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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS
No. 9/1987

BUCURESTI

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February 1987

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Bishop's condition (β) and rich extensions of linear operators

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Introduction

Our aim is to characterize those Banach space operators which occur as restrictions of operators with rich invariant subspace lattices. The starting point for our interest was the observation that each restriction of a decomposable operator onto one of its closed invariant subspaces satisfies Bishop's condition (β).

In the first part of the paper we show that conversely each operator with Bishop's property (β) admits extensions with sufficiently rich spectral decompositions. In the second part we prove that a natural strengthening of Bishop's property (β) can be used to characterize those operators ocurring as restrictions of generalized scalar operators.

Property (β) was introduced by E. Bishop [1] nearly thirty years ago in connection with a general duality theory for spectral decompositions. In our terminology a continuous linear operator T on a Banach space X is said to possess Bishop's property (β), if the operator

$$T_z : \mathcal{O}(U,X) \longrightarrow \mathcal{O}(U,X) \; ; \; f \longmapsto T_z f = (z-T)f$$

is a topological monomorphism for each open set U in $\mathbb{C}$. Here $\mathcal{O}(U,X)$ denotes the Fréchet space of all X-valued analytic functions on U.

The probably most interesting result concerning Bishop's property (β) known so far is the observation [4] that a continuous linear operator on a complex Banach space X is decomposable in the sense of C. Foiaş, if and only if both T and its adjoint T' possess property (β) (for a proof in the case of reflexive Banach spaces see [7]).

Obviously property (β) is inherited on restrictions to closed invariant subspaces. So each subdecomposable operator, i.e. an operator which is up to similarity the restriction of a decomposable operator onto a closed invariant subspace, satisfies property (β). We are going to show that conversely, each operator with property (β) can modulo similarity be extended to a decomposable operator on a strict (LF)-space.

In analogy a continuous linear operator T on a Banach space X is said to possess property $(\beta)_{\mathcal{E}}$, if for each open set U in $\mathbb{C}$ the operator

$$T_z : \mathcal{E}(U,X) \longrightarrow \mathcal{E}(U,X), \; f \longmapsto (z-T)f$$

is a topological monomorphism. Here $\mathcal{E}(U,X)$ denotes the Fréchet space of all X-valued C^∞-functions on U. A property coming very close to condition $(\beta)_{\mathcal{E}}$ was used in [13] to show that each hyponormal operator on a Hilbert space is subscalar, i.e. is up to similarity the restriction of a generalized scalar operator onto a closed invariant subspace.

The second main result of this paper is the observation that a continuous linear operator on a complex Banach space is subscalar, if and only if it satisfies condition $(\beta)_{\mathcal{E}}$.

The plan of the paper is as follows. In section 1 we show that a Banach space operator T has property (β), if and only if for each open covering of $\mathbb{C}$ there is an extension of T to a Banach space operator which admits a spectral decomposition with respect to this open covering. This result is used in section 2 to extend a Banach space operator with property (β) to a decomposable operator on a strict (LF)-space. Section 3 contains several equivalent descriptions of property $(\beta)_{\mathcal{E}}$, which for instance allow to show that M-hyponormal operators satisfy this condition. The announced characterization for subscalar operators is given in section 4. Section 5 is devoted to a few applications concerning division problems for distributions.

We thank Nicolae Popa for his valuable advice concerning topological tensor products.

1. Direct consequences of property (β)

As explained in the introduction a bounded linear operator T on a complex Banach space X is said to possess Bishop's property (β), if the map

$$T_z : \mathcal{O}(U,X) \longrightarrow \mathcal{O}(U,X) , \quad f \longmapsto T_z f = (z-T)f$$

is a topological monomorphism between Fréchet spaces for all open sets U in $\mathbb{C}$. It is quite obvious that property (β) for T implies property (β) for each restriction onto a closed invariant subspace. On the other hand, it is well-known that each operator which has a sufficiently rich spectral decomposition fulfills (β), see [17] and the proof of Theorem 1.1 below. To give the following complete characterization of property (β) denote for each open set U in $\mathbb{C}$ by $A^2(U,X)$ the Banach space of all square integrable analytic functions on U with values in X, equipped with its canonical norm $\| \|_{2,U}$.

Theorem 1.1 <u>For a continuous linear operator</u> T <u>on a Banach space</u> X, <u>the following are equivalent</u>:

(i) T <u>has property</u> (β).

(ii) <u>For each open cover</u> $\sigma(T) \subset U_1 \cup \ldots \cup U_n$ <u>the canonical map</u>

$$J : X \longrightarrow \bigoplus_{i=1}^{n} A^2(U_i,X)/\overline{T_z A^2(U_i,X)} , \quad x \longmapsto \bigoplus_{i=1}^{n} [x]$$

<u>is a topological monomorphism</u>.

(iii) <u>For each open cover</u> $\mathbb{C}=U_1 \cup U_2$ <u>there is an extension</u> $\hat{T}$ <u>of</u> T <u>onto a suitable Banach space</u> $\hat{X}$, <u>which admits a decomposition of the form</u>

$$\hat{X} = X_1 + X_2, \quad \hat{T}X_i \subset X_i, \quad \sigma(\hat{T}|X_i) \subset U_i, \quad i=1,2.$$

Proof. (iii)$\Rightarrow$(i) Let U be an open set in $\mathbb{C}$ and let (f_n) be a sequence in $\mathcal{O}(U,X)$ with $\lim_{n\to\infty} T_z(f_n) = 0$.

Let D_0, D be open discs in $\mathbb{C}$ with $\overline{D}_0 \subset D \subset U$ and choose an extension $\hat{T} \in L(\hat{X})$, $X_1, X_2 \in \mathrm{Lat}(\hat{T})$ as described in (iii) with respect to the covering

$$\mathbb{C} = U_1 \cup U_2 \ , \ U_1 = D, \ U_2 = \mathbb{C} \setminus \overline{D}_0 .$$

Since

$$\sigma(\hat{T},\hat{X}/X_2) = \sigma(\hat{T},X_1/X_1 \cap X_2) \subset \sigma(\hat{T},X_1) \cup \sigma(\hat{T},X_1 \cap X_2) \subset D$$

and since

$$f_n/X_2 = R(z,\hat{T}/X_2)(T_z(f_n)/X_2)$$

tends to zero in $\mathcal{O}(U \setminus \sigma(\hat{T},\hat{X}/X_2),\hat{X}/X_2)$ the maximum principle implies that $\lim_{n\to\infty} (f_n/X_2) = 0$ in $\mathcal{O}(U,\hat{X}/X_2)$. Due to the exactness of

$$0 \longrightarrow \mathcal{O}(U,X_2) \longrightarrow \mathcal{O}(U,\hat{X}) \longrightarrow \mathcal{O}(U,\hat{X}/X_2) \longrightarrow 0,$$

where the first map is the inclusion and the second is induced by the quotient map, we may conclude that $(f_n - g_n)$ tends to zero in $\mathcal{O}(U,\hat{X})$ for a suitably chosen sequence (g_n) in $\mathcal{O}(U,X_2)$. Therefore $\lim_{n\to\infty} \hat{T}_z(g_n) = 0$ in $\mathcal{O}(U,X_2)$ and because of $\sigma(\hat{T},X_2) \subset \mathbb{C} \setminus \overline{D}_0$, the same is true for $(g_n | D_0)$. Thus we have shown that (f_n) tends to zero in $\mathcal{O}(D_0,X)$.

(i)$\Rightarrow$(ii). Since for each open set U in $\mathbb{C}$ the Banach space topology of $A^2(U,X)$ is stronger than the topology induced from $\mathcal{O}(U,X)$, it suffices to show that for each open cover $\sigma(T) \subset U_1 \cup \ldots \cup U_n$ the corresponding map

$$\tilde{J} : X \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}(U_i,X)/\overline{T_z \mathcal{O}(U_i,X)}, \quad x \longmapsto \bigoplus_{i=1}^{n} [x]$$

is a topological monomorphism. Let (x_k) be a sequence in X such that $(\tilde{J}x_k)$ tends to zero. In other words, for each $i=1,\ldots,n$ there is a sequence $(f_{i,k})_k$ in $\mathcal{O}(U_i,X)$ with

$$(x_k - T_z(f_{i,k})) \xrightarrow[k]{} 0 \quad \text{in } \mathcal{O}(U_i,X).$$

Due to the property (β) this implies

$$(f_{i,k}-f_{j,k}) \xrightarrow[k]{} 0 \quad \text{in } \mathcal{O}(U_i \cap U_j, X)$$

for $1 \leq i,j \leq n$. The Čech resolution with respect to the open cover $\mathcal{U}=(U_i)_{i=1}^{n}$ of $U=U_1 \cup \ldots \cup U_n$

$$0 \longrightarrow \mathcal{O}(U,X) \longrightarrow \mathcal{C}^0(\mathcal{U},\mathcal{O}\hat{\otimes} X) \xrightarrow{\delta} \mathcal{C}^1(\mathcal{U},\mathcal{O}\hat{\otimes} X) \xrightarrow{\delta} \ldots$$

is an exact sequence of continuous linear operators between Fréchet spaces. Since the sequence

$$(F_k)_{k=0}^{\infty} = ((f_{i,k}-f_{j,k})_{1\leq i,j\leq n})_{k=0}^{\infty}$$

of 1-cocycles converges to zero, there is a sequence

$$(G_k)_{k=0}^{\infty} = ((g_{i,k})_{1\leq i\leq n})_{k=0}^{\infty}$$

of 0-cochains, which converges to zero and satisfies $\delta G_k=F_k$, i.e.

$$g_{i,k}-g_{j,k} = f_{i,k}-f_{j,k} \quad , 1\leq i,j\leq n, k\geq 0.$$

Hence there is a sequence (f_k) in $\mathcal{O}(U,X)$ with

$$f_k|U_i = f_{i,k}-g_{i,k} \quad , 1\leq i\leq n, k\geq 0.$$

Since $\lim_{k\to\infty}(x_k-T_z(f_k)) = 0$ in $\mathcal{O}(U,X)$, a standard argument:

$$(2\pi i)x_k = \int_\Gamma R(z,T)x_k dz = \int_\Gamma R(z,T)(x_k-T_z f_k(z))dz \xrightarrow[k]{} 0$$

completes the proof.

(ii) $\Rightarrow$ (iii). If $\sigma(T)\subset U_1\cup U_2$ is an open covering by bounded open sets in $\mathbb{C}$, define

$$X = \bigoplus_{i=1,2} A^2(U,X)/\overline{T_z A^2(U,X)}$$

and notice that the operator $\hat{T}\in L(\hat{X})$ acting as

$$([f],[g]) \longmapsto ([zf],[zg])$$

can be regarded as an extension of T via the embedding $J:X\longrightarrow \hat{X}$.

For i=1,2 the spectrum of the multiplication operator

$$M_z : \overline{A^2(U_i,X)} \longrightarrow A^2(U_i,X), \; f\longmapsto zf$$

is given by $\sigma(M_z) = \overline{U}_i$. Since for each $w\in \mathbb{C}\setminus\overline{U}_i$ the resolvent of M_z in w maps $T_z A^2(U,X)$ into itself, we conclude that

$$\sigma(\hat{T}, A^2(U_i,X)/\overline{T_z A^2(U_i,X)}) \subset \overline{U}_i .$$

Thus the proof of Theorem 1.1 is complete.

With minor modifications the same equivalences hold for a continuous linear operator on a Fréchet space with compact spectrum (in the sense of Waelbroeck) in $\mathbb{C}$. The number two of open sets in the statement (iii) may obviously replaced by an arbitrary finite number.

By the above equivalences we learn that normal as well as subnormal operators possess property (β). Other distinguished classes of operators with property (β) will appear in the following sections.

As an application of Theorem 1.1 we present a different proof for a statement proved for the first time in [12] even in the case of commutative n-tuples of operators.

Corollary 1.2 <u>If</u> $T\in L(X)$ <u>has property</u> (β), <u>then</u> f(T) <u>has property</u> (β) <u>for each analytic function</u> $f\in \mathcal{O}(\sigma(T))$.

<u>Proof</u>. Assume that T has property (β) and consider an analytic function f defined on a bounded open neighbourhood U of $\sigma(T)$. If $\mathbb{C}=V_1\cup V_2$ is an open covering, define $U_i=f^{-1}(V_i)$, $X_i = A^2(U_i,X)/\overline{T_z A^2(U_i,X)}$, i=1,2, and notice that with the notations of Theorem 1.1

$$J : X \longrightarrow \hat{X} = X_1\oplus X_2, \; x\longmapsto [x]\oplus[x]$$

is a topological monomorphism which intertwines T and $\hat{T} \in L(\hat{X})$.

Since $\sigma(\hat{T}) \subset \sigma(T) \subset U$ and $\sigma(\hat{T};X_i) \subset \sigma(T) \cap \overline{U_i}$ for $i=1,2$, it follows that

$$Jf(T) = f(\hat{T})J, \quad \sigma(f(\hat{T})|X_i) = \sigma(f(\hat{T}|X_i)) \subset \overline{V_i}$$

for $i=1,2$. Thus we have verified condition (iii) of Theorem 1.1 for the operator $f(T)$.

The converse of Corollary 1.2 is almost true.

Corollary 1.3 <u>Let</u> $T \in L(X)$ <u>be a continuous linear operator and let</u> $f \in \mathcal{O}(U)$ <u>be analytic in an open neighbourhood</u> U <u>of</u> $\sigma(T)$.

<u>If</u> $f(T)$ <u>has property</u> (β) <u>and</u> f <u>is not constant on each component of</u> U, <u>then</u> T <u>has property</u> (β).

<u>Proof</u>. Consider an open set G in $\mathbb{C}$ and a sequence (g_n) in $\mathcal{O}(G,X)$ such that $\lim_{n\to\infty} T_z(g_n) = 0$. Since property (β) is local, we may of course assume that $G \subset U$. For each point $a \in G$ there is a closed disc $D=\overline{D_r(a)} \subset G$ with radius r, such that $K=\partial D$ has empty intersection with the set $\{z \in U;\ f'(z)=0\}$. Hence there are open sets $U_1,\ldots,U_k$ and $V_1,\ldots,V_k$ such that

$$K \subset U_1 \cup \ldots \cup U_k \subset G$$

and such that each restriction $f_i=f|U_i : U_i \longrightarrow V_i$ is biholomorphic.

If $F \in \mathcal{O}(U \times U)$ satisfies $f(z)-f(w)=(z-w)F(z,w)$, $z,w \in U$, then $f(z)-f(T)= F(z,T)(z-T)$ holds on U and

$$(f(z)-f(T))(g_n|U_i) \xrightarrow[n]{} 0$$

in $\mathcal{O}(U_i,X)$ for each $i=1,\ldots,k$. This implies that

$$(w-f(T))g_n \circ f_i^{-1} \xrightarrow[n]{} 0$$

in $\mathcal{O}(V_i,X)$ for each $i=1,\ldots,k$. Since $f(T)$ has property (β) we conclude that $\lim_{n\to\infty}(g_n|U_i) = 0$ in $\mathcal{O}(U_i,X)$ for each $i=1,\ldots,k$. Now it suffices to apply the maximum principle to complete the proof

$$\sup_{z\in D} \|g_n(z)\| = \sup_{z\in K} \|g_n(z)\| \xrightarrow[n]{} 0.$$

We notice at the end of this section that property (β) is highly unstable with respect to perturbations. Unstability with respect to perturbations which are small in norm is obvious since each neighbourhood of zero contains operators without property (β). In [15] there is an example which demonstrates the unstability with respect to compact perturbations.

2. Property (β) and decomposability

An important class of operators which satisfy condition (β) is that of decomposable, or more generally, of subdecomposable operators. The purpose of this section is to prove that any operator with property (β) is subdecomposable. However to construct a decomposable extension we leave the category of Banach spaces.

The original notion of decomposable operator was introduced by C. Foias in 1963, but after some recent progress in that field we may adopt the following apparently weaker definition.

A continuous linear operator T on a separated locally convex space E is said to be decomposable, if for every fifite open cover $(U_i)_{i=1}^n$ of the Riemann sphere $\tilde{C}=C\cup\{\infty\}$ there are closed invariant subspaces E_i for T such that

$$E = \sum_{i=1}^n E_i \ ; \quad \sigma(T|E_i) \subset U_i, 1\leq i\leq n.$$

Here for a continuous linear operator T on the separated locally convex space E we denote by $\sigma(T)$ the spectrum in the sense of Waelbroeck, i.e. the complement in $\tilde{C}$ of the set of those points $\lambda\in\tilde{C}$, for which there exists a neighbourhood U of λ in $\tilde{C}$ such that

(i) z-T has a continuous inverse for every $z\in U\cap C$.

(ii) the set of operators $(z-T)^{-1}$, $z\in U\cap C$, is bounded in the space L(E) of all continuous linear operators on E equipped with the topology of pointwise convergence.

Let Ω be a bounded open set in $\mathbb{C}$. As examples of decomposable operators we mention the multiplication operator M_z with the coordinate on each of the spaces

$$\mathcal{E}(\Omega), \mathcal{E}'(\Omega), \mathcal{D}(\Omega), \mathcal{D}'(\Omega), H^s(\Omega), H^s_0(\Omega), \quad s \in \mathbb{R},$$

where $H^s(\Omega), H^s_0(\Omega)$ are the ordinary Hilbertian Sobolev spaces of order s. Since all these spaces are in a natural way $\mathcal{E}(\mathbb{C})$-modules, the decomposability of M_z follows easily.

In fact, the above examples belong to a subclass of decomposable operators, which will be discussed in the next section. The decomposable operators may behave much worse than our examples, see for instance [17].

As an application of the equivalence (i)$\Leftrightarrow$(iii) in Theorem 1.1 we state for the convenience of the reader the following well-known result.

Proposition 2.1 _Each decomposable operator_ T _on a Fréchet space with_ $\sigma(T) \subset \mathbb{C}$ _satisfies property_ (β).

In [4] we have proved that a Banach space operator T is decomposable if and only if both T and its adjoint T' possess property (β). The result was previously known in the case of reflexive Banach spaces [7].

Before we state the main result of this section we recall the notion of a sheaf model for an operator with property (β). Let X be a Banach space and let $T \in L(X)$ be an operator with property (β). For every open set there is an exact sequence of Fréchet spaces

$$0 \longrightarrow \mathcal{O}(U,X) \xrightarrow{T_z} \mathcal{O}(U,X) \longrightarrow \mathcal{F}(U) \longrightarrow 0,$$

where $\mathcal{F}(U) = \mathcal{O}(U,X)/T_z\mathcal{O}(U,X)$ and the last map is the quotient map.

It is routine to check that the presheaf $U \mapsto \mathcal{F}(U)$ is in fact an analytic Fréchet sheaf with the topologically free resolution

$$0 \longrightarrow \mathcal{O}\hat{\otimes} X \longrightarrow \mathcal{O}\hat{\otimes} X \longrightarrow \mathcal{F} \longrightarrow 0.$$

Here we denote by $\mathcal{O}$ the sheaf of analytic functions on $\mathbb{C}$ and by $\mathcal{O}\hat{\otimes} X$

the sheaf of X-valued analytic functions.

In [12] it was shown that $\mathcal{F}(\mathbb{C})\cong X$ canonically. In particular, the operator T on X corresponds to the multiplication operator with z via this identification. A universal property insures the uniqueness of the sheaf $\mathcal{F}$ associated to T, which was called the <u>sheaf model</u> of T, see [12], [4], [3]. We finally remark that $\operatorname{supp}(\mathcal{F})=\sigma(T)$.

Theorem 2.2 <u>If a continuous linear operator</u> T <u>on a Banach space</u> X <u>satisfies condition</u> (β), <u>then it extends to a decomposable operator on a strict (LF)-space</u>.

<u>Proof</u>. Let T be a Banach space operator which satisfies condition (β) and let $\mathcal{F}$ denote its sheaf model.

Inductively one can define a sequence of finite open coverings $\mathcal{U}_n$, $n\geq 1$, of the compact set $\sigma(T)$ such that for each integer $n\geq 1$:

(i) $1/2^{n+1}< \operatorname{diam}(U)< 1/2^n$, $\operatorname{diam}(U\cap V)< 1/2^{n+2}$ for $U,V\in\mathcal{U}_n$,

(ii) each element of $\mathcal{U}_{n+1}$ is contained in an element of $\mathcal{U}_n$ (which is uniquely determined by (i)),

(iii) each element of $\mathcal{U}_n$ is a union of elements of $\mathcal{U}_{n+1}$.

Write $\mathcal{U}_0$ for the covering consisting of the single set

$$W=\bigcup(U;\ U\in\mathcal{U}_1)$$

and define

$$E_n=\bigoplus_{U\in\mathcal{U}_n}\mathcal{F}(U)\qquad ,n\in\mathbb{N}.$$

Notice that for each n there is a natural map induced by restriction

$$j_n: E_n\longrightarrow E_{n+1}$$

which is a topological monomorphism by a reasoning similar to that used to prove the implication (i)$\Rightarrow$(ii) of Theorem 1.1. The topological isomorphism

$$X \xrightarrow{\sim} \mathcal{F}(W),\quad x\longmapsto [x]\ (\text{cf. } [12])$$

allows to identify X and E_0. Multiplication with the coordinate functions induces a continuous linear operator $\hat{T}_n$ on each E_n. The family $(\hat{T}_n)_n$ determines an operator $\hat{T}$ on the strict (LF)-space

$$E = \operatorname{ind}_n (E_n, j_n).$$

The canonical mappings $E_n \longrightarrow E$ are topological monomorphisms, in particular X can be regarded as a subspace of E. Relative to this identification T is the restriction of $\hat{T}$ onto X.

It is our aim to prove that $\hat{T}$ is a decomposable operator. First, notice that

$$\sigma(\hat{T}) \subset \bigcup_{n \in \mathbb{N}} \sigma(\hat{T}_n) \subset \sigma(T).$$

Let $\mathcal{W} = \{W_1, \ldots, W_k\}$ be an open covering of $\mathbb{C}$. With no loss of generality we may suppose that $W_j \subset W$ for $j=1,\ldots,k$. For a sufficiently small $\varepsilon > 0$ there is a covering $V_1, \ldots, V_k$ of $\sigma(T)$ such that $V_j \Subset W_j$, $1 \le j \le k$, and

$$\min_{j=1}^{k} \left[\operatorname{dist}(V_j, \mathbb{C} \setminus W_j) \right] > \varepsilon .$$

Choose a positive integer n with $1/2^n < \varepsilon$. For each $U \in \mathcal{U}_n$ with $\mathcal{F}(U) \neq 0$ we choose an integer $j_n \in \{1, \ldots, k\}$ such that $U \cap V_{j_n} \neq \emptyset$. If we define for each $p=1,\ldots,k$

$$\mathcal{U}_{n,p} = \{ U \in \mathcal{U}_n ;\ \mathcal{F}(U) \neq 0 \text{ and } j_n = p \} ,$$

$$E_{n,p} = \bigoplus_{U \in \mathcal{U}_{n,p}} \mathcal{F}(U),$$

then we obtain a spectral decomposition for $\hat{T}_n$ relative to $\mathcal{W}$:

$$E_n = \bigoplus_{p=1}^{k} E_{n,p} \ ; \ \sigma(\hat{T}_n | E_{n,p}) \subset \bigcup_{U \in \mathcal{U}_{n,p}} \bar{U} \subset W_p \ , \ 1 \le p \le k.$$

For $m > n$ we define $\mathcal{U}_{m,p}$ inductively according to

$$\mathcal{U}_{m,p} = \{V \in \mathcal{U}_m;\ V \subset U \text{ for some } U \in \mathcal{U}_{m-1,p}\}.$$

Then for $E_{m,p} = \bigoplus_{U \in \mathcal{U}_{m,p}} \mathcal{F}(U)$ we obtain as above

$$E_m = \bigoplus_{p=1}^{k} E_{m,p}\ ,\quad \sigma(\hat{T}_m | E_{m,p}) \subset W_p\ ,\quad m > n, 1 \leq p \leq k.$$

Each $E_{m,p}$ has a natural direct complement $F_{m,p}$ in E_m such that the decomposition $E_m = E_{m,p} \oplus F_{m,p}$ reduces the inductive spectrum $(E_m, j_m)_{m \geq n}$, i.e.

$$j_m(E_{m,p}) \subset E_{m+1,p}\ ,\quad j_m(F_{m,p}) \subset F_{m+1,p}\ .$$

Therefore the canonical map $X_p = \operatorname{ind}_{m \geq n} E_{m,p} \longrightarrow E$ is a topological monomorphism. If we identify X_p and its image, we obtain

$$E = \bigoplus_{p=1}^{k} X_p\ ,\quad \sigma(\hat{T}|X_p) \subset \bigcup_{m \geq n} \sigma(\hat{T}_m | E_{m,p}) \subset W_p\ ,\quad 1 \leq p \leq k,$$

and the proof is complete.

Notice that we even obtained a direct sum decomposition of E relative to the given open covering of $\sigma(\hat{T})$. Let us remark, that the above proof works equally well for continuous linear operators on Fréchet spaces with property (β) and Waelbroeck spectrum contained in $\mathbb{C}$.

3. Property $(\beta)_{\mathcal{E}}$

Our aim is to show that property $(\beta)_{\mathcal{E}}$ plays the same role for generalized scalar operators as Bishop's property (β) does for decomposable operators. To this end we first collect some equivalent descriptions of property $(\beta)_{\mathcal{E}}$.

Let us recall from the introduction that an operator $T \in L(X)$ is said to possess property $(\beta)_{\mathcal{E}}$, if for each open set U in $\mathbb{C}$ the map

$$T_z : \mathcal{E}(U,X) \longrightarrow \mathcal{E}(U,X)$$

is a topological monomorphism. Since there are sufficiently many C^∞-functions with compact support, this condition is equivalent to the fact that

$$T_z: \mathcal{E}(\mathbb{C},X) \longrightarrow \mathcal{E}(\mathbb{C},X)$$

is a topological monomorphism.

For an open set Ω in $\mathbb{C}$ and an integer $n \geq 0$ let $W^n(\Omega,X)$ be the Sobolev type space

$$W^n(\Omega,X) = \left\{ f \in L^2(\Omega,X);\ \bar\partial^j f \in L^2(\Omega,X) \text{ for } j=0,\dots,n \right\},$$

where the derivatives with respect to $\bar z$ are formed in the sense of distributions. It is a Banach space with respect to the norm

$$\| f \|_{W^n,\Omega} = \Big(\sum_{j=0}^{n} \| \bar\partial^j f \|_{2,\Omega}^2 \Big)^{1/2}.$$

If X is a Hilbert space, then $W^n(\Omega,X)$ is a Hilbert space in a natural way.

In the same way as the usual Sobolev embedding theorem is proved one can show that $\mathcal{E}(\Omega,X) \supset \bigcap_{n\in\mathbb{N}} W^n(\Omega,X)$. Consequently $\mathcal{E}(\Omega,X)$ is a Fréchet space together with the seminorms

$$\| \bar\partial^n f \|_{2,U} \quad , n \in \mathbb{N}, U \subset\subset \Omega .$$

The closed graph theorem implies that the topology induced by these seminorms coincides with the usual topology of $\mathcal{E}(\Omega,X)$. The same argument shows that $\mathcal{E}(\Omega,X)$ has the natural representation

$$\mathcal{E}(\Omega,X) = \operatorname*{proj}_{\substack{U \subset\subset \Omega \\ n \in N}} W^n(U,X).$$

In the following proposition we use the notation

$$K_\varepsilon = \{ z \in \mathbb{C};\ \operatorname{dist}(z,X) < \varepsilon \}$$

for a subset K of $\mathbb{C}$.

Proposition 3.1 Let $T \in L(X)$ and let $\Omega \supset \sigma(T)$ be a bounded open set. The following are equivalent:

(i) T satisfies condition $(\beta)_{\mathcal{E}}$.

(ii) For every open disc D in $\mathbb{C}$ and every $\varepsilon > 0$ there are $C > 0$ and $n \in \mathbb{N}$ such that

$$\|f\|_{2,D} \leq C \sum_{k=0}^{n} \|T_z \bar{\partial}^k f\|_{2,D_\varepsilon} \quad , \; f \in \mathcal{E}(\mathbb{C},X).$$

(iii) There are $C > 0$ and $n \in \mathbb{N}$ such that

$$\|f\|_{2,\Omega} \leq C \, \|T_z f\|_{W^n,\Omega} \quad , f \in \mathcal{D}(\Omega, X).$$

Proof. (i)$\Rightarrow$(ii). Fix $\varepsilon > 0$ and let D be an open disc in $\mathbb{C}$. Since $T_z : \mathcal{E}(D_\varepsilon, X) \longrightarrow \mathcal{E}(D_\varepsilon, X)$ is a topological monomorphism, we can find a constant $C > 0$ and an open disc D_δ with $D_\delta \subset\subset D_\varepsilon$ such that

$$\|f\|_{2,D} \leq C \sum_{k=0}^{n} \|T_z \bar{\partial}^k f\|_{2,D_\delta}$$

holds for all $f \in \mathcal{E}(D_\varepsilon, X)$.

(ii)$\Rightarrow$(iii). Simply fix an open disc D which contains Ω, and choose $C > 0, n \in \mathbb{N}$ for $\varepsilon = 1$ as described in (ii).

(iii)$\Rightarrow$(i). Let (f_n) be a sequence in $\mathcal{E}(\mathbb{C},X)$ with $\lim_{n\to\infty} T_z f_n = 0$. If U is an open set with $U \subset\subset \Omega$, choose θ in $\mathcal{D}(\Omega)$ with $\theta = 1$ on U. From $\|f\|_{2,U} \leq \|\theta f\|_{2,\Omega}$ for $f \in L^2(\Omega, X)$ and the estimate described in (iii), we conclude that $\|f_n\|_{2,U}$ tends to zero. Replacing f_n by $\bar{\partial}^k f_n$ we obtain, that $(\|\bar{\partial}^k f_n\|_{2,U})_n$ tends to zero for all $k \in \mathbb{N}, U \subset\subset \Omega$ open. Therefore $\lim_{n\to\infty} f_n = 0$ holds in $\mathcal{E}(\Omega, X)$. But since $\sigma(T) \subset \Omega$ this certainly implies that (f_n) tends to zero in $\mathcal{E}(\mathbb{C},X)$.

Condition (iii) turns out to be useful thanks to the following lemma from [13].

Lemma 3.2 Let Ω be a bounded domain with smooth boundary. There is a constant C_Ω such that for every operator $T \in L(X)$ and every function .

$f \in \mathcal{D}(\Omega, X)$ the following estimate holds:

$$\|f\|_{2,\Omega} \leq C_\Omega \left(\|T_{\bar z} \bar\partial f\|_{2,\Omega} + \|T_{\bar z} \bar\partial^2 f\|_{2,\Omega} \right).$$

The ideea of the proof is the following. The Cauchy-Pompeiu formula yields

$$(f - T_{\bar z} \bar\partial f)(z) = (2\pi i)^{-1} \int_\Omega \frac{T_{\bar\zeta} \bar\partial^2 f(\zeta)}{\zeta - z} d\zeta \wedge d\bar\zeta \quad , z \in \Omega .$$

Then a familiar estimate for the convolution próduct gives the desired inequality.

By combining Lemma 3.2 with proposition 3.1 we can state the following.

Próposition 3.3 Let $T, S \in L(X)$. If there is a constant $C > 0$ such that for every $x \in X$ and $z \in \mathbb{C}$

$$\|(\bar z - S)x\| \leq C \, \|(z-T)x\|$$

holds, then T has property $(\beta)_{\mathcal{E}}$.

An immediate application of Proposition 3.3 is offered by the class of normal operators. Indéed, if N is a normal operator on a Hilbert space H, then $\|N_z^* h\| = \|N_z h\|$ for every $h \in H$ and $z \in \mathbb{C}$. More generally, the proposition applies to M-hyponormal operators T on H, because by definition $\|T_z^* h\| \leq M \cdot \|T_z h\|$ in this case.

4. Subscalar operators

The aim of this section is to show that the class of subscalar operators coincides with the class of operators with property $(\beta)_{\mathcal{E}}$.

Let $T \in L(X)$ with X a Banach space. The operator T is said to be generalized scalar, if there exists a continuous algebra homomorphism

$$\Phi: \mathcal{E}(\mathbb{C}) \longrightarrow L(X)$$

with $\Phi(1)=I$ and $\Phi(z)=T$. The map Φ, while not necessarily uniquely determined by T, is called a spectral distribution for T. Due to the fact that there are smooth partitions of unity, every generalized scalar operator is decomposable, in particular satisfies condition (β). For a thourough discussion of the properties of generalized scalar operators see [2].

Due to the continuity of Φ there are $n \in \mathbb{N}, C > 0$ and a bounded open set $\Omega \supset \sigma(T)$ in $\mathbb{C}$ such that

$$\|\Phi(f)\| \leq C \|f\|_{W^n,\Omega} \quad , f \in \mathcal{E}(\mathbb{C},X).$$

Since each function in $W^n(\Omega)$ which vanishes outside a compact subset of Ω is the limit in $W^n(\Omega)$ of a sequence of functions belonging to $\mathcal{D}(\Omega)$ (cf. Lemma 31.1 in [16]) and since the support of Φ is precisely the set $\sigma(T)$, the spectral distribution Φ induces canonically a continuous linear operator

$$\Phi: W^n(\Omega) \longrightarrow L(X)$$

again denoted by Φ. The unique continuous linear operator

$$\Psi: W^n(\Omega)\hat{\otimes}_\pi X \longrightarrow X$$

with $\Psi(f \otimes x) = \Phi(f)x$ satisfies the relation

$$\Psi(zf) = T\Psi(f) \quad \text{for all } f \in W^n(\Omega)\hat{\otimes}_\pi X.$$

As examples of generalized scalar operators we mention the multiplication operators M_z with the coordinate function on the function spaces $W^n(\Omega), C^p(\bar{\Omega}), L^p(\Omega)$ or any Banach function space which is in a natural way an $\mathcal{E}(\mathbb{C})$-module.

An operator similar to the restriction of a generalized scalar operator to a closed invariant subspace is called subscalar. Notice that every subscalar operator has property (β).

Theorem 4.1 <u>A continuous linear operator on a Banach space is subscalar if and only if it satisfies condition</u> $(\beta)_{\mathcal{E}}$.

<u>Proof. The sufficiency.</u>

Let $T \in L(X)$ be an operator with $(\beta)_{\mathcal{E}}$. First we prove that the canonical map

$$J : X \longrightarrow \mathcal{E}(\mathbb{C},X)/\overline{T_z \mathcal{E}(\mathbb{C},X)} \quad ,x \mapsto [x]$$

is a topological isomorphism. To this end let (x_n) be a sequence in X such that $\lim_{n\to\infty} (x_n + T_z f_n) = 0$ in $\mathcal{E}(\mathbb{C},X)$ for a sequence (f_n) in $\mathcal{E}(\mathbb{C},X)$. It follows that $\lim_{n\to\infty}(T_z \bar{\partial} f_n) = 0$ and hence by assumption that $\lim_{n\to\infty} \bar{\partial} f_n = 0$. Consequently, there exists a sequence (g_n) in $\mathcal{O}(\mathbb{C},X)$ with the property $\lim_{n\to\infty} (f_n - g_n) = 0$, which implies $\lim_{n\to\infty} (x_n + T_z g_n) = 0$ in $\mathcal{O}(\mathbb{C},X)$. Let

$$\gamma : \mathcal{O}(\mathbb{C},X) \longrightarrow X$$

be the unique continuous linear map with $\gamma(f \otimes x) = f(T)x$ for $f \in \mathcal{O}(\mathbb{C})$ and $x \in X$. Since $\gamma(T_z g) = 0$ for all $g \in \mathcal{O}(\mathbb{C},X)$, it follows that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \gamma(x_n + T_z g_n) = 0.$$

On the other hand, if J is a topological monomorphism, then we can find a bounded open set $\Omega \supset \sigma(T)$ and $n \in \mathbb{N}, C > 0$ with

$$(1) \quad \|x\| \le C.\inf\left\{\|x + T_z f\|_{W^n,\Omega} ; f \in \mathcal{E}(\mathbb{C},X)\right\}$$

for all $x \in X$. Then the canonical map

$$\tilde{J} : X \longrightarrow \hat{X} = W^n(\Omega,X)/\overline{T_z W^n(\Omega,X)}, \quad x \mapsto [x]$$

is a topological monomorphism. To see this fix $\theta \in \mathcal{E}(\mathbb{C})$ with $\theta = 0$ near $\sigma(T)$ and $1 - \theta \in \mathcal{D}(\Omega)$. Regard $H = I - T_z \theta R(z,T)$ as a function in $\mathcal{D}(\Omega, L(X))$ and notice that $\lim_{n\to\infty} (x_n - T_z f_n) = 0$, $f_n \in W^n(\Omega,X)$ for all n, implies that

$$\lim_{n\to\infty} (x_n - T_z(\theta R(z,T)x_n + Hf_n)) = 0$$

in $W^n(\Omega,X)$. But Hf_n vanishes outside a compact subset of Ω and therefore belongs to the closure of $\mathcal{D}(\Omega,X)$ in $W^n(\Omega,X)$. Hence (1) implies that $\lim_{n\to\infty} x_n = 0$. Since $\tilde{J}$ intertwines T and the generalized scalar operator $\hat{T}$ induced on $\hat{X}$ by the multiplication with the coordinate, we have thus shown T to be subscalar.

The necessity.

It suffices to rpove that a generalized scalar operator $T \in L(X)$ satisfies condition $(\beta)_{\mathcal{E}}$. Since the adjoint $T' \in L(X')$ is also generalized scalar, the arguments from the beginning of this section applied to T' yield a bounded open set Ω in $\mathbb{C}$ with $\sigma(T) \subset \Omega$ and an integer $n \geq 0$ such that there is a continuous linear operator

$$\Psi: W^n(\Omega)\hat{\otimes}_\pi X' \longrightarrow X'$$

satisfying

$$\Psi(zf) = T'\Psi(f) \quad \text{for all } f \in W^n(\Omega)\hat{\otimes}_\pi X'.$$

Let us consider a smaller open set ω with the properties $\sigma(T) \subset \omega \subset\subset \Omega$. It suffices to show that the map

$$T_W : \mathcal{E}(\omega,X) \longrightarrow \mathcal{E}(\omega,X)$$

is a topological monomorphism, or equivalently that its dual

$$T'_W : \mathcal{E}(\omega)'\hat{\otimes} X' \longrightarrow \mathcal{E}(\omega)'\hat{\otimes} X'$$

is onto (for the identification $\mathcal{E}(\omega,X)' \cong \mathcal{E}'(\omega)\hat{\otimes} X'$ see Th.12 in Ch.2, § 3, n°2 of [5]).

In order to prove that T'_W is onto we shall look at the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E}'(\omega)\hat{\otimes} W^n(\Omega)\hat{\otimes}_\pi X' & \xrightarrow{(w-z)\otimes I} & \mathcal{E}'(\omega)\hat{\otimes} W^n(\Omega)\hat{\otimes}_\pi X' \\ \downarrow I\otimes\Psi & & \downarrow I\otimes\Psi \\ \mathcal{E}'(\omega)\hat{\otimes} X' & \xrightarrow{T'_W} & \mathcal{E}'(\omega)\hat{\otimes} X' \end{array}$$

The map ψ has $X' \longrightarrow W^n(\Omega)\hat{\otimes}_\pi X'$, $x' \mapsto 1\otimes x'$, as a right inverse, hence $I\otimes\psi$ is onto. If we manage to prove that $(w-z)\otimes I$ is onto, then the proof will be finished. First we isolate the following observation.

Lemma 4.2 <u>Let $\omega\subset\mathbb{C}$ be open, H a Hilbert space and</u> $h\in L(\mathcal{E}(\omega)\hat{\otimes}H')$. <u>If h' is surjective, then the same is true for</u>

$$h'\otimes I : \mathcal{E}'(\omega)\hat{\otimes}H\hat{\otimes}_\pi X' \longrightarrow \mathcal{E}'(\omega)\hat{\otimes}H\hat{\otimes}_\pi X',$$

<u>where</u> X <u>is an arbitrary Banach space.</u>

<u>Proof</u>. The surjectivity of h' is equivalent to the fact that h is a topological monomorphism, and therefore implies that

$$h\otimes I : \mathcal{E}(\omega)\hat{\otimes}H'\hat{\otimes}_\varepsilon X \longrightarrow \mathcal{E}(\omega)\hat{\otimes}H'\hat{\otimes}_\varepsilon X$$

is a topological monomorphism [6], §44.4.(6). But the dual of the Banach space $H'\hat{\otimes}_\varepsilon X$ is isometrically isomorphic to $H\hat{\otimes}_\pi X'$ [6], §45.6.(5). Therefore the surjectivity of h' implies that of $h'\otimes I$ on $\mathcal{E}'(\omega)\hat{\otimes}H\hat{\otimes}_\pi X'$.

Thus we have reduced the proof of Theorem 4.1 to the proof of the surjectivity of the map $(w-z)$: $\mathcal{E}'(\omega)\hat{\otimes}W^m(\Omega) \longrightarrow \mathcal{E}'(\omega)\hat{\otimes}W^m(\Omega)$ for any non-negative integer m. To solve this problem we make use of a standard representation of $\mathcal{E}'(\omega)$.

Assume that E is a nuclear locally convex space which is represented as the limit of a reduced countable projective system

$$E = \operatorname{proj}_k H_k$$

of Hilbert spaces. For each Hilbert space K there are unique topological isomorphisms [6], §41.6.(3) and §44.5.(5)

$$E\hat{\otimes}K \xrightarrow{\sim} \operatorname{proj}_k (H_k\hat{\otimes}_\pi K) \text{ (resp. } \operatorname{proj}_k (H_k\hat{\otimes}_\varepsilon K))$$

acting as $(x_k)\otimes y \mapsto (x_k\otimes y)$ on elementary tensors. In particular, the composition of the following natural maps

$$E\hat{\otimes}K \longrightarrow \underset{k}{\text{proj}}\, H_k\hat{\otimes}_\pi K \longrightarrow \underset{k}{\text{proj}}\, H_k\hat{\otimes}_\sigma K \longrightarrow \underset{k}{\text{proj}}\, H_k\hat{\otimes}_\varepsilon K,$$

where $\hat{\otimes}_\sigma$ denotes the Hilbertian tensor product, is a topological isomorphism. Hence the last last map is surjective. Since the Hilbertian tensor product is faithful, it is also injective.

It follows that the canonical map

$$E\hat{\otimes}K \longrightarrow \underset{k}{\text{proj}}\,(H_k\hat{\otimes}_\sigma K)$$

is a topological isomorphism. Since the projective system on the right is reduced and defines a reflexive space, by standard duality results

$$E'\hat{\otimes}K' = \underset{k}{\text{ind}}\,(H_k'\hat{\otimes}_\sigma K')$$

holds topologically [14] Th.IV.4.4 .

In our case K' will be the Hilbert space $A^2(\Omega)$ of all square integrable functions on an open set Ω in $\mathbb{C}$. Then the above identification becomes

$$E'\hat{\otimes}A^2(\Omega) = \underset{k}{\text{ind}}\, A^2(\Omega, H_k'),$$

because $H_k'\hat{\otimes}_\sigma A^2(\Omega) = A^2(\Omega, H_k')$ in a canonical way.

We need the following Banach space variant of a result in [12]:

Lemma 4.3 <u>Let</u> $T \in L(X)$ <u>be a Banach space operator and let</u> U <u>be a bounded open neighbourhood of</u> $\sigma(T)$ <u>in</u> $\mathbb{C}$. <u>Then</u>

$$J : X \longrightarrow A^2(U,X)/T_z A^2(U,X), \quad x \mapsto [x]$$

<u>is a topological isomorphism, which is the inverse of</u>

$$\hat{\Phi}: A^2(U,X)/T_z A^2(U,X) \longrightarrow X, \quad [f] \mapsto \gamma(f).$$

<u>Proof.</u> Since the composition

$$X \xrightarrow{J} A^2(U,X)/T_z A^2(U,X) \xrightarrow{i} \mathcal{O}(U,X)/T_z\mathcal{O}(U,X),$$

where $i[f]=[f]$ for $f \in A^2(U,X)$, is a topological isomorphism by [12], the operator J is at least a topological monomorphism. Because of $\sigma(T)\subset U$ the map i is injective, which in turn implies the surjectivity of J. But $\hat{\phi}$ is obviously a left inverse for J and hence also a right inverse.

Now as before, consider bounded open sets ω,Ω in $\mathbb{C}$ with $\omega\subset\subset\Omega$. We make use of the representation:

$$\mathcal{E}(\omega) = \operatorname*{proj}_{\substack{k\in\mathbb{N}\\ U\subset\subset\omega}} W^k(U).$$

As explained above this leads to

$$\mathcal{E}'(\omega)\hat{\otimes}A^2(\Omega) = \operatorname*{ind}_{\substack{k\in\mathbb{N}\\ U\subset\subset\omega}} A^2(\Omega,W^k(U)').$$

For each k and U the spectrum of the multiplication operator

$$T=T_k : W^k(U)\longrightarrow W^k(U);\quad f\longmapsto wf$$

is contained in Ω. Therefore all the sequences (cf. Lemma 4.3)

$$0\longrightarrow A^2(\Omega,W^k(U)')\xrightarrow{z-T'} A^2(\Omega,W^k(U)')\longrightarrow W^k(U)'\longrightarrow 0$$

are exact. Forming the inductive limit we obtain the exact sequence

$$0\longrightarrow \mathcal{E}'(\omega)\hat{\otimes}A^2(\Omega)\xrightarrow{w-z}\mathcal{E}'(\omega)\hat{\otimes}A^2(\Omega)\longrightarrow\mathcal{E}'(\omega)\longrightarrow 0.$$

Lemma 4.4 <u>For each pairs of bounded open sets $\omega\subset\subset\Omega$ and every integer $m\geq 0$, the map</u>

$$w-z : \mathcal{E}'(\omega)\hat{\otimes}W^m(\Omega)\longrightarrow\mathcal{E}'(\omega)\hat{\otimes}W^m(\Omega)$$

<u>is onto.</u>

<u>Proof.</u> The assertion is proved by induction on m. For m=0 the statement

follows from the observation that normal operators have property $(\beta)_{\mathcal{E}}$ using duality, see Proposition 3.3.

For the proof of the induction step we consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K_3 & \longrightarrow & \mathcal{E}'(\omega)\hat{\otimes} W^{m-1}(\Omega) & \xrightarrow{w-z} & \mathcal{E}'(\omega)\hat{\otimes} W^{m-1}(\Omega) & \longrightarrow & 0 \\
 & & & & \uparrow \bar{\partial}_z & & \uparrow \bar{\partial}_z & & \\
0 & \longrightarrow & K_2 & \longrightarrow & \mathcal{E}'(\omega)\hat{\otimes} W^{m}(\Omega) & \xrightarrow{w-z} & \mathcal{E}'(\omega)\hat{\otimes} W^{m}(\Omega) & \longrightarrow & C_2 \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{E}'(\omega)\hat{\otimes} A^{2}(\Omega) & \xrightarrow{w-z} & \mathcal{E}'(\omega)\hat{\otimes} A^{2}(\Omega) & \longrightarrow & C_1 \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Here C_1 stands for $\mathcal{E}'(\omega)$.

By the well-known serpent's lemma there is a natural exact sequence

$$K_2 \longrightarrow K_3 \xrightarrow{d} C_1 \longrightarrow C_2 \longrightarrow 0.$$

Our aim is to prove that the coboundary operator d is surjective. This suffices to finish the proof of the induction step, since then $C_2=0$.

We make use of the natural topological identification

$$\mathcal{E}'(\omega)\hat{\otimes} W^{m-1}(\Omega) = L_b(\mathcal{E}(\omega), W^{m-1}(\Omega)),$$

see [16] p.525. For given elements $a \in W^{m-1}(\Omega)$, $k \in \mathbb{N}$, $\varkappa \in \mathcal{D}(\omega)$ the operator

$$\alpha: \mathcal{E}(\omega) \longrightarrow W^{m-1}(\Omega), \quad \varphi \longmapsto \bar{\partial}^k(\varkappa\varphi)a$$

belongs modulo the above identification to the space K_3:

$$((w-z)\alpha)\varphi = \alpha(w\varphi) - z\alpha(\varphi) = 0.$$

If $\beta \in \mathcal{E}'(\omega)\hat{\otimes} W^m(\Omega)$ is a solution of $\bar{\partial}_z \beta = \alpha$, then $d\alpha$ is the image of $(w-z)\beta$ in C_1. It is standard to check that

$$(\beta(\varphi))(z) = (2\pi i)^{-1} \int_{\mathbb{C}} \frac{\bar{\partial}^k(\varkappa\varphi)(\zeta)a(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

is well defined for almost all $z \in \Omega$ and defines an element $\beta \in \mathcal{E}'(\omega)\hat{\otimes} W^m(\Omega)$ such that $\bar{\partial}_z \beta = \alpha$.

For almost all $t \in \Omega$ we obtain

$$[((w-z)\beta)(\varphi)](t) = (2\pi i)^{-1} \int_{\mathbb{C}} \frac{(\zeta - t)\bar{\partial}^k(\varkappa\varphi)(\zeta)a(\zeta)}{\zeta - t}\, d\zeta \wedge d\bar{\zeta} =$$

$$= (-1)^{k+1}\pi^{-1}(\varkappa\bar{\partial}^k a)(\varphi),$$

where the last equality is just the definition of the distribution $\varkappa\bar{\partial}^k a \in \mathcal{E}'(\omega)$. Using tensor product notation this means

$$(w-z)\beta = (-1)^{k+1}\pi^{-1}(\varkappa\bar{\partial}^k a) \otimes 1 \in \mathcal{E}'(\omega)\hat{\otimes} W^m(\Omega).$$

The definitions preceding Lemma 4.3 therefore show that

$$d\alpha = (-1)^{k+1}\pi^{-1}(\varkappa\bar{\partial}^k a).$$

But each distribution $u \in \mathcal{E}'(\omega)$ is of the form $u = \varkappa\bar{\partial}^k a$ for suitable $\varkappa$, k and a as above, hence the coboundary operator d is onto and the proof of Theorem 4.1 is complete.

We summarize below for the convenience of the reader some of the equivalences proved in the last two sections.

Proposition 4.5 *Let* $T \in L(X)$. *The following are equivalent:*

a) T *is subscalar.*

b) $J : X \longrightarrow \mathcal{E}(\mathbb{C},X)/T_z\,\mathcal{E}(\mathbb{C},X)$, $x \mapsto [x]$, *is a topological monomorphism.*

c) *For each* $x' \in X'$ *there is a distribution* $u \in \mathcal{E}(\mathbb{C},X)'$ *such that*

$(z-T')u = 0$ *and* $u(1) = x'$.

d) *T satisfies* $(\beta)_{\mathcal{E}}$.

e) *For every bounded neighbourhood* Ω *of* $\sigma(T)$ *there are* $C>0$ *and* $n\in\mathbb{N}$ *such that*

$$\|f\|_{2,\Omega}\leq C\,\|T_z f\|_{W^n,\Omega}\ ,\quad f\in\mathcal{D}(\Omega,X).$$

Proof. The implications d)$\Rightarrow$b) and b)$\Rightarrow$a) are contained in the proof of Theorem 4.1.

We have to explain the notation in part c). The adjoint of the embedding of X into $\mathcal{E}(\mathbb{C},X)$ is simply the map $1\otimes I:\mathcal{E}'(\mathbb{C})\hat{\otimes}X'\longrightarrow\mathbb{C}\hat{\otimes}X'\cong X'$. For $u\in\mathcal{E}'(\mathbb{C})\hat{\otimes}X'$ we write $u(1)$ instead of $1\otimes I(u)$. Thus part c) is just the dualized version of part b).

Remarks. 1) The proof of the sufficiency part of Theorem 4.1 shows that a subscalar operator on a Hilbert space can be extended to a generalized scalar operator on a Hilbert space.

2) If T is a subscalar operator on a Banach space X and $\hat{T}\in L(\hat{X})$ is a generalized scalar extension constructed as in the proof of the sufficiency part of Theorem 4.1, then

$$\partial\sigma(T)\subset\sigma_{\pi}(T)\subset\sigma_{\pi}(\hat{T})\subset\sigma(\hat{T})\subset\sigma(T).$$

Here $\sigma_{\pi}(S)$ denotes the approximate point spectrum of an operator S.

3) We have already noticed that a Banach space operator T is decomposable, if and only if T and T' both possess property (β). It would be interesting to know whether the corresponding result for property $(\beta)_{\mathcal{E}}$ is true, i.e. whether T is generalized scalar, if and only if T and T' both possess property $(\beta)_{\mathcal{E}}$.

4) Notice the equality between the order of a generalized scalar extension of a subscalar operator and the integer n in the estimate e) in Proposition 4.5.

5. Applications

I). The following theorem of Malgrange [11] is well-known:

Let $\mathcal{F}$ be an analytic coherent sheaf on a complex manifold. Then $\mathcal{F}\otimes_{\mathcal{O}}\mathcal{E}$ is a sheaf of Fréchet spaces.

Our aim is to prove that this statement is no longer true for non-coherent sheaves, even they are generalized coherent in a natural topological sense. More precisely, we prove the following.

Lemma 5.1 There is an analytic function $f \in \mathcal{O}(\mathbb{C}, L(H))$ with H a Hilbert space, such that for every open set $V \subset \mathbb{C}$ the induced multiplication operator $F: \mathcal{O}(V,H) \longrightarrow \mathcal{O}(V,H)$ is one to one with closed range, but there are open sets U in $\mathbb{C}$ such that $F: \mathcal{E}(U,H) \longrightarrow \mathcal{E}(U,H)$ has no longer closed range.

If we consider the cokernel $\mathcal{F}$ of the map F at the level of sheaves, then one gets the exact sequence

$$0 \longrightarrow \mathcal{O}\hat{\otimes} H \longrightarrow \mathcal{O}\hat{\otimes} H \longrightarrow \mathcal{F} \longrightarrow 0.$$

Thus $\mathcal{F}$ is a Banach coherent analytic Fréchet sheaf in the terminology of Leiterer [8] or quasicoherent in the terminology of the French school. Lemma 5.1 states that a natural topological tensor product $\mathcal{F}\hat{\otimes}_{\mathcal{O}}\mathcal{E}$ which extends the algebraic tensor product, when $\mathcal{F}$ is coherent, is not a sheaf of Fréchet spaces.

Proof of Lemma 5.1. Let T be a quasinilpotent operator on the Hilbert space H which is not nilpotent. We define $f(z)=z-T$ for $z \in \mathbb{C}$.

The statement is equivalent to the assertion that T has property (β), but not property $(\beta)_{\mathcal{E}}$.

Since $\sigma(T)=\{0\}$, condition (β) is trivially satisfied. If T would possess property $(\beta)_{\mathcal{E}}$, then it would have a quasinilpotent generalized scalar extension. As every quasinilpotent generalized scalar operator is nipotent, T would be nilpotent, too.

Of course the Volterra operator defined on $L^2[0,1]$ by

$$(Tf)(x) = \int_0^x f(t)dt$$

can be choosen for T in the above proof.

II) Theorem 4.1 provides in its dual version a very general result concerning the division of vector valued distributions by certain linear functions. It is our aim to present in the sequel a couple of particular cases of this abstract division theorem. Thus we find again by this way the division theorem of distributions in a domain of $\mathbb{C}^n$ by complex analytic functions. The reader will easily imagine other similar applications of Theorem 4.1 which may be of an independent interest.

Proposition 5.2 Let Ω be a domain of $\mathbb{R}^n, n \geq 1$, and let $H^d(\Omega)$ denote the Sobolev space of order $d \in \mathbb{Z}$. For any function $f \in C^{|d|}(\overline{\Omega})$, the map

$$z-f(w) : \mathcal{D}'(\mathbb{C}) \hat{\otimes} H^d(\Omega) \longrightarrow \mathcal{D}'(\mathbb{C}) \hat{\otimes} H^d(\Omega)$$

is onto.

Proof. The multiplication operator M_f is generalized scalar operator on $H^d(\Omega)$. Moreover, M_f is obviously the dual operator of the generalized scalar operator M_f' acting on the (pre)dual of the Hilbert space $H^d(\Omega)$. By Theorem 4.1 the application

$$z-f(w) : \mathcal{E}'(\mathbb{C}) \hat{\otimes} H^d(\Omega) \longrightarrow \mathcal{E}'(\mathbb{C}) \hat{\otimes} H^d(\Omega)$$

is onto.

An argument based on the smooth partition of unity ends the proof.

Let us point out that the space $\mathcal{D}'(\mathbb{C})$ cannot be replaced in the above proposition by a smaller space. For instance the application

$$w-z : H^{-r}_{loc}(\mathbb{C})\hat{\otimes}_\pi H^s(\Omega) \longrightarrow H^{-r}_{loc}(\mathbb{C})\hat{\otimes}_\pi H^s(\Omega).$$

is not onto whenever $s \gg r > 0$. Indeed, in that case the range of the map $w-z$ lies into the kernel of the natural restriction and multiplication application

$$H^{-r}_{loc}(\mathbb{C})\hat{\otimes}_\pi H^s(\Omega) \longrightarrow H^{-r}(\Omega), \quad u\otimes f \longmapsto f(u|_\Omega),$$

which is not trivial.

Proposition 5.3 <u>Let</u> $K \subset \mathbb{R}^n$ <u>be a compact set</u>, $1 \leq p \leq \infty$ <u>and let</u> P <u>be a monic polynomial with coefficients in</u> $L^\infty(K)$:

$$P(z,w) = z^m + a_1(w)z^{m-1} + \dots + a_m(w), \quad a_j \in L^\infty(K), 1 \leq j \leq m.$$

<u>Then the multiplication map</u>

$$\mathcal{D}'(\mathbb{C})\hat{\otimes} L^p(K) \xrightarrow{P} \mathcal{D}'(\mathbb{C})\hat{\otimes} L^p(K)$$

<u>is onto.</u>

<u>Proof</u>. First assume that the polynomial P has order one, namely $P(z,w)= z-f(w)$ where $f \in L^\infty(K)$. The operator M_f is generalized scalar on $L^p(K)$ and it is the dual of a generalized scalar operator acting on the predual of the Banach space $L^p(K)$. Then we conclude as before that the map

$$z-f(w) : \mathcal{D}'(\mathbb{C})\hat{\otimes} L^p(K) \longrightarrow \mathcal{D}'(\mathbb{C})\hat{\otimes} L^p(K)$$

is onto.

Since the polynomial P can be decomposed into factors:

$$P(z,w) = (z-f_1(w))\dots(z-f_m(w)),$$

with $f_j \in L^\infty(K)$, $1 \leq j \leq m$, the proof of the proposition reduces to the above case.

The decomposition of P into linear factors runs as follows. Consider the application

$$\sigma:\mathbb{C}^m \longrightarrow \mathbb{C}^m,\ \sigma(z)=(\sigma_1(z),\ldots,\sigma_m(z)),$$

where σ_k denote the fundamental symmetric polynomials in $z_1,\ldots,z_m$. The local structure of the finite algebraic coverings, as σ above (see for instance [19]), shows that the application σ has a measurable right inverse. This inverse map gives the parametrized roots $f_j(w), 1 \leq j \leq m$.

Thus the proof of Proposition 5.2 is complete.

Our next aim is to derive from the last proposition a proof of the following well-known division theorem, see [9] and [10] for further information and the original proofs even in the real analytic case.

Corollary 5.4 <u>Let Ω be a domain of $\mathbb{C}^n, n \geq 1$, and let $f \in \mathcal{O}(\Omega)$ be an analytic function, not identically equal to zero. Then the multiplication map</u>

$$\mathcal{D}'(\Omega) \xrightarrow{\ f\ } \mathcal{D}'(\Omega)$$

<u>is onto.</u>

<u>Proof</u>. The problem is obviously local. Fix a point $a \in \Omega$. By Weierstrass Preparation Lemma [19] Ch.1 §5, there exists a linear change of coordinates and an open neighbourhood U of a in Ω, such that in the new coordinates

$$f(z,w) = g(z,w)P(z,w)\ ,\quad (z,w) \in U \cap \mathbb{C}\times\mathbb{C}^{n-1},$$

where $g \in \mathcal{O}(U)$ is nowhere vanishing on U and P is a monic polynomial in z:

$$P(z,w) = z^m + a_1(w)z^{m-1} + \ldots + a_m(w).$$

Since our problem is invariant to linear changes of coordinates, it remains to be proved that the multiplication with P is onto on the space $\mathcal{D}'(U)$.

After shrinking U to a domain like $U=V\times W$, such that the coefficients a_j belong to $L^\infty(W)$ for every $1\le j\le m$, we take a distribution $u\in\mathcal{D}'(U)$. By the local solvability and the regularity of the $\bar\partial$-operator, we may express u as a linear combination of distributions of the form

$$\bar\partial_1^{\alpha_1}\dots\bar\partial_n^{\alpha_n} g,$$

with $g\in L^2_{loc}(U)$ and $\alpha_1+\dots+\alpha_n$ large enough.

But the function g can be regarded equally as an element of the space $\mathcal{D}'(V)\hat\otimes L^2(K)$, where K is an arbitrary compact subset of W. Thus Proposition 5.3 applies and yields a solution of the equation $Pf=g$ on $V\times K$. Since the multiplication with P commutes with the operators $\bar\partial_j$, $1\le j\le n$, by putting together the distributions $\bar\partial_1^{\alpha_1}\dots\bar\partial_n^{\alpha_n}f$ we get a solution $v\in\mathcal{D}'(V\times K)$ of the equation $Pv=u$. This completes the proof of the corollary.

Let us remark that, since only the space $L^2(K)$ has been involved in the proof of the corollary, only Lemma 3.2 is needed in this proof.

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