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by

Dan Tudor VUZA

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Dan Tudor VUZA*)

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*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, ROMANIA

THE COMPONENTS OF A POSITIVE OPERATOR

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0. Introduction

In his paper [6], de Pagter proved the following theorem:

THEOREM 0.1. Let E,F be order complete Riesz spaces such that $F^{\times}$ (the order continuous dual of F) separates F. Then the set of components C_V of any positive operator $V:E \longrightarrow F$ can be obtained from the set $\mathcal{A}_V$ of simple components by a three - steps up - down process; more precisely,

$$C_V = SDS_{\aleph_0} \mathcal{A}_V \tag{1}$$

(see § 1 and § 4 for definitions and notations).

Soon after, Aliprantis and Burkinshaw gave in [1] a shorter proof for de Pagter's result. Actually, they asked for E only the principal projection property instead of order completeness. However, they still kept the hypothesis that $F^{\times}$ is separating.

In this paper we show that theorem 0.1 still holds when F belongs to the so - called class $\mathcal{C}_0$, which strictly includes the class of order complete Riesz spaces with separating order continuous dual. Moreover, we give a version of theorem 0.1 which holds for an arbitrary order complete Riesz space F. More precisely, we introduce the classes $\mathcal{C}_\alpha$ (α being an arbitrary ordinal) and we show that the three - steps up - down process in (1) is replaced by an up - down process depending on the index α of the class to which F belongs; each order complete Riesz space is a member of some $\mathcal{C}_\alpha$.

The paper is divided into four sections.

§ 1 is devoted to basic definitions and notations.

§ 2 has a technical character. Its aim is to construct the theory of Φ- systems, which is a basic tool to be used in § 4 for the proof of our results.

In § 3 the classes $\mathcal{C}_\alpha$ are introduced and some stability properties of them are given. It is shown that $\mathcal{C}_0$ strictly contains the class of order complete

Riesz spaces with separating order continuous dual. For every ordinal α, an example of an order complete Riesz space not in $\mathcal{C}_\alpha$ is produced.

§ 4 contains the main results of the paper. Beside the extensions of theorem 0.1 we present some results concerning the order approximation of the operators in the order interval $[-|V|,|V|]$ by elements in the submodule generated by V (V being an order bounded not necessarily positive operator); these results are useful in the case when E lacks order projections, but instead is a so - called "principal module". We also give an up - down theorem in the center of $L_r(E,F)$ which represents an extension on the lines of the previous discussion of the corresponding result proved by Buskes, Dodds, de Pagter and Schep [2] only for Riesz spaces with separating order continuous dual.

1. Preliminaries

1_E will be the identity map of a set E.

C(X) will be the Riesz space of all continuous real - valued functions on the compact space X.

For any Riesz space E and any $x \in E$, we denote by E_x the order ideal generated by x and by B_x, the band generated by x. The set of all components of $x \in E_+$ (that is, the elements $y \in E$ verifying $y \wedge (x-y) = 0$) is denoted by C_x.

Whenever E is Archimedean, we shall consider the norm $\| \, \|_x$ on E_x given by

$$\|y\|_x = \inf \{ a \mid a \in \mathbb{R}_+ , |y| \leq a|x| \}.$$

Let $(x_\delta)_{\delta \in \Delta}$ be a net in E and let $x \in E$. We write $x_\delta \xrightarrow{\omega} x$ if there is a net $(y_\delta)_{\delta \in \Delta} \subset E$ such that $|x_\delta - x| \leq y_\delta$ and $y_\delta \downarrow 0$. We write $x_\delta \xrightarrow{\rho} x$ if there is $y \in E$ such that $\|x_\delta - x\|_y \longrightarrow 0$.

A map T between two Riesz spaces is called order continuous if $x_\delta \xrightarrow{\omega} x$ implies $T(x_\delta) \xrightarrow{\omega} T(x)$.

We shall denote by $E^\times$ the Riesz space of all order continuous order bounded linear forms on the Riesz space E. We say that $E^\times$ is separating if for every $x \in E \setminus \{0\}$ there is $f \in E^{\times}$ with $f(x) \neq 0$.

If E,F are Riesz spaces with F order complete, $L_r(E,F)$ will be the Riesz space of all order bounded linear maps from E into F.

A band B in a Riesz space E is called a projection band if every $x \in E$ can be written as $x_1 + x_2$ with $x_1 \in B$ and $x_2 \in B^{\perp}$; the map $x \longmapsto x_1$ is called the order projection onto B. By an order projection we shall always mean a projection associated with a projection band; the set of all order projections on E will be denoted by $\mathcal{P}(E)$.

Whenever B_x is a projection band, we shall denote by $[x]$ the projection associated with it. A Riesz space E is said to have the principal projection property if B_x is a projection band for every $x \in E$.

For the properties of order projections, see [5]. We shall especially need the following ones:

i) Each order projection is an order continuous Riesz homomorphism.

ii) $\mathcal{P}(E)$ is a Boolean algebra; we have

$$P_1P_2(x) = (P_1 \wedge P_2)(x) = P_1(x) \wedge P_2(x),$$
$$(P_1 \vee P_2)(x) = P_1(x) \vee P_2(x)$$

for every $P_1, P_2 \in \mathcal{P}(E)$ and $x \in E_+$.

iii) Whenever B_x is a projection band and $P \in \mathcal{P}(E)$, then $P[x] = [P(x)]$.

Recall that an f - algebra is a Riesz space A endowed with a multiplication such that $A_+A_+ \subset A_+$ and $ac \wedge b = ca \wedge b = 0$ for any $a, b, c \in A_+$ with $a \wedge b = 0$. In this paper, however, the word "f - algebra" will be exclusively employed to design an Archimedean f - algebra admitting an element e as an algebraic unit as well as a strong order unit.

As examples of f - algebras which will be used, we mention the following (E is an Archimedean Riesz space):

i) The center Z(E) of E. It is the set of all linear operators U on E for which there is $a \in \mathbb{R}_+$ such that $-a1_E \leq U \leq a1_E$.

ii) The algebra $Z_p(E)$ generated by all $P \in \mathcal{P}(E)$.

If A is an f - algebra, $\Sigma(A)$ will be the subalgebra in A generated by C_e. For instance, we have $Z_p(E) = \Sigma(Z(E))$.

Let A be an f - algebra. By a Riesz A - module we shall mean an Archimedean Riesz space E which is an algebraic module over A such that $A_+E_+ \subset E_+$. It is well known that in every Riesz A - module the equality $|ax| = |a||x|$ holds for every $a \in A$ and $x \in E$ (see for instance [7]); from this it follows in particular that the maps $y \mapsto ay$ and $b \mapsto bx$ are Riesz homomorphisms for any $a \in A_+$ and $x \in E_+$.

For any Riesz A – module E, any $x \in E$ and $M \subset A$, Mx will denote the set $\{ax \mid a \in M\}$.

If X, Y are preordered sets, a map $f: X \to Y$ will be called increasing (respectively decreasing) if $f(x) \leq f(y)$ (respectively $f(x) \geq f(y)$) whenever $x \leq y$.

We refer the reader to [4] for the theory of cardinal and ordinal numbers.

For every ordinals α, β we shall denote by $[\alpha, \beta)$ the set of all ordinals γ verifying $\alpha \leq \gamma < \beta$. In §2 we shall assume, for technical reasons, that -1 (not 0) is the least ordinal; hence, symbols as $[-1, \alpha)$ will make sense.

The symbol card M will be used to denote the cardinal number of a set M. If $\alpha \geq 0$ is any ordinal, we shall write card α instead of card $[0, \alpha)$.

Recall that an ordinal α is called initial if card $\beta <$ card α whenever $\beta \in [0, \alpha)$. For every ordinal $\alpha \geq 0$ we let ω_α be the α – th initial ordinal and $\aleph_\alpha =$ card ω_α. It is well known that the map $\alpha \mapsto \omega_\alpha$ is strictly increasing and continuous. We shall assume the axiom of choice to hold, so that any cardinal number is an $\aleph_\alpha$.

We shall now define a list of symbols to be used throughout all the text. Let E be any Riesz space and let M be a subset of E. If $\mathfrak{m}$ is any cardinal number, define $S_{\mathfrak{m}}M$ (respectively $D_{\mathfrak{m}}M$) to be the set of those $x \in E$ for which there is $N \subset M$ with card $N \leq \mathfrak{m}$ and $x = \sup N$ (respectively $x = \inf N$). Put

$$SM = S_{\text{card } M} M, \qquad DM = D_{\text{card } M} M.$$

Also, let $L_{\mathfrak{m}}M$ be the set of those $x \in E$ for which there is a net $(x_\delta)_{\delta \in \Delta} \subset M$ with card $\Delta \leq \mathfrak{m}$ and $x_\delta \xrightarrow{\omega} x$. If in the definition above we impose no restriction on card Δ, we obtain the set LM.

With the aid of transfinite induction, define the sets $C_\alpha M$ and $L^2_\alpha M$ as follows:

$$C_0 M = D_{\aleph_0} S_{\aleph_0} M, \qquad L^2_0 M = L_{\aleph_0} L_{\aleph_0} M,$$

$$C_{\alpha+1} M = D_{\aleph_{\alpha+1}} S_{\aleph_{\alpha+1}} C_\alpha M, \quad L^2_{\alpha+1} M = L_{\aleph_{\alpha+1}} L_{\aleph_{\alpha+1}} L^2_\alpha M$$

and

$$C_\alpha M = D_{\mathrm{card}\,\alpha}\ S_{\mathrm{card}\,\alpha} \bigcup_{\beta\in[0,\alpha)} C_\beta M\ ,$$

$$L^2_\alpha M = L_{\mathrm{card}\,\alpha}\ L_{\mathrm{card}\,\alpha} \bigcup_{\beta\in[0,\alpha)} L^2_\beta M.$$

if α is a limit ordinal.

In case M is a sublattice, then all the sets $C_\alpha M$ and $L^2_\alpha M$ are sublattices; consequently, the equalities $x = \sup N$ and $x = \inf N$ in the definition of the symbols $S_{\mathfrak{m}}$ and $D_{\mathfrak{m}}$ used throughout the construction of the C_α's can be changed into $N \uparrow x$ and $N \downarrow x$ (these symbols mean that N is upwards (respectively downwards) directed and $x = \sup N$ (respectively $x = \inf N$)). It follows in particular that $C_\alpha M \subset L^2_\alpha M$ for any sublattice M and any ordinal α.

2. Φ - systems

Throughout all the section, E and F will be two fixed order complete Riesz spaces and $T: E \longrightarrow F$ will be a fixed order continuous map.

A Φ - system is a triple (X, Φ, Ψ) formed by a preordered set X and two maps $\Phi: X \longrightarrow E$, $\Psi: X \longrightarrow F$ such that the following hold:

i) $\Phi(X)$ is order bounded.

ii) Ψ is decreasing.

iii) $T(\bigvee_{i=1}^{n} \Phi(x_i)) \leq \Psi(x_1)$ whenever $x_1, \ldots, x_n \in X$ and $x_1 \leq x_2 \leq \ldots \leq x_n$.

For any preordered set X and any infinite ordinal α we let X^α be the set of all increasing maps $f: [-1, \alpha) \longrightarrow X$. The set X^α is preordered in the following way: $f \leq g$ if for every $\beta \in [-1, \alpha)$ there is $\gamma \in [-1, \alpha)$ such that $f(\beta) \leq g(\gamma)$.

LEMMA 2.1. Let (X, Φ, Ψ) be a Φ - system and let α be an infinite ordinal. Define $\Phi^\alpha: X^\alpha \longrightarrow E$ and $\Psi^\alpha: X^\alpha \longrightarrow F$ by

$$\Phi^\alpha(f) = \bigwedge_{\beta\in[-1,\alpha)} \ \bigvee_{\gamma\in[\beta,\alpha)} \Phi(f(\gamma))\ ,$$

$$\Psi^\alpha(f) = \bigwedge_{\beta\in[-1,\alpha)} \Psi(f(\beta)).$$

Then $(X^\alpha, \Phi^\alpha, \Psi^\alpha)$ is a Φ - system.

PROOF. Properties i) and ii) in the definition of a Φ - system are easily verified. In order to verify iii), let us make the following notation: if M,N are two subsets of a preordered set, we write $M \leq N$ if every element in M is less or equal to any element in N.

Let $f_1,\dots,f_n \in X^{\alpha}$ be such that $f_1 \leq f_2 \leq \dots \leq f_n$. We shall prove that for every $k \in \{0,\dots,n\}$, every $\beta_0 \in [-1,\alpha)$ and every finite subsets $M_1,\dots,M_k \subset$ $\subset [-1,\alpha)$ such that $\{\beta_0\} \leq M_1 \leq \dots \leq M_k$ and $f_1(M_1) \leq \dots \leq f_k(M_k)$ we have

$$(1) \qquad T\Big(\bigvee_{i=1}^{k} \sup \Phi(f_i(M_i)) \vee \bigvee_{i=k+1}^{n} \Phi^{\alpha}(f_i)\Big) \leq \Psi(f_1(\beta_0)) .$$

We argue by induction on $n-k$. Indeed, for $n-k=0$, (1) follows from the definition of a Φ - system. Now suppose that (1) is true for k and let us prove it for $k-1$. As $f_{k-1} \leq f_k$, there is $\beta_k \in [-1,\alpha)$ such that $M_{k-1} \leq$ $\leq \{\beta_k\}$ and $f_{k-1}(M_{k-1}) \leq \{f_k(\beta_k)\}$. If Δ_β denotes the set of all finite subsets of $[\beta,\alpha)$, then $(\sup \Phi(f_k(M))_{M\in\Delta_\beta}$ is a net increasing to $\bigvee_{\gamma\in[\beta,\alpha)} \Phi(f_k(\gamma))$. As we have by the induction hypothesis

$$(2) \qquad T\Big(\bigvee_{i=1}^{k-1} \sup \Phi(f_i(M_i)) \vee \sup \Phi(f_k(M)) \vee \bigvee_{i=k+1}^{n} \Phi^{\alpha}(f_i)\Big) \leq \Psi(f_1(\beta_0))$$

for any $\beta \in [\beta_k,\alpha)$ and any $M \in \Delta_\beta$ and T as well as the lattice operations are order continuous, it follows from (2) that

$$(3) \qquad T\Big(\bigvee_{i=1}^{k-1} \sup \Phi(f_i(M_i)) \vee \bigvee_{\gamma\in[\beta,\alpha)} \Phi(f_k(\gamma)) \vee \bigvee_{i=k+1}^{n} \Phi^{\alpha}(f_i)\Big) \leq \Psi(f_1(\beta_0))$$

for any $\beta \in [\beta_k,\alpha)$. Now the net $\Big(\bigvee_{\gamma\in[\beta,\alpha)} \Phi(f_k(\gamma))\Big)_{\beta\in[\beta_k,\alpha)}$ is decreasing to $\Phi^{\alpha}(f_k)$; consequently, it follows from (3) that (1) is true for $k-1$.

For $k=0$, (1) gives

$$T\Big(\bigvee_{i=1}^{n} \Phi^{\alpha}(f_i)\Big) \leq \Psi(f_1(\beta)), \quad \beta \in [-1,\alpha) ;$$

as β is arbitrary, we obtain that the left side is less than $\Psi^{\alpha}(f_1)$ and the proof is complete.

We say that $(X_\alpha, j_{\beta\alpha})$ is an inductive system of preordered sets (α,β are running over an upwards directed set) if it is an inductive system in the set - theoretic sense and, in addition, each $j_{\beta\alpha}$ is increasing. We shall consider the set - theoretic inductive limit $X = \varinjlim_{\alpha} X_\alpha$ as preordered in the following way: $x \leq y$ if there are an index α and $x_\alpha, y_\alpha \in X_\alpha$ such that $x_\alpha \leq y_\alpha$, $x = j_\alpha(x_\alpha)$ and $y = j_\alpha(y_\alpha)$, where $j_\alpha : X_\alpha \to X$ denotes the canonical map; clearly, each j_α is increasing for the preorder relation so defined.

We say that $(X_\alpha, \Phi_\alpha, \Psi_\alpha, j_{\beta\alpha})$ is an inductive system of Φ - systems if the following hold :

i) Each $(X_\alpha, \Phi_\alpha, \Psi_\alpha)$ is a Φ - system.

ii) $(X_\alpha, j_{\beta\alpha})$ is an inductive system of preordered sets.

iii) Whenever $\alpha \leq \beta$ we have $\Phi_\alpha = \Phi_\beta j_{\beta\alpha}$ and $\Psi_\alpha = \Psi_\beta j_{\beta\alpha}$.

iv) $\bigcup_\alpha \Phi_\alpha(X_\alpha)$ is order bounded.

The proof of the following lemma is straightforward and will be ommited.

LEMMA 2.2. Let $(X_\alpha, \Phi_\alpha, \Psi_\alpha, j_{\beta\alpha})$ be an inductive system of Φ - systems and let $X = \varinjlim_\alpha X_\alpha$, $\Phi = \varinjlim_\alpha \Phi_\alpha$ and $\Psi = \varinjlim_\alpha \Psi_\alpha$. Then (X, Φ, Ψ) is a Φ - system, called the inductive limit of the above inductive system.

Let $\alpha \geqslant 0$ be any ordinal number. We say that a preordered set X has the α - majoration property if every $M \subset X$ with card $M \leq \aleph_\alpha$ is bounded from above. Let us agree that the (-1) - majoration property means that X is upwards directed.

In the rest of the section we shall fix an upwards directed preordered set X and we shall construct the sets X_α and the maps $j_{\beta\alpha}: X_\alpha \longrightarrow X_\beta$, $p_{\alpha\beta}: X_\beta \longrightarrow X_\alpha$ $(\alpha, \beta \geqslant -1,\ \alpha \leq \beta)$ by transfinite induction. Put $X_{-1} = X$ and define X_α for $\alpha \geqslant 0$ as follows:

Case a): $X_{\alpha+1} = X_\alpha^{\omega_{\alpha+1}}$,

$$j_{\alpha+1,\alpha+1} = p_{\alpha+1,\alpha+1} = 1_{X_{\alpha+1}},$$
$$j_{\alpha+1,\alpha}(x)(\gamma) = x,$$
$$p_{\alpha,\alpha+1}(f) = f(-1),$$
$$j_{\alpha+1,\beta} = j_{\alpha+1,\alpha}\, j_{\alpha\beta}, \quad \beta < \alpha$$
$$p_{\beta,\alpha+1} = p_{\beta\alpha}\, p_{\alpha,\alpha+1}, \quad \beta < \alpha.$$

Case b): α is a limit ordinal.

X_α is the set of those $f: [-1, \alpha) \longrightarrow \bigcup_{\beta \in [-1,\alpha)} X_\beta$ with the following properties:

i) $f(\beta) \in X_\beta$ for $\beta \in [-1, \alpha)$.

ii) $j_{\gamma\beta}(f(\beta)) \leq f(\gamma)$ and $f(\beta) = p_{\beta\gamma}(f(\gamma))$ whenever $\beta, \gamma \in [-1, \alpha)$ and $\beta \leq \gamma$.

The preorder relation on X_α is defined by: $f \leq g$ if for every $\beta \in [-1, \alpha)$ there is $\gamma \in [-1, \alpha)$ with $\beta \leq \gamma$ and $j_{\gamma\beta}(f(\beta)) \leq g(\gamma)$.

As concerns the maps $j_{\alpha\beta}$ and $p_{\beta\alpha}$ for $\beta \leq \alpha$, define them as follows:

$$p_{\alpha\alpha} = j_{\alpha\alpha} = 1_{X_\alpha}$$

and for $\beta < \gamma$:

$$p_{\beta\alpha}(f) = f(\beta),$$

$$j_{\alpha\beta}(x)(\gamma) = j_{\gamma\beta}(x) \text{ if } \gamma \geq \beta ,$$

$$= p_{\gamma\beta}(x) \text{ if } \gamma < \beta .$$

The next lemma presents the properties of the sets and maps so constructed; it ensures in particular that the definition of $j_{\alpha\beta}$ in case b) is correct (its values are indeed contained in X_α).

LEMMA 2.3.

i) Each $j_{\beta\alpha}$ is increasing.

ii) $p_{\alpha\gamma} = p_{\alpha\beta} p_{\beta\gamma}$, $j_{\gamma\alpha} = j_{\gamma\beta} j_{\beta\alpha}$ whenever $\alpha \leq \beta \leq \gamma$.

iii) $p_{\alpha\beta} j_{\beta\alpha} = 1_{X_\alpha}$, $j_{\beta\alpha} p_{\alpha\beta}(x) \leq x$ whenever $\alpha \leq \beta$ and $x \in X_\beta$.

iv) X_α has the α - majoration property for any $\alpha \geq -1$.

v) For any $\alpha \geq -1$ and any $x \in X_\alpha$, the set $p^{-1}_{\alpha,\alpha+1}(\{x\})$ is cofinal in $X_{\alpha+1}$.

PROOF. The proof is done by transfinite induction: we assume that all the statements are true for $\beta < \alpha$ and we prove them for α .

i) to iii) are straightforward computations and we shall ommit them. We shall verify only iv) and v).

As X is upwards directed, iv) is true for $\alpha = -1$. We shall prove it for an arbitrary α by distinguishing two cases, according to the fact whether α is a limit ordinal or not.

Case a): The proof for $\alpha + 1$.

Let $M \subset X_{\alpha+1}$, card $M \leq \aleph_{\alpha+1}$. Then the elements of M may be written as a net $(f_\gamma)_{\gamma \in [-1, \omega_{\alpha+1})}$. Define $f: [-1, \omega_{\alpha+1}) \longrightarrow X_\alpha$ by transfinite induction:

$f(-1)$ is an arbitrary element of X_α .

$f(\beta)$ is an upper bound for the set

$$\{ f(\gamma) \mid -1 \leq \gamma < \beta \} \cup \{ f_\gamma(\beta) \mid -1 \leq \gamma \leq \beta \} .$$

Observe that an upper bound for the above set exists as, by the induction hypothesis, X_α has the α - majoration property.

Clearly $f \in X_{\alpha+1}$ and is an upper bound for M.

Case b): α is a limit ordinal.

Let $M \subset X_\alpha$, card $M \leq \aleph_\alpha$. Then the elements of M may be written as a set $(f_\gamma)_{\gamma \in [-1, \omega_\alpha)}$. Define a map $f: [-1, \alpha) \longrightarrow \bigcup_{\beta \in [-1,\alpha)} X_\beta$ so that conditions i) and ii) in the definition of X_α hold. Let $f(-1)$ be an arbitrary element in X and then define $f(\beta)$ by transfinite induction by distinguishing two cases:

Case a): $f(\beta+1)$ is choosen in $X_{\beta+1}$ so that it is an upper bound for the set

$$\{ j_{\beta+1,\beta}(f(\beta))\} \cup \{ f_\gamma(\beta+1) \mid -1 \leq \gamma \leq \omega_{\beta+1} \}$$

and verifies the relation

$$p_{\beta,\beta+1}(f(\beta+1)) = f(\beta) .$$

The existence of such an element follows from the $(\beta+1)$ – majoration property of $X_{\beta+1}$ and from v) in the statement of the lemma.

Case b): let $\beta \in [-1, \alpha)$ be a limit ordinal and suppose that f was already defined on $[-1, \beta)$. But then f, as a map on $[-1,\beta)$, produces an element of X_β, which will be taken as $f(\beta)$.

The map f so constructed is an element of X_α. To see that is an upper bound for M, let $\gamma \in [-1, \omega_\alpha)$ and let $\delta \in [-1, \alpha)$. As $\alpha \longmapsto \omega_\alpha$ is a continuous map, there is $\beta \in [-1, \alpha)$ such that $\beta \geq \delta$ and $\gamma \leq \omega_\beta$. We have

$$j_{\beta+1,\delta}(f_\gamma(\delta)) \leq f_\gamma(\beta+1) \leq f(\beta+1);$$

as δ was arbitrary, $f_\gamma \leq f$.

Finnaly we prove v). Let $x \in X_\alpha$ and $f \in X_{\alpha+1}$. Define $g: [-1, \omega_{\alpha+1}) \longrightarrow X_\alpha$ by transfinite induction:

$g(-1) = x$;

$g(\beta)$ is an upper bound for the set $\{f(\beta)\} \cup g([-1,\beta))$.

The existence of such an upper bound follows from the α – majoration property of X_α.

Clearly $p_{\alpha,\alpha+1}(g) = x$ and $f \leq g$.

The main result in the section is the following theorem:

THEOREM 2.1. Let (X, Φ, Ψ) be a Φ – system with X upwards directed. Then for every $M \subset X$ with card $M \leq \aleph_\alpha$ there is $z \in C_\alpha \Phi(X)$ such that $T(z) \leq$

PROOF. Consider the sets X_α associated to X and define $\Phi_\alpha : X_\alpha \longrightarrow E$ and $\Psi_\alpha : X_\alpha \longrightarrow F$ as follows ($(X_{-1}, \Phi_{-1}, \Psi_{-1})$ is taken to be equal to (X, Φ, Ψ)):

Case a):

$$\Phi_{\alpha+1}(f) = \bigwedge_{\beta \in [-1, \omega_{\alpha+1})} \bigvee_{\gamma \in [\beta, \omega_{\alpha+1})} \Phi_\alpha(f(\gamma)),$$

$$\Psi_{\alpha+1}(f) = \bigwedge_{\beta \in [-1, \omega_{\alpha+1})} \Psi_\alpha(f(\beta)).$$

Case b): α is a limit ordinal.

$$\Phi_\alpha(f) = \bigwedge_{\beta \in [-1, \alpha)} \bigvee_{\gamma \in [\beta, \alpha)} \Phi_\gamma(f(\gamma)).$$

$$\Psi_\alpha(f) = \bigwedge_{\beta \in [-1, \alpha)} \Psi_\beta(f(\beta)).$$

It is proved by transfinite induction that $(X_\alpha; \Phi_\alpha, \Psi_\alpha)$ is a Φ - system and that $(X_\beta, \Phi_\beta, \Psi_\beta, j_{\gamma\beta})$ is an inductive system of Φ - systems when γ and β run over $[-1, \alpha)$. In case a), this is a consequence of lemma 2.1. In case b), we argue as follows: let $(\hat{X}_\alpha, \hat{\Phi}_\alpha, \hat{\Psi}_\alpha)$ be the inductive limit of the inductive system $(X_\beta, \Phi_\beta, \Psi_\beta, j_{\gamma\beta})$ $(\gamma, \beta \in [-1, \alpha))$, which exists by lemma 2.2; let also $\hat{j}_{\alpha\beta} : X_\beta \longrightarrow \hat{X}_\alpha$ be the canonical map. By lemma 2.1, $(\hat{X}^\alpha_\alpha, \hat{\Phi}^\alpha_\alpha, \hat{\Psi}^\alpha_\alpha)$ is a Φ - system. Define $J_\alpha : X_\alpha \longrightarrow \hat{X}^\alpha_\alpha$ by

$$J_\alpha(f)(\beta) = \hat{j}_{\alpha\beta}(f(\beta)).$$

It is readily seen that J_α is increasing and that $\Phi_\alpha = \hat{\Phi}^\alpha_\alpha J_\alpha$, $\Psi_\alpha = \hat{\Psi}^\alpha_\alpha J_\alpha$; consequently, $(X_\alpha, \Phi_\alpha, \Psi_\alpha)$ is also a Φ - system.

It is easily proved (by transfinite induction) that $\Phi_\alpha(X_\alpha) \subset C_\alpha \Phi(X)$.

Now let $M \subset X$, card $M \leq \aleph_\alpha$. By lemma 2.3, X_α has the α - majoration property; therefore, there is an upper bound $f \in X_\alpha$ for $j_{\alpha,-1}(M)$. Consequently,

$$T(\Phi_\alpha(f)) \leq \Psi_\alpha(f) \leq \Psi_\alpha(j_{\alpha,-1}(x)) = \Psi(x)$$

for any $x \in M$. As $\Phi_\alpha(f) \in C_\alpha \Phi(X)$, the theorem is proved.

3. The classes $\mathcal{C}_\alpha$

Let $\alpha \geq 0$ be any ordinal. A Riesz space E is said to belong to the class $\mathcal{C}_\alpha$ iff it is order complete and if $\bigcap_{x \in D_{\aleph_\alpha} M} B_x = \{0\}$ for any subset M of E with inf M = 0.

PROPOSITION 3.1. An order complete Riesz space E belongs to $\mathcal{C}_\alpha$ iff for every $M \subset E$ with $M \downarrow 0$ there are a net $(P_\delta) \subset \mathcal{P}(E)$ and a net (M_δ) of subsets of M such that $P_\delta \uparrow 1_E$, card $M_\delta \leq \aleph_\alpha$ and $P_\delta(M_\delta) \downarrow 0$ for each δ.

PROOF. Suppose that $E \in \mathcal{C}_\alpha$ and let $M \subset E$ with $M \downarrow 0$. For any $x \in D_{\aleph_\alpha} M$ let $P_x = 1_E - [x]$. As M is downwards directed, $D_{\aleph_\alpha} M$ also is; consequently, (P_x) is a net. By hypothesis, $P_x \uparrow 1_E$. For any $x \in D_{\aleph_\alpha} M$ there is $N_x \subset M$ such that card $N_x \leq \aleph_\alpha$ and $x = \inf N_x$; therefore,

$$\inf P_x(N_x) = P_x(\inf N_x) = P_x(x) = 0.$$

The first part of the proof will be complete if we show that there is a downwards directed set M_x such that $N_x \subset M_x \subset M$ and card $M_x \leq \aleph_\alpha$. To this purpose, define inductively the subsets N_x^n of M as follows. Set $N_x^0 = N_x$. Suppose N_x^n is defined and let $\mathcal{F}(N_x^n)$ be the set of all finite subsets of N_x^n. Let $f_n : \mathcal{F}(N_x^n) \longrightarrow M$ be a map with the property that $f_n(F) \leq \inf F$ for every $F \in \mathcal{F}(N_x^n)$ (the existence of f_n is ensured by the fact that M is downwards directed). Set $N_x^{n+1} = N_x^n \cup f_n(\mathcal{F}(N_x^n))$. The set $M_x = \bigcup_{n=0}^{\infty} N_x^n$ satisfies all the requirements.

Conversely, let E satisfy the requirements in the statement of the proposition. Let $M \subset E$ with $\inf M = 0$. If we set

$$N = \left\{ \bigwedge_{i=1}^{m} x_i \mid m \geq 1,\ x_i \in M \right\}$$

then $N \downarrow 0$; consequently, there are a net $(P_\delta) \subset \mathcal{P}(E)$ and a net (N_δ) of subsets of N such that $P_\delta \uparrow 1_E$, card $N_\delta \leq \aleph_\alpha$ and $P_\delta(N_\delta) \downarrow 0$. It follows from the definition of N that $\inf N_\delta \in D_{\aleph_\alpha} M$. Then if $y \in \bigcap_{x \in D_{\aleph_\alpha} M} B_x$, we must have $P_\delta(|y|) = 0$ as $P_\delta(\inf N_\delta) = 0$. But $P_\delta \uparrow 1_E$, hence $y = 0$.

PROPOSITION 3.2. An order complete Riesz space E does not belong to $\mathcal{C}_\alpha$ iff there are $\beta > \alpha$ and a decreasing net $(x_\gamma)_{\gamma \in [0, \omega_\beta)}$ in $E_+ \setminus \{0\}$ such that $x_\gamma \downarrow 0$ and $B_{x_\gamma} = B_{x_\delta}$ for every $\gamma, \delta \in [0, \omega_\beta)$.

PROOF. Suppose first that E satisfies the condition in the statement of the proposition. If M is any subset of $[0, \omega_\beta)$ with card $M \leq \aleph_\alpha$, then M is bounded from above by some $\delta \in [0, \omega_\beta)$. It follows that $\inf_{\gamma \in M} x_\gamma \geq x_\delta$; as $\bigcap_{\delta \in [0, \omega_\beta)} B_{x_\delta} \neq \{0\}$ we obtain that $E \notin \mathcal{C}_\alpha$.

Conversely, suppose that $E \notin \mathcal{C}_\alpha$ and let β be the least ordinal for which there is $M \subset E$ with card $M = \aleph_\beta$, $\inf M = 0$ and $\bigcap_{x \in D_{\aleph_\alpha} M} B_x \neq \{0\}$. Let also M be choosen according to the property of β. Clearly $\alpha < \beta$. We shall prove that

$$(1) \qquad \bigcap_{\substack{N \subset M \\ \operatorname{card} N < \aleph_\beta}} B_{\inf N} \neq \{0\}.$$

Indeed, suppose the contrary. Denote by Δ the set $\{N \mid N \subset M, \operatorname{card} N < \aleph_\beta\}$. For every $N \in \Delta$, set $P_N = 1_E - [\inf N]$. We have $\inf P_N(N) = 0$ and $\operatorname{card} P_N(N) < \aleph_\beta$; consequently, by the choice of β, we must have

$$(2) \qquad \bigcap_{\substack{N' \subset N \\ \operatorname{card} N' \leq \aleph_\alpha}} B_{\inf P_N(N')} = \{0\}, \quad N \in \Delta.$$

For every $N \in \Delta$ and every $N' \subset N$ with $\operatorname{card} N' \leq \aleph_\alpha$ set $Q_{NN'} = 1_E - [P_N(\inf N')]$. We have

$$P_N Q_{NN'}(\inf N') = P_N(\inf N') - P_N[\inf N'](\inf N') = 0.$$

Consequently, if $y \in \bigcap_{x \in D_{\aleph_\alpha} M} B_x$, then $Q_{NN'} P_N(|y|) = P_N Q_{NN'}(|y|) = 0$. By (2), $\bigvee_{\substack{N' \subset N \\ \operatorname{card} N' \leq \aleph_\alpha}} Q_{NN'} = 1_E$; hence, $P_N(|y|) = 0$. As we have supposed that (1) is false, it follows that $\bigvee_{N \in \Delta} P_N = 1_E$; hence, $y = 0$. Thus, we have obtained that $\bigcap_{x \in D_{\aleph_\alpha} M} B_x = \{0\}$, a contradiction; therefore, (1) holds.

Let $(z_\gamma)_{\gamma \in [0, \omega_\beta)}$ be any enumeration of M and define the net $(y_\gamma)_{\gamma \in [0, \omega_\beta)}$ by $y_\gamma = \bigwedge_{0 \leq \delta \leq \gamma} z_\delta$. Clearly $y_\gamma \downarrow 0$. As $y_\gamma \in D_{\operatorname{card} \gamma} M$ and $\operatorname{card} \gamma < \aleph_\beta$ for every $\gamma \in [0, \omega_\beta)$, it follows from (1) that $B = \bigcap_{\gamma \in [0, \omega_\beta)} B_{y_\gamma} \neq \{0\}$. If we let P be the order projection on B, then the net $(x_\delta)_{\delta \in [0, \omega_\beta)}$ defined by $x_\delta = P(y_\delta)$ has all the required properties.

The next proposition gives some stability properties of the classes $\mathscr{C}_\alpha$

PROPOSITION 3.3.

i) Let E be an order complete Riesz space with the property that for every $x \in E_+ \setminus \{0\}$ there is $F \in \mathscr{C}_\alpha$ and a positive order continuous linear map $T: E \longrightarrow F$ such that $T(x) \neq 0$. Then $E \in \mathscr{C}_\alpha$.

ii) Any (finite or not) product of Riesz spaces of class $\mathscr{C}_\alpha$ is a Riesz space of class $\mathscr{C}_\alpha$.

iii) If $F \in \mathscr{C}_\alpha$, then $L_r(E,F) \in \mathscr{C}_\alpha$ for any Riesz space E.

PROOF. To prove i), let $M \subset E$ with $\inf M = 0$. Replacing M by $\{\bigwedge_{i=1}^{n} x_i \mid n \geq 1, x_i \in M\}$ we may assume that M is a lower sublattice. Suppose that $y \in \bigcap_{x \in D_{\aleph_\alpha} M} B_x$ and $y \neq 0$. Then there is $F \in \mathscr{C}_\alpha$ and a positive order continuous

$T:E \longrightarrow F$ such that $T(|y|) \neq 0$. As $T(M)\downarrow 0$ there are, by proposition 3.1, a net (P_δ) in $\mathcal{P}(F)$ and a net (M_δ) of subsets of M such that $P_\delta \uparrow 1_F$ and $\inf P_\delta T(M_\delta)=0$ for each δ; we may assume that the M_δ's are also lower sublattices. Then

$$P_\delta T(\inf M_\delta) = \inf P_\delta T(M_\delta) = 0,$$

which implies that $P_\delta T(|y|) = 0$. As $P_\delta \uparrow 1_F$, we obtain $T(|y|) = 0$, a contradiction. Hence $y = 0$ and the conclusion follows.

ii) is an immediate consequence of i) (use the projections on each factor) as well as iii) (use the maps $U \longmapsto U(x)$ for $x \in E_+$).

The class $\mathcal{C}_0$ is of particular interest as the conclusion of theorem 0.1 is still true when F belongs to $\mathcal{C}_0$ (see the next section). In view of this fact, we shall indicate some subclasses of $\mathcal{C}_0$. First, recall some definitions.

A Riesz space E is called order separable if $DM = D_{\aleph_0} M$ for any $M \subset E$.

A Riesz space E is called weakly(σ,∞) – distributive (see [9]) if it is order complete and for every sequence (M_n) of upwards directed subsets of E such that $\bigcup_{n=0}^{\infty} M_n$ is order bounded, we have

$$\inf_{n\geq 0} \sup M_n = \sup \left\{ \inf_{n\geq 0} x_n \mid (x_n)_{n\geq 0} \in \prod_{n\geq 0} M_n \right\}.$$

PROPOSITION 3.4. Any order complete order separable Riesz space belongs to $\mathcal{C}_0$. Any weakly (σ,∞) – distributive Riesz space belongs to $\mathcal{C}_0$.

PROOF. The first assertion is obvious. To prove the second, let E be a weakly (σ,∞) – distributive Riesz space and let $M \subset E$ be such that $M\downarrow 0$. We may assume that M is bounded from above by $x \in E_+$. There is a stonean space X and an order isomorphism T of E_x onto $C(X)$. As $M\downarrow 0$, the set $Y = \{t \mid t\in X, \inf_{y\in M} T(y)(t) > 0\}$ is meagre; as E is weakly (σ,∞) – distributive, it follows that Y is nowhere dense (see the proof of lemma L in [8]). Let Δ be the set of all closed – open subsets of X which do not intersect the closure of Y. For every $K\in\Delta$, set $P_K = 1_E - [x] + [T^{-1}(\chi_K)]$, where χ_K denotes the characteristic function of K. Clearly $P_K \uparrow 1_E$ as $K\in\Delta$. On the other side, by Dini's theorem, the set of functions $T(M)$ converges uniformly to 0 on every $K\in\Delta$; therefore, there is, for every $K\in\Delta$, a subset $M_K \subset M$ such that card $M_K \leq \aleph_0$ and $P_K(M_K)\downarrow 0$. By proposition 3.1, it follows that $E \in \mathcal{C}_0$.

By proposition 3.3, any order complete Riesz space with a separating order continuous dual belongs to $\mathcal{C}_0$ (in fact, such a space is weakly (σ,∞) - distributive). We shall see by three examples that the class of order complete Riesz spaces with separating order continuous dual is strictly contained in $\mathcal{C}_0$.

The first example is provided by the Dedekind extension E of $C([0,1])$. As E is order separable, it belongs to $\mathcal{C}_0$; however, it is well known that $E^{\times} = \{0\}$.

The second example is provided by C(X) where X is Dixmier's stonean space from [3]. The space X has the property that every meagre subset of it is nowhere dense; hence, C(X) is weakly (σ,∞) - distributive and, consequently, it belongs to $\mathcal{C}_0$. However, it is proved in [3] that every Radon measure on X has a nowhere dense support; therefore, $C(X)^{\times} = \{0\}$.

The third example is obtained by taking an uncountable product of copies of the first example; one obtains a Riesz space of class $\mathcal{C}_0$ which is neither order separable, nor weakly (σ,∞) - distributive.

We close this section by showing that for every ordinal α there is an order complete Riesz space which does not belong to $\mathcal{C}_\alpha$.

To this purpose, let X be the set of all decreasing functions $t:[0,\omega_{\alpha+1}) \longrightarrow [0,1]$; X is a closed subspace of $[0,1]^{\aleph_{\alpha+1}}$, hence a compact space. For every $\beta\in[0,\omega_{\alpha+1})$, let $x_\beta \in C(X)$ be given by $x_\beta(t) = t(\beta)$. It is easy to see that $(x_\beta)_{\beta\in[0,\omega_{\alpha+1})}$ is a decreasing net such that $x_\beta \downarrow 0$ in C(X) and $B_{x_\beta} = C(X)$ for every $\beta\in[0,\omega_{\alpha+1})$ (as the set on which x_β vanishes is nowhere dense in X). Therefore, if we let E be the Dedekind extension of C(X), it follows that E is order complete and $(x_\beta)_{\beta\in[0,\omega_{\alpha+1})}$ is a decreasing net in E with $x_\beta\downarrow 0$ and $B_{x_\beta} = E$ for every $\beta\in[0,\omega_{\alpha+1})$. By proposition 3.2, $E\notin\mathcal{C}_\alpha$.

4. The main results

Throughout the section we shall be concerned with a Riesz A - module E and an order complete Riesz space F.

Consider the multiplication on $A\otimes Z_p(F)$ (the algebraic tensor product) defined by

$$(a\otimes\pi)(a'\otimes\pi') = aa'\otimes\pi\pi' \quad , a,a'\in A,\ \pi,\pi'\in Z_p(E)$$

and the order relation defined by the convex cone generated by $\{a\otimes\pi \mid a\in A_+,$ $\pi\in Z_p(F)_+\}$. In this way, $A\otimes Z_p(F)$ becomes an f - algebra; to verify this, note that every $c\in A\otimes Z_p(F)$ may be written as

$$\sum_{i=1}^{n} a_i\otimes P_i$$

where $a_i\in A$ and the P_i's are mutually disjoint order projections. It is then readily seen that the modulus of c is given by

$$|c| = \sum_{i=1}^{n} |a_i|\otimes P_i .$$

The unit of $A\otimes Z_p(F)$ is $e\otimes 1_F$, where e denotes the unit of A.

We define a structure of Riesz $A\otimes Z_p(F)$ - module on $L_r(E,F)$ by

$$((a\otimes\pi)U)(x) = \pi U(ax)$$

for $a\in A$, $\pi\in Z_p(F)$, $U\in L_r(E,F)$ and $x\in E$.

The proof of the main theorems will rely on the group of lemmas below. We begin with that one which, in association with theorem 2.1, provide the basic tool for the proof of the mentioned theorems.

Before stating the lemma we describe a construction. Denote by D_A the set of all systems $(a_1,\dots,a_n)$ (n is running over $\mathbb{N}$) of elements in A_+ such that $\sum_{i=1}^{n} a_i = e$. A preorder relation is defined on D_A by : $(a_1,\dots,a_m)\leq$ $\leq(a'_1,\dots,a'_n)$ if there is a partition $(M_i)_{1\leq i\leq m}$ of $\{1,\dots,n\}$ such that $a_i =$ $= \sum_{j\in M_i} a'_j$ for $1\leq i\leq m$. The Riesz decomposition property ensures that D_A is upwards directed.

Now fix $U,U'\in L_r(E,F)_+$ and $x\in E_+$. To every $\Delta = (a_1,\dots,a_n)\in D_A$ attach the system $P_\Delta = (P_1,\dots,P_n)$ of order projections on F given by

$$P_i = [(U(a_ix) - U'(a_ix))_+] .$$

Then define the element $\Delta\otimes P_\Delta$ of $A\otimes Z_p(F)$ by

$$\Delta\otimes P_\Delta = \sum_{i=1}^{n} a_i\otimes P_i .$$

Consider the maps $\Phi : D_A \longrightarrow L_r(E,F)$ and $\Psi : D_A \longrightarrow F$ given by

$$\Phi(\Delta) = (\Delta\otimes P_\Delta)(U + U') - U ,$$

$$\Psi(\Delta) = 2\sum_{i=1}^{n} U(a_ix)\wedge U'(a_ix)$$

for $\Delta = (a_1, \ldots, a_n)$. As

$$0 \leq \Delta \otimes P_\Delta \leq e \otimes 1_F$$

it follows that $\Phi(D_A)$ is contained in the order bounded sublattice $[0, e \otimes 1_F](U + U') - U$.

LEMMA 4.1. (D_A, Φ, Ψ) is a Φ – system ; the map T in the definition of a Φ – system is given here by $T(V) = |V|(x)$.

PROOF. Conditions i) and ii) in the definition of a Φ – system are obviously satisfied. The verification of condition iii) will be divided into three steps.

STEP 1).

$$((e \otimes 1_F - \Delta \otimes P_\Delta)U)(x) \leq 2^{-1}\Psi(\Delta) \ , \quad \Delta \in D_A .$$

PROOF. Let $\Delta = (a_1, \ldots, a_n)$ and $P_\Delta = (P_1, \ldots, P_n)$. We have

$$((e \otimes 1_F - \Delta \otimes P_\Delta)U)(x) = \sum_{i=1}^{n} (1_F - P_i)U(a_i x) =$$

$$= \sum_{i=1}^{n} (1_F - P_i)((U(a_i x) - U'(a_i x))_+) + \sum_{i=1}^{n} (1_F - P_i)(U(a_i x) \wedge U'(a_i x)) \leq$$

$$\leq \sum_{i=1}^{n} U(a_i x) \wedge U'(a_i x) = 2^{-1}\Psi(\Delta) .$$

STEP 2). Let $\Delta_1, \ldots, \Delta_n \in D_A$ be such that $\Delta_1 \leq \Delta_2 \leq \ldots \leq \Delta_n$. Then

$$((\bigvee_{i=1}^{n} \Delta_i \otimes P_{\Delta_i})U')(x) \leq 2^{-1}\Psi(\Delta_1) .$$

PROOF. We may change the order of elements in each Δ_i and complete, if necessary, with zeros; hence we may assume that

$$\Delta_1 = (a_{j_1})_{1 \leq j_1 \leq m} ,$$

$$\Delta_2 = (a_{j_1 j_2})_{1 \leq j_1, j_2 \leq m} ,$$

$$\ldots\ldots$$

$$\Delta_n = (a_{j_1 \ldots j_n})_{1 \leq j_1, \ldots, j_n \leq m} ,$$

the relation $\Delta_i \leq \Delta_{i+1}$ being expressed by

$$a_{j_1 \ldots j_i} = \sum_{j_{i+1}=1}^{m} a_{j_1 \ldots j_{i+1}} .$$

Set

$$P_{\Delta_i} = (P_{j_1 \ldots j_i})_{1 \leq j_1, \ldots, j_i \leq m}$$

and

$$c = \sum_{j_1,\ldots,j_n=1}^{m} a_{j_1\ldots j_n} \otimes (P_{j_1} \vee P_{j_1 j_2} \vee \ldots \vee P_{j_1\ldots j_n}).$$

We first prove the relation

$$(1) \qquad (cU')(x) = \sum_{j_1=1}^{m} P_{j_1}(U(a_{j_1}x) \wedge U'(a_{j_1}x)) +$$

$$+ \sum_{j_1,j_2=1}^{m} (P_{j_1} \vee P_{j_1 j_2} - P_{j_1})(U(a_{j_1 j_2}x) \wedge U'(a_{j_1 j_2}x)) + \ldots$$

$$+ \sum_{j_1,\ldots,j_n=1}^{m} (P_{j_1} \vee \ldots \vee P_{j_1\ldots j_n} - P_{j_1} \vee \ldots \vee P_{j_1\ldots j_{n-1}})(U(a_{j_1\ldots j_n}x) \wedge U'(a_{j_1\ldots j_n} x$$

This is done by induction on n. For $n = 0$ there is nothing to prove. Supposing (1) true for $n - 1$, let us prove it for n. We have

$$(2) \qquad (P_{j_1} \vee \ldots \vee P_{j_1\ldots j_n})(U'(a_{j_1\ldots j_n}x)) =$$

$$= (P_{j_1} \vee \ldots \vee P_{j_1\ldots j_n})((U'(a_{j_1\ldots j_n}x) - U(a_{j_1\ldots j_n}x))_+) +$$

$$+ (P_{j_1} \vee \ldots \vee P_{j_1\ldots j_n})(U(a_{j_1\ldots j_n}x) \wedge U'(a_{j_1\ldots j_n}x)) .$$

But

$$P_{j_1\ldots j_n}((U'(a_{j_1\ldots j_n}x) - U(a_{j_1\ldots j_n}x))_+) = 0 ,$$

hence the first term in the right side of (2) is equal to

$$(P_{j_1} \vee \ldots \vee P_{j_1\ldots j_{n-1}})((U'(a_{j_1\ldots j_n}x) - U(a_{j_1\ldots j_n}x))_+) =$$

$$= (P_{j_1} \vee \ldots \vee P_{j_1\ldots j_{n-1}})(U'(a_{j_1\ldots j_n}x) - U(a_{j_1\ldots j_n}x) \wedge U'(a_{j_1\ldots j_n}x)).$$

Therefore

$$(3) \qquad (cU')(x) = \sum_{j_1,\ldots,j_n=1}^{m} (P_{j_1} \vee \ldots \vee P_{j_1\ldots j_n})(U'(a_{j_1\ldots j_n}x)) =$$

$$= \sum_{j_1,\ldots,j_n=1}^{m} (P_{j_1} \vee \ldots \vee P_{j_1\ldots j_{n-1}})(U'(a_{j_1\ldots j_n}x)) +$$

$$+ \sum_{j_1,\ldots,j_n=1}^{m} (P_{j_1} \vee \ldots \vee P_{j_1\ldots j_n} - P_{j_1} \vee \ldots \vee P_{j_1\ldots j_{n-1}})(U(a_{j_1\ldots j_n}x) \wedge U'(a_{j_1\ldots j_n}$$

Taking into account the fact that

$$\sum_{j_n=1}^{m} U'(a_{j_1\ldots j_n}x) = U'(a_{j_1\ldots j_{n-1}}x)$$

and the induction hypothesis, it follows from (3) that (1) holds for n.

Next observe that the right side of (1) is less than

$$(4) \qquad \sum_{j_1=1}^{m} P_{j_1}(U(a_{j_1}x)\wedge U'(a_{j_1}x)) + \dots$$

$$+ \sum_{j_1,\dots,j_{n-1}=1}^{m} (P_{j_1}\vee\dots\vee P_{j_1\dots j_{n-1}} - P_{j_1}\vee\dots\vee P_{j_1\dots j_{n-2}})(U(a_{j_1\dots j_{n-1}}x)\wedge U'(a_{j_1\dots j_{n-1}}x)) +$$

$$+ \sum_{j_1,\dots,j_n=1}^{m} (1_F - P_{j_1}\vee\dots\vee P_{j_1\dots j_{n-1}})(U(a_{j_1\dots j_n}x)\wedge U'(a_{j_1\dots j_n}x)) .$$

The above element is in turn less than $\sum_{j_1=1}^{m} U(a_{j_1}x)\wedge U'(a_{j_1}x) = 2^{-1}\Psi(\Delta_1)$. We see this by induction on n as follows : we have

$$\sum_{j_1,\dots,j_n=1}^{m} (1_F - P_{j_1}\vee\dots\vee P_{j_1\dots j_{n-1}})(U(a_{j_1\dots j_n}x)\wedge U'(a_{j_1\dots j_n}x)) =$$

$$= \sum_{j_1,\dots,j_{n-1}=1}^{m} (1_F - P_{j_1}\vee\dots\vee P_{j_1\dots j_{n-1}}) \sum_{j_n=1}^{m} (U(a_{j_1\dots j_n}x)\wedge U'(a_{j_1\dots j_n}x)) \le$$

$$\le \sum_{j_1,\dots,j_{n-1}=1}^{m} (1_F - P_{j_1}\vee\dots\vee P_{j_1\dots j_{n-1}})(U(a_{j_1\dots j_{n-1}}x)\wedge U'(a_{j_1\dots j_{n-1}}x)) .$$

The rightmost term of the above inequality adds with the last but one term in (4) up to

$$\sum_{j_1,\dots,j_{n-1}=1}^{m} (1_F - P_{j_1}\vee\dots\vee P_{j_1\dots j_{n-2}})(U(a_{j_1\dots j_{n-1}}x)\wedge U'(a_{j_1\dots j_{n-1}}x)) .$$

From this remark and the induction hypothesis the conclusion is obtained.

Finally get the announced inequality by observing that $\bigvee_{i=1}^{n} \Delta_i \otimes P_{\Delta_i} \le c$.

STEP 3). The proof of condition iii).

Let $\Delta_1,\dots,\Delta_n \in D_A$ be such that $\Delta_1 \le \dots \le \Delta_n$. We have by steps 1) and 2)

$$T(\bigvee_{i=1}^{n}\Phi(\Delta_i)) = |(\bigvee_{i=1}^{n}\Delta_i\otimes P_{\Delta_i})(U+U') - U|(x) \le$$

$$\le ((e\otimes 1_F - \bigvee_{i=1}^{n}\Delta_i\otimes P_{\Delta_i})U)(x) + ((\bigvee_{i=1}^{n}\Delta_i\otimes P_{\Delta_i})U')(x) \le$$

$$\le ((e\otimes 1_F - \Delta_1\otimes P_{\Delta_1})U)(x) + ((\bigvee_{i=1}^{n}\Delta_i\otimes P_{\Delta_i})U')(x) \le$$

$$\le 2^{-1}\Psi(\Delta_1) + 2^{-1}\Psi(\Delta_1) = \Psi(\Delta_1) .$$

LEMMA 4.2. Let G,F be order complete Riesz spaces, M be an order bounded subset of G, $\mathcal{S}\subset\mathcal{P}(G)$ and $\mathcal{T}\subset L_r(G,F)_+$ be an upwards directed set of order continuous maps such that the following hold:

i) $(1_G - P)(x) + P(y) \in M$ whenever $P\in\mathcal{S}$ and $x,y\in M$.

ii) For every $T \in \mathcal{T}$ there are nets $(P_\delta) \subset \mathcal{S}$ and $(x_\delta) \subset M$ such that $P_\delta \uparrow 1_G$ and $T(P_\delta(|x_\delta|)) = 0$ for each δ.

iii) $\bigcap_{T \in \mathcal{T}} T^{-1}(\{0\}) = \{0\}$.

Then $0 \in SDM$.

PROOF. Consider a fixed $T \in \mathcal{T}$ and a fixed $P \in \mathcal{S}$ and assume that the set

$$N = \{x \mid x \in DM, \quad T(P(|x|)) = 0\}$$

is non void; then let $y = \inf N$. As T and P are order continuous, it follows that $T(P(|y|)) = 0$; as $y \in DDM = DM$, it is the least element of N. We shall prove that $y \leq 0$. By condition iii), it suffices to show that $T_1(y_+) = 0$ for any $T_1 \in \mathcal{T}$. So let $T_1 \in \mathcal{T}$ be given. There is $T_2 \in \mathcal{T}$ such that $T, T_1 \leq T_2$. By condition ii) there are nets $(P_\delta) \subset \mathcal{S}$ and $(x_\delta) \subset M$ such that $P_\delta \uparrow 1_G$ and $T_2(P_\delta(|x_\delta|)) = 0$. Consider the element $y_\delta = (1_G - P_\delta)(y) + P_\delta(x_\delta)$. By condition i), $y_\delta \in DM$. We also have

$$T(P(|y_\delta|)) = T(P(1_G - P_\delta)(|y|)) + T(PP_\delta(|x_\delta|)) \leq$$

$$\leq T(P(|y|)) + T_2(P_\delta(|x_\delta|)) = 0 .$$

Consequently, $y_\delta \in N$ and hence $y \leq y_\delta$. It follows that

$$P_\delta(y) \leq P_\delta(x_\delta)$$

which implies

$$P_\delta(y_+) = P_\delta(y)_+ \leq P_\delta(x_\delta)_+ = P_\delta((x_\delta)_+) \leq P_\delta(|x_\delta|) ,$$

$$T_1(P_\delta(y_+)) \leq T_2(P_\delta(y_+)) \leq T_2(P_\delta(|x_\delta|)) = 0 .$$

As $P_\delta \uparrow 1_G$, we obtain $T_1(y_+) = 0$.

The above reasoning together with condition ii) show that the set $DM \cap (-G_+)$ is non void ; let z be the supremum of this set. Clearly $z \in SDM$; the proof will be concluded if we show that $z = 0$. This will be done by proving that $T(z) = 0$ for any $T \in \mathcal{T}$. So let $T \in \mathcal{T}$ be given. By condition ii) and the first part of the proof, we can find the nets $(P_\delta) \subset \mathcal{S}$ and $(x_\delta) \subset DM \cap (-G_+)$ such that $P_\delta \uparrow 1_G$ and $T(P_\delta(x_\delta)) = 0$. As $x_\delta \leq z \leq 0$ it follows that $T(P_\delta(z)) = 0$. As $P_\delta \uparrow 1_G$ we have $T(z) = 0$, and the proof is complete.

In [7] we have defined a principal module as a Riesz A - module E endowed with a locally solid topology such that for any $x \in E$, Ax is dense in the principal order ideal E_x . We shall now need the concept of a principal module in a non topological situation. For our present purposes, the following definition will be convenient: the Riesz A – module E is called principal if whenever $x \in E$ and $y \in E_x$ there is a sequence $(a_n) \subset A$ such that $a_n x \xrightarrow{\rho} y$. Every principal module in the sense of [7] which is metrizable and complete is principal for the above definition (for instance, any Banach lattice with a quasi interior element is a principal module over its center); also, every Riesz space E with the principal projection property is a principal module over $Z_p(E)$ (this follows from Freudenthal's spectral theorem ; see [5]).

LEMMA 4.3. Let E be a principal A – module, F be an order complete Riesz space and $U, U' \in L_r(E,F)$ be such that $U \wedge U' = 0$. Then

$$\left\{ \sum_{i=1}^{n} U(a_i x) \wedge U'(a_i x) \;\middle|\; (a_1, \ldots, a_n) \in D_A \right\} \downarrow 0$$

for any $x \in E_+$.

PROOF. It is based on the fact (which is established in the same way as lemma 2.1 in [7]) that whenever $x_1, \ldots, x_n \in E_+$ verify $\sum_{i=1}^{n} x_i = x$, there is a sequence $(a_{1m}, \ldots, a_{nm}) \subset D_A$ such that $a_{im} x \xrightarrow{\rho} x_i$ as $m \to \infty$ for $1 \le i \le n$.

We are now in position to prove the first main result in the paper.

THEOREM 4.1. Let E be a principal A – module and let F be a Riesz space of class $\mathcal{C}_\alpha$. Then

(1) $$[0,V] = SDC_\alpha [0, e \otimes 1_F] V \quad ,$$

(2) $$[-V,V] = SDC_\alpha [-e \otimes 1_F \;,\; e \otimes 1_F] V$$

for any $V \in L_r(E,F)_+$.

PROOF. Denote by $\mathcal{M}$ the right side of (1). We shall first prove that each component of V belongs to $\mathcal{M}$. To this purpose, let $U \in C_V$ and let $U' = V - U$. Fix for the moment an $x \in E_+$ and construct the maps Φ and Ψ as described before the statement of lemma 4.1. Then lemma 4.3 combined with proposition 3.1 produce a net $(P_\delta) \subset \mathcal{P}(F)$ and a net $(\Delta_\delta) \subset D_A$ such that $P_\delta \uparrow 1_F$, card $\Delta_\delta \le \aleph_\alpha$ and $\inf P_\delta \Psi(\Delta_\delta) = 0$ for each index δ. Now by lemma 4.1, (D_A, Φ, Ψ) is a Φ – system with respect to the map $T: L_r(E,F) \to F$ given by $T(S) = |S|(x)$; hence $(D_A, \bar{P}_\delta \Phi, P_\delta \Psi)$ is a Φ – system with respect to the map $T\bar{P}_\delta$; here $\bar{P}$ denotes the order projection on $L_r(E,F)$ given by $S \mapsto PS$. Consequently, theorem 2.1 yields for each index δ, an $S_\delta \in C_\alpha([0,e\otimes 1_F]V - U)$ such that $P_\delta|S_\delta|(x) = T\tilde{P}_\delta(S_\delta) = 0$.

Now consider the set $\mathcal{S} = \{\tilde{P} \mid P \in \mathcal{P}(F)\}$ and the set $\mathcal{T}$ of all maps from $L_r(E,F)$ to F of the form $S \mapsto S(x)$ with $x \in F_+$; clearly $\mathcal{T}$ is upwards directed and consists of order continuous positive linear maps. The above reasoning shows that $C_\alpha([0,e\otimes 1_F]V - U)$, $\mathcal{S}$ and $\mathcal{T}$ verify condition ii) in the statement of lemma 4.2; as the other two conditions in that lemma are easily seen to hold, it follows that $0 \in SDC_\alpha([0,e\otimes 1_F]V - U) = \mathcal{M} - U$, that is, $U \in \mathcal{M}$.

For an arbitrary $U \in [0, V]$, let $\mathcal{N}$ be the set of operators in $[0,U]$ of the form $\bigvee_{i=1}^{n} c_i U_i$ with $U_i \in C_V$ and $c_i \in \mathbb{R}$, $0 \le c_i \le 1$. As $\mathcal{M}$ is a sublattice closed for multiplication by scalars in $[0,1]$, it follows by the first part of the proof that $\mathcal{N} \subset \mathcal{M}$. By Freudenthal's spectral theorem, $\mathcal{N} \uparrow U$; hence $U \in S\mathcal{M} = \mathcal{M}$ and the proof of (1) is complete.

Finally, (2) follows from (1): indeed,

$$[-V,V] = [0,2V] - V = SDC_\alpha[0,e\otimes 1_F]2V - V =$$

$$= SDC_\alpha([0,e\otimes 1_F]2V - V) = SDC_\alpha[-e\otimes 1_F, e\otimes 1_F]V \quad .$$

We pass now to a variant of theorem 4.1 in which $[0,V]$ is replaced by C_V. To this purpose, let us introduce the notion of a $\mathcal{P}$ - simple component.

Consider a principal A - module E and an order complete Riesz space F. Suppose that $A = \sum(A)$ and let $\mathcal{P}$ be an ideal in the Boolean algebra C_e (e being the unit of A) with the property that for every $x \in E$ there is $p \in \mathcal{P}$ such that $px = x$. Any component of $V \in L_r(E,F)_+$ of the form

$$\bigvee_{i=1}^{n} (p_i \otimes P_i)V$$

where $p_i \in \mathcal{P}$ and $P_i \in \mathcal{P}(F)$ will be called $\mathcal{P}$ - simple; the set of all such components is a sublattice of $L_r(E,F)$ and will be denoted by $\mathcal{P}_V$.

Let us consider two examples. The first is provided by the case when E is a Riesz space with the principal projection property. In this case, take $A = Z_p(E)$ and take $\mathcal{P}$ to be the set of all principal order projections.

For the second example, let $E = C(X)$ with X a totally disconnected compact space, let $\mathcal{P}$ be the set of characteristic functions of closed - open subsets of X and let A be the subalgebra of E generated by $\mathcal{P}$. The fact that X is totally disconnected ensures that E is a principal A - module. Observe that in case when

X is not σ - stonean, then E has not the principal projection property and hence, it does not satisfy the hypothesis of Aliprantis' and Burkinshaw's improvement of de Pagter's result. However, it does satisfy the hypothesis of our next theorem.

LEMMA 4.4. Let E,F and $\mathcal{P}$ be as above and let $x \in E_+$. Fix a $p \in \mathcal{P}$ for which $px = x$ and let $\mathcal{P}_x$ be the set of all systems $(p_1, \ldots, p_n)$ of mutually disjoint elements in $\mathcal{P}$ such that $\sum_{i=1}^{n} p_i = p$ (n is running over $\mathbb{N}$). Then for every $U, U' \in L_r(E,F)$ such that $U \wedge U' = 0$ we have

$$\{\sum_{i=1}^{n} U(p_i x) \wedge U'(p_i x) \mid (p_1, \ldots, p_n) \in \mathcal{P}_x\} \downarrow 0 .$$

PROOF. By lemma 4.3 we have

$$\{\sum_{i=1}^{m} U(a_i x) \wedge U'(a_i x) \mid (a_1, \ldots, a_m) \in D_A\} \downarrow 0 .$$

It will then suffice to show that for every $(a_1, \ldots, a_m) \in D_A$ there is $(p_1, \ldots, p_n) \in \mathcal{P}_x$ such that $\sum_{j=1}^{n} U(p_j x) \wedge U'(p_j x) \leq \sum_{i=1}^{m} U(a_i x) \wedge U'(a_i x)$.

Indeed, if $(a_1, \ldots, a_m) \in D_A$ there are mutually disjoint $e_1, \ldots, e_n \in C_e \setminus \{0\}$ such that $\sum_{i=1}^{n} e_i = e$ and each a_i has the form

$$a_i = \sum_{j=1}^{n} c_{ij} e_j$$

for some $c_{ij} \in \mathbb{R}_+$ (recall that $A = \Sigma(A)$). As $\sum_{i=1}^{m} a_i = e$ we have $\sum_{i=1}^{m} c_{ij} = 1$ for $1 \leq j \leq n$.

Consider the system $(a_i e_j)_{1 \leq i \leq m,\ 1 \leq j \leq n}$. It is an element of D_A greater than $(a_1, \ldots, a_m)$. Consequently,

$$\sum_{j=1}^{n} U(e_j x) \wedge U'(e_j x) = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} U(e_j x) \wedge U'(e_j x) =$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} U(a_i e_j x) \wedge U'(a_i e_j x) \leq \sum_{i=1}^{m} U(a_i x) \wedge U'(a_i x) .$$

As $\mathcal{P}$ is an ideal in C_e we have $e_j p \in \mathcal{P}$ for $1 \leq j \leq n$. Therefore, $(e_1 p, \ldots, e_n p) \in \mathcal{P}_x$ and

$$\sum_{j=1}^{n} U(e_j px) \wedge U'(e_j px) \leq \sum_{i=1}^{m} U(a_i x) \wedge U'(a_i x) .$$

THEOREM 4.2. Let E,F and $\mathcal{P}$ be as above and suppose that $F \in \mathcal{C}_\alpha$. Then

$$C_V = SDC_\alpha \mathcal{P}_V$$

for every $V \in L_r(E,F)_+$.

PROOF. Let $U \in C_V$. Consider an $x \in E_+$ and define a Φ - system in the same way as in the discussion preceding lemma 4.1 with the only exception that D_A is replaced by the set $\mathcal{P}_x$ of lemma 4.4 ($\mathcal{P}_x$ is ordered in the same way as D_A and is upwards directed). Observe that the proof of lemma 4.1 still works in order to show that we have indeed obtained a Φ - system. Then the same argument as in the proof of theorem 4.1 (using lemma 4.4 instead of lemma 4.3) yields a net (P_δ) in $\mathcal{P}(F)$ and a net (S_δ) in $C_\alpha(\mathcal{P}_V - U)$ such that $P_\delta \uparrow 1_F$ and $P_\delta |S_\delta|(x) = 0$. As x was arbitrary, the proof is completed by using lemma 4.2.

It is worthwhile to note that for Riesz spaces of class $\mathcal{C}_0$, the conclusion of theorem 4.2 can be restated to become identical to the conclusion of theorem 0.1 :

COROLLARY 4.1. Let E,F and $\mathcal{P}$ be as above and suppose that $F \in \mathcal{C}_0$. Then

$$C_V = SDS_{\aleph_0} \mathcal{P}_V$$

for any $V \in L_r(E,F)_+$.

PROOF. By theorem 4.2,

$$C_V = SDC_0 \mathcal{P}_V = SDD_{\aleph_0} S_{\aleph_0} \mathcal{P}_V = SDS_{\aleph_0} \mathcal{P}_V .$$

It should be mentioned that in the general case too, the statement of theorems 4.1 and 4.2 could be improved by remarking that, as in the proof of corollary 4.1, the final $D_{\aleph_\alpha}$ is "absorbed" by D; we leave to the reader the formulation of the precise statement.

We give now an up - down theorem in the center of $L_r(E,F)$ which extends to arbitrary Riesz spaces the corresponding result of Buskes, Dodds, de Pagter and Schep [2] proved only for Riesz spaces with separating order continuous dual.

Before giving the theorem let us remark that, whenever G is a Riesz A - module, its center Z(G) can be turned into a Riesz A - module by defining

$$(a\pi)(x) = \pi(ax) = a\pi(x)$$

for $a \in A$, $\pi \in Z(G)$ and $x \in G$. In particular, whenever E is a Riesz A - module and F is an order complete Riesz space, $Z(L_r(E,F))$ will be considered as a Riesz $A \otimes Z_p(F)$ - module. We shall denote by e the unit of A.

THEOREM 4.3. Let E be a principal A - module and let F belong to $\mathcal{C}_\alpha$.

Then

(1) $$[0,\pi] = SDC_{\alpha}[0,e\otimes 1_F]\pi \ ,$$

(2) $$[-\pi,\pi] = SDC_{\alpha}[-e\otimes 1_F, e\otimes 1_F]\pi$$

for every $\pi \in Z(L_r(E,F))_+$.

PROOF. We shall prove only (1), as (2) is deduced from (1) as in the proof of theorem 4.1.

For every $P \in \mathcal{P}(F)$ let $\bar{P} \in \mathcal{P}(Z(L_r(E,F)))$ be defined by

$$\bar{P}(\pi) = (e\otimes P)\pi \ .$$

Denote by $\mathcal{S}$ the set $\{\bar{P} \mid P \in \mathcal{P}(F)\}$.

Consider also the set $\mathcal{T}$ of all maps from $Z(L_r(E,F))$ into F of the form $\pi \longmapsto \pi(U)(x)$ with $U \in L_r(E,F)_+$ and $x \in E_+$; clearly $\mathcal{T}$ is upwards directed and consists of positive order continuous linear maps.

Now let $\sigma \in [0,\pi]$. Fix for the moment $U \in L_r(E,F)_+$ and $x \in E_+$. As $\sigma(U) \in [0,\pi(U)]$, there are, according to the proof of theorem 4.1, a net (P_δ) in $\mathcal{P}(F)$ and a net (S_δ) in $C_{\alpha}[0,e\otimes 1_F]\pi(U)$ such that $P_\delta \uparrow 1_F$ and $P_\delta|S_\delta - \sigma(U)|(x) = 0$ for each δ . The map $\tau \longmapsto \tau(U)$ is an order continuous Riesz homomorphism from $Z(L_r(E,F))$ into $L_r(E,F)$, hence it takes $C_{\alpha}[0,e\otimes 1_F]\pi$ onto $C_{\alpha}[0,e\otimes 1_F]\pi(U)$; consequently, $S_\delta = \tau_\delta(U)$ for some $\tau_\delta \in C_{\alpha}[0,e\otimes 1_F]\pi$. We have

$$\bar{P}_\delta(|\tau_\delta - \sigma|)(U)(x) = ((e\otimes P_\delta)|\tau_\delta - \sigma|)(U)(x) =$$

$$= (P_\delta|\tau_\delta - \sigma|(U))(x) = P_\delta|\tau_\delta(U) - \sigma(U)|(x) = P_\delta|S_\delta - \sigma(U)|(x) = 0.$$

Clearly $\bar{P}_\delta \uparrow 1_{Z(L_r(E,F))}$.

As U and x were arbitrary, the preceding reasoning shows that the sets $\mathcal{S}$, $\mathcal{T}$ and the order bounded sublattice $C_{\alpha}[0,e\otimes 1_F]\pi - \sigma$ satisfy to condition ii) in the statement of lemma 4.2; as conditions i) and iii) are obviously satisfied, it follows that $0 \in SD(C_{\alpha}[0,e\otimes 1_F]\pi - \sigma)$, which implies the conclusion.

The corresponding variant for principal components is proved in an analogous way (with the same modifications as in the proof of theorem 4.2) :

THEOREM 4.4. Let E,F and $\mathcal{P}$ be as in the statement of theorem 4.2. Define the set $\mathcal{P}_\pi$ of $\mathcal{P}$ - simple components of $\pi \in Z(L_r(E,F))_+$ by

$$\mathcal{P}_\pi = \left\{ \bigvee_{i=1}^{n} (p_i \otimes P_i)\pi \;\middle|\; n \geq 1,\ p_i \in \mathcal{P},\ P_i \in \mathcal{P}(F) \right\} .$$

Then

$$C_\pi = SDC_\alpha \mathcal{P}_\pi$$

for every $\pi \in Z(L_r(E,F))_+$.

The proof is based on the remark that $\sigma(U) \in C_{\pi(U)}$ whenever $\sigma \in C_\pi$ and $U \in L_r(E,F)_+$.

Theorem 4.3 will be used in order to obtain a variant of theorem 4.1 for non positive operators. We shall need a lemma whose proof is straightforward:

LEMMA 4.5. Let E,F be Riesz spaces and let $T:E \longrightarrow F$ be order continuous. Then

$$T(LM) \subset LT(M) , \quad T(L_\alpha^2 M) \subset L_\alpha^2 T(M)$$

for every $M \subset E$ and every ordinal α.

THEOREM 4.5. Let E be a principal A - module and let F belong to $\mathcal{C}_\alpha$. Then

$$[-|V|, |V|] = LLL_\alpha^2 [-e \otimes 1_F , e \otimes 1_F] V$$

for every $V \in L_r(E,F)$.

PROOF. Let $U \in [-|V|, |V|]$. There is $\sigma \in [-1_{L_r(E,F)} , 1_{L_r(E,F)}]$ such that $U = \sigma(V)$. By theorem 4.3 we have

$$\sigma \in SDC_\alpha [-e \otimes 1_F , e \otimes 1_F] 1_{L_r(E,F)} \subset$$
$$\subset LLL_\alpha^2 [-e \otimes 1_F , e \otimes 1_F] 1_{L_r(E,F)} .$$

Applying lemma 4.5 to the map $\tau \mapsto \tau(V)$ we obtain

$$U = \sigma(V) \in LLL_\alpha^2 [-e \otimes 1_F , e \otimes 1_F] V .$$

For spaces of class $\mathcal{C}_0$, theorem 4.5 takes the following form :

COROLLARY 4.2. Let E be a principal A - module and let F belong to $\mathcal{C}_0$. Then

$$[-|V|, |V|] = LLL_{\aleph_0} [-e \otimes 1_F , e \otimes 1_F] V$$

for every $V \in L_r(E,F)$.

The last corollary is a consequence of theorems 4.1, 4.2 and 4.5, observing that every order complete Riesz space belongs to some class $\mathcal{C}_\alpha$. Recall that the order topology on a Riesz space is the topology whose closed sets are the sets M with M = LM.

COROLLARY 4.3. Let E be a principal A - module and let F be an order complete Riesz space. Consider the order topology on $L_r(E,F)$ and let the bar denote the closure of a set with respect to this topology. Then we have

$$[0,V] = \overline{[0,e\otimes 1_F]V}$$

for every $V \in L_r(E,F)_+$ and

$$[-|V|,|V|] = \overline{[-e\otimes 1_F\ ,\ e\otimes 1_F]V}$$

for every $V \in L_r(E,F)$.

If moreover E and $\mathcal{P}$ are as in the statement of theorem 4.2, then

$$C_V = \overline{\mathcal{P}_V}$$

for every $V \in L_r(E,F)_+$.

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