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The equations of the abelian surfaces embedded in $\mathbb{P}^4(C)$

by Nicolae Manolache at Bucharest

The aim of this note is to present the equations of the Horrocks-Mumford surfaces (HM surfaces, for short), thus being called the zero-sets of the sections of the Horrocks-Mumford bundle E. In general they are nonsingular, hence abelian surfaces (cf. [2]) and, up to automorphisms of $\mathbb{P}^4 = \mathbb{P}^4(C)$, all abelian surfaces in $\mathbb{P}^4$ are obtainable in this way. Some singular HM surfaces were already mentioned in [2], others were found in [4] and their complete classification was given in [1]. We shall put the eqoations in such a form as to be able to write explicitely the binary sextic which, by [1], gives the classification.

We remark among the equations given here the "determinantal" hypersurfaces given by $s \wedge t \in \Gamma(\Lambda^2 E) = \Gamma(O(5))$, where $s, t \in \Gamma(E)$. The general member of this family has 1oo nodes as the singular locus. A remarkable determinantal hypersurfaces of this type, with 125 nodes, was thoroughly studied in [7].

This note is a natural continuation of [6], so that we shall use the notations from there (which are congruent with the notations from [2]), recalling only those which are strictly necessary and using freely some of the others.

I want to express my thanks to C.Borcea and H.Lange who asked whether I can give not only the shape of the minimal resolutions, as in [6], but actually the equations.

The method runs as follows :

1) One makes explicit the minimal resolution of the HM bundle E given in [6], namely one writes down the homomorphisms from there as matrices of polynomials. In particular one obtains E as the

cokernel of an explicit homomorphism

$$u_1 : 35\, O(-2) \longrightarrow 4\, O \oplus 15\, O(-1) \quad .$$

2) One obtains the equations of the zero set $V(s)$, for any $s \in \Gamma(E)$, via the composition :

$$4O \oplus 15O(-1) \xrightarrow{u_0} E \xrightarrow{\wedge s} \Lambda^2 E = O(5),$$

where u_0 is the surjection from 1).

3) The equations are arranged by chosing a convenient basis in $\Gamma(E)$, so that one can write the classifying binary sextic of [1] in terms of the chosen homogeneous coordinates in $\mathbb{P}(\Gamma(E)) = \mathbb{P}^3$.

The minimal N-invariant resolution of E is :

$$o \to W' O(-5) \to T^{\#} V_1 O(-3) \to (L+W) V_3 O(-2) \to TO \oplus UV_2 O(-1) \to E \to o \; ,$$

where, as in [2], N is the normalizer of the Heisenberg group H of level 5 (in fact $N \approx H \rtimes SL_2(\mathbb{Z}_5)$), T, U, V_2 etc. being certain irreducible representations and the signs for tensor products being omitted. It must be observed that all the homomorphisms are prinicipially known, as they are induced by certain inclusions of representations of N .

From the resolution of E one deduces the minimal G-invariant resolution of the zero set of any section $s \in \Gamma(E)$, where $G \approx H \rtimes Z_2 < N$:

$$o \to 2SO(-10) \to 4V_1^{\#} O(-8) \to (5V_3 + 2V_3^{\#}) O(-7) \to 3O(-5) \oplus 3V_2 O(-6) \to I_{V(s)} \to o$$

If one doesn't take into account the symmetries, one writes the minimal resolutions simply:

$$o \to 2O(-5) \xrightarrow{u_3} 20O(-3) \xrightarrow{u_2} 35O(-2) \xrightarrow{u_1} 4O \oplus 15O(-1) \xrightarrow{u_0} E \to o$$

$$o \to 2O(-10) \xrightarrow{u_3} 20O(-8) \xrightarrow{u_2} 35O(-7) \xrightarrow{u_1(s)} 3O(-5) \oplus 15O(-6) \xrightarrow{u_0(s)} I_{V(s)} \to o,$$

where, modulo a shifting in grading, the superior syzygies coincide. Moreover, $u_1(s)$ differs from u_1 only in the lines of degree 2, in the sense that a linear combination of the first 4 lines of u_1 goes to zero when composed with $(\ldots \wedge s) \circ u_0$.

In order to write down the rather big polynomial matrices which give the homomorphisms in these resolutions, we make some notations :

Let Y_i , $i \in \mathbb{Z}_5$ be the homogeneous coordinates in $\mathbb{P}^4$; for any $\ell, m \in \mathbb{Z}_5$ consider the 5x5 matrix :

$$A_{\ell m} := (\delta_{i+\ell, j}\, Y_{i+m})_{i,j} \quad (0 \leq i,j \leq 4)$$

($i+\ell$, $i+m$ being sums modulo 5) ; for any $k = 1,2,3,4$ and any $\ell, m \in \mathbb{Z}_5$ consider the 4x5 matrix :

$$B_{k\ell m} := (\delta_{ik} Y_{j+\ell} Y_{j+m})_{i,j} \quad (1 \leq i \leq 4,\ 0 \leq j \leq 4) ;$$

for any $\ell, m \in \mathbb{Z}_5$ consider the column :

$$C_{\ell m} := (Y_{i+\ell} Y_{i+m})_i \quad (0 \leq i \leq 4) .$$

Recall also the following basis of the space $\Gamma_H(O(5))$ of H-invariant quintics : $Y = \prod Y_i$, $S = \sum Y_i^5$, $Q = \sum Y_i^3 Y_{i+1} Y_{i+4}$, $Q' = \sum Y_i^3 Y_{i+2} Y_{i+3}$, $R = \sum Y_i^2 Y_{i+2}^2 Y_{i+1}$, $R' = \sum Y_i^2 Y_{i+1}^2 Y_{i+3}$.

<u>Theorem 1</u>. The Horrocks-Mumford vector bundle E admits the following minimal resolution :

$$0 \to 2O(-5) \xrightarrow{u_3} 20O(-3) \xrightarrow{u_2} 35O(-2) \xrightarrow{u_1} 4O \oplus 15O(-1) \xrightarrow{u_0} E \to 0$$

where :

$$u_1 = \begin{pmatrix} (B_{112}+B_{203}) & (-B_{211}) & (-B_{244}) & (-B_{122}) & (-B_{133}) & (B_{314}-B_{223}) & (B_{423}-B_{114}) \\ (-A_{10}) & (-A_{21}) & (-A_{34}) & (-A_{42}) & (-A_{13}) & 0 & 0 \\ 0 & A_{02} & A_{03} & 0 & 0 & (-A_{00}) & (-A_{24}-A_{31}) \\ 0 & 0 & 0 & A_{04} & A_{01} & (-A_{12}-A_{43}) & -A_{00} \end{pmatrix}$$

$$u_2 = \begin{pmatrix} (-A_{10}+A_{23}) & (A_{02}-A_{31}) & 0 & 0 \\ A_{03} & 0 & -A_{00} & A_{01} \\ -A_{02} & 0 & A_{20} & -A_{34} \\ 0 & A_{01} & A_{32} & A_{20} \\ 0 & -A_{04} & -A_{43} & -A_{10} \\ 0 & 0 & (A_{23}-A_{02}) & 0 \\ 0 & 0 & 0 & (A_{24}-A_{11}) \end{pmatrix}$$

$$u_3 = \begin{pmatrix} C_{14} & C_{oo} \\ C_{oo} & -C_{23} \\ o & C_{o3} \\ -C_{34} & o \end{pmatrix}$$

Proof. A method to obtain the homomorphisms in the minimal resolution could be to see precisely the action of N on the various representations which appear. This path seems very laborious. Instead of that we shall determine firstly the minimal resolution for special zero-sets $V(s)$, $s \in \Gamma(E)$. This task is made easier by the fact that we already know the shape of the minimal resolution and all what we have to do is to produce at every step the required number of independent vectors of polynomials of the prescribed degrees such that to be in the kernel of the previously obtained u_i. To make the beginning we must write $u_o(s)$, namely 3 independent equations of degree 5 in $I(V(s))$ and 15 of degree 6, linearly independent modulo the ideal generated by the first ones.

We shall take as special $V(s)$ the union of 5 quadrics $Q_i(\alpha)$, given by the equations : $Y_i = o$, $Y_{i+1}Y_{i+4} - \alpha Y_{i+2}Y_{i+3} = o$ (cf. [2] and [1]). Then we find the following minimal set of generators of the ideal $V(s)$:

$$L := Y, \quad M := R' - \alpha Q, \quad N := Q' - \alpha R$$

$$P_i := Y_{i+1}Y_{i+2}Y_{i+3}Y_{i+4}(Y_{i+1}Y_{i+4} - \alpha Y_{i+2}Y_{i+3})$$

$$Q_i := Y_iY_{i+1}Y_{i+4}(Y^2_{i+1}Y_{i+3} + Y^2_{i+4}Y_{i+2} - \alpha Y_iY_{i+1}Y_{i+4})$$

$$R_i := Y_iY_{i+2}Y_{i+3}(Y_iY_{i+2}Y_{i+3} - \alpha Y^2_{i+3}Y_{i+4} - \alpha Y^2_{i+2}Y_{i+1}),$$

where $i \in \mathbb{Z}_5$.

By the method already explained, one proves :

<u>Lemma 1.</u> The union $V = V(s)$ of the quadrics $Q_i(\alpha)$ admits the following minimal resolution :

$$o \to 2O(-1o) \to 2oO(-8) \to 35O(-7) \to 3O(-5) \oplus 15O(-6) \to I_V \to o$$

where :

$u_o(s) = (L, M, N, (P_i)_i, (Q_i)_i, (R_i)_i)$.

$$u_1(s) = \begin{pmatrix} (B'_{103}-\alpha B'_{112}) & (-B'_{111}) & (-B'_{144}) & (\alpha B'_{122}) & (\alpha B'_{133}) & (B'_{214}-B'_{123}) & (B'_{323}+\alpha B'_{114}) \\ (-A_{1o}) & (-A_{21}) & (-A_{34}) & (-A_{42}) & (-A_{13}) & o & o \\ o & A_{o2} & A_{o3} & o & o & (-A_{oo}) & (-A_{24}-A_{31}) \\ o & o & o & A_{o4} & A_{o1} & (-A_{12}-A_{43}) & -A_{oo} \end{pmatrix}$$

and u_2 , u_3 are those from Theorem 1.(Here $B'_{k\ell m}$ are 3x5 matrices $B'_{k\ell m} = (\delta_{ik} Y_{j+\ell} Y_{j+m})_{i,j}$, $1 \le i \le 3$, $j \in \mathbb{Z}_5$,similar to $B_{k\ell m}$.)

Proof of Theorem 1. All is done if we obtain u_1 from $u_1(s)$. One shows that,modulo the 18 lines of $u_1(s)$, there is only one (up to a scalar factor) line vector v such that $vu_2 = o$. This gives a unique u_1 ,up to an isomorphism. In fact we chosed as the first line in u_1 the first line from $u_1(s)$ for $\alpha = \infty$ and as the second, the same line in $u_1(s)$ for $\alpha = o$.

<u>Lemma 2.</u> If s_1 , s_2 , s_3 , s_4 are the sections of E from Theorem 1, then the ideals of $V(s_i)$ are respectively given by the vectors of polynomials $K^i = (L^i, P^i, Q^i, R^i)$, $L^i = (L^i_j)_j \in 4\Gamma(O(5))$, $P^i, Q^i, R^i \in 5\Gamma(O(6))$, as follows :

$L^1 = (o , Y , R', Q')$, $P^1 = (Y^2_{i+1} Y_{i+2} Y_{i+3} Y^2_{i+4})_i$,

$Q^1 = (Y_i Y_{i+1} Y_{i+2} Y^3_{i+4} + Y_i Y^3_{i+1} Y_{i+3} Y_{i+4})_i$, $R^1 = (Y^2_i Y^2_{i+2} Y^2_{i+3})_i$;

$L^2 = (-Y, o , -Q , -R)$, $P^2 = (-Y_{i+1} Y^2_{i+2} Y^2_{i+3} Y_{i+4})_i$,

$Q^2 = (-Y^2_i Y^2_{i+1} Y^2_{i+4})_i$, $R^2 = (-Y_i Y_{i+1} Y^3_{i+2} Y_{i+3} - Y_i Y_{i+2} Y^3_{i+3} Y_{i+4})_i$

$L^3 = (-R', Q, o , S-5Y)$, $P^3 = (Y^2_i Y^2_{i+1} Y^2_{i+4} + Y^4_{i+1} Y_{i+2} Y_{i+4} + Y_{i+1} Y_{i+3} Y^4_{i+4} - Y_i Y^2_{i+1} Y_{i+2} Y^2_{i+3} - Y_i Y^2_{i+2} Y_{i+3} Y^2_{i+4} - Y^3_{i+2} Y^3_{i+3})_i$,

$$Q^3 = (Y_iY^5_{i+1} + Y_iY^5_{i+4} + Y^3_{i+1}Y^2_{i+2}Y_{i+3} + Y_{i+1}Y^3_{i+2}Y^2_{i+4} + Y_{i+2}Y^2_{i+3}Y^3_{i+4} - Y^2_iY_{i+1}Y_{i+2}Y_{i+3}Y_{i+4})_i\ ,\ R^3 = (Y^4_iY_{i+2}Y_{i+3} - Y^2_iY_{i+1}Y^3_{i+3} - Y^2_iY^3_{i+2}Y_{i+4} - Y^2_{i+1}Y^4_{i+2} - 2Y_{i+1}Y^2_{i+2}Y^2_{i+3}Y_{i+4} - Y^4_{i+3}Y^2_{i+4})_i\ ;$$

$$L^4 = (-Q', R, 5Y-S, o)\ ,\ P^4 = (Y_iY^2_{i+1}Y^2_{i+2}Y_{i+4} + Y_iY_{i+1}Y^2_{i+3}Y^2_{i+4} + Y^3_{i+1}Y^3_{i+4} - Y^2_iY^2_{i+2}Y^2_{i+3} - Y_{i+1}Y_{i+2}Y^4 - Y^4_{i+2}Y_{i+3}Y_{i+4})_i\ ,$$

$$Q^4 = (Y^2_iY^3_{i+1}Y_{i+2} + Y^2_iY_{i+3}Y^3_{i+4} + 2Y^2_{i+1}Y_{i+2}Y_{i+3}Y^2_{i+4} + Y^4_{i+1}Y^2_{i+3} + Y^2_{i+2}Y^4_{i+4} - Y^4_iY_{i+1}Y_{i+4})_i\ ,\ R^4 = (Y^2_iY_{i+1}Y_{i+2}Y_{i+3}Y_{i+4} - Y_iY^5_{i+2} - Y_iY^5_{i+3} - Y^3_{i+1}Y^2_{i+2}Y_{i+3} - Y_{i+1}Y^3_{i+2}Y^2_{i+4} - Y^2_{i+1}Y^3_{i+3}Y_{i+4} - Y_{i+2}Y^2_{i+3}Y^3_{i+4})_i\ .$$

Proof. One shows that $K^i = (L^i, P^i, Q^i, R^i)$ are a basis for the space of homogeneous homomorphisms $K : 40 \oplus 150(-1) \to O(5)$ such that $K \circ u_1 = o$. But these homomorphisms are those which factor through $E \to \Lambda^2 E \simeq O(5)$ and any nonzero map $E \to O(5)$ corresponds to exterior multiplication by a section $s \in \Gamma(E)$. A section s decomposes in $s = \sum \alpha_i s_i$ in our basis iff $\sum \alpha_i L_i = o$, where L_i are the components of L in $K = (L, P, Q, R)$. An examination of the sign distribution in the 5 degree part L will convince us that K^i give ... generators for $I_{V(s_i)}$ and these generators are those obtained via the composition :

$$40 \oplus 150(-1) \xrightarrow{(s_1, \dots s_4,\ t_1, \dots, t_{15})} E \xrightarrow{\Lambda s_i} O(5)\ .$$

<u>Remark.</u> S-5Y is the determinantal hypersurface $s_2 \wedge s_3 = o$ studied in 7, swept by the pencil $\lambda s_2 + \mu s_3 = o$, the generic member of which is an abelian surface with real multiplication in $Q(\sqrt{5})$. They are special Comessatti surfaces (cf. [5] for new proofs about their existence and their geometry), namely Jacobians A with $Q(\sqrt{5}) \subset End_Q(A)$. The automorphisms of S-5Y = o produce 6 copies of the line spanned by s_2, s_3 in $\mathbb{P}(\Gamma(E))$. For the interpretation of these lines as certain lines on a cubic surface

in $\mathbb{P}(\Gamma(E))$ see [3], there being given the interpretation for all 27 lines on that cubic surface (which is the unique cubic in $\mathbb{P}(\Gamma(E))$ invariant under the icosahedral group $\mathfrak{A}_5$, cf. [1]).

From Lemma 2 one obtains directly :

Theorem 2. If $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{P}(\Gamma(E))$, then the ideal of $V(\alpha) = V(s_\alpha)$, where $s_\alpha = \sum \alpha_i s_i$, is generated by the components of $\sum \alpha_i K^i =: (L(\alpha), P(\alpha), Q(\alpha), R(\alpha))$.

Remark. It is clear that we have a linear relation among the 4 elements of degree 5 , namely $\sum \alpha_i L_i(\alpha) = o$, so that only three of them are essential .

Corollary 1. The schemes $V(\alpha), \alpha \in \mathbb{P}(\Gamma(E))$ are classified via the binary sextic $\sigma(\alpha) = \alpha_1 Y_1^4 Y_2^2 - \alpha_2 Y_1^2 Y_2^4 + \alpha_3 Y_2(2Y_1^5 - Y_2^5) + \alpha_4(Y_1^5 + 2Y_2^5)$, in the sense that $V(\alpha)$ is one of the schemes :

abelian surface
↓
translation scroll → 5 quadrics
translation scroll → tangent scroll
translation scroll ↓ (double) quintic scroll
tangent scroll → (double) 5 planes
(double) quintic scroll ↓ (double) 5 planes
5 quadrics → (double) 5 planes

according to the following multiplicities of the zeroes of ( ) :

(1,1,1,1,1,1)
↓
(2,1,1,1,1)
(2,1,1,1,1) — (3,1,1,1)
(2,1,1,1,1) ↓ (2,2,2)
(2,1,1,1,1) — (2,2,1,1)
(3,1,1,1) → (4,2)
(2,2,2) ↓ (4,2)
(2,2,1,1) → (4,2)

Proof. In [1] it is shown that the classification of $V(s)$, $s \in \Gamma(E)$ can be done as follows : if $\mathbb{L}$ is the line $Y_o = Y_1 + Y_4 = Y_2 + Y_3 = o$, then $E|_{\mathbb{L}} \simeq O_{\mathbb{L}}(6) \oplus O_{\mathbb{L}}(-1)$ and $\Gamma(E)$ is identified as an $SL_2(\mathbb{Z}_5)$ - module with the space of binary sextics generated by $Y_1^4 Y_2^2$, $Y_1^2 Y_2^4$, $Y_1(Y_1^5 + 2Y_2^5)$, $Y_2(-2Y_1^5 + Y_2^5)$. Then the multiplicities of the roots of $s|_{\mathbb{L}}$ correspond in the way recalled in the corollary to the geometric significance of $V(s)$ (cf. [1], 8.2.). In order to obtain explicitely $\sigma(\alpha)$, we restrict the exact sequence of vector bundles

$$35O(-2) \longrightarrow 4O \oplus 15O(-1) \longrightarrow E$$

to the line $\mathbb{L}$ and reobtain $E|_{\mathbb{L}} \simeq O_{\mathbb{L}}(6) \oplus O_{\mathbb{L}}(-1)$ and besides : $s_1|_{\mathbb{L}} = Y_1^4 Y_2^2$, $s_2|_{\mathbb{L}} = -Y_1^2 Y_2^4$, $s_3|_{\mathbb{L}} = Y_2(2Y_1^5 - Y_2^5)$, $s_4|_{\mathbb{L}} = Y_1(Y_1^5 + 2Y_2^5)$.

Corollary 2. For $\alpha,\beta \in \mathbb{P}(\Gamma(E))$, the "determinantal" quintic $s(\alpha) \wedge s(\beta)$ is given by : $Q_{\alpha\beta} = (\alpha_1\beta_2 - \alpha_2\beta_1)Y + (\alpha_1\beta_3 - \alpha_3\beta_1)R' + (\alpha_1\beta_4 - \alpha_4\beta_1)Q' - (\alpha_2\beta_3 - \alpha_3\beta_2)Q - (\alpha_2\beta_4 - \alpha_4\beta_2)R + (\alpha_3\beta_4 - \alpha_4\beta_3)(S-5Y)$.

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