used in order to construct a certain object (as orthogonal polynomials) in the nonstationary case by sifting it from the corresponding (and known) one in the stationary case. But, in view of the precedent remark, it is desirable that such kind of objects be directly obtained.

(3) In [24] are also investigated some indefinite moment matrices, but by imposing a certain signature. We treated similar questions , but without imposing the signature , in [5]. The formalism using the parameter $\mathcal{G}$ works well in the matrix case, but in the operatorial (nonstationary) case, some care is necessary in connection with the applicability of Proposition 1.1[5] . However, in the matrix case the considerations include singular matrices,

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