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by

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Lucian Bădescu

Introduction

Throughout this paper we shall fix an algebraically closed field k. Consider the following:

Problem. Let (Y,L) be a normal polarized variety over k, i.e. a normal projective variety Y over k together with an ample line bundle L on Y. Then one may ask under which conditions the following statement holds:

(*) Every normal projective variety X containing Y as an ample Cartier divisor such that the normal bundle of Y in X is L, is isomorphic to the projective cone over Y.

Recall that the projective cone over (Y,L) is by definition the projective variety $C(Y,L) = \mathrm{Proj}(S(Y,L)[T])$, where $S(Y,L) = \bigoplus_{i\geq 0} H^0(Y,L^i)$ is the graded k-algebra associated to (Y,L), and the polynomial k-algebra $S(Y,L)[T]$ (T is one variable) is graded by the condition that $\deg(sT^i) = \deg(s) + i$ whenever $s \in S$ is homogeneous.

Note that (*) is not true in one of the most obvious situations, namely when Y is a smooth hyperplane section of dimension ≥ 1 of a smooth subvariety Z of a projective space P^n, and $L = O_Z(1)\otimes O_Y$.

Examples of polarized varieties (Y,L) satisfying (*) may already be found in the classical literature (see e.g. [17], [18], or also theorem A below). Further such examples may be found in [2], [3], [4], [6] and [9]. Let us mention the following criterion given by Fujita in [9]: if (Y,L) is a smooth polarized variety of dimension ≥ 2 such that $H^1(Y,T_Y\otimes L^i) = o$ for every $i < o$ and $\mathrm{char}(k) = o$, then (Y,L) satisfy (*). The main idea of all these papers was to use the Grothendieck-Lefschetz theory [10] for (X,Y) in order to deduce that $X \cong C(Y,L)$.

In this paper, following an entirely different approach, we are going to show how the theory of rigid singularities (see [15], [16]) can be used in order

to get a criterion for a (possibly singular) polarized variety (Y,L) to satisf $(*)$. Let us first recall two already known results. The first one (which is directly connected with the above problem) is classical.

Theorem A (C. Segre, G. Scorza, see [17], [18]). The statement $(*)$ holds if either $Y = P^n$ $(n \geqslant 2)$ and $L = O(s)$ $(s > o)$, or $Y = P^m \times P^n$ $(m \geqslant 1,\ n \geqslant 2)$ and $L = O(1,1)$ (the hyperplane section bundle of $P^m \times P^n$ with respect to its Segre embedding in P^{mn+m+n}).

Theorem A was rediscovered in [2], where a modern proof of it is given. Note that the classical formulation of it is slightly different (see the discussion in [4], §3).

On the other hand, one knows the following:

Theorem B (Grauert, Kerner, Schlessinger, see e.g. [15]). Let z be the vertex of the cone $C(Y,L)$, where (Y,L) is either $(P^n, O(s))$ $(n \geqslant 2,\ s > o)$, or $(P^m \times P^n, O(1,1))$ $(m \geqslant 1,\ n \geqslant 2)$. Then the singularity $(C(Y,L),z)$ is infinitesimally rigid in the sense of [15].

Our theorem 1 below implies immediately that theorem A is then a consequenc of theorem B. Note also that some kind of converse is also true. Namely, theorem A implies the rigidity of the projective cone $C(Y,L)$ (with (Y,L) as in theorem A, see [4]), and by [13] we know that the rigidity of $C(Y,L)$ and the rigidity of $(C(Y,L),z)$ are closely related.

Theorem 1. Let S be a normal graded k-algebra of finite type of dimension $\geqslant 3$, and denote by P the irrelevant ideal S_+ and by $Y = \mathrm{Proj}(S)$. Assume that the following three conditions hold:

i) $O_Y(1)$ is invertible.

ii) $H^1(Y, O_Y(i)) = o$ for every $i \in \mathbb{Z}$ (this is true when $\mathrm{depth}(S_P) \geqslant 3$).

iii) The local k-algebra S_P is infinitesimally rigid.

Then the polarized variety (Y,L) satisfies $(*)$.

Theorem 1 will be used to get the following improvement of the main result of Buium's paper [6]:

Theorem 2. Let Y be a weighted projective space of dimension $\geqslant 2$. If $\mathrm{char}(k) = o$ and $\mathrm{codim}_Y(\mathrm{Sing}(Y)) \geqslant 3$ (where $\mathrm{Sing}(Y)$ denotes the singular locus of Y), then Y satisfies $(*)$ with respect to any ample line bundle $L \in \mathrm{Pic}(Y)$.

Buium's proof does not require any restriction on char(k), but uses the stronger hypothesis that $\mathrm{codim}_Y(\mathrm{Sing}(Y)) \geqslant 4$. On the other hand, we show by examples that theorem 2 fails to be true if $\mathrm{codim}_Y(\mathrm{Sing}(Y)) = 2$. A more prec form of theorem 2 is given in corollary 3 of §2.

The paper is divided in two sections. In the first one we prove theorem 1 and in the second we indicate some applications of theorem 1 (the most import being corollary 2 which implies theorem 2), give examples and make some remar

Unless otherwise specified, the terminology and notations used are standa

§1. Proof of theorem 1

Let X be a normal projective variety containing Y as an ample Cartier div sor such that the normal bundle of Y in X is $\mathcal{O}_Y(1)$. Since S is normal, a lemm of Grothendieck (see e.g. [11], proposition 5.1) shows that $S \cong S(Y,\mathcal{O}_Y(1)) = \bigoplus_{i\geqslant 0} H^0(Y,\mathcal{O}_Y(i))$. Put $S' = S(X,\mathcal{O}_X(Y)) = \bigoplus_{i\geqslant 0} H^0(X,\mathcal{O}_X(iY))$, $P' = S'_+$ (the irrele vant ideal of S'), $A = S_P$ and $B = S'_{P'}$. Let $t \in S'_1 = H^0(X,\mathcal{O}_X(Y))$ be a global eq tion of the Cartier divisor Y on X, i.e. t is such that $\mathrm{div}_X(t) = Y$.

Step 1. There is a natural homomorphism of graded k-algebras $S'/tS' \cong S$, and hence $B/tB \cong A$.

Proof of step 1. Consider the standard exact sequence ($i \in \mathbb{Z}$)

$$(1) \qquad 0 \longrightarrow \mathcal{O}_X((i-1)Y) \xrightarrow{\ t\ } \mathcal{O}_X(iY) \longrightarrow \mathcal{O}_Y(i) \longrightarrow 0.$$

Since X is a normal projective variety of dimension $\geqslant 2$ and Y is an ampl Cartier divisor on X, a result of Serre (see [19], or also [10]) shows that $H^1(X,\mathcal{O}_X(iY)) = 0$ for every $i \ll 0$. Then the cohomology sequence associated to (1) together with hypothesis ii) shows by induction on i that $H^1(X,\mathcal{O}_X(iY)) =$ for every $i \in \mathbb{Z}$. This latter fact and the cohomology sequence of (1) yields the assertion.

Step 2. There is an isomorphism of local rings $\hat{B} \cong A[[T]]$ such that $t \in \hat{B}$ is mapped to T under this isomorphism, where $\hat{B}$ is the t-adic completion of B, and $A[[T]]$ is the formal power series A-algebra in one variable T.

Proof of step 2. By step 1 we know that $B/tB \cong A$. It is a general fact t $\hat{B}/t\hat{B} \cong B/tB$ (see [1]). Thus $\hat{B}/t\hat{B} \cong A$. Now, we can view our problem as a proble

of deformation theory, because we have the commutative diagram

$$\begin{tikzcd} \hat{B} \arrow[r] & A \cong \hat{B}/t\hat{B} \cong B/tB \\ R = k[[T]] \arrow[u, "\varphi"] \arrow[r] & k \arrow[u] \end{tikzcd}$$

where $k[[T]]$ is the formal power series k-algebra in the variable T, the horizontal maps are the natural surjections, and φ is the homomorphism of k-algebras such that $\varphi(T) = t$ ($\hat{B}$ is t-adically complete). Since t is a non-zero divisor in $\hat{B}$, φ is flat. Thus $\hat{B}$ is a deformation of the k-algebra A over R. Set $R_i = R/t^iR \cong k[T]/(T^i)$, $i \geqslant 1$. Since by iii) A is an infinitesimally rigid k-algebra, there is an isomorphism of R_i-algebras $u_i : B/t^iB \xrightarrow{\sim} A \otimes_k R_i \cong$ $\cong A[T]/(T^i)$. If we could choose the sequence $\{u_i\}_{i \geqslant 1}$ of such isomorphisms such that the following diagrams

$$\begin{tikzcd} \hat{B}/t^{i+1}\hat{B} \arrow[r, "u_{i+1}", "\sim"'] \arrow[d] & A[T]/(T^{i+1}) \arrow[d] \\ \hat{B}/t^i\hat{B} \arrow[r, "u_i", "\sim"'] & A[T]/(T^i) \end{tikzcd}$$

commute for every $i \geqslant 1$ (where the vertical arrows are the natural surjections) then we get the conclusion of step 2 by passing to inverse limits.

But this is indeed the case since A is infinitesimally rigid over k, and hence we may apply lemma 3.8 of [14] to show that every automorphism of the R_i-algebra $A[T]/(T^i)$ can be lifted to an automorphism of the R_{i+1}algebra $A[T]/(T^{i+1})$. This allows one to proceed by induction on i and replace (if necessary) u_{i+1} with a composition of a suitable automorphism of the R_{i+1}-algebra $A[T]/(T^{i+1})$ with u_{i+1} such that the above diagram becomes commutative.

<u>Step 3</u> (Conclusion). X is isomorphic to the projective cone $C(Y, O_Y(1))$.

<u>Proof of step 3</u>. Consider the filtrations $\{F^iS\}_{i \geqslant 0}$ and $\{F^iS'\}_{i \geqslant 0}$ of S and S' respectively, defined by $F^iS = \bigoplus_{j \geqslant i} S_j$ and $F^iS' = \bigoplus_{j \geqslant i} S'_j$, and let $\{F^iA\}_{i \geqslant 0}$, $\{F^iB\}_{i \geqslant 0}$ and $\{F^i\hat{B}\}_{i \geqslant 0}$ be the filtrations of the local k-algebras A, B and $\hat{B}$ induced by $\{F^iS\}_{i \geqslant 0}$ and $\{F^iS'\}_{i \geqslant 0}$ respectively. For every $i \geqslant 0$ we have

$$F^iA/F^{i+1}A \cong F^iS/F^{i+1}S \cong S_i \quad \text{and} \quad F^i\hat{B}/F^{i+1}\hat{B} \cong F^iB/F^{i+1}B \cong F^iS'/F^{i+1}S' \cong S'_i ,$$

and hence, denoting by $gr(B)$ (resp. $gr(\hat{B})$, resp. $gr(A)$) the associated graded rings, we get $gr(\hat{B}) \cong gr(B) \cong S'$ and $gr(A) \cong S$.

On the other hand, we have $F^i\hat{B}/tF^{i-1}\hat{B} \cong F^iB/tF^{i-1}B \cong F^iA$ (step 1), and the identification from step 2 easily shows that $F^i\hat{B} \cong F^iA \oplus TF^{i-1}A \oplus \ldots \oplus T^{i-1}F^1A \oplus \oplus T^i\hat{B}$, and therefore

$$F^i\hat{B}/F^{i+1}\hat{B} \cong (F^iA \oplus TF^{i-1}A \oplus \ldots \oplus T^{i-1}F^1A \oplus T^i\hat{B})/(F^{i+1}A \oplus TF^iA \oplus \ldots \oplus T^iF^1A \oplus T^{i+1}\hat{B})$$

$$\cong (F^iA/F^{i+1}A) \oplus T(F^{i-1}A/F^iA) \oplus \ldots \oplus T^{i-1}(F^1A/F^2A) \oplus T^i(\hat{B}/(F^1A \oplus T\hat{B})) \cong$$

$$\cong S_i \oplus TS_{i-1} \oplus \ldots \oplus T^{i-1}S_1 \oplus T^i(\hat{B}/F^1\hat{B}) \cong$$

$$\cong S_i \oplus TS_{i-1} \oplus \ldots \oplus T^{i-1}S_1 \oplus T^iS_o ,$$

or else, $S'_i \cong S_i \oplus TS_{i-1} \oplus T^2S_{i-2} \oplus \ldots \oplus T^{i-1}S_1 \oplus T^iS_o$ for every $i \geqslant o$. This proves that S' is isomorphic (as a graded k-algebra) to $S[T]$, where the polynomial S-algebra $S[T]$ is graded by $\deg(sT^i) = \deg(s) + i$ for every homogeneous element $s \in S$.

Step 3, and thereby theorem 1, is proved. ∎

<u>Remark.</u> Let I be an ideal of $A = S_P$ such that $V(I)$ is the singular locus of $Spec(A)$, and let T_U be the tangent bundle of $U = Spec(A)-V(I)$. Denote by theorem 1' the statement of theorem 1 except that the hypothesis iii) is replaced by the following one:

iii') $I\text{-depth}(A) \geqslant 3$ and $H^1(U,T_U) = o$.

Recall that $I\text{-depth}(A) \geqslant 3$ means that one of the following equivalent conditions holds (see [10]):

a) There are three elements $a_1, a_2, a_3 \in I$ which form an A-sequence, or

b) For every prime ideal p of A containing I one has $\text{depth}(A_p) \geqslant 3$.

Then Schlessinger proved in [15] (see also [16]) that if A satisfies iii') then A is an infinitesimally rigid k-algebra. Therefore theorem 1 implies theorem 1'.

On the other hand, since the formulation of theorem 1' is -a priori - independent of deformation theory, one may ask whether theorem 1' could be proved without using deformation theory. And indeed, theorem 1' can be proved in char. zero using the methods of [10] (especially exposé IX, proposition 1.4), and thus avoiding the deformation theory (in fact, only step 2 requires such a proof).

§2. Applications

Let $Y \subset P^N$ be a smooth projective subvariety of P^N given by the homogeneous ideal $I(Y)$ of the polynomial algebra $k[T_o,\dots,T_N]$, and put $S = k[T_o,\dots,T_N]/I(Y)$. Assume that the following conditions are satisfied:

a) The Serre map (see [19]) $\alpha : S \longrightarrow \bigoplus_{i \geq o} H^o(Y, O_Y(i))$ is an isomorphism of graded k-algebras (in other words, Y is projectively normal in P^N),

b) $H^1(Y, O_Y(i)) = o$ for every $i \in \mathbb{Z}$ (note that a) and b) together mean - via a lemma of Grothendieck (see e.g. [11], §5) - that $\mathrm{depth}(S_P) \geq 3$, with $P = S_+$), and

c) $H^1(Y, T_Y(i)) = o$ for every $i \in \mathbb{Z}$, where T_Y is the tangent bundle of Y.

These three conditions are verified if either $(Y, O_Y(1)) = (P^n, O(s))$ (with $n \geq 2$, $s > o$ and $(n,s) \neq (2,3)$), or $(Y, O_Y(1)) = (P^m \times P^n, O(1,1))$ (with $m \geq 1$ and $n \geq 2$). Then the first consequence of theorem 1 is the following:

Corollary 1. If $Y \subset P^N$ is a smooth projective subvariety of P^N satisfying the conditions a), b) and c) above, then the condition (*) (from the introduction) holds for $(Y, O_Y(1))$.

Proof. Indeed, if a), b) and c) hold, then by a criterion of Schlessinger [15], the k-algebra S_P is infinitesimally rigid, and hence we may apply theorem

Remark. Corollary 1 requires much stronger hypotheses than Fujita's result quoted in the introduction (see [9]). However, it was mentioned only because it holds in positive char. as well (while Fujita's result is proved in [9] only in char. zero).

Corollary 2. Let $G \subset GL(n,k)$ be a finite subgroup of linear transformations of k^n $(n \geq 3)$ such that the order of G is prime to char(k), and denote by S the k-algebra $k[T_1,\dots,T_n]^G$ of G-invariant polynomials in $T_1,\dots,T_n$. Assume that:

i) $O_Y(1)$ is invertible, where $Y = \mathrm{Proj}(S)$, and

ii) $\mathrm{Codim}_Y(\mathrm{Sing}(Y)) \geq 3$.

Then the property (*) holds for $(Y, O_Y(1))$.

Proof. The fact that $S = k[T_1,\dots,T_n]^G$ is a graded k-algebra is a general assertion, and thus it makes sense to consider $Y = \mathrm{Proj}(S)$. Let H be the subgroup of G generated by all reflections of G (in the sense of [2o]). Then by a theorem of Chevalley-Shephard-Todd, the k-algebra $k[T_1,\dots,T_n]^H$ of H-invari-

ant polynomials is generated by n algebraically independent homogeneous element $U_1,\dots,U_n$ (see [20], theorem 4.2.5). Moreover, H is a normal subgroup of G and G/H acts on $k[T_1,\dots,T_n]^H = k[U_1,\dots,U_n]$ in such a way that $k[U_1,\dots,U_n]^{G/H} = k[T_1,\dots,T_n]^G = S$ (the action of G/H is by automorphisms of $k[U_1,\dots,U_n]$ of graded k-algebras). Since only the reflections of G give rise to one-codimensi nal components of the ramification locus of the canonical surjection $f:k^n \longrightarrow k^n/G = \mathrm{Spec}(S)$, it follows that f factors as

$$\begin{tikzcd} k^n \arrow[rr, "\text{finite}"] \arrow[dr, "f"'] & & k^n = k^n/H \arrow[dl, "f'"] \\ & k^n/G = \mathrm{Spec}(S) & \end{tikzcd}$$

where the ramification locus of f' is of codimension $\geqslant 2$.

On the other hand, let $p:\mathrm{Spec}(S)-\{S_+\} \longrightarrow Y = \mathrm{Proj}(S)$ be the canonical morphism. Then p is the canonical projection of the G_m-bundle (G_m being the mul tiplicative group k^*) associated to the vector bundle $V(O_Y(1))$ (and ontained by deleting the zero section of $V(O_Y(1))$, because by i) we know that $O_Y(1)$ is inve tible). By ii), $\mathrm{codim}_Y(\mathrm{Sing}(Y)) \geqslant 3$, and therefore it follows that $\mathrm{codim}_{\mathrm{Spec}(S)}(\mathrm{Sing}(\mathrm{Spec}(S))) \geqslant 3$. By the theorem of Zariski-Nagata of the purity of the branch locus (see e.g. [10], éxposé X, théorème 3.4) it follows that the ramification locus of the restriction of f' to $k^n - f'^{-1}(\mathrm{Sing}(\mathrm{Spec}(S)))$ is eithe a divisor, or empty. The first possibility was excluded by the construction of It follows that the ramification divisor of f' is contained in $f'^{-1}(\mathrm{Sing}(\mathrm{Spec}(S$ and hence is of codimension $\geqslant 3$ in k^n. Since G/H acts on k^n leaving the origin fix (because G/H acts on $k[U_1,\dots,U_n]$ by automorphisms of graded k-algebras) and ac freely outside some G/H-invariant closed algebraic subset of codimension $\geqslant 3$, a fundamental result of Schlessinger (see [15], and also [16], page 155) shows th the k-algebra S_p (corresponding to the vertex of Spec(S)) is infinitesimally ri gid. Since S_p is also Cohen-Macaulay (loc. cit.), we can apply theorem 1 to get the conclusion. ∎

Now, we shall use corollary 2 to prove theorem 2. First, recall the defini- tion and some elementary properties of weighted projective spaces, for which th reader is referred to [7] or [8] (or also to the more recent survey [5]). Let $k[T_o,\dots,T_r]$ be the polynomial k-algebra in the indeterminates $T_o,\dots,T_r$, grade

by fixing a system $Q = (q_o,\dots,q_r)$ of r+1 positive integers (called weights) and putting $\deg(T_i) = q_i$, $i = o,\dots,r$. Denote by $S(Q)$ the k-algebra $k[T_o,\dots,T$ endowed with this gradation. Then the projective variety $P(Q) = P(q_o,\dots,q_r)$: := $\mathrm{Proj}(S(Q))$ is called the weighted projective space of dimension r and weigh $Q = (q_o,\dots,q_r)$. Then it is obvious that $P(Q)$ is a normal variety. One can prc that one can always modify Q in such a way that $P(Q)$ remains unchanged (up to isomorphism), and moreover Q is normalized, i.e. $\mathrm{g.c.d.}(q_o,\dots,\hat{q}_i,\dots,q_r) = 1$ for every $i = o,\dots,r$ (where the notation " $\hat{q}_i$ " means that q_i is to be droppe From now on we shall assume that Q is normalized. The Picard group of $P(Q)$ is isomorphic to $\mathbb{Z}$ and a generator of $\mathrm{Pic}(P(Q))$ is the class of $O_{P(Q)}(m)$ for so $m > o$. As a matter of notation, if S is a graded k-algebra and $n > o$ is a positi integer, we shall denote by $S^{(n)}$ the graded k-algebra such that $S_i^{(n)} = S_{in}$ for every $i \in \mathbb{Z}$ (where the multiplication of $S^{(n)}$ is induced by that of S).

<u>Proof of theorem 2.</u> Let X be a normal projective variety containing the weighted projective space $Y = P(Q)$ (with $r \geqslant 2$) as an ample Cartier divisor. Si $\mathrm{Pic}(P(Q))$ is generated by $O_{P(Q)}(m)$ for some $m > o$, the normal bundle of Y in X : isomorphic to $O_{P(Q)}(n)$ for some $n > o$. Let μ_n be the subgroup of G_m of all n^{th} roots of 1. Then μ_n acts on $S(Q) = k[T_o,\dots,T_r]$ by $\xi.T_i = \xi^{q_i} T_i$ for every $\xi \in \mu_n$ and $i = o,\dots,r$. Then it is well known (and easy to see) that the μ_n-in variant k-subalgebra $k[T_o,\dots,T_r]$ coincides to $S = k[T_o,\dots,T_r]^{(n)}$. It follow that Y is isomorphic to $\mathrm{Proj}(S)$ in such a way that $O_{P(Q)}(n)$ corresponds under this isomorphism to $O_{\mathrm{Proj}(S)}(1)$. In particular, $O_{\mathrm{Proj}(S)}(1)$ is invertible, and since $\mathrm{codim}_Y(\mathrm{Sing}(Y)) \geqslant 3$, the conclusion of theorem 2 follows from corollary 2.

We note that our proof works in positive char. as well, provided that the integer n (such that the normal bundle of Y in X is $O_{P(Q)}(n)$) is prime to char(

Now we want to state a slightly more explicit form of theorem 2. In order t do it we need some further elementary facts about weighted projective spaces. Let $P(Q)$ be a weighted projective space with $Q = (q_o,\dots,q_r)$ normalized and $r \geqslant$ One can define the Mori's locus of $P(Q)$ in the following way (see [12], or also [5]). For every prime number p denote by $M_p(Q)$ the closed subscheme of $P(Q)$ defined by the homogeneous ideal of $S(Q)$ generated by those T_i's such that p does not divide q_i (and written $p \nmid q_i$). Put also $g_p(Q) = \#\{i \in \{o,\dots,r\} / p \nmid q_i\}$ and $g(Q) = \min\{g_p(Q) / p \text{ prime}\}$. If p does not divide any q_i $(i = o,\dots,r)$ the

closed subscheme $\bigcup_{p \text{ prime}} M_p(Q)$ of $P(Q)$ with the reduced structure. Then:

a) One has $g(Q) \geqslant 2$ (Q is as always normalized) and $\operatorname{codim}_{P(Q)}(M(Q)) = g(Q)$ (obvious).

b) One has $g(Q) \geqslant 3$ if and only if $\text{g.c.d.}(q_o, \ldots, \hat{q}_i, \ldots, \hat{q}_j, \ldots, q_r) = 1$ for every $o \leqslant i < j \leqslant r$ (obvious).

c) The open subset $P^o(Q) = P(Q) - M(Q)$ of $P(Q)$ is smooth (see [12]).

d) For every $i \in \mathbb{Z}$ the sheaf $O_{P(Q)}(i)/P^o(Q)$ is invertible on $P^o(Q)$ and $P^o(Q)$ is the largest open subset of $P(Q)$ with this property (see [12], or also [5]).

e) For every $i \in \mathbb{Z}$ the sheaf $O_{P(Q)}(i)$ is reflexive of rank one, and the divisor class group of $P(Q)$ is generated by $[O_{P(Q)}(1)]$ (see [6], or also [5]).

f) The Serre map $\alpha : S(Q) \longrightarrow \bigoplus_{i \geqslant o} H^o(P(Q), O_{P(Q)}(i))$ is an isomorphism of graded k-algebras (see [7], or [8], or [5]).

g) One has $\operatorname{Sing}(P(Q)) = M(Q)$ (easy consequence of a), c), d) and e)).

h) For every $n \in \mathbb{Z}$, $O_{P(Q)}(n)$ is invertible on $P(Q)$ if and only if n is a common multiple of $q_o, \ldots, q_r$ (see e.g. [5], theorem 7.1; however, an earlier reference is Al. Amrani, Classes d'idéaux et groupe de Picard des fibrés projectifs tordus, Publications de l'I.R.M.A., Strasbourg (1983)).

<u>Corollary 3</u>. <u>Let $P(Q)$ be a weighted projective space with $Q = (q_o, \ldots, q_r)$ such that for every $o \leqslant i < j \leqslant r$, $\text{g.c.d.}(q_o, \ldots, \hat{q}_i, \ldots, \hat{q}_j, \ldots, q_r) = 1$, and assume $r \geqslant 2$ and $\operatorname{char}(k) = o$. Let X be a normal projective variety containing $P(Q)$ as an ample Cartier divisor. Then there is a positive integer q_{r+1} which is a common multiple of $q_o, \ldots, q_r$, such that, putting $Q' = (q_o, \ldots, q_r, q_{r+1})$, X is isomorphic to the weighted projective space $P(Q')$, and the inclusion $Y \subset X$ corresponds to the surjection of graded k-algebras $S(Q') \longrightarrow S(Q)$ which leaves T_i fixed ($o \leqslant i \leqslant r$) and maps T_{r+1} to zero</u> ($\deg(T_i) = q_i$, $i = o, \ldots, r, r+1$).

<u>Proof</u>. By a), b) and g) above, our hypothesis on the q_i's means exactly that $\operatorname{codim}_{P(Q)}(\operatorname{Sing}(P(Q))) \geqslant 3$. Let $O_{P(Q)}(n)$ be the normal bundle of $P(Q)$ in X. Then $n > o$, and putting $q_{r+1} = n$, q_{r+1} is a common multiple of $q_o, \ldots, q_r$ by the property h) above. By theorem 2, X is isomorphic to the cone $C(P(Q), O_{P(Q)}(n))$, and taking into account the property f) above we get $X \cong \operatorname{Proj}(S(Q)^{(n)}[T])$ (with $\deg(T) = 1$). If $S(Q') = k[T_o, \ldots, T_r, T_{r+1}]$ (with $\deg(T_i) = q_i$, $i = o, \ldots, r, r+1$), then it

clear that $S(Q')^{(n)} \cong S(Q)^{(n)}[T]$, whence $X \cong \mathrm{Proj}(S(Q)^{(n)}[T]) \cong \mathrm{Proj}(S(Q')^{(n)})$ $\cong P(Q')$. ∎

Remarks. 1) Corollary 3 is an improvement (in char. zero) of the main resul of Buium's paper [6]. Buium's result is proved in [6] in arbitrary char., but with the stronger hypothesis that $\mathrm{codim}_{P(Q)}(\mathrm{Sing}(P(Q))) \geqslant 4$ (and without the information that q_{r+1} is a common multiple of $q_o,\dots,q_r$).

2) If the normal bundle $O_{P(Q)}(n)$ of $P(Q)$ in X is such that n is prime to $\mathrm{char}(k)$, our proof works in positive char. as well.

2) One may ask whether corollary 3 (or theorem 2) remains still true if $\mathrm{codim}_{P(Q)}(\mathrm{Sing}(P(Q))) = 2$. The answer is negative because of the following:

Example. Take the surface $S = P^1 \times P^1$, and let B be any curve belonging to the complete linear system $|O(1,n)|$ for some $n \geqslant 1$. Then B is a connected smooth rational curve, and if $S = P^1 \times P^1 \hookrightarrow P^{2n+1}$ is the embedding associated to $|O(1,n)|$ (given by $(x_o,x_1;y_o,y_1) \longrightarrow (x_oy_o^n, x_oy_o^{n-1},\dots,x_oy_1^n, x_1y_o^n, x_1y_o^{n-1}y_1,\dots,x_1y_1^n)$), there is a hyperplane H of P^{2n+1} such that $B = S \cap H$. Embed P^{2n+1} in P^{2n+2} as a hyperplane and take the pecil $\{H_t\}_{t \in P^1}$ of hyperplanes of P^{2n+2} containing the 2-codimensional linear subspace H of P^{2n+2}. We may assume that for $t = o$, H_o is just the hyperplane containing H and passing through the vertex of the cone $C(S) \subset P^{2n+2}$ over S. Putting $X_t = C(S) \cap H_t$, we get a flat family $\{X_t\}_{t \in P^1}$ of surfaces such that for $t \neq o$, $X_t \cong S$, and since S is projectively normal in P^{2n+1}, X_o is just the cone in P^{2n+1} over $B \cong P^1$ with respect to the Veronese embedding $P^1 \hookrightarrow P^{2n}$ (given by $|O(2n)|$). In other words, X_o is nothing but the weighted projective plane $P(1,1,2n)$. This example (cf. [13], page 46) shows in particular that $P(1,1,2n)$ is a hyperplane section on the normal projective variety $C(S) = C(P^1 \times P^1, O(1,n))$. Since the divisor class group of $C(S)$ is easily seen to be isomorphic to $\mathbb{Z} \times \mathbb{Z}$, $C(S)$ cannot be isomorphic to a weighted projective space in view of the property e) above. This shows that in theorem 2 and corollary 3 one cannot drop the hypothesis that $\mathrm{codim}_{P(Q)}(\mathrm{Sing}(P(Q))) \geqslant 3$.

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