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Stefan PAPADIMA

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Stefan PAPADIMA*)

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*) *Department of Mathematics, National Institute for Scientific and Technical Creation, Bdul Pacii 220, 79622 Bucharest, Romania.*

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1. INTRODUCTION AND STATEMENT OF RESULTS

Consider $K \subset G$, a proper pair of equal rank compact connected Lie groups. It is known (see [6], proof of Theorem 1.1) that the group of self-homotopy equivalences of the rationalization of G/K is (anti)isomorphic, in a natural way, with the group of graded algebra automorphisms of $H^*(G/K; \mathbb{Q})$. On one side of the matter there is the fact that Aut $H^*(G/K; \mathbb{Q})$ almost gives the integral picture of the self-homotopy equivalences of G/K, that is up to a finite ambiguity and up to grading automorphisms of $H^*(G/K; \mathbb{Q})$ (i.e. those acting on H^{2i} as λ^i.id, for some nonzero $\lambda \in \mathbb{Q}$)-see also [6]. On the other side there is the classical description by Borel [2] of $H^*(G/K; \mathbb{Q})$ in terms of invariants of Weyl groups, which gives hope for a satisfactory understanding of Aut $H^*(G/K; \mathbb{Q})$.

Denoting by $\mathbb{F}$ a field of characteristic zero, by T a common maximal torus, by V its Lie algebra and by $\Gamma \subset V$ the exponential lattice, one has a graded algebra isomorphism $H^*(G/K; \mathbb{F}) \sim I_K/I_K \cdot I_G^+$, where I_G stands for the invariants of the natural action of the Weyl group W_G on the polynomials on $\Gamma \otimes \mathbb{F}$, and similarly for I_K. Denote by N_G the normalizer of W_G in $GL(\Gamma \otimes \mathbb{F})$, and similarly for N_K. Set $N(\mathbb{F}) = N_G \cap N_K$ and notice that this group naturally acts on $H^*(G/K; \mathbb{F})$. Finally denote by Lieaut $H^*(G/K; \mathbb{F})$ the subgroup of Aut $H^*(G/K; \mathbb{F})$ coming from $N(\mathbb{F})$. This paper investigates the structure of Lieaut, some properties of Lie-type cohomology automorphisms and some geometric consequences. Besides the fact that restricting one's attention to Lieaut is a most natural choice (which covers the examples coming from geometric symmetry), it often happens that all cohomology automorphisms are of Lie type, e.g. when $K = T$ [13]; the case of complex flag manifolds was intensively studied and there seems to be enough evidence to believe that the same result holds in that case too, as conjectured (in a different formulation - see Example 6.9) in [7], [12]. Though the existence of examples as simple as $SO(7)/U(3)$ (see Example 6.9) prevents one from being too optimistic about the above coincidences, it seems necessary to have a good understanding of automorphisms of Lie type in order to have a good general guess. It is the aim of this paper to prove that the Lie type part of the cohomological symmetry may be very explicitely described, producing useful geometric information.

Obviously the elements of W_K induce the identity of $H^*(G/K)$. Our first main result makes precise this observation in the case G/K is in normal form (which represents no loss of generality, as far as the topology of G/K is concerned, see § 2 for the precise definition; it tries to avoid embarrassing redundancies such as $G \times G_1/K \times G_1$.)

THEOREM 1.1. Suppose G/K is in normal form. For $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$, one has

$$\text{Lieaut}\, H^*(G/K;\mathbb{F}) \sim N(\mathbb{F})/W_K$$

The structure of N is discussed in detail in Section 6.

The elements of N_G act on the reflecting hyperplanes of W_G and consequently on Weyl chambers, and similarly for N_K. Our next main result relates the action of N on the Weyl groups and Weyl chambers to the computation of Lefschetz numbers of the induced cohomology automorphisms. If $n \in N$, $[n]$ will denote its class mod W_K, and in the statement below each such class will be <u>normalized</u>, i.e. supposed to leave some fixed W_K-chamber invariant.

THEOREM 1.2. Fix a W_K-chamber and consider a finite order normalized Lie-type cohomology automorphism $[n] \in N(\mathbb{R})/W_K$. Then: the Lefschetz number $L([n])$ is nonzero if and only if n leaves some W_G-chamber invariant, and in this case $L([n]) = [C_{W_G}(n) : C_{W_K}(n)]$.

In the last statement C stands for centralizers. This computation should be compared with the one in [11]. It follows from [13] that, for G simple, N_G modulo grading automorphisms is finite, so the finite order elements of Lieaut do indeed represent the interesting part of Lieaut.

A second kind of results is related to the existence of isometry-invariant geodesics on G/K. Given $f \in \mathrm{Isom}(M)$, where the manifold M carries an arbitrary metric, it was recognized [8] that the existence of f-invariant geodesics is intimately related to the existence of fixed points of f in $\pi_*(M)$ and then that this last problem can be successfully dealt with, using rational homotopy theory methods [10, 9]. Here our main result is the following computation of rational homotopy groups (to be compared with [5], Corollary 1.3 and [11], Theorem 1):

THEOREM 1.3. Suppose G is simple and denote by h the Coxeter number of G. Then

$$\pi_i(G/K)\otimes\mathbb{Q} = 0 \ \text{ for } \ i > 2h-1 \ \text{ and } \dim \pi_{2h-1}(G/K)\otimes\mathbb{Q} = 1$$

Recall that a geodesic curve c is called <u>f-invariant</u> if it is nonconstant and there exists a period t such that $f(c(x)) = c(x+t)$, any x. Putting together the information on the action of f on the rational homotopy of G/K deduced from the knowledge of $H^*(f;\mathbb{Q})$ with the preceding result, we will prove:

COROLLARY 1.4. Given $f \in \mathrm{Isom}(G/K)$, there exists an f-invariant geodesic whenever

either (i) G is simple or (ii) G/K is in normal form and H^*f

is of Lie type.

This complements the main result of [9], which asserts that every isometry of a 1-connected closed manifold M has an invariant geodesic, provided $\dim \pi_*(M)\otimes\mathbb{Q} = \infty$ or M is odd-dimensional.

The paper is organized as follows. Sect. 2 contains preparatory material: we set up the root system framework and the notations and we recall some useful facts on harmonic polynomials [16] and normalizers of Weyl groups [13]. In Sect. 3 we derive the Lefschetz number formula (Theorem 1.2) using the relationship between $H^*(G/K; \mathbb{R})$ and harmonic polynomials. This formula is used for the proof of Theorem 1.1 (Sect. 4). In Sect. 5 rational homotopy theory methods are used for the proofs of Theorem 1.3 and Corollary 1.4 (actually we prove a slightly stronger statement than Corollary 1.4, see Proposition 5.1). Finally, we analyse in Sect. 6 the structure of N, relying on the detailed knowledge of N_G [13] and on root systems theory [4], and give many examples.

2. HARMONIC POLYNOMIALS

Consider $\Phi_K^* \subset \Phi_G^* \subset V^*$, the roots of the adjoint action of the common maximal torus [1], and use an Ad-invariant metric in order to identify V with V^*. This gives our basic framework: a pair of root systems, denoted by $\Phi = (\Phi_K \subset \Phi_G \subset V)$. A single root system Φ_G will be considered as a pair, by setting $\Phi_K = \emptyset$. This is (almost) the situation considered in [13]; here we will no more insist on Φ_G being <u>normalized</u>, i.e. satisfying $V = \mathbb{R} - \operatorname{span}\{\Phi_G\}$, that is our root systems are considered in the sense of [16], Appendix 4.15. We will say that G/K is <u>in normal form</u> if $G = \times_{i \in A} G_i$, with each G_i simple and 1-connected, and $K = \times_{i \in A} K_i$, each K_i being a closed connected proper subgroup of maximal rank of G_i. Any equal rank homogenous space may be put in normal form, and this choice normalizes Φ_G; however Φ_K will not be normalized, in general. Given a root system $(\Phi \subset V)$, we may plainly consider the induced orthogonal decomposition of V, $V = V^W \oplus V_W$, with V^W = fixed points of the Weyl group W and $V_W = \mathbb{R} - \operatorname{span}\{\Phi\}$, and thus get an associated normalized system $(\Phi \subset V_W)$. Starting with a pair $\Phi = (\Phi_K \subset \Phi_G)$ we will write $\Phi_G = \coprod_{i \in A} \Phi_G^i$ for the decomposition into irreducible components, we will consider the natural orthogonal decomposition $V = V^{W_G} \oplus (\oplus_{i \in A} V^i)$, denote $\Phi_K \cap \Phi_G^i$ by Φ_K^i and write $\Phi = \coprod_{i \in A} \Phi^i$ for <u>the irreducible decomposition of the pair</u> Φ, where $\Phi^i = (\Phi_K^i \subset \Phi_G^i \subset V^i)$ is a pair of root systems with $(\Phi^i_G \subset V^i)$ normalized and irreducible, for any i. According to the geometric picture, we will say that Φ is in normal form if $(\Phi_G \subset V)$ is normalized and Φ^i_K is a proper subsystem of Φ^i_G, for any i.

Our notations will follow in general those of [13]. The guiding rule is that by setting $\Phi_K = \emptyset$ one recovers the constructions and notations used in [13] for the case of a single root system. The characteristic zero field coefficients for the cohomology will be in general $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q}$; when there is no special mention of $\mathbb{F}$, it should be understood that $\mathbb{F} = \mathbb{R}$.

Consider a root system $(\Phi \subset V)$. The first statement of the proposition below says that the normalizer of the Weyl group is essentially determined by the knowledge of the associated normalized situation and the second is a useful device for reducing various considerations to the case when Φ_G is irreducible.

PROPOSITION 2.1. Let $(\Phi \subset V)$ be a root system, not necessarily normalized. Set $N = N_{GL(V)}(W)$.

(i) $N = GL(V^W) \times N_{GL(V_W)}(W)$.

(ii) Suppose that Φ is normalized and write $\Phi = \coprod_{i \in A} \Phi^i$, $V = \bigoplus_{i \in A} V^i$ for the decomposition into irreducible components. There is a group morphism $\pi_0 : N \to \Sigma(A)$ with the property that, for any $n \in N$ and $i \in A$, setting $\pi_0(n)i = j$, n induces by restriction an isomorphism $n \;: V^i \xrightarrow{\sim} V^j$ which maps the Weyl group W^i isomorphically onto W^j.

Proof. (i) Start with $n \in GL(V)$ and write it in matrix form, $n = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, according to the decomposition $V = V^W \oplus V_W$. Then $n \in N$ if and only if for any $w \in W$ there exists $w' \in W$ such that $nw = w'n$, which simply means that $dw = w'd$, ker c contains image(w-id) and image b is contained in ker(w-id), for any $w \in W$. Our assertion follows.

(ii) By [4], p.146, $V = \bigoplus_{i \in A} V^i$ represents a decomposition of V into simple $\mathbb{R}[W]$-submodules. One has next to notice that the decomposition $W = \times_{i \in A} W^i$ implies that these are pairwise nonisomorphic $\mathbb{R}[W]$-modules. For general reasons the V^i's, $i \in A$, must then exhaust the collection of simple $\mathbb{R}[W]$-submodules of V, on which N naturally acts. Use this action to define π_0 and check the remaining assertions. Finally note that this construction is consistent with that given by (1) in the proof of Proposition 2.8, [13].

Given a pair of root systems, $\Phi = (\Phi_K \subset \Phi_G \subset V)$, we shall denote by $H^*(\Phi;\mathbb{F})$ the graded algebra $I_K/\text{ideal}(I_G^+)$, where the invariants of the Weyl groups are considered relative to the natural right action of $GL(\Gamma \otimes \mathbb{F})$ on the polynomials on $\Gamma \otimes \mathbb{F}$ given by $p \in \mathbb{F}[\Gamma \otimes \mathbb{F}] =: P_{\mathbb{F}}$, $a \in GL(\Gamma \otimes \mathbb{F}) \to p \cdot a =: p^a$. As a matter of convention, the grading of $H^*(\Phi)$ will be induced by the natural grading of P obtained by assigning the degree one to the linear polynomials, whenever dealing with pairs of root systems. As soon as these come from pairs of equal rank Lie groups $K \subset G$, we shall identify $H^*(G/K;\mathbb{F})$ and $H^*(\Phi;\mathbb{F})$ by simply doubling the degrees.

The proofs of Theorems 1.1 and 1.2 are based on approaching $H^*(G/K)$ via <u>harmonic polynomials</u>. We are going now to recall from [16], Appendix 4.15, especially pp. 415-416, some useful (elementary) properties of harmonic polynomials.

Let $(\Phi \subset V)$ be a root system. The canonical identification of V^* with V given by the metric extends to an isomorphism between the polyomial and the symmetric algebras on V, denoted by $p \to \tilde{p}$. Recalling that the symmetric algebra naturally acts on the polynomial algebra, one obtains an euclidean structure on P by setting $(p, q) = \partial_{\tilde{q}} p(0)$, for $p, q \in P$. It is easy to see that if $a \in GL(V)$ is an isometry then the induced algebra automorphism of P, denoted by a^*, is also an isometry. Denoting by $\mathbb{H}^*(\Phi)$ the graded vector space of <u>harmonic polynomials</u>, defined by $\mathbb{H}^*(\Phi) = (P \cdot I^+)^{\perp}$, which is clearly invariant with respect to the action of the Weyl group, one has a natural graded vector space isomorphism $\mathbb{H}^*(\Phi) \xrightarrow{\sim} H^*(\Phi)$ which is compatible with the obvious W-actions.

Given any $x \in V$, one obtains a linear map $h_x : \mathbb{H}^*(\Phi) \to \mathbb{R}[W]$, defined by:

$$h_x(p) = \sum_{w \in W} p^w(x) \cdot w , \quad p \in \mathbb{H}^*(\Phi) \qquad (2.2)$$

One says that x is regular, written $x \in V'$, if x does not belong to any of the reflecting hyperplanes defined by Φ. Finally, suppose that we are given an orthogonal W-invariant decomposition $V = \oplus V_j$, with $V_o = V^W$. Setting $\Phi_j = \Phi \cap V_j$, one obtains a root system decomposition $\Phi = \amalg \Phi_j$, with $\Phi_j \subset V_j$ normalized excepting $\Phi_o = \emptyset$.

PROPOSITION 2.3. Let $(\Phi \subset V)$ be a root system.

(i) If $x \in V'$ then the linear map defined by (2.2), $h_x : \mathbb{H}^*(\Phi) \to \mathbb{R}[W]$ is a linear isomorphism, compatible with the natural right action of W on $\mathbb{H}^*(\Phi)$ and the right action of W or $\mathbb{R}[W]$ induced by the left regular representation.

(ii) $\mathbb{H}^*(\Phi)$ is invariant with respect to the action of N on P and the graded I-module map $I \otimes \mathbb{H}^*(\Phi) \to P$ defined by $q \otimes p \to q \cdot p$ gives an isomorphism which is compatible with the obvious N-actions.

(iii) An orthogonal W-invariant decomposition $V = \oplus V_j$ (with $V_o = V^W$) gives rise to a root system decomposition $\Phi = \amalg \Phi_j$ (with $\Phi_o = \emptyset$) and to a graded algebra isomorphism $P \sim \otimes P_j$ which is compatible with constructions I, H^* and $\mathbb{H}^*$.

Proof. (i) See [16], p.416.

(iii) Compatibility with I and H^* is clear. Use the alternative definition of $\mathbb{H}^*$ given in [16], p. 415, and the tensor product splitting of the W-invariants, in order to deduce that $\otimes \mathbb{H}^*(\Phi_j) \subset \mathbb{H}^*(\Phi)$. Finally use a dimension argument together with (i).

(ii) Theorem 4.15.28 [16] takes care of almost everything, excepting the statement on the N-invariance of $\mathbb{H}^*(\Phi)$. This in turn is clear, as far as the part of N consisting of isometries is concerned. But we may always reduce to this situation, by suitably multiplying a given element of N by an element of $GL(V^W)$, in order to first normalize the root system, see Proposition 2.1 (i), and then multiplying by an element of $GL(V_W)$ which is diagonal with respect to the decomposition of V_W given by the irreducible components, see Lemma 2.2 [13]. Use (iii) and write $\mathbb{H}^*(\Phi) = \mathbb{H}^*(\Phi_{normalized})$ in order to get the $GL(V^W)$-invariance of $\mathbb{H}^*(\Phi)$. Finally, given $d \in GL(V_W)$ which acts as a scalar d_j on each irreducible component V_j, use (iii) again and notice that d^* acts on each P_j as a grading automorphism, thus leaving any graded subspace invariant, which concludes the proof.

One more notational convention: given a pair of root systems, $\Phi = (\Phi_K \subset \Phi_G \subset V)$, we shall denote by $H_*^G(n; \mathbb{F})$ the action on $H^*(\Phi_G; \mathbb{F})$ induced by $n \in N_G(\mathbb{F})$, and similarly for K, and by $H_*(n; \mathbb{F})$ the action of $n \in N(\mathbb{F})$ on $H^*(\Phi; \mathbb{F})$.

3. LEFSCHETZ NUMBERS

Consider a pair of root systems, $\Phi = (\Phi_K \subset \Phi_G \subset V)$. We shall also need to consider W_G-chambers (resp. W_K-chambers), denoted by C_G (resp. C_K) and pairs of Weyl chambers, denoted by $C = (C_G \subset C_K)$. For the proof of Theorem 1.2 we shall fix a W_K-chamber C_K and recall that the cosets $[n] \in N/W_K$ are supposed to be <u>normalized</u>, i.e. $n(C_K) = C_K$.

To start with, we are going to prove first, for the case $\Phi_K = \emptyset$, a statement which is stronger than the theorem, next to deduce from it Theorem 1.2 (for K = T) in a straightforward manner, and finally to settle the general case. A little bit more notation will be convenient: given a (set) mapping $\varphi : W \to W$, we shall denote by $\mathbb{R}[\varphi]$ the linear map $\mathbb{R}[\varphi] : \mathbb{R}[W] \to \mathbb{R}[W]$ which is defined on the group ring by

$$\mathbb{R}[\varphi](w) = \varphi(w), \text{ any } w \in W \tag{3.1}$$

Here are the statements.

PROPOSITION 3.2. Let $(\Phi \subset V)$ be a root system. Fix a Weyl chamber C. Given $n \in N$, ord $n < \infty$, the characteristic polynomials of $H_* n$ and of $\mathbb{R}[\varphi_n]$ coincide, where φ_n is defined by $\varphi_n(v) = n^{-1} v w^{-1} n$, for $v \in W$, and $w \in W$ is determined by $n(C) = w(C)$.

COROLLARY 3.3. Theorem 1.2 holds for K = T.

PROPOSITION 3.4. If $\Phi = (\Phi_K \subset \Phi_G \subset V)$ is a pair of root systems, then there exists a graded isomorphism $H^*(\Phi)\otimes H^*(\Phi_K) \xrightarrow{\sim} H^*(\Phi_G)$, which is compatible with the $H_* \otimes H_*^K$ and H_*^G actions of N.

Assuming these, we shall quickly derive Theorem 1.2: by Proposition 3.4 we may write $L(H_*[n]) \cdot L(H_*^K n) = L(H_*^G n)$; moreover, Corollary 3.3 guarantees that $L(H_*^K n) = \# C_{W_K}(n)$. One more application of Corollary 3.3 (this time for G/T) helps to conclude.

3.5. Proof of Proposition 3.2. Remember that the harmonic polynomials are N-invariant (Proposition 2.3 (ii)), which means that the characteristic polynomials of H_*n and of $n^*|_{\mathbb{H}^*(\Phi)}$ coincide. On the other hand, it is easy to check that, given $x \in V$, $n \in N$ and $w \in W$, one has $h_x n^* = \mathbb{R}[c_n]h_{nx}$ and $h_{wx} = \mathbb{R}[r_w]h_x$, where the linear maps h are defined by (2.2) and c_n (resp. r_w), defined by $c_n(v) = n^{-1}vn$, resp. $r_w(v) = vw^{-1}$, for $v \in W$, give rise to self-maps of $\mathbb{R}[W]$ as explained in (3.1). Since $w^{-1}n(C) = C$ and $\mathrm{ord}(w^{-1}n) < \infty$, the (standard) trick used in the proof of Lemma 3.2 [13] gives the existence of $x \in C$ such that $nx = wx$. Notice that $x \in V'$ and apply Proposition 2.3 (i) in order to finish the proof.

3.6. Proof of Corollary 3.3. We know that $L(n) = \mathrm{trace}\,\mathbb{R}[\varphi_n] = \# \{v \in W \mid nv^{-1}n^{-1}v = w\} = \# \{v \in W \mid n(vC) = vC\}$, which gives the proof of the theorem for K = T.

3.7. Proof of Proposition 3.4. This is nothing else but the algebraic form of the Leray-Hirsch theorem for the fibration $K/T \to G/T \to G/K$. What we really want to emphasize is that the Leray-Hirsch isomorphism can be made N-equivariant in our case. One constructs a graded $H^*(\Phi)$-module map $\Psi : H^*(\Phi)\otimes H^*(\Phi_K) \to H^*(\Phi_G)$ by choosing, as usual, a degree zero section of the natural graded algebra surjection $H^*(\Phi_G) = P/P \cdot I_G^+ \to P/P \cdot I_K^+ = H^*(\Phi_K)$. Here we may consider the natural composition $\mathbb{H}^*(\Phi_K) \to H^*(\Phi_G) \to H^*(\Phi_K)$, which is a degree zero, N_K-equivariant, isomorphism (by Proposition 2.3 (ii)) in order to obtain an N-equivariant section $H^*(\Phi_K) \to H^*(\Phi_G)$.

It follows in a standard way that Υ is an N-equivariant surjection. On the other hand the W_G-equivariant isomorphism $I_G \otimes \mathbb{H}^*(\Phi_G) \xrightarrow{\sim} P$ (Proposition 2.3 (ii)) gives an isomorphism $H^*(\Phi) \sim \mathbb{H}^*(\Phi_G)^{W_K}$, and the W_G-equivariant isomorphism $h_x : \mathbb{H}^*(\Phi_G) \xrightarrow{\sim} \mathbb{R}[W_G]$ (Proposition 2.3 (i)) gives $\dim \mathbb{H}^*(\Phi_G)^{W_K} = [W_G : W_K]$. Using 2.3 (i) again, a dimension argument shows Υ to be an isomorphism. The proof of Proposition 3.4, and consequently the proof of Theorem 1.2, are completed.

4. PROOF OF THEOREM 1.1

We are dealing with a pair of root systems, $\Phi = (\Phi_K \subset \Phi_G \subset V)$, which is supposed to be in normal form, and we have to show that, given $n \in N(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{Q}$) such that $H_*(n; \mathbb{F}) = \mathrm{id}$, then necessarily $n \in W_K$. It plainly suffices to see this for $\mathbb{F} = \mathbb{R}$.

We shall first reduce the discussion to the case when Φ_G is irreducible. Write, as in § 2, $\Phi = \coprod_{i \in A} \Phi^i$ (the irreducible pair decomposition) and notice that the natural isomorphism $P \sim \bigotimes_{i \in A} P_i$ induces a graded algebra isomorphism $H^*(\Phi) \sim \bigotimes_{i \in A} H^*(\Phi^i)$ (see Proposition 2.3 (iii)). Having this geometric decomposition of $H^*(\Phi)$ in mind, we shall consider the subgroup $\mathcal{G}$ of $\mathrm{Aut}\, H^*(\Phi)$ of "geometric" graded algebra automorphisms, defined by the requirement of preserving this decomposition (see the definition below). We shall encounter them again in the next section (see Proposition 5.1).

DEFINITION 4.1. $\mathcal{G} = \{ a \in \mathrm{Aut}\, H^*(\Phi) \mid \forall\, i \in A, \exists\, j \in A \text{ s.t. } a\, H^*(\Phi^i) = H^*(\Phi^j) \}$.

It follows from Proposition 2.1 (ii) that if $n \in N$ then $H_* n \in \mathcal{G}$ (and $\pi_0(n) j = i$ - which makes sense since $N \subset N_G$), which motivates the definition. If $H_* n = \mathrm{id}$ then obviously $\pi_0(n) = \mathrm{id}$, and it follows (invoking 2.1 (ii) again) that we may reduce the matter to the case Φ_G = irreducible, which we shall from now on suppose throughout this section.

LEMMA 4.2. If $H_* n = \mathrm{id}$ then $\mathrm{ord}\, n < \infty$.

Proof: It follows from Proposition 2.5 [13] (when applied to N_G) that we may write $n = \lambda n_1$, with $\lambda \in \mathbb{R}^+$ and $n_1 \in N$, ord $n_1 < \infty$, hence the grading automorphism $H_* \lambda$ must be of finite order. On the other hand, Φ_K being a proper subsystem of Φ_G, one may see, using Proposition 3.4 and Propositoin 2.3 (i), that ncessarily $H^+(\Phi) \neq 0$ (otherwise $H^*(\Phi_G) \sim H^*(\Phi_K)$, which would imply $W_G = W_K$, hence $\Phi_G = \Phi_K$), which forces $\lambda = 1$ and shows that ord $n < \infty$.

This first step of the proof of Theorem 1.1 opens the way for the use of Theorem 1.2. We shall therefore fix an arbitrary pair of Weyl chambers $C = (C_G \subset C_K)$ and set $W_C = \{ u \in W_G \mid u(C_G) \subset C_K \}$. The next lemma, which is an application of Theorem 1.2, represents the key step in the proof of Theorem 1.1.

LEMMA 4.3. The following statement implies Theorem 1.1.

If $H_* n = \mathrm{id}$ and $n(C_G) = C_G$, then $W_C \subset C_{W_G}(n) \Rightarrow C_{W_G}(n) = W_G$ (4.4)

Proof. Given $n \in N$ such that $H_* n = \mathrm{id}$, we may well normalize $[n]$ as in Theorem 1.2, supposing that $n(C_K) = C_K$ (as far as the proof of Theorem 1.1 is concerned). Using 4.2 and 1.2, it follows at once that we may even suppose that $n(C_G) = C_G$, eventually for some other choice of W_G-chamber C_G (i.e. of a pair C of Weyl chambers). Moreover, 1.2, 3.4 and 2.3 (i) together imply that $[C_{W_G}(n) : C_{W_K}(n)] = L(\mathrm{id}) = [W_G : W_K]$. This equality is immediately seen to be equivalent (by simply looking at the natural embedding of finite coset spaces $W_K/C_{W_K}(n) \to W_G/C_{W_G}(n)$) with the fact that $W_G \subset W_K \cdot C_{W_G}(n)$. It is a routine exercise to see that this inclusion is equivalent with the inclusion $W_C \subset C_{W_G}(n)$, if $n \in N$ and $n(C_G) = C_G$. By (4.4) we know that the whole Weyl group W_G centralizes n, hence leaves the fixed points of n in V invariant. By the argument in Lemma 3.2 [13] this fixed point subspace of V is nontrivial, hence it must coincide with V (remember that Φ_G is irreducible !), which concludes the proof of Theorem 1.1.

We are now moving towards the proof of (4.4). The choice of C determines a choice of positive roots, denoted by Φ_G^+, respectively by $\Phi_K^+ = \Phi_K \cap \Phi_G^+$, and of simple roots, denoted by S_G, respectively S_K. We recall ([13], Proposition 2.5) that any $n \in N_G$ may be uniquely written in the form $n = \lambda w \sigma(g)$, where $\lambda \in \mathbb{R}^+$, $w \in W_G$ and $g \in$ Graphaut (C_G), the automorphism group of the Coxeter graph having S_G as set of vertices, and that $\sigma(g)(C_G) = C_G$, for any $g \in$ Graphaut (C_G). If $n \in N$ is as in (4.4) then it follows from Lemma 4.2 and the above discussion that $n = \sigma(g)$, for some $g \in$ Graphaut (C_G).

LEMMA 4.5. If $g \in$ Graphaut (C_G) and $W_C \subset C_{W_G}(\sigma(g))$, then g must have a fixed point on S_G.

LEMMA 4.6. If Φ_G has one root length then W_G is generated by W_C.

Assuming these two lemmas, we are going to prove (4.4). An easy inspection of the connected Coxeter graphs and of their automorphism groups reveals the fact that, if $g \in$ Graphaut (C_G) has a fixed point on S_G, then either $g = \mathrm{id}$ (hence $\sigma(g) = \mathrm{id}$) or Φ_G has one root length. Lemmas 4.5, 4.6 and the discussion preceding them clearly give the proof of (4.4).

We ought to point out that in (4.4), which is essentially a statement about the size of W_C, the two cases coming from the classification (Φ_G has one, respectively two root lengths) must somehow be considered separately, as the example of the long roots of $B_2 = A_1 \times A_1 \subset B_2$ shows, where the group generated by W_C coincides with W_C and is a proper subgroup of W_G.

We turn now to the proofs of Lemmas 4.5 and 4.6. They both depend on the following inductive test (in what follows s_α will denote the orthogonal reflection corresponding to the root α).

LEMMA 4.7. If $u \in W_C$ and $\alpha \in S_G$ then $us_\alpha \in W_C$ if and only if $u(\alpha) \notin \Phi_K^+$.

Proof of Lemma 4.7. Given $w \in W_G$, it follows directly from the definition of

W_C that $w \in W_C$ if and only if $w^{-1}(\Phi_K^+) \subset \Phi_G^+$. On the other hand it is well-known (see e.g. [16]) that if $\alpha \in S_G$ then $s_\alpha(\Phi_G^+ \setminus \{\alpha\}) = \Phi_G^+ \setminus \{\alpha\}$. Putting these two facts together, our statement follows.

4.8. Proof of Lemma 4.5. It is time to recall that our root system pair Φ is supposed to be in normal form, in particular Φ_K is properly included in Φ_G. It follows that there must be some $\alpha \in S_G$ such that $\alpha \notin \Phi_K^+$ (use the fact that W_G is generated by the reflections corresponding to the simple roots). Put $u = \mathrm{id}$ in Lemma 4.7 and deduce that $s_\alpha \in W_C$, hence $\sigma(g)$ centralizes s_α. By the construction of $\sigma(g)$, see (1) in the proof of Proposition 2.5 [13], $\sigma(g)s_\alpha\sigma(g)^{-1} = s_{g(\alpha)}$, for any $\alpha \in S_G$. This implies $g(\alpha) = \alpha$, as desired.

4.9. Proof of Lemma 4.6. As in the previous proof, it follows that there exists some $\alpha \in S_G \setminus \Phi_K$. Such simple roots will cause no difficulties, by Lemma 4.7, and our claim will follow as soon as we prove that s_β lies in the group generated by W_C, for any $\beta \in S_G \cap \Phi_K$. Given such β , connect it to α by a straight edge-path of the Coxeter graph of Φ_G (which is connected and contains no multiple bonds) in order to arrive at the following situation (eventually after choosing some other $\alpha \in S_G \setminus \Phi_K$)

where $\beta_i \in S_G \cap \Phi_K$, $i = 1, \ldots, r$, and $\beta_r = \beta$. It will be enough to show that $s_\alpha s_{\beta_1} \ldots s_{\beta_i} \in W_C$, for any $1 \leq i \leq r$, since we already know that $s_\alpha \in W_C$. We will use induction on i. For $i = 1$, if $\alpha \in S_G \setminus \Phi_K$, $\beta \in S_G \cap \Phi_K$ and $(\alpha, \beta) \neq 0$, then (use Lemma 4.7 with $u = s_\alpha$). $s_\alpha s_\beta \in W_C$ if and only if $s_\alpha(\beta) \notin \Phi_K^+$. Suppose on the contrary that $s_\alpha(\beta) \in \Phi_K$. But (as it is easily seen) $s_\alpha(\beta) = s_\beta(\alpha)(= \alpha + \beta)$, for any distinct non-perpendicular simple roots of a system with one root length, which contradicts the fact that $\beta \in \Phi_K$ and $\alpha \notin \Phi_K$. Supposing $s_\alpha s_{\beta_1} \ldots s_{\beta_{i-1}} \in W_C$, use Lemma 4.7 again and deduce that $s_\alpha s_{\beta_1} \ldots s_{\beta_i} \in W_C$ if and only if $s_\alpha s_{\beta_1} \ldots s_{\beta_{i-1}}(\beta_i) = \alpha + \beta_1 + \ldots + \beta_i \notin \Phi_K$. Set $\alpha + \beta_1 + \ldots + \beta_{i-1} = \gamma$,

$\beta_i = \delta$, and suppose on the contrary that $\gamma + \delta \in \Phi_K$, while $\gamma \notin \Phi_K$ (by induction) and $\delta \in \Phi_K$. Since $(\gamma, \delta) = (\beta_{i-1}, \beta_i) < 0$, the fact that Φ_G has one root length implies again that $\gamma + \delta = s_\gamma(\delta) = s_\delta(\gamma)$, a contradiction. This closes the induction and the proof.

5. EXISTENCE OF INVARIANT GEODESICS

We are going first to derive Corollary 1.4 from Theorem 1.3. The underlying idea is quite simple. If $f \in \mathrm{Isom}(M)$, M a 1-connected closed Riemannian manifold, and there exists no f-invariant geodesic, then id-$\pi_k(f)$ must be an isomorphism, for any k, [8]. If, by chance, it happens that $\dim \pi_k(M) \otimes \mathbb{Q} = 1$, for some k, it is obvious that it is impossible for $\pi_k(f) \otimes \mathbb{Q}$ and id-$\pi_k(f) \otimes \mathbb{Q}$ to be simultaneously unimodular which simply proves Corollary 1.4 (i). However, if $M = M_1 \times M_1$, $\pi_k(M) \otimes \mathbb{Q}$ is either zero or at least two-dimensional. In our case, this difficulty may be circumvented as follows. Write $G/K = \underset{i \in A}{\times} G_i/K_i$, where each G_i is simple. Recall that $\pi_*(f) \otimes \mathbb{R}$ is entirely determined by $H^*(f; \mathbb{R})$, which is to be denoted in the sequel by g. If $g \in \mathcal{G}$ (see Definition 4.1), that is it respects the geometric decomposition $H^*(G/K) = \underset{i \in A}{\otimes} H^*(G_i/K_i)$, it is not surprising that $\pi_*(f) \otimes \mathbb{R}$ will respect the similar homotopy decomposition, which brings us back essentially to the case when G is simple.

Due to the remark following Definition 4.1, the next proposition clearly implies Corollary 1.4 (ii).

PROPOSITION 5.1. Suppose G/K is in normal form and consider $f \in \mathrm{Isom}(G/K)$. If $H^*(f; \mathbb{R}) =: g \in \mathcal{G}$ (See Definition 4.1 for the meaning of $\mathcal{G}$) then there exists an f-invariant geodesic.

<u>Proof.</u> The proof uses Theorem 1.3 and rational homotopy theory [15]. Our first task is to clarify the relationship between the action of g on $\underset{i \in A}{\otimes} H^*(G_i/K_i; \mathbb{R})$ and the action of $\pi_*(f) \otimes \mathbb{R}$ on $\underset{i \in A}{\oplus} \pi_*(G_i/K_i) \otimes \mathbb{R}$.

. Given G/K, with G and K compact and connected and of the same rank, the isomorphism $H^*(G/K;\mathbb{R}) \sim I_K/I_K \cdot I_G^+$ describes $H^*(G/K;\mathbb{R})$ as the quotient of a graded polynomial algebra by the ideal generated by a regular sequence ([4], p.107 and p.137), which shows G/K to be intrinsically formal, see [15], § K. In particular the minimal model of $H^*(G/K;\mathbb{R})$, considered as a differential graded algebra with trivial differential, will coincide with the minimal model of G/K (from now on, the ground field will be $\mathbb{F} = \mathbb{R}$).

Pick a minimal model $\rho_i : \mathcal{M}_i \to H^*(G_i/K_i)$, for any $i \in A$. Then the tensor product $\rho = \otimes \rho_i : \mathcal{M} = \otimes \mathcal{M}_i \to \otimes H^*(G_i/K_i) = H^*(G/K)$ will give the minimal model of G/K. Also notice that, if Φ is the root system pair corresponding to G/K, then in the irreducible decomposition $\Phi = \coprod_{i \in A} \Phi^i$, Φ^i will correspond to G_i/K_i, for any i.

Given $g \in \mathcal{G}$, denote by $\bar{g} \in \Sigma(A)$ the naturally induced permutation, defined by $gH^*(G_i/K_i) = H^*(G_j/K_j)$ if and only if $\bar{g}(i) = j$. For any $i \in A$, construct a minimal model map $\tilde{g}_i : \mathcal{M}_i \xrightarrow{\sim} \mathcal{M}_{\bar{g}(i)}$, with the property that $\rho_{\bar{g}(i)} \tilde{g}_i = g \rho_i$. Setting $\tilde{g} = \otimes \tilde{g}_i : \mathcal{M} \xrightarrow{\sim} \mathcal{M}$, one clearly has $\rho \tilde{g} = g \rho$, which implies, if $g = H^* f$, that $\tilde{g}$ represents the minimal model of f. Writing $A = \coprod_j A_j$ for the decomposition into $\bar{g}$-orbits, one first immediate consequence is the fact that the rational homotopy type of G_i/K_i is constant along each orbit A_j. Second (and equally easy) it follows that $\pi_*(f) \otimes \mathbb{R}$ sends $\pi_*(G_i/K_i) \otimes \mathbb{R}$ isomorphically onto $\pi_*(G_{\bar{g}^{-1}(i)}/K_{\bar{g}^{-1}(i)}) \otimes \mathbb{R}$, for any $i \in A$, in particular leaving the subspaces $\pi_*^{A_j} =: \bigoplus_{i \in A_j} \pi_*(G_i/K_i) \otimes \mathbb{R}$ invariant.

Going back to the proof of Proposition 5.1 and supposing that there is no f-invariant geodesic, it follows from [8] that both $\pi_k(f)$ and $id - \pi_k(f)$ are isomorphisms, for any k. It is a straightforward exercise to infer that the restrictions to $\pi_k^{A_j}$ of both $\pi_k(f) \otimes \mathbb{R}$ and $id - \pi_k(f) \otimes \mathbb{R}$ are unimodular, for any k,j.

We shall now restrict our attention to a single orbit, on which $\bar{g}^{-1}$ acts as a cycle, say $(i_1 i_2 \dots i_r)$. Denote by h the Coxeter number of G_{i_1} and set $k = 2h - 1$, $\pi_k(f) \otimes \mathbb{R} = b$. By Theorem 1.3, $\dim \pi_k(G_{i_1}/K_{i_1}) \otimes \mathbb{R} = 1$, which, combined with our previous information on b, gives that b acts on a basis $X_1, \dots, X_r$ of $\pi_k^{A_j}$ in a cyclic manner, by $b(X_i) = \lambda_i \cdot X_{i+1}$, for some $\lambda_i \in \mathbb{R}$. But it is then impossible to have simultaneously det(b) = ± 1 and det(id-b) = ± 1, and this contradiction finishes the proof of Proposition 5.1.

The proof of Theorem 1.3 uses again rational homotopy theory. As we have already mentioned, the minimal model of G/K is the same as the minimal model of $H^*(G/K)$. On the other hand there is a simple model of $H^*(G/K)$, which may be described as follows ([15]): denote by Q_G (respectively Q_K) the graded vector space of the indecomposable invariants, $Q_G = I_G^+/I_G^+ \cdot I_G^+$ (and similarly for Q_K); denote by $\Sigma^{-1} Q_G$ the desuspension of Q_G and choose a degree one linear section, denoted by $d : \Sigma^{-1} Q_G \to I_G^+ \subset I_K$, of the natural projection $I_G^+ \to Q_G$. Use this to obtain a model $\rho : (I_K \otimes \Lambda \Sigma^{-1} Q_G, d) \to (H^*(G/K; \mathbb{F}), 0)$, where the differential d is the extension of the previously chosen section defined by $d(I_K) = 0$, and the map ρ is defined on I_K so as to coincide with the natural projection $I_K \to I_K/I_K \cdot I_G^+$, and is defined on the exterior part by $\rho(\Sigma^{-1} Q_G) = 0$. The only trouble comes from the fact that in general this model is not minimal. However, the general theory ([15]), § S and § R) shows that one can still obtain from it homotopy information on G/K, namely one has an isomorphism between $\mathrm{Hom}(\pi_*(G/K) \otimes \mathbb{F}, \mathbb{F})$ and $H^*(Q_K \oplus \Sigma^{-1} Q_G, Qd)$, where the differential Qd is nothing else but the linear part of d, and in our case is entirely determined (modulo a dimension shift) by the map induced between indecomposables by the inclusion $I_G \subset I_K$, to be denoted by $Q : Q_G \to Q_K$. In particular one has

$$\dim \pi_{2k-1}(G/K) \otimes \mathbb{F} = \dim \ker Q^k, \text{ for any } k \qquad (*)$$

Lemma 2.5 [5] implies that the top degrees of $\pi_*(G/K) \otimes \mathbb{F}$ and of $\pi_{odd}(G/K) \otimes \mathbb{F}$ coincide, which shows, via (*), that our theorem is in fact a statement about ker Q. Given G, we shall denote by k_G the top degree of the graded vector space Q_G; if G is simple, set h_G = Coxeter number of Φ_G.

Suppose from now on that G is simple. Using equality (*), Theorem 1.3 is clearly implied by the following lemmas

LEMMA 5.2. $k_G = h_G$ and $\dim Q_G^{h_G} = 1$.

LEMMA 5.3. $k_K < k_G$.

5.4. Proof of Lemma 5.2. This is well-known. We indicate the proof, for reader's convenience. Denoting by $k_1 \leq k_2 \leq \ldots \leq k_r$ the degrees of the basic invariants of W_G, notice that $k_G = k_r$, by definition, and that our statements are equivalent with $k_{r-1} < k_r = h$. The equality $k_r = h$ follows from [4], p. 119 and p.121, while the inequality $k_{r-1} < k_r$ follows from [4], p.169.

5.5. Proof of Lemma 5.3. This is our key lemma. Its proof uses results of [14] on Jacobians, so it will be convenient to work with $\mathbb{F} = \mathbb{C}$.

Write $I_G = \mathbb{C}[p_1, \ldots, p_r]$ and $I_K = \mathbb{C}[q_1, \ldots, q_r]$, as in [4], where $p_i, q_i \in \mathbb{C}[V]$ are basic invariants of Weyl groups satisfying $\deg p_1 \leq \ldots \leq \deg p_r$ (and similarly for K), whence $k_G = \deg p_r$ and $k_K = \deg q_r$. Denote by Z_G the zero set of $p_1, \ldots, p_{r-1}$ in $V \otimes \mathbb{C} = \mathbb{C}^r$, and define Z_K similarly. Since their defining equations are regular seq uences, the arguments of [14] show them to be one-dimensional, more precisely each one being a finite union of lines through the origin.

Suppose now that $k_K \geq k_G$. If $k_K > k_G$, it follows that each p_i $(1 \leq i \leq r)$ is a polynomial in $q_1, \ldots, q_{r-1}$, hence Z_K is included in the zero set of $p_1, \ldots, p_r$, which consists of the origin alone, a contradiction. Suppose then $k_G = k_K$ and write $p_r = \sum_{i=1}^{r} c_i q_i + q$, where $c_i \in \mathbb{C}$ and $q \in \mathbb{C}[q_1, \ldots, q_{r-1}]$. By Lemma 5.2 $\deg p_{r-1} < \deg p_r$, hence $p_i \in \mathbb{C}[q_1, \ldots, q_{r-1}]$, for $i < r$. The preceding argument shows that necessarily $c_r \neq 0$, which implies that $I_K = \mathbb{C}[q_1, \ldots, q_{r-1}, p_r]$, i.e. we may also suppose that $q_r = p_r$ from now on. The same discussion also shows that $Z_K \subset Z_G$.

Write now $p_i = \sum_{j=1}^{r} f_{ij} q_j + \bar{q}_i$, for any $1 \leq i \leq r$, where $f_{ij} \in \mathbb{C}$ and $\bar{q}_i$ is a polynomial in $q_1, \ldots, q_r$ with no linear part. It is clear that the p_i's (respectively q_i's) modulo decomposables give a basis of Q_G (respectively Q_K) and that with respect to these bases Q is given by $Q(p_i) = \sum f_{ij} q_j$. We infer that $\det(f_{ij}) = 0$. Indeed, otherwise the discussion preceding (*) would imply that $\pi_*(G/K) \otimes \mathbb{C} = 0$, hence $H^+(G/K; \mathbb{C}) = 0$ (use Hurewicz), which is absurd (remember that K is a proper subgroup of G!).

On the other hand, denote by z the coordinates in $V \otimes \mathbb{C} = \mathbb{C}^r$, set $J_G = \det(\partial p_i / \partial z_j)$, $J_K = \det(\partial q_i / \partial z_j)$ and $J = \det(\partial p_i / \partial q_j)$, notice that $J_G = J \cdot J_K$, and recall the main result of [14] concerning Jacobians: J_G is not identically zero, when restricted to any irreducible component of Z_G. We are going to contradict this fact, using $\det(f_{ij}) = 0$, by showing $J(z) = \det(f_{ij})$, for any $z \in Z_K$. But this is clear: our assumption $k_K = k_G$ gave us $p_i \in \mathbb{C}[q_1, \ldots, q_{r-1}]$, for $i < r$, and $p_r = q_r$, therefore $(\partial p_i / \partial q_j)(qz) = f_{ij}$ for any $i,j \leq r$ and for any $z \in Z_K$. This final contradiction ends the proof of Lemma 5.3 and consequently of Theorem 1.3.

REMARK 5.6. An alternative proof of Theorem 1.3, using the classification, would go as follows: by the classical result of [3] one even knows that $\Phi_K \subset \Phi_G$ is a <u>closed</u> sub-root system (in the sense of [4]) and moreover it is again classical (and easy to write down) how to classify maximal proper closed sub-root systems of a given irreducible root system (see [3], also [4], p.229). Case-by-case checking gives then Lemma 5.3 for the case of a maximal proper closed sub-root system. Given an arbitrary proper closed sub-root system $\Phi_K \subset \Phi_G$, intercalate a maximal one and deduce $Q^{k_G} = 0$. Use then equality (*) and Lemma 5.2 as before in order to derive Theorem 1.3.

6. STRUCTURE OF N. EXAMPLES

In this section we are going to discuss the structure of $N(\mathbb{F})$, for $\mathbb{F} = \mathbb{R}, \mathbb{Q}$. Start then with a root system pair, $\Phi = (\Phi_K \subset \Phi_G \subset V)$, supposed to be in normal form. Moreover (see [3]) we may, and we shall from now on suppose, that Φ_K is <u>closed</u> in Φ_G (see [4], p.160, for a definition). We shall fix a pair of Weyl chambers, denoted by $C = (C_G \subset C_K)$ and consider the associated positive roots $(\Phi_K^+ \subset \Phi_G^+)$ and simple roots (S_G, S_K).

As a first remark, denoting by $\Phi = \coprod_{i \in A} \Phi^i$ the decomposition into irreducible components, we get from Proposition 2.1 (ii) the following exact sequence: $1 \to \times_{i \in A} N^i \to N \xrightarrow{\pi_0} \Sigma(A)$, where each N^i corresponds to Φ^i, which splits the discussion according to the irreducible decomposition. Therefore, we shall also suppose from now on that Φ_G is irreducible.

Next, we recall from [13], Propositions 2.5 and 2.8, the existence of an exact sequence

$$1 \to \mathbb{F}^+ \times W_G \to N_G(\mathbb{F}) \xrightarrow{\gamma} \text{Graphaut}\,(C_G) \to 1 \tag{6.1}$$

which is split, for $\mathbb{F} = \mathbb{R}$, by Graphaut $(C_G) \xrightarrow{\sigma} N_G$. This restricts to the basic exact sequence

$$1 \to \mathbb{F}^+ \times N_{W_G}(W_K) \to N(\mathbb{F}) \xrightarrow{\gamma} \text{Graphaut}^{\mathbb{F}}(C) \to 1 \tag{6.2}$$

LEMMA 6.3. Given $g \in \text{Graphaut}\,(C_G)$, $g \in \text{Graphaut}^{\mathbb{F}}(C)$ if and only if there exists $u \in W_G$ such that $u^{-1}\sigma(g) \in N_K$ and leaves C_K invariant.

<u>Proof</u>. Clear, for $\mathbb{F} = \mathbb{R}$. Suppose that $\mathbb{F} = \mathbb{Q}$ and $u^{-1}\sigma(g) \in N_K$ leaving C_K invariant, for some $u \in W_G$. On the other hand (see the proof of Proposition 2.8 [13]) there exists $\lambda \in \mathbb{R}^+$ such that $\lambda\sigma(g) \in GL(\Gamma \otimes \mathbb{Q})$. It follows that $\lambda u^{-1}\sigma(g) \in N(\mathbb{Q})$ and $g \in \text{Graphaut}^{\mathbb{Q}}(C)$.

As a consequence, $\text{Graphaut}^{\mathbb{Q}}(C) = \text{Graphaut}^{\mathbb{R}}(C)$, and in what follows we are left with the determination of this subgroup of Graphaut (C_G), to be denoted simply by Graphaut (C), working with $\mathbb{F} = \mathbb{R}$. The groups Graphaut (C_G) are small and easy to handle, [13], and the criterion of Lemma 6.3 may be improved in order to become effective enough. Our next aim is to produce evidence to support this statement, by explicitly determining Graphaut (C) for the case when Φ_K is a <u>maximal</u> proper closed sub-root system of a (normalized) irreducible root system Φ_G.

PROPOSITION 6.4. Let $\Phi = (\Phi_K \subset \Phi_G \subset V)$ be a root system pair, with Φ_G normalized and irreducible and Φ_K maximal proper closed. Then Graphaut (C) = Graphaut (C_G), excepting the following cases: $A_3 \subset D_4$, $A_{2m-1} \subset D_{2m}$ $(m > 2)$,

$A_1 \subset B_2$ or $A_1 \times A_1 \subset B_2$, $A_2 \subset G_2$, $B_4 \subset F_4$ or $A_1 \times C_3 \subset F_4$. In the first case, Graphaut (C) = $\mathbb{Z}_2$, in rest it is trivial.

Some remarks are in order, before starting the proof. The group of root system automorphisms of Φ_G, denoted by Aut(Φ_G), acts (in an order-preserving manner) on the lattice of closed sub-root systems of Φ_G and, as it is easy to see, preserves everything related to N and Graphaut. We are going to use the classification; this means that, after fixing Φ_G, we are really dealing with Φ_K modulo Aut(Φ_G). This greatly simplifies things, since the lists for Φ_K = maximal (modulo Aut(Φ_G)) are quite explicit and reasonably small [3] (see also [4], p.229). The notations related to the classification will follow those of [4]; one notable exception: we shall denote the highest root by α_o.

As a preliminary remark, let us see that the statement of 6.4 is clear, excepting the cases $\Phi_G = B_2, G_2, F_4$, or D_{2m} $(m \geq 2)$. We assert that with these exceptions, we have Graphaut (C) = Graphaut (C_G), for any proper sub-root system $\Phi_K \subset \Phi_G$. This is trivial if $\Phi_G = B_r, C_r$ $(r \geq 3)$, E_7 or E_8, since in these cases Graphaut (C_G) = $\{1\}$, and also trivial if $\Phi_K = \emptyset$. Finally, if $\Phi_G = A_r$ $(r \geq 2)$, D_{2m+1} $(m \geq 2)$, or E_6, one knows that Graphaut (C_G) = $\mathbb{Z}_2$ (say g is the nontrivial element) and that $-1 \notin W_G$. But, since -1 is an element of finite order of $N \subset N_G$, we must have $-1 = u\sigma(g)$, for some $u \in W_G$, and consequently $g \in$ Graphaut(C).

On our way on improving Lemma 6.3, let us notice that, given $n \in N_G$, n naturally acts on the reflecting hyperplanes of Φ_G, and $n \in N_K$ if and only if this action leaves the hyperplanes of Φ_K invariant. More formally, define $\tilde{n} : \Phi_G \tilde{\to} \Phi_G$ by $ns_\alpha n^{-1} = s_{\tilde{n}(\alpha)}$, where $\tilde{n}(\alpha) \in \Phi_G^+$, if $\alpha \in \Phi_G^+$, and then $\tilde{n}(-\alpha) = -\tilde{n}(\alpha)$. This action of N_G on Φ_G is related to the action on V by

LEMMA 6.5. Given $n \in N_G$, there exists $\mu_n : \Phi_G \to \mathbb{R}^*$ such that $n(\alpha) = \mu_n(\alpha) \cdot \tilde{n}(\alpha)$, for any $\alpha \in \Phi_G$.

Proof. By Lemma 2.2 [13], there is some $\lambda \in \mathbb{R}^+$ such that λn is an isometry.

Since obviously $\widetilde{\lambda n} = \tilde{n}$, we may suppose that n is isometric, which implies that $s_{n(\alpha)} = s_{\tilde{n}(\alpha)}$, for any $\alpha \in \Phi_G$, whence the lemma.

With these preliminaries, we can formulate the improved version of Lemma 6.3. With the aid of (i) below we shall be able to clarify the cases $\Phi_G = B_2, G_2$ or F_4 and (ii) will be used to take care of the case $\Phi_G = D_{even}$, thus finishing the proof of Proposition 6.4.

LEMMA 6.6. Given $g \in$ Graphaut (C_G), $g \in$ Graphaut (C) is equivalent to each of the following

(i) there exists $u \in W_G$ such that $\widetilde{\sigma(g)}(\Phi_K) = \tilde{u}(\Phi_K)$

(ii) there exists $u \in W_G$ such that $\widetilde{\sigma(g)}(S_K) = \tilde{u}(S_K)$ and $\tilde{u}^{-1}\widetilde{\sigma(g)} \in$ Graphaut (S_K).

Proof of Lemma 6.6. Statement (i) is just a reformulation of Lemma 6.3, given the construction of $\tilde{n}$. For the same reasons, this is also equivalent to the existence of $u \in W_G$ with the property that $\tilde{u}^{-1}\widetilde{\sigma(g)}(S_K) \subset \Phi_K^+$. Set $n = u^{-1}\sigma(g)$. If $n \in N_K$ and $n(C_K) = C_K$, then n permutes the walls of C_K, and consequently $\tilde{n}$ must leave S_K invariant. Using Lemma 2.2 [13] in the same way as in the proof of Lemma 6.5, we may also suppose that n is an isometry. A simple application of Lemma 6.5 shows then that $\tilde{n}(\alpha)$ is a positive multiple of $n(\alpha)$, for any $\alpha \in \Phi_K^+$, hence the restriction of $\tilde{n}$ to Φ_K^+ preserves angles, which implies that $\tilde{n}|_{S_K} \in$ Graphaut (S_K). The proof of (ii) is complete.

The basic technicalities needed for handling $\tilde{n}$ are provided by

LEMMA 6.7. (i) If $g \in$ Graphaut (C_G), then $\widetilde{\sigma(g)}(\alpha) = g(\alpha)$, for any $\alpha \in S_G$.

(ii) If $f \in$ Aut (Φ_G) then $\tilde{f}(\alpha) = \pm f(\alpha)$, for any $\alpha \in \Phi_G$.

(iii) If $g \in$ Dgraut (C_G) then $\widetilde{\sigma(g)}(\alpha) = \sigma(g)(\alpha)$, for any $\alpha \in \Phi_G$, where Dgraut $(C_G) \subset$ Graphaut (C_G) is the subgroup of the Dynkin diagram automorphisms.

Proof of Lemma 6.7. (i) immediately follows, by simply comparing the construction of $\widetilde{n}$ and the construction of $\sigma(g)$ ([13], p.643). Statement (ii) follows from Lemma 6.5. If $g \in \mathrm{Dgraut}\ (C_G)$ then $\sigma(g) \in \mathrm{Aut}(\Phi_G)$ ([13], p.645) and $\sigma(g)(S_G) = S_G$ ([13], p.644). Since obviously $\widetilde{\sigma(g)}(\Phi_G^+) = \Phi_G^+$ and $\sigma(g)(\Phi_G^+) = \Phi_G^+$, (iii) also follows, using 6.5 again.

6.8. End of proof of Proposition 6.4. For the beginning, let $\Phi_G = B_2, G_2$ or F_4. We know that Graphaut $(C_G) = \mathbb{Z}_2$ and, denoting by g the nontrivial graph-automorphism, we also know that g turns the graph end for end. We assert that if $g \in \mathrm{Graphaut}\ (C)$ then the number of long and the number of short roots of Φ_K must coincide. This follows, via Lemma 6.6 (i) and Lemma 6.7 (ii), from the fact that $\widetilde{\sigma(g)}$ sends long roots of Φ_G to short roots, and conversely. To see this fact, let us choose a pair of simple roots of Φ_G (α = short, β = long) such that $(\alpha, \beta) \neq 0$. Denoting by q the maximum number of bonds appearing in the Dynkin diagram of Φ_G, it is an easy exercise to compute (following the recipe of [13], p.643) $\sigma(g)(\alpha) = (1/\sqrt{q}) \cdot \beta$ and $\sigma(g)(\beta) = \sqrt{q} \cdot \alpha$. Given any $w \in W_G$, write $\sigma(g)(w\alpha) = \sigma(g) w\, \sigma(g)^{-1}(\sigma(g)\alpha) =$ $= (1/\sqrt{q}) \cdot \sigma(g) w\, \sigma(g)^{-1}(\beta) = \mu_{\sigma(g)}(w\alpha) \cdot \widetilde{\sigma(g)}(w\alpha)$ and deduce that $\widetilde{\sigma(g)}(w\alpha)$ is long. Since all roots of the same length are conjugate under W_G, this proves our claim (the computation showing $\widetilde{\sigma(g)}(w\beta)$ = short being similar).

Now we separately check the various cases. As a word of caution, the recipe given in [4], p.229, for listing (mod W_G) the maximal proper closed sub-root systems Φ_K of a given normalized irreducible rank r root system Φ_G, equipped with a choice of simple roots S_G, provides a choice of simple roots, say S'_K, for Φ_K, which coincides with the canonical choice (denoted by S_K and determined by $C_G \subset C_K$, in terms of associated Weyl chambers) if rank $\Phi_K = r - 1$, but will be different otherwise. Nevertheless, given $n \in N_G$, $n \in N$ if and only if $\widetilde{n}(S'_K) \subset \Phi_K$, where $\Phi_K \subset \Phi_G$ is an arbitrary sub-root system with an arbitrary choice of simple roots S'_K; this is immediate, recalling that $n s_\alpha n^{-1} = s_{\widetilde{n}(\alpha)}$, for any $\alpha \in \Phi_G$, and that W_K is generated by s_β, $\beta \in S'_K$.

If $\Phi_G = B_2$ then ([4], p.252) $\Phi_K = A_1$ or $\Phi_K = A_1 \times A_1$ = long roots of Φ_G. In both cases, the numbers of short and long roots of Φ_K are different and it follows that Graphaut (C) $= \{1\}$.

If $\Phi_G = G_2$ then ([4], p.274) $\Phi_K = A_2$ (in which case Graphaut (C) $=\{1\}$) or $\Phi_K = A_1 \times A_1$, having as simple roots $S_K = \{\alpha_1 \text{(short)}, \alpha_o = 3\alpha_1 + 2\alpha_2 \text{ (long)}\}$. In the second case, the restriction of $\widetilde{\sigma(g)}$ to S_K is given by $\widetilde{\sigma(g)}(\alpha_1) = \alpha_2$, $\widetilde{\sigma(g)}(\alpha_o) = 2\alpha_1 + \alpha_2$ (use $\sigma(g)(\alpha_1) = (1/\sqrt{3}) \cdot \alpha_2$, $\sigma(g)(\alpha_2) = \sqrt{3} \cdot \alpha_1$ and Lemma 6.5). Denoting by $u \in W_G$ the direct $\pi/3$ rotation, an application of Lemma 6.7 (ii) immediately leads to $\widetilde{\sigma(g)}(S_K) = \tilde{u}(S_K)$, by simply inspecting the picture of G_2 (see [4], p.276). As previously noticed, this implies that $u^{-1}\sigma(g) \in N$, hence $g \in$ Graphaut (C) and Graphaut (C) = Graphaut (C_G), in this case.

If $\Phi_G = F_4$ then ([4], p.272) $\Phi_K = B_4, A_1 \times C_3$, in which cases the numbers of short and long roots are different, and consequently Graphaut (C) $= \{1\}$, or $\Phi_K = A_2 \times A_2$ and $S'_K = \{-\alpha_o, \alpha_1 \text{ (long)}, \alpha_3, \alpha_4 \text{ (short)}\}$. By construction, $\sigma(g)(\alpha_1) = \sqrt{2} \cdot \alpha_4$, $\sigma(g)(\alpha_2) = \sqrt{2} \cdot \alpha_3$, $\sigma(g)(\alpha_3) = (1/\sqrt{2}) \cdot \alpha_2$ and $\sigma(g)(\alpha_4) = (1/\sqrt{2}) \cdot \alpha_1$, which gives, as explained before, $\widetilde{\sigma(g)} | S'_K$, namely $\widetilde{\sigma(g)}(-\alpha_o) = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4$, $\widetilde{\sigma(g)}(\alpha_1) = \alpha_4$, $\widetilde{\sigma(g)}(\alpha_3) = \alpha_2$ and $\widetilde{\sigma(g)}(\alpha_4) = \alpha_1$. Denoting by $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ the canonical basis of V, consider $u \in W_G$ defined by $u(\varepsilon_1) = \varepsilon_4$, $u(\varepsilon_4) = \varepsilon_1$, $u(\varepsilon_2) = -\varepsilon_3$ and $u(\varepsilon_3) = -\varepsilon_2$ and check (using Lemma 6.7 (ii)) that $\tilde{u}(-\alpha_o) = -\alpha_2$, $\tilde{u}(\alpha_1) = \alpha_1$, $\tilde{u}(\alpha_3) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ and $\tilde{u}(\alpha_4) = \alpha_4$. Since plainly $\tilde{u}^{-1}\widetilde{\sigma(g)}(S'_K) \subset \Phi_K$, it follows that $u^{-1}\sigma(g) \in N$, hence $g \in$ Graphaut (C) and Graphaut (C) = Graphaut (C_G) in this case too.

Next, let $\Phi_G = D_r$ $(r \geq 4)$ and denote by $g \in$ Graphaut (C_G) = Dgraut (C_G) the permutation of S_G defined by $g(\alpha_i) = \alpha_i$, for $1 \leq i \leq r-2$, $g(\alpha_{r-1}) = \alpha_r$ and $g(\alpha_r) = \alpha_{r-1}$ (cf. [4], p.256). Since $\sigma(g) \in \mathrm{Aut}(\Phi_G)$ and $\sigma(g)(C_G) = C_G$ ([13]) we have $\sigma(g)(\alpha_o) = \alpha_o$. Using Lemma 6.7 (iii) and (i), we deduce that $\widetilde{\sigma(g)}(-\alpha_o) = -\alpha_o$, $\widetilde{\sigma(g)}(\alpha_i) = \alpha_i$, $1 \leq i \leq r-2$, $\widetilde{\sigma(g)}(\alpha_{r-1}) = \alpha_r$ and $\widetilde{\sigma(g)}(\alpha_r) = \alpha_{r-1}$. We have two possibilities for Φ_K ([4], p.256) : either $\widetilde{\sigma(g)}(S'_K) = S'_K$ (which implies $g \in$ Graphaut (C), as before) or $S'_K = S_K = \{\alpha_1, \dots, \alpha_{r-2}, \alpha_{r-1}\}$ or $\{\alpha_1, \dots, \alpha_{r-2}, \alpha_r\}$ (which are conjugate by $\sigma(g) \in \mathrm{Aut}(\Phi_G)$).

Suppose then that $\Phi_K = A_{r-1}$, $S_K = \{\alpha_1, \dots, \alpha_{r-2}, \alpha_{r-1}\}$. We claim that if $r = 2m$ $(m \geq 2)$ then $g \notin$ Graphaut (C). Supposing the contrary, we know from Lemma 6.6 (ii) that there exists $u \in W_G = \{$permutations on r letters followed by an even number of changes of sign$\}$ such that $\widetilde{\sigma(g)}(S_K) = \tilde{u}(S_K)$ and $\tilde{u}^{-1}\widetilde{\sigma(g)} \in$ Graphaut $(S_K) = \mathbb{Z}_2$. A straightforward computation, which uses Lemma 6.7 (ii) for handling $\tilde{u}$ (and whose details will be omitted), shows that this contradicts the assumption $r = 2m$.

Since Graphaut $(C_G) = \mathbb{Z}_2$, if $\Phi_G = D_{2m}$, $m > 2$, the preceding discussion completely clarifies this case. If $\Phi_G = D_4$, then there are three rank 3 possibilities for Φ_K, all being conjugate under $\mathrm{Aut}(\Phi_G)$ to $\Phi_K = A_3$, $S_K = \{\alpha_1, \alpha_2, \alpha_3\}$, and one more case, namely $\Phi_K = A_1 \times A_1 \times A_1 \times A_1$, $S_K = \{\alpha_0, \alpha_1, \alpha_3, \alpha_4\}$. In the last case we have $\widetilde{\sigma(g)}(\alpha_i) = \alpha_i$, $i = 0$ or 2, for any $g \in$ Graphaut $(C_G) = \Sigma_3$ ([4], p.256-257), hence $\widetilde{\sigma(g)}(S_K) = S_K$, any $g \in$ Graphaut (C_G) and Graphaut (C) = Graphaut (C_G). Finally, if $\Phi_K = A_3$, then we already know that $g = (34) \notin$ Graphaut (C), but $g = (1\ \ 3)$ $\in$ Graphaut (C), since in this last case we clearly have $\widetilde{\sigma(g)}(S_K) = S_K$. These show that Graphaut (C) = $\mathbb{Z}_2$, being generated by $(1\ \ 3)$, if $\Phi = (A_3 \subset D_4)$. The proof of Proposition 6.4 is thus completed.

EXAMPLE 6.9. It would be tempting to conjecture that Aut $H^*(G/K; \mathbb{F})$ = Lieaut $H^*(G/K; \mathbb{F})$, for any proper pair of equal rank compact connected Lie groups and any characteristic zero coefficient field $\mathbb{F}$, as suggested by the cases $K = T$([13]) and $G/K = U(n)/U(n_1) \times \dots \times U(n_k)$, $n_1 + \dots + n_k = n$, ([7], [12]). In the first case the guess turns out to be true (for $\mathbb{F} = \mathbb{Q}, \mathbb{R}$, see Theorem 1.1. [13]). In the second case it was conjectured (after checking a number of particular situations, e.g. Grassmann manifolds, see [12] for more details) that Aut $H^*(G/K; \mathbb{Q})$ is generated by

$\mathbb{Q}^*$ (grading automorphisms) and automorphisms coming from $N_G(K)/K$ [7] or equivalently from $N_{W_G}(W_K)$ [12]. The exact sequence (6.2) shows, as indicated in the proof of Proposition 6.4 for $\Phi_G = A_{n-1}$, that $\mathbb{F}^*$ and $N_{W_G}(W_K)$ generate $N(\mathbb{F})$ in this case (for both $\mathbb{F} = \mathbb{Q}$ and $\mathbb{R}$), thus the above mentioned conjecture may be simply restated as Aut = Lieaut.

Unfortunately things are more complicated in general. We shall next present a simple example, namely $U(3) \subset SO(7)$, where not all $\mathbb{F}$-cohomology automorphisms are of Lie type, $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q}$ (actually it may be shown that the same thing happens also for $\mathbb{F} = \mathbb{C}$, but we shall not touch this here).

In our case $\Phi = (A_2 \subset B_3 \subset \mathbb{R}^3)$ and the embeddings are standard (see [1], [4]). Moreover $N(\mathbb{F}) = \mathbb{F}^* \cdot N_{W_G}(W_K)$, for both $\mathbb{F} = \mathbb{R}$ and $\mathbb{Q}$, as follows from (6.2). It is also straight forward to see that $N_{W_G}(W_K)/W_K = \mathbb{Z}_2$ and is generated by -1 (recall that $-1 \in W_G$ and $-1 \notin W_K$). We infer that $N(\mathbb{F})/W_K$ is generated by $\mathbb{F}^*$, hence Aut = Lieaut is equivalent to the fact that Aut $H^*(\Phi; \mathbb{F})$ consists of grading automorphisms alone.

The well-known relations between Pontrjagin and Chern classes imply, on the other hand, that $H^*(\Phi; \mathbb{F}) = \mathbb{F}[c_1, c_3]/(c_1^4 - c_1 c_3, c_3^2)$ as a graded algebra (where $|c_i| = i$). Consider then the graded algebra automorphism of $\mathbb{F}[c_1, c_3]$ defined by $h(c_1) = c_1$ and $h(c_3) = -c_3 + 2c_1^3$. It is easy to check that it induces an automorphism of $H^*(\Phi; \mathbb{F})$ which is not a grading automorphism.

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Department of Mathematics
INCREST, B-dul Pacii 220
79622 Bucharest, Romania