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# A TIME DEPENDENT SCATTERING THEORY FOR STRONGLY PROPAGATIVE SYSTEMS WITH PERTURBATIONS OF SHORT-RANGE CLASS

by

ARSU GRUIA

## 1. INTRODUCTION

In this paper we prove the asymptotic completeness for strongly propagative systems with perturbations of short-range class by means of time dependent methods. Using a suitable modification of the thechniques, developed by E. Mourre in [3], we introduce a similar decomposition of the identity $1 = P^{+} + P^{-}$, and we prove a basic estimate which enable us to prove the asymptotic completeness and the discreteness of point spectrum in $\mathbb{R}\setminus\{0\}$.

We shall formulate the problem to be discussed here with several assumptions. The operators to be considered are given in the following form:

$$\Lambda = E(x)^{-1} \sum_{j=1}^{n} A_j D_j \ , \tag{1.1}$$

$$\Lambda_o = \sum_{j=1}^{n} A_j D_j \ , \tag{1.2}$$

where $D_j = -i\partial/\partial x_j$, $E(x)$ and $A_j$, $j=1,\dots,n$ are $m \times m$ Hermitian matrices satisfying the following assumptions:

(A.1) $E(x)$ and the derivatives $D_1E(x),\dots,D_jE(x)$ are continuous and bounded on $\mathbb{R}^n$. Moreover, there exist positive constants $c$ and $c'$ such that

(1.3) $$cI \leqq E(x) \leqq c'I \quad \text{for all} \quad x\in\mathbb{R}^n;$$

(A.2) There exists $\varepsilon>0$ such that $|E(x)-I|=0(|x|^{-1-\varepsilon})$ as $|x|\to\infty$ ;

(A.3) The symbol $\Lambda(x,\xi)=E(x)^{-1}\sum_{j=1}^{n} A_j\xi_j$ satisfies

(1.4) $$\operatorname{rank}\Lambda(x,\xi)=m-k \quad \text{for all } x\in\mathbb{R}^n \text{ and } \xi\in\mathbb{R}^n\setminus\{0\}$$

Let $\mathcal{H}$ denote the Hilbert space of all measurable $\mathbb{C}^k$ valued functions $u$, defined on $\mathbb{R}^n$, such that

$$||u||^2_{\mathcal{H}} = \int_{\mathbb{R}^n} E(x)u(x)\cdot u(x)dx<\infty$$

and $\mathcal{H}_o$ the Hilbert space $L^2(\mathbb{R}^n)^k$ with the usual norm. Then we can easily show that the operators $\Lambda$ and resp. $\Lambda_o$ defined by (1.1) and resp. (1.2) have some natural self-adjoint realizations (denoted by the same symbols $\Lambda$ and resp. $\Lambda_o$) in $\mathcal{H}$ and resp. $\mathcal{H}_o$ with the domains given by $\mathcal{D}(\Lambda)=\mathcal{D}(\Lambda_o)=\{u\in\mathcal{H}_o;\Lambda_o u\in\mathcal{H}_o\}$ (see [3]) .

Let $U(t)=e^{-i\Lambda t}$ resp. $U_o(t)=e^{-i\Lambda_o t}$ be the one-parameter unitary groups in $\mathcal{H}$ resp. $\mathcal{H}_o$ generated by $\Lambda$ resp. $\Lambda_o$. The wave operators $W_+$, $W_-$ associated with the groups $U_o(t)$ and $U(t)$ are defined by

(1.5) $$W_{\pm}=s\text{-}\lim_{t\to\pm\infty} U(-t)JU_o(t)P^o$$

where $P^o=I-P_o$ is the projection of $\mathcal{H}_o$ onto $(\ker\Lambda_o)^{\perp}$ and $J$ is the identification operator of $\mathcal{H}_o$ onto $\mathcal{H}$:Ju=u. It has been shown in [1] that $(\ker \Lambda_o)^{\perp} = \mathcal{H}^o_{ac}$, the subspace of absolute continuity of $\Lambda_o$ in $\mathcal{H}_o$ .

The main result is the following,

THEOREM 1.1. Assume that the hypotheses (A.1)-(A.3) are satisfied. Then

(i) The wave operators $W_{\pm}$ exist,

(ii) Range $W_{\pm} = \mathcal{H}_c(\Lambda)$, the continuous subspace of $\Lambda$ in $\mathcal{H}$ ,

(iii) In $R\setminus\{0\}$ the eigenvalues of $\Lambda$ are discrete and of finite multiplicity with possible accumulating points 0 and $\pm\infty$ .

Proof. (i) The existence of $W_{\pm}$ was proved in [2] under more general hypotheses.

The other parts of Theorem 1.1 will be proved below by means of time-dependent methods.

## 2. DECOMPOSITION OF THE IDENTITY AND THE BASIC ESTIMATE

Let A be the well known dilatation group generator on $\mathcal{H}_o$. Denote by $P^+$ and $P^-$ the spectral projectors of A on the positive and negative parts of its spectrum.

Let $\Lambda_o$ be a self-adjoint operator on $\mathcal{H}_o$ such that

(2.1) $\quad e^{+iAt}\Lambda_o e^{-iAt} = e^{-\alpha t}\Lambda_o \quad$ for some $\alpha > 0$.

Denote by $\chi^{\pm} = \chi_{\mathbb{R}^{\pm}\setminus\{0\}}$.

THEOREM 2.1. Let $g \in C_o^{\infty}(\mathbb{R}^+\setminus\{0\})$ and $0 \leq \mu' < \mu$. Then there is a constant c (depending on g, $\mu, \mu'$) such that

(2.2) $\quad ||\chi^{\pm}(t)|A+i|^{-\mu} e^{-i\Lambda_o t} g(\Lambda_o) P^{\pm}|| \leq c|t|^{-\mu'}$

Proof

1°. Let $0<r<R$ such that $\operatorname{supp}(g)\subset(0,r)$. By Cauchy's integral representation, we obtain

$$e^{-i\Lambda_o t}g(\Lambda_o)=(2\pi i)^{-1}\int_C e^{-izt}(z-\Lambda_o)^{-1}g(\Lambda_o)dz=$$

$$=(-1)^{m'}m'!(it)^{-m'}(2\pi i)^{-1}\int_C e^{-izt}(z-\Lambda_o)^{-m'-1}g(\Lambda_o)dz$$

where the path C of integration is composed of the segments

$[-R-i\delta,\ R-i\delta]$, $[R-i\delta,\ R+i\varepsilon]$ ,

$[R+i\varepsilon,\ -R+i\varepsilon]$, $[-R+i\varepsilon,\ -R-i\delta]$ .

Hence, by letting $R\to\infty$ , we obtain

$$e^{-\Lambda_o t}g(\Lambda_o)=(-1)^{m'}m'!(it)^{-m'}e^{-\delta t}(2\pi i)^{-1}\int_{-\infty}^{\infty}e^{-iEt}(E-i\delta-\Lambda_o)^{-m'-1}\cdot$$

$$\cdot g(\Lambda_o)dE+(-1)^{m'+1}m'!(it)^{-m'}e^{\varepsilon t}(2\pi i)^{-1}\int_{-\infty}^{\infty}e^{-iEt}(E+i\varepsilon-\Lambda_o)^{-m'-1}g(\Lambda_o)dE\ .$$

Letting $\delta\to\infty$ when $t>0$ and letting $\varepsilon\to\infty$ when $t<0$ we obtain

$$(2.3)\quad e^{-i\Lambda_o t}g(\Lambda_o)=\pm m'!(it)^{-m'}e^{\pm\varepsilon t}(2\pi i)^{-1}\int_{-\infty}^{\infty}e^{-iEt}(\Lambda_o-E\mp i\varepsilon)^{-m'-1}g(\Lambda_o)dE,\quad t\gtrless 0$$

2°. If $t>0$, from (2.3) it follows

$$(2.4)\quad |A+i|^{-m}e^{-i\Lambda_o t}g(\Lambda_o)P^{+}=m'!(it)^{-m'}e^{\varepsilon t}(2\pi i)^{-1}\ \cdot$$

$$\cdot\int_{-\infty}^{\infty}e^{-iEt}|A+i|^{-m}(\Lambda_o-E-i\varepsilon)^{-m'-1}g(\Lambda_o)P^{+}dE$$

For $0<a<b$ such that $\operatorname{supp}(g)\subset(a,b)$ we have

$$(2.5)\quad \Big\| \int_{\mathbb{R}\setminus[a,b]} e^{-iEt}|A+i|^{-m}(\Lambda_o-E-i\varepsilon)^{-m'-1}g(\Lambda_o)P^+dE\Big\| \leqq$$

$$\leqq c_o=c_o(A,g,m,m'),\qquad 0<\varepsilon\leqq 1.$$

3°. We shall estimate the norm of the operator

$$|A+i|^{-m}(\Lambda_o-E-i\varepsilon)^{-m'-1}g(\Lambda_o)P^+$$

uniformly for $E\in[a,b]$ and $\varepsilon\in(0,1]$.

The problem can be reduced to the study of

$$\{|A+i|^{-m}(\Lambda_o-E-i\varepsilon)^{-m'-1}P^+\}_{\substack{E\in[a,b]\\ \varepsilon\in(0,1]}}$$

because $|A+i|^{-m}g(\Lambda_o)A^m$ is clearly a bounded operator for $m\in N$:

$$i[g(\Lambda_o),A]=\alpha\Lambda_o g'(\Lambda_o)\qquad \text{by (2.1)}$$

Using complex interpolation, this property can be extended to real values of m.

4°. If $m\in\mathbb{R}$, $n\in\mathbb{N}$, $m>n$, $a\leqq\varepsilon\leqq b$, $0<\varepsilon\leqq 1$, $0\leqq\theta\leqq\pi/2\alpha$, we define

$$F(\varepsilon,E,\theta)=|A+i|^{-m}(\Lambda_o e^{-i\alpha\theta}-E-i\varepsilon)^{-n}e^{-A\theta}P^+$$

with

$$F(\varepsilon,E,0)=|A+i|^{-m}(\Lambda_o-E-i\varepsilon)^{-n}P^+$$

$$s\text{-}\lim_{\theta\to 0+}F(\varepsilon,E,\theta)=F(\varepsilon,E,0).$$

$F(\varepsilon,E,\theta)$ is the restriction to the positive pure imaginary axis of the analytic function of $z=\theta_o+i\theta$

$$F(\varepsilon,E,\theta_o+i\theta)=|A+i|^{-m}(\Lambda_o e^{-\alpha(\theta_o+i\theta)}-E-i\varepsilon)^{-n}e^{iA(\theta_o+i\theta)}P^+$$

$$=e^{iA\theta_o}|A+i|^{-m}(\Lambda_o e^{-i\alpha\theta}-E-i\varepsilon)^{-n}e^{-A\theta}P^+\qquad \text{by (2.1).}$$

Then the Cauchy-Riemann equations imply

$$(d/d\theta)F(\varepsilon,E,\theta)=-A|A+i|^{-m}(\Lambda_o e^{-i\alpha\theta}-E-i\varepsilon)^{-n}e^{-A\theta}P^+$$

This implies that

(2.6) $$||(d/d\theta)F(\varepsilon,E,\theta)||\leqq|||A+i|^{-m+1}(\Lambda_o e^{-i\alpha\theta}-E-i\varepsilon)^{-n}e^{-A\theta}P^+||$$

Since $|A+i|^{-z}$ is analytic for $Re(z)>0$, we can give by interpolation an estimate for (2.6):

$$m-1=0\cdot(1/m)+m\cdot((m-1)/m)$$

$Re\ (z)=0$

$$|||A+i|^{-z}(\Lambda_o e^{-i\alpha\theta}-E-i\varepsilon)^{-n}e^{-A\theta}P^+||\leqq$$

$$\leqq c(A)(E\sin\alpha\theta+\varepsilon\cos\alpha\theta)^{-n}\leqq c(A,g,n)\theta^{-n}$$

$Re(z)=m$

$$|||A+i|^{-z}(\Lambda_o e^{-i\alpha\theta}-E-i\varepsilon)^{-n}e^{-A\theta}P^+||=||F(\varepsilon,E,\theta)||$$

Then we get

(2.7) $$||(d/d\theta)F(\varepsilon,E,\theta)||\leqq c(A,g,n,m)\theta^{-n/m}||F(\varepsilon,E,\theta)||^{1-1/m}$$

5°. The differential inequality (2.7) implies that

(2.8) $$\sup\{||F(\varepsilon,E,\theta)||;\ 0<\varepsilon\leqq 1,\ 0\leqq\theta\leqq\pi/2\alpha,\ a\leqq E\leqq b\}=s<\infty\quad .$$

For $\nu\in\mathbb{N}$ sufficiently large we define

$$s_\nu=\sup\{||F(\varepsilon,E,\theta)||;\ \nu^{-1}\leqq\varepsilon\leqq 1,\ \nu^{-1}\leqq\theta\leqq\pi/2\alpha,\ a\leqq E\leqq b\}$$

Then $\lim s_\nu=s$ and there exists a sequence $\{(\varepsilon_\nu,E_\nu,\theta_\nu)\}$ such that

$$\nu^{-1}\leqq\varepsilon_\nu,\ \theta_\nu,\quad s_\nu=||F(\varepsilon_\nu,E_\nu,\theta_\nu)||$$

If $s=\infty$ then $\lim \varepsilon_\nu = \lim \theta_\nu = 0$ (if this is not true then the Uniform Boundeness Theorem implies that the sequence $\{s_\nu\}$ is bounded).

From (2.7) we can deduce for $0 \leqq \theta \leqq \theta' \leqq \pi/2\alpha$ that

$$||F(\varepsilon,E,\theta)-F(\varepsilon,E,\theta')|| \leqq c(A,g,m,n)(\theta'^{1-n/m}-\theta^{1-n/m}) \cdot$$

$$\cdot \sup_{\theta \leqq \eta \leqq \theta} ||F(\varepsilon,E,\eta)||^{1-1/m}$$

By choosing $\varepsilon=\varepsilon_\nu$, $\theta=\theta_\nu$, $\theta'=\pi/2\alpha$, $E=E_\nu$ it follows that there exist two constants $c$, $c_1>0$ (not depending on $\nu$) such that

$$s_\nu \leqq c s_\nu^{1-1/m} + c_1$$

From this relation we conclude that the sequence $\{s_\nu\}$ is bounded, contrary to $\lim s_\nu = \infty$.

In particular it follows from (2.8) that

$$|||A+i|^{-m}(\Lambda_o-E-i\varepsilon)^{-n}P^+|| \leqq s < \infty \qquad E \in [a,b] \quad , \quad \varepsilon \in (0,1] .$$

6°. Let

$$L(t)=|A+i|^{-m}e^{-i\Lambda_o t}g(\Lambda_o)P^+$$

with $m \in \mathbb{R}$, $m' \in \mathbb{N}$, $m>m'+1$. Then the steps 2° and 5° of the proof imply that

$$||L(t)|| \leqq m'!t^{-m'}e^{\varepsilon t}[c_o(A,g,m,m')+$$

$$+c(A,g,m)s(A,g,m,m')(b-a)] , \qquad \forall \ \varepsilon \in (0,1] .$$

This implies that for every $(m,m') \in \mathbb{R} \times \mathbb{N}$, $m>m'+1$ there exists $c=c(g,m,m')$ such that

$$|||A+i|^{-m}e^{-i\Lambda_o t}g(\Lambda_o)P^+|| \leqq ct^{-m'}$$

Furthermore, we have

$$||e^{-i\Lambda_o t}g(\Lambda_o)P^+|| \leq ||g(\Lambda_o)||$$

Now the theorem follows by interpolation with respect to Re(m) (If $0<\mu'<\mu$ one takes $m'=1+[\max\{\mu',\mu'/(\mu-\mu')\}]$, $m=(\mu/\mu')m'$, $p=m'/\mu'$, $1/q=1-1/p$. Then one applies Hadamard's three lines theorem in the strip $\{z;\ 0<\mathrm{Re}(z)<m\}$ to the analytic function $h(z)=|A+i|^{-z}e^{-i\Lambda_o t}\cdot g(\Lambda_o)P^+$).

Q.E.D.

COROLLARY 2.2. Let $g\in C_o^\infty(\mathbb{R}^-\setminus\{0\})$ and $0\leq\mu'<\mu$. Then there is a constant c (depending on g, $\mu$, $\mu'$) such that

(2.2)' $$||\chi^{\mp}(t)|A+i|^{-\mu}e^{-i\Lambda_o t}g(\Lambda_o)P^{\pm}|| \leq c|t|^{-\mu'}$$

Proof

We apply Theorem 2.1 to the operator $-\Lambda_o$ and to the function $\check{g}\in C_o^\infty(\mathbb{R}^+\setminus\{0\})$, $\check{g}(x)=g(-x)$.

Q.E.D.

## 3. ASYMPTOTIC COMPLETENESS

As a preliminary, we note the following result.

LEMMA 3.1. For $0\leq\beta\leq 2$

$$|A+i|^{\beta}(\Lambda_o+i)^{-1}(1+|x|^2)^{-\beta/2}$$

is a bounded operator.

Proof

We need only to prove the case $\beta=2$, and then use complex interpolation. Thus we need to prove that

$$A^2(\Lambda_0+i)^{-1}(1+|x|^2)^{-1}$$

is bounded. Since $\Lambda_0$ satisfies (2.1) with $\alpha=1$ we find for suitable $g$ that

$$Ag(\Lambda_0)=i\Lambda_0 g'(\Lambda_0)+g(\Lambda_0)A$$

By iterating this formula we get

$$A^2 g(\Lambda_0)=-\Lambda_0 g'(\Lambda_0)-\Lambda_0^2 g''(\Lambda_0)+i\Lambda_0 g'(\Lambda_0)A+g(\Lambda_0)A^2$$

By taking $g(\lambda)=(\lambda+i)^{-1}$ we obtain the conclusion of Lemma 3.1 by using the explicit formula for $A$ i.e. $A=1/2(D\cdot x+x\cdot D)$.

Q.E.D.

LEMMA 3.2. Suppose that the assumptions (A.1)-(A.3) are fulfiled. Then for every $g\in C_0^\infty(\mathbb{R}^+\setminus\{0\})$

$$(W_\pm-1)g(\Lambda_0)P^\pm$$

are compact operators on $\mathcal{H}_0$.

<u>Proof</u>

$$(W_+-1)g(\Lambda_0)P^+=i\int_0^\infty e^{i\Lambda s}(\Lambda J-J\Lambda_0)e^{-i\Lambda_0 s}g(\Lambda_0)P^+ds$$

$$=i\int_0^\infty e^{i\Lambda s}(E^{-1}-I)e^{-i\Lambda_0 s}g(\Lambda_0)P^+ds$$

The operator $(E^{-1}-I)e^{-i\Lambda_0 s}g(\Lambda_0)P^+$ is a compact operator for any $s>0$, as follows from the diagram

$$\mathcal{H}_0 \xrightarrow{P^+} \mathcal{H}_0 \xrightarrow{e^{-i\Lambda_0 s}g(\Lambda_0)} \mathcal{D}(\Lambda_0) \xrightarrow{P^0} \mathcal{H}^1(\mathbb{R}^n) \xrightarrow{E^{-1}-I} \mathcal{H}^1_{1+\varepsilon}(\mathbb{R}^n) \hookrightarrow \mathcal{H}_0$$

In the papers [5], [6] it was proved that the third arrow is a bounded operator. The last arrow is a compact operator by Rellich's Theorem. Furthermore the integral $\int_0^\infty ||(E^{-1}-I)e^{-i\Lambda_o s}g(\Lambda_o)P^+||ds$ is well defined since

$$||(E^{-1}-I)e^{-i\Lambda_o s}g(\Lambda_o)P^+|| \leqq$$

$$\leqq ||(E^{-1}-I)(\Lambda_o+i)^{-\beta}e^{-i\Lambda_o s}g(\Lambda_o)(\Lambda_o+i)^{\beta}P^+|| \leqq$$

$$\leqq ||(E^{-1}-I)(\Lambda_o+i)^{-\beta}|A+i|^{1+\varepsilon}||\ \ |||A+i|^{-1-\varepsilon}e^{-i\Lambda_o s}\cdot$$

$$\cdot g(\Lambda_o)(\Lambda_o+i)^{\beta}P^+||$$

From Theorem 2.1 it is sufficient to verify that $\beta>0$ can be chosen such that $(E^{-1}-I)(\Lambda_o+i)^{-\beta}|A+i|^{1+\varepsilon}$ is bounded on $\mathcal{H}_o$. By Lemma 3.1 this is true for $\beta=1$ because we may suppose $\varepsilon\leqq 1$ in (A.2).

Q.E.D.

COROLLARY 3.3. Suppose that the assumptions (A.1)-(A.3) are fulfiled. Then for every $g\in C_o^\infty(\mathbb{R}^-\setminus\{0\})$

$$(W_{\mp}-i)g(\Lambda_o)P^{\pm}$$

are compact operators on $\mathcal{H}_o$.

<u>Proof</u>

We apply the above results to the operators $-\Lambda_o$, $-\Lambda$ and to the function $\check{g}\in C_o^\infty(\mathbb{R}^+\setminus\{0\})$, $\check{g}(t)=g(-t)$. Hence Lemma 3.2 implies that

$$(W_{\pm}(-\Lambda,\ -\Lambda_o)-1)g(-\Lambda_o)P^{\pm}$$

are compact operators ($-\Lambda_o$ still satisfies (2.1)), which means that

$$(W_{\mp}-1)g(\Lambda_o)P^{\pm}$$

are compact operators on $\mathcal{H}_o$.

Q.E.D.

For the proof of Theorem 1.1 we need one more elementary result whose proof can be found in [9], [7].

LEMMA 3.4. Let $g\in C_o^{\infty}(\mathbb{R}\setminus\{0\})$. Then

$$g(\Lambda)-g(\Lambda_o)$$

is a compact operator in $\mathcal{H}_o$.

END OF THE PROOF OF THEOREM 1.1. (ii) We give the proof for the positive sign, i.e. Range $(W_+)=\mathcal{H}_c$. Assume on the contrary that Range $(W_+)\neq\mathcal{H}_c$. Then the subspace $\mathcal{H}_c \ominus$ Range $(W_+)$ reduces the operator $\Lambda$ and hence there exists an element $u\in\mathcal{H}_c \ominus$ Range$(W_+)$, $u\neq 0$, such that $E(I_o)u=u$ for some compact interval which is disjoint from zero. Let $I_o\subset$Int I, where I is another compact interval disjoint from zero, and $g\in C_o^{\infty}(I)$ such that $g(\lambda)=1$ for $\lambda\in I_o$. Then $g(\Lambda)u=u$. Since I is an interval disjoint from zero we have either $I\subset\mathbb{R}^-\setminus\{0\}$ or $I\subset\mathbb{R}^+\setminus\{0\}$. Let us consider for the definitiness that $I\subset\mathbb{R}^-\setminus\{0\}$. Then using compactness properties of operators in Corollary 3.3 and Lemma 3.4, we can find a sequence $t_n \to +\infty$ (Lemma 2 in [9]) such that

$$
\begin{aligned}
&||(g(\Lambda)-g(\Lambda_o))e^{-i\Lambda t_n}u|| \to 0 ;\\
(3.1)_-\quad &||(W_- -1)g(\Lambda_o)P^+e^{-i\Lambda t_n}u|| \to 0 ;\\
&||(W_+ -1)g(\Lambda_o)P^-e^{-i\Lambda t_n}u|| \to 0 .
\end{aligned}
$$

Finally we get

$$(3.2)\quad \begin{aligned} 0\neq ||u||^2 &= \lim_{n\to\infty} ||g(\Lambda)e^{-i\Lambda t_n}u||^2 = \\ &= \lim_{n\to\infty}(g(\Lambda)e^{-i\Lambda t_n}u,\ W_-g(\Lambda_o)P^+e^{-i\Lambda t_n}u) + \\ &+ \lim_{n\to\infty}(g(\Lambda)e^{-i\Lambda t_n}u,\ W_+g(\Lambda_o)P^-e^{-i\Lambda t_n}u) \end{aligned}$$

The second right side term is equal to zero by hypothesis. The first one is the limit of the following term:

$$(W_-^*g(\Lambda)u,\ e^{i\Lambda_o t_n}g(\Lambda_o)P^+e^{-\Lambda t_n}u)$$

which tends to zero when $t_n\to+\infty$, because $W_-^*g(\Lambda)u$ can be approached in norm sense by vectors belonging to the range of $|A+i|^{-\mu}$ (Corollary 2.2), so we get a contradiction.

(iii) The proof of this assertion is quite similar to that of (ii). Suppose to the contrary. Then we can find an orthonormal family $\{u_n\}$ with $\Lambda u_n=\lambda_n u_n$ and $\lambda_n\to\lambda\in\mathbb{R}\setminus\{0\}$. By throwing out finitely many $u_n$'s we can suppose that each $\lambda_n$ belongs to a compact interval $I_o$ disjoint from zero. Thus $E(I_o)u_n=u_n$. Then there is $g\in C_o^\infty(\mathbb{R}^+\setminus\{0\})$ (if $I_o\subset\mathbb{R}^+\setminus\{0\}$) such that $g(\lambda)u_n=u_n$. Since $u_n \overset{w}{\to} 0$, we find

$$(3.1)'_+\quad \begin{aligned} &(g(\Lambda)-g(\Lambda_o))u_n \xrightarrow{s} 0; \\ &(W_+-1)g(\Lambda_o)P^+u_n \xrightarrow{s} 0; \\ &(W_--1)g(\Lambda_o)P^-u_n \xrightarrow{s} 0; \end{aligned}$$

Similary to (3.2) we obtain

$$(3.2)'\quad \begin{aligned} 1=||u_n||^2 &= \lim_{n\to\infty}(u_n,W_+g(\Lambda_o)P^+u_n) + \\ &+ \lim_{n\to\infty}(u_n,W_-g(\Lambda_o)P^-u_n) \end{aligned}$$

Since $u_n$ , as an eigenfunction, is orthogonal to Range $W_+ \cup$ Range $W_-$ we get a contradiction.

Q.E.D.

REMARK 3.5. One can use the abve arguments for proving similar results concerning the asymptotic completeness for the operators $D_1^2 - D_2^2 + (1+|x|)^{-1-\varepsilon}$ and $D_1 D_2 + (1+|x|)^{-1-\varepsilon}$ on $L^2(\mathbb{R}^2)$.

Let $h_o(\xi) = \xi_1 \xi_2$ or $h_o(\xi) = \xi_1^2 - \xi_1^2$ for $\xi = (\xi_1, \xi_2)$ and let $V: R^2 \to R$ be a measurable function such that for some $c>0$, $\varepsilon>0$ we have

$$|V(x)| \leq C(1+|x|)^{-1-\varepsilon} \quad \forall\ x \in \mathbb{R}^2$$

Let $H_o = h_o(D)$ and $H = H_o + V$ be the self-adjoint realization in $L^2(\mathbb{R}^2)$. Here we use Lemma 3.1 from [4] which implies that $V(H_o + i)^{-1}$ is a compact operator. This result is also used in the proof of Lemma 3.2, so one can prove in the same way the following.

THEOREM 1.1'. (i) The wave operators $W_{\pm}(H, H_o)$ exists.

(ii) Range $(W_{\pm}) = \mathcal{H}_c(H)$, the continuous subspace of H in $L^2(\mathbb{R}^2)$.

(iii) In $\mathbb{R} \setminus \{0\}$ the eigenvalues of H are discrete and of finite multiplicity, with possible accumulating points 0 and $\pm\infty$.

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