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by

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A two-dimensional moment problem

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Introduction

The purpose of this note is to characterize the moments

$$(1) \qquad a_{mn} = \int_{\mathbb{C}} z^m \bar{z}^n \, d\nu(z), \quad n,m \in \mathbb{N},$$

of an arbitrary finite positive measure $d\nu$, which is compactly supported on $\mathbb{C}$ and is absolutely continuous, with a bounded weight $d\nu/d\mu \in L^\infty(\mathbb{C})$, with respect to the planar Lebesgue measure $d\mu$. Our approach uses the theory of the principal function of a hyponormal operator.

The solution of the two-dimensional moment problem for finite positive measures goes back to the thirties, see Haviland [7]. The characterization of the moments of a finite positive measure, which in addition is compactly supported on $\mathbb{C}$, has been more recently treated by different modern methods by Devinatz [5], Atzmon [1], Szafraniec [11] and others. Let us state, for later use, the following form of the solution of this last moment problem.

THEOREM 1 *The sequence $(a_{mn})_{m,n=0}^{\infty}$ represents the moments* (1) *of a finite positive measure* $d\nu$ *on* $\mathbb{C}$, *with* $\mathrm{supp}(d\nu)$ *compact, if and only if*

a) *the kernel* $k : \mathbb{N}^2 \times \mathbb{N}^2 \longrightarrow \mathbb{C}$, $k(p,q;r,s) = a_{p+s,q+r}$, $p,q,r,s \in \mathbb{N}$ *is positive definite, and*

b) *the shift operator corresponding to* (1,0) *is bounded in the norm associated to* k.

For a proof of the theorem see [1] or [11]. The terminology used in condition b) will be explained in the sequel.

The solution presented in this note to the moment problem stated at the beginning of this introduction is perfectly similar to Theorem 1, except the form of the kernel k, which is replaced by some polynomial expressions (depending on p,q,r,s) in the entries a_{mn}. Similarly to previous works on two-parameter moment problems, we derive our solution from an analysis of a pair A_1, A_2

of self-adjoint operators. This time however, they are subject to the commutator condition $i[A_1, A_2] \geq 0$.

The paper is organized as follows. Section 1 contains the statement of the main result, together with a few remarks on it, Section 2 gives a brief recall of the needed facts concerning hyponormal operators and Section 3 is devoted to the proof of the main result.

1. The formal transform of the moments sequence

Let $(a_{mn})_{m,n=0}^{\infty}$ be a double sequence of complex numbers with the property

$$a_{mn} = \bar{a}_{nm}, \quad n,m \in \mathbb{N},$$

and let δ be a positive real. We associate to these data a function

$$K_\delta : \mathbb{N}^2 \times \mathbb{N}^2 \longrightarrow \mathbb{C}$$

which will be the analogous of the kernel k in Theorem 1.

Let consider two commuting indeterminates X and Y, and the formal series

$$(2) \quad \sum_{m,n=0}^{\infty} b_{mn} X^{m+1} Y^{n+1} = 1-\exp\left(-\frac{1}{\delta} \sum_{k,l=0}^{\infty} a_{kl} X^{k+1} Y^{l+1}\right).$$

Notice that the expression under the exponential belongs to the maximal ideal $\underline{m}$ of the formal series ring $\mathbb{C}[[X,Y]]$, so that the exponential function converges in the $\underline{m}$-adic topology.

Let $\iota = (1,0)$ and $\varkappa = (0,1)$ denote the generators of the semigroup $\mathbb{N}^2$, and $\theta = (0,0)$ its neutral element.

The kernel K_δ will be recursively defined according to the following rules:

(i) $K_\delta(\theta, \alpha) = K_\delta(m\varkappa, n\varkappa) = b(\alpha)$ for any $\alpha = (m,n) \in \mathbb{N}^2$

(ii) $K_\delta(\alpha,\beta) = \overline{K_\delta(\beta,\alpha)}$ for any $\alpha,\beta\in\mathbb{N}^2$, and

(iii) $K_\delta(\alpha+\iota,\beta) - K_\delta(\alpha,\beta+\varkappa) = \sum_{r=0}^{\infty} K_\delta(\alpha,r\iota)b(\alpha-(r+1)\iota)$, for $\alpha,\beta\in\mathbb{N}^2$.

We put $b(\alpha)=b_{mn}$ for $\alpha=(m,n)$ and we take by convention $b(\alpha)$ to be zero if at least one of the entries of α is negative.

Since the matrix (a_{mn}) was supposed to be hermitian, the rules (i), (ii) and (iii) are consistent and sufficient for the definition of K_δ on $\mathbb{N}^2\times\mathbb{N}^2$.

Next we recall some terminology needed in the statement of Theorem 2 below. Let $\mathcal{S}$ denote an abstract commutative semigroup. By a positive definite kernel K on $\mathcal{S}$ we mean a map $K: \mathcal{S}\times\mathcal{S}\longrightarrow\mathbb{C}$, such that

$$\sum_{s,t\in\mathcal{S}} K(s,t)f_s\overline{f}_t \geqslant 0,$$

for every function $f: \mathcal{S}\longrightarrow\mathbb{C}$ with finite support. Let $\mathcal{F}$ be the space of all those function. If the kernel K is positive definite, then it endows the vector space $\mathcal{F}$ with a hermitian scalar product

$$\langle f,g\rangle_K = \sum_{s,t\in\mathcal{S}} K(s,t)f_s\overline{g}_t \quad , f,g\in\mathcal{F}.$$

The shift operator S_u associated to an element u is defined on $f\in\mathcal{F}$ as follows:

$$(S_u f)(s) = \begin{cases} f(s-u), & \text{if } s\in u+\mathcal{S}, \\ 0 & ,\text{otherwise.}\end{cases}$$

We may assume for our purposes that the element s-u is uniquely determined by s and u.

The linear operator $S_u: \mathcal{F}\longrightarrow\mathcal{F}$ extends up to a bounded operator in the separate Hilbert space completion of $\mathcal{F}$ with respect to the norm $\|\cdot\|_K$ if and only if there is a constant $C>0$, such that:

$$\sum_{s,t\in\mathcal{S}} K(s+u,t+u)f_s\overline{f}_t \leqslant C \sum_{s,t\in\mathcal{S}} K(s,t)f_s\overline{f}_t ,$$

for every $f\in\mathcal{F}$. In this case we simply say that the shift S_u is bounded with respect to the kernel K. See [11] for more details.

Now we can state the main result of this note.

THEOREM 2 The sequence $(a_{mn})_{m,n=0}^{\infty}$ corresponds to the moments (1) of a finite positive measure $d\nu$, compactly supported on $\mathbb{C}$ and absolutely continuous, with bounded weight, with respect to the Lebesgue measure on $\mathbb{C}$, if and only if $a_{mn}=\bar{a}_{nm}$ for any $n,m\in\mathbb{N}$ and if there is a constant $\delta>0$, so that:

a) the associated kernel K_δ to (a_{mn}) is positive definite, and

b) the shift $S_{(1,0)}$ is bounded with respect to K_δ.

Moreover, in this case $d\nu/d\mu\leq\delta/\pi$ and supp$(d\nu)$ is contained in a ball centered at 0, of radius $\|S_{(1,0)}\|_{K_\delta}$.

Remarks. 1) The reason for the condition $a_{mn}=\bar{a}_{nm}$ to be stated explicitly in Theorem 2 is only the aesthetics of the definition rules of the kernel K_δ. This condition may de dropped after an alternative choice of the generating rule (ii). However, the new form of (ii) looks a bit more complicated.

2) The nonlinear nature of the entries of the kernel K_δ, regarded as functions of a_{mn}'s, takes away from K_δ the important feature of the kernel k appearing in Theorem 1 to be of the form

$$k(\alpha,\beta) = l(\alpha+\beta^*), \qquad \alpha,\beta\in\mathbb{N}^2,$$

with suitable involution "$*$" on $\mathbb{N}^2$ and function l.

3) It follows from the proof of Theorem 2 that, if the kernel K_δ satisfies conditions a) and b), then, for any $\gamma\geq\delta$, the kernel K_γ satisfies them too. Moreover, it will be also proved in the last section that

$$\operatorname*{ess-sup}_{z\in\mathbb{C}}(d\nu/d\mu(z)) = \inf\{\delta/\pi, K_\delta \text{ satisfies a) and b)}\}.$$

4) We ignore if there exists a direct proof of Theorem 2, or at least an explanation of the form of the kernel K_δ, not resorting to outer objects as hyponormal operators.

2. The background of hyponormal analysis

Let H be a complex separable Hilbert space. A linear bounded operator T acting on H is said to be hyponormal if its selfcommutator is non-negative:

$$[T^*,T] = T^*T - TT^* \geqslant 0.$$

If the operator T has not a normal operator as a direct summand, then T is said to be pure hyponormal.

By a straighforward combinatorics with commutator identities, one proves that the complex numbers

$$\langle T^mT^{*n}\xi, T^pT^{*q}\eta\rangle \quad ; n,m,p,q \in \mathbb{N}, \quad , \xi,\eta \in \operatorname{Ran}[T^*,T],$$

form a complete system of unitary invariants of the pure hyponormal operator T, see for instance [8]. We have denoted by RanA the closure of the range of the operator A.

Another type of unitary invariant of a pure hyponormal operator T, subject to the additional assumption that $[T^*,T]$ is trace class, is the principal function g_T, introduced by Pincus. It can be defined by the next formula, established by Helton and Howe [6]:

$$(3) \quad \operatorname{trace}[P(T,T^*),Q(T,T^*)] = \pi^{-1}\int (\bar{\partial}P\partial Q - \partial P\bar{\partial}Q)g_T \, d\mu,$$

where P and Q are complex polynomials of two variables.

The order of the factors T and T^* doesn't affect the trace of the commutator, by the assumption on $[T^*,T]$. The principal function g_T of the operator T is supported by the spectrum of T, is non-negative and summable, hence it is completely determined by fomula (3). Its invariance under unitary equivalence is immediate. For these topics see Pincus [9] and the books of Clancey [3] and Xia [12].

For the rest of this section we assume that T is a pure hyponormal operator with rank$[T^*,T]=1$. This is the best understood class of hyponormal operators. In this case $g_T \leqslant 1$ and it provides a complete unitary invariant for T, see [9]. Moreover, any integrable, compactly supported function g defined on $\mathbb{C}$, such that $0 \leqslant g \leqslant 1$ is the principal function of a pure hyponormal operator with one dimensional selfcommutator, see [2]. More exactly, the class of g in L^1 determines T, and conversely.

Let us denote $[T^*,T]=\xi\otimes\xi$. Recently, Clancey has proved the following formula [3]

$$(4)\quad \langle (T^*-\bar{w})^{-1}\xi, (T^*-\bar{z})^{-1}\xi\rangle = 1-\exp\left(-\frac{1}{\pi}\int \frac{g_T(\zeta)}{(\zeta-z)(\bar{\zeta}-\bar{w})}\,d\mu(\zeta)\right),$$

which is valid in an extended sense for every pair $(z,w)\in\mathbb{C}^2$. However, we need (4) only for large values of $|z|$ and $|w|$, where the inverses exist and the integrand is obviously summable. For the significance of the kernel appearing in (4) see [4] and [10].

The last identity can be transformed into:

$$(5)\quad \sum_{m,n=0}^{\infty}\langle T^mT^{*n}\xi,\xi\rangle z^{-m-1}\bar{w}^{-n-1} = 1-\exp\left(-\frac{1}{\pi}\sum_{k,l=0}^{\infty}\left(\int \zeta^k\bar{\zeta}^l g_T(\zeta)\,d\mu\right)z^{-k-1}\bar{w}^{-l-1}\right)$$

which holds for large values of $|z|$ and $|w|$, when the two series are absolutely convergent.

The coefficients of the left side series are among the discrete unitary invariants described at the beginning of this section. Since $[T^*,T]=\xi\otimes\xi$, the complex numbers

$$N_T(\alpha,\beta)=\langle T^mT^{*n}\xi, T^pT^{*q}\xi\rangle,$$

where $\alpha=(m,n)$ and $\beta=(p,q)$ run over $\mathbb{N}^2$, form a complete system of unitary invariants of T. In fact only the values

$$\Lambda_T(\alpha)=\langle T^mT^{*n}\xi,\xi\rangle,\ \alpha=(m,n)\in\mathbb{N}^2,$$

appearing in (5) completely determine N_T, and hence T. This fact follows from the next proposition, which is a particular case of Theorem 5.2 in [8].

PROPOSITION 1 Let $N:\mathbb{N}^2\times\mathbb{N}^2\to\mathbb{C}$ be a positive definite kernel with the properties

1) the shift S_ι is bounded with respect to N,

2) *There is a function* $\Lambda:\mathbb{N}^2 \to \mathbb{C}$, *so that* $N(\theta,\alpha)=N(m\varkappa,n\varkappa)=\Lambda(\alpha)$, *and*

$$N(\alpha+\iota,\beta)-N(\alpha,\beta+\varkappa)=\sum_{r=0}^{\infty} N(\alpha,r\iota)\Lambda(\beta-(r+1)\iota),$$

for any $\alpha=(m,n)$ *and* β in $\mathbb{N}^2$.

Then and only then there exists a pure hyponormal operator T, *with one dimensional selfcommutator, such that* $N=N_T$ *and* $\Lambda=\Lambda_T$.

The proof of the proposition uses only commutator identities and the well-known Kolmogorov facorization theorem of a positive definite kernel.

Thus we have two complete unitary invariants for pure hyponormal operators with one dimensional selfcommutator: the double sequence Λ_T and the principal function g_T. They are related by identity (5).

The ideea of the proof of Theorem 2 is to exploit in both senses this relationship, by knowing the ranges of the two parametrizations of the operators T. More precisely, g_T may be any integrable function with compact support and such that $0\leq g_T\leq 1$, and Λ_T may be any sequence with the property that the hermitian kernel, builded on Λ_T by the rules in point 2) of Proposition 1, is positive definite and has property 1).

3. The proof of Theorem 2

Let f be an integrable function on $\mathbb{C}$, with supp(f) compact and $0\leq f\leq M$, a.e., with $M<\infty$. Let $g=f/M$ and take by Carey and Pincus theorem [2] a pure hyponormal operator T with $[T^*,T]=\xi\otimes\xi$ and principal function g_T, so that $g_T=g$ in L^∞.

In view of relations (5) and (2), compared in the formal series ring, we get

$$b_{mn}=\langle T^mT^{*n}\xi,\xi\rangle \ , (m,n)\in\mathbb{N}^2,$$

or, in other notations $b(\alpha)=\Lambda_T(\alpha)$ for any $\alpha\in\mathbb{N}^2$.

Then Proposition 1 and the rules (i), (ii) and (iii) imply

$$K_{M\pi}(\alpha,\beta)=N_T(\alpha,\beta) \quad , \quad \alpha,\beta\in\mathbb{N}^2.$$

Above b_{mn} and $K_{M\pi}$ are associated to the moments of the function f.

Concluding, the kernel $K_{M\pi}$ satisfies conditions a) and b) in Theorem 2.

Conversely, assume that the sequence (a_{mn}) satisfies $a_{mn}=\overline{a}_{nm}$ for any $m,n\in\mathbb{N}$, together with conditions a) and b), for a given constant $\delta>0$.

Then the kernel K_{δ} fulfills the hypothesis of Proposition 1, whence there exists a pure hyponormal operator T, with $[T^*,T]=\xi\otimes\xi$, such that

$$K_{\delta}(\alpha,\beta)=N_T(\alpha,\beta) \quad \text{and} \quad b(\alpha)=\Lambda_T(\alpha),$$

for any $\alpha,\beta\in\mathbb{N}^2$. According to (2) and (5) we obtain the relation

$$\exp\left(\frac{1}{\delta}\sum_{k,l=0}^{\infty} a_{kl}z^{-k-1}\overline{w}^{-l-1}\right)=\exp\left(\frac{1}{\pi}\int\frac{g_T(\zeta)}{(\zeta-z)(\overline{\zeta}-\overline{w})}\,d\mu(\zeta)\right),$$

which holds for $|z|$ and $|w|$ large.

By taking z=w, both expressions under exponential become positive (the matrix (a_{mn}) is positive definite by a part of condition a)!), therefore

$$a_{kl}=\frac{\delta}{\pi}\int\zeta^k\overline{\zeta}^l g_T(\zeta)\,d\mu(\zeta)\quad,\ k,l\in\mathbb{N}.$$

This proves the converse implication in Theorem 2.

It remains to remark that the norm of the shift S_{ι} relative to the kernel N in Proposition 1 is precisely $\|S_{\iota}\|_N=\|T\|$, see [8]. On the other hand, $supp(g_T)=\sigma(T)\subset B(0,\|T\|)$, and the proof of Theorem 2 is complete.

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