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February 1988

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Introduction

In the fundamental paper [26], J.W.Morgan and P.B.Shalen give, among other nice things, an original procedure of compactification of an arbitrary affine variety defined over the field C of complex numbers or the field R of real numbers. Reading by chance their paper, the author of the present work realized that their construction can be equivalently described in terms of Robinson's non-standard analysis, by replacing the sequences of points on the variety by non-standard points on an enlargement of the given variety. The non-standard point of view has the advantage to put in evidence an elementary extension of the base field, a sort of universal domain, equipped with a canonic non-archimedean valuation, playing the role of a "generic point" for the Riemann spaces of valuations on the fields of rational functions on the irreducible components of the variety. The author also realized that the main Theorems

I.3.7, I.4.4 given in Section I of [26] can be alternatively proved by using basic model theoretic methods. It seemed natural to try to extend the afore mentioned results to the case when the base field is a local field, i.e., a Cauchy complete discrete valued field with finite residue field.

The task to give an unitary model theoretic approach of the Morgan-Shalen compactification over arbitrary locally compact fields of characteristic zero is one of the main goals of the present paper. The another one, intimately related to the first one, but having a more general character, is to treat in an unitary model theoretic way some significant classes of spectral spaces induced by first order theories of fields.

The paper is organized in seven sections as follows. Section 1 introduces the reader to some basic model theoretic results concerning the algebraically closed, the real closed and the p-adically closed fields. As an original contribution we mention the explicit description of the substructures, called K-domains, of the p-adically closed p-valued field extensions of a p-adically closed p-valued base field K (see Definition 1.11, Lemma 1.12, Theorem 1.14, ii). Sections 2 and 3 are devoted to the spectral spaces induced by theories of fields. Based on Stone's representation theorem for distributive lattices, the Lindenbaum Boolean algebra of a first order theory and Gödel's completeness theorem, a general theory of spectral spaces asigned to commutative rings is developed, containing as particular cases the Zariski spectrum, the Coste-Roy real spectrum and the p-adic spectrum of a ring. Some basic algebraic-geometric facts concerning the affine varieties defined over arbitrary fields are discussed in this general frame. Let us mention as significant results the p-adic analogues of the Artin-Lang homomorphism theorem and of the finiteness theorem for open semialgebraic sets (see Theorems 3.15 and 3.16). The general scheme developed in Sections 2 and 3 is applied in Section 4 in order to extend the basic concept of Riemann space of a field to adequate Riemann spaces of commutative rings, including real and p-adic versions. Robinson's theorem on elimination of quantifiers for non-trivial valued algebraically closed fields and its real and p-adic analogues Theorems 4.3 and 4.6 play here an important role. The section ends with a density theorem on Riemann spaces (Theorem 4.7), which is a basic tool for the last part of the paper. Some natural continuous maps on the Riemann space of a field and a density theorem (Proposition 5.2) are considered in Section 5. Section 6 is devoted to the non-standard description of the Morgan-Shalen procedure of compactification over arbitrary locally compact fields. Finally, the main results concerning the compactification of affine varieties over arbitrary locally compact fields of charactheristic zero make the object of Section 7.

1. Some model theory for fields

In this section, having a preliminary character, some basic model theoretic results concerning the algebraically closed, real closed and p-adically closed fields are stated. These results will play a key role in the rest of the paper.

The basic notions from model theory used in this paper can be found in books like [9], [12] or [33].

Denote by $\underline{\underline{L}}$ the customary first order language of rings whose vocabulary contains besides the logical symbols and the variables $\{x_i\}_{i<\omega}$, two binary function symbols + and · standing for addition and multiplication, a unary function symbol - standing for the map $x \mapsto -x$, and two constants for the neutral elements 0, 1.

Let $\underline{\underline{D}}$, respectively $\underline{\underline{ACF}}$, be the universal $\underline{\underline{L}}$-theory of integral domains, respectively the inductive $\underline{\underline{L}}$-theory of algebraically closed fields. The main theorem concerning the $\underline{\underline{L}}$-theory $\underline{\underline{ACF}}$, due to Tarski, Chevalley and Robinson reads as follows:

Theorem 1.1. $\underline{\underline{ACF}}$ admits elimination of quantifiers. Equivalently, in geometric terms, the projection map $K^{n+1} \to K^n$ (K algebraically closed) maps a constructible subset onto a constructible one. Equivalently, in model theoretic terms $\underline{\underline{ACF}}$ is the model completion of $\underline{\underline{D}}$.

An analogous result due to Tarski, Seidenberg and Robinson holds for real closed fields.

Definition 1.2. An ordered domain is a pair $\underline{\underline{A}}=(A, P)$ where A is an integral domain and P is a subset of A, called order, subject to the conditions: i) $P+P \subset P$; ii) $P.P \subset P$; iii) $P \cup -P = A$ and iv) $P \cap -P = \{0\}$. The ordered domain $\underline{A}$ is an ordered field if A is a field.

Definition 1.3. The field K is real closed if $\underline{K}=(K, K^2)$ is an ordered field and each polynomial $f \in K[X]$ of odd degree has a root in K.

Let L_2 be the language $\underline{L}$ augmented with a one-place relation symbol P, standing for order. Denote by OD the universal L_2-theory of ordered domains. Let RCF be the inductive L_2-theory obtained from the $\underline{L}$-theory of real closed fields by adding the axiom - definition $P(x_1) \leftrightarrow (\exists x_2)\ (x_1 = x_2^2)$. Now the analogue of Theorem 1.1 reads as follows:

Theorem 1.4. RCF admits elimination of quantifiers. Equivalently, the projection map $K^{n+1} \to K^n$ (K real closed) maps a semialgebraic subset onto a semialgebraic one. Equivalently, RCF is the model completion of OD.

A quite similar result, due to MacIntyre [24] and Prestel Roquette [28], holds for the p-adically closed fields. More general results concerning the relative elimination of quantifiers for Henselian valued fields of characteristic zero are proved by Weispfenning [39] by primitive recursive techniques and by the author [5] by algebraic and basic model theoretic methods.

Definition 1.5. Given a prime number p and the positive integers e and f, a p-valued field of type (e, f) is a valued field $\underline{K} = (K, v)$ satisfying i) K is of characteristic zero, while the residue field K_v is of characteristic p;

ii) the maximal ideal $\underline{\underline{m}}_v$ of the valuation ring O_v is principal, say $\underline{\underline{m}}_v=\pi O_v$; puting $v(\pi)=1$, the ordered group of integers $\underline{Z}$ is identified with a convex subgroup of the value group vK;

iii) the p-<u>ramification index</u> v(p) is e and the <u>residue degree</u> is f, i.e., $K_v \cong \underline{F}_q$ with $q=p^f$.

<u>Definition 1.6.</u> A p-valued field $\underline{\underline{K}}=(K, v)$ of <u>type (e, f)</u> is called p-<u>adically closed</u> if $\underline{\underline{K}}$ does not admit any proper algebraic p-valued field extension of the same type.

By Zorn's lemma, if $\underline{\underline{K}}$ is p-valued of type (e, f) there exists a maximal p-valued algbraic extension $\tilde{\underline{\underline{K}}}$ of the same type. Any such valued field $\tilde{\underline{\underline{K}}}$ is called a p-<u>adic closure</u> of $\underline{\underline{K}}$. The p-adic closure of a p-valued field $\underline{\underline{K}}$ is not necessarily unique. The most prominent examples of p-adically closed fields are those which are locally compact with respect to the given valuation. They can be characterized as the completions of finite algebraic number fields with respect to a nonarchimedean valuation.

The next lemma is very useful in the following.

<u>Lemma</u> 1.7. Let (K, v) be a Henselian valued field such that the value group vK has a smallest positive element 1, say $1=v(\pi)$ for some $\pi \in K$. Let $n \geq 2$ be a natural number which is prime to the residue characteristic and G be a multiplicative subgroup of K^X such that $K^{Xn} \subset G$ and $\pi \notin G$. Then the valuation ring O_v admits the description: $O_v = \{a \in K : 1+\pi a^n \in G\}$.

Proof. For $i \geq 1$, let us put $\mathcal{O}_i = \{a \in K : 1 + \pi^i a^n \in G\}$. By Hensel's lemma we get $1 + \pi^i \mathcal{O}_v^n \subset \mathcal{O}_v^{\times n} \subset G$, and hence $\mathcal{O}_v \subset \mathcal{O}_i$ for $i \geq 1$. In particular, $\mathcal{O}_v \subset \mathcal{O}_1$. Conversely, let $a \in \mathcal{O}_1$ and assume that $v(a) < 0$. As $\pi^{-1} a^{-1} \in \mathcal{O}_v \subset \mathcal{O}_{n-1}$, we get $(1+\pi a^n)^{n-1}(1+\pi^{n-1}(\pi^{-1}a^{-1})^n) = \pi^{-1} a^{-n}(1+\pi a^n)^n \in G$, and hence $\pi \in G$, a contradiction. Consequently, $\mathcal{O}_v = \mathcal{O}_1$, as required. ∎

Lemma 1.8. Given a p-valued field $\underline{\underline{K}} = (K, v)$ of type (e, f) and a field extension F of K, the next statements are equivalent:

i) There exists a valuation w of F such that $\underline{\underline{F}} = (F, w)$ is a p-adically closed p-valued field extension of $\underline{\underline{K}}$ of type (e, f).

ii) There exists a unique valuation w of F satisfying the condition above.

iii) Let $\pi \in K$ be such that $v(\pi) = 1$ and let $\mathcal{O} = \{a \in F : 1 + \pi a^2 \in F^2\}$ if $p \neq 2$, respectively $\mathcal{O} = \{a \in F : 1 + \pi a^3 \in F^3\}$ if $p = 2$. Then $\mathcal{O}$ is a Henselian valuation ring of F lying over $\mathcal{O}_v$, the canonic morphism $\mathcal{O}_v/p\mathcal{O}_v \to \mathcal{O}/p\mathcal{O}$ is an isomorphism (equivalently, $\pi\mathcal{O}$ is the maximal ideal of $\mathcal{O}$ and the residue field $\mathcal{O}/\pi\mathcal{O}$ is isomorphic to $\bar{K}_v \simeq \underline{F}_q$, $q = p^f$) and the value group $F^{\times}/\mathcal{O}^{\times}$ is a $\underline{Z}$-group, i.e., $F^{\times}/\pi^{Z}\mathcal{O}^{\times}$ is divisible.

Proof. Let $\underline{\underline{F}} = (F, w)$ be a p-valued field extension of $\underline{\underline{K}}$ of type (e, f). According to [28] Theorem 3.1, the necessary and sufficient condition for $\underline{\underline{F}}$, to be p-adically closed is that $\underline{\underline{F}}$ is Henselian and its value group wF is a $\underline{Z}$-group, i.e.,

$wF/_{\underline{Z}}$ is divisible. Thus, the lemma is immediate, thanks to Lemma 1.7. ∎

Remark. According to [28] Theorem 6.15, we may take $\mathcal{O}=\gamma F=\{\gamma(a):a\in F\}$, where $\gamma(X)=\frac{1}{\pi}\frac{X^q-X}{(X^q-X)^2-1}$ is the Kochen-Roquette operator.

The previous lemma shows that the class of p-adically closed p-valued fields of type (e, f) extending the given p-valued field $\underline{K}=(K, v)$ of type (e, f) is axiomatizable in terms of the language $\underline{L}_{\underline{K}}$ of rings augmented by constants standing for the elements of K.

Definition 1.9. ([25], 4.2). A valued domain is a pair $\underline{A}=(A, R)$, where A is an integral domain and R is a binary relation satisfying i) R is transitive; ii) aRb or bRa; iii) aRb and aRc $\Rightarrow$ aR(b+c); iv) $c\neq 0 \Rightarrow$ (aRb $\Leftrightarrow$ acRbc); v) not 0R1.

Given an integral domain A there is a canonical bijection between the structures of valued domains on A and the valuation rings of the quotient field K=Q(A) of A, given by $R\mapsto \mathcal{O}_R=\{\frac{a}{b}: a,b\in A,\ b\neq 0,\ bRa\}$, whose inverse is given by $\mathcal{O}_v\mapsto R_v=\{(a,b)\in A\times A: v(a)\leq v(b)\}$.

Definition 1.10. A p-valued domain of type (e, f) is a valued domain $\underline{A}=(A, R)$ satisfying i) $\underline{K}=(Q(A), v_R)$ is a p-valued field of type (e, f) and ii) the canonic map $\mathcal{O}_A\to \mathcal{O}_{v_R}/p\mathcal{O}_{v_R}$ is onto, where $\mathcal{O}_A=\{a\in A: 1\,Ra\}=\mathcal{O}_{v_R}\cap A$.

Definition 1.11. Given a p-adically closed p-valued field $\underline{\underline{K}}=(K, v)$ of type (e, f), a $\underline{\underline{K}}$-domain is a structure $\underline{\underline{A}}=(A, P_n: n\geq 2)$, where A is an integral domain extending K and $P_n \subset A$ for $n\geq 2$ such that the next conditions are satisfied:

i) $A^n \subset P_n$ for $n\geq 2$;

ii) $P_n^X = P_n \setminus \{0\}$ is a monoid with respect to multiplication and $(P_n^X)^{-1} \cdot P_n \cap A = P_n$ for $n\geq 2$, i.e., $(a\in A,\ b\in P_n,\ c\in P_n^X$ and $ac=b) \Rightarrow a\in P_n$;

iii) $K\cap P_n = K^n$ for $n\geq 2$;

iv) $A=\bigcup_{i=1}^{\ell_n} c_i P_n$, where $c_1,\dots,c_{\ell_n}\in K^X$ is a system of representatives of $K^X/_{K^{Xn}}$;

v) $m|n \Rightarrow P_n \subset P_m$;

vi) Let $R=\{(a, b)\in A\times A: a^2+\pi b^2 \in P_2\}$ if $p\neq 2$, respectively $R= =\{(a,b)\in A\times A: a^3+\pi b^3 \in P_3\}$ if $p=2$, where $\pi\in K$ satisfies $v(\pi)=1$; then (A, R) is a p-valued domain of type (e, f) extending $\underline{\underline{K}}$;

vii) if $a\in A$, $b\in P_n^X$ and $(\pi n^2 b)R\, a$ then $a+b\in P_n$ for $n\geq 2$.

The next lemma is immediate.

Lemma 1.12. Let $\underline{\underline{K}}=(K, v)$ be a p-adically closed p-valued field of type (e, f), F be a field extension of K and $(P_n)_{n\geq 2}$ be a family of subsets of F. The necessary and sufficient condition for $\underline{\underline{F}}=(F, P_n: n\geq 2)$ to be a $\underline{\underline{K}}$-domain (then we say that $\underline{\underline{F}}$ is a $\underline{\underline{K}}$-field) is that

i) $F^n \subset P_n$ for $n\geq 2$;

ii) P_n^X is a subgroup of F^X for $n\geq 2$;

iii) the canonic morphism $K^X/_{K^{Xn}} \rightarrow F^X/_{P_n^X}$ is an isomorphism for $n\geq 2$;

iv) $m|n \Rightarrow P_n \subset P_m$;

v) let $\mathcal{O}=\{a\in F: 1+\pi a^2 \in P_2\}$ if $p\neq 2$, respectively $\mathcal{O}=\{a\in F: 1+\pi a^3 \in P_3\}$ if $p=2$; then $\mathcal{O}$ is a valuation ring of F with corresponding valuation w and (F, w) is a p-valued field extension of (K, v) of the same type as (K, v);

vi) $1+\underline{\underline{m}}_w^{2v(n)+1} \subset P_n$ for $n\geq 2$.

<u>Lemma</u> 1.13. Let $\underline{\underline{K}}=(K, v)$ be a p-adically closed p-valued field of type (e, f) and A be an integral domain extending K. The map $(P_n)_{n\geq 2} \mapsto ((P_n^{\times})^{-1}P_n = \{\frac{a}{b}: a\in P_n,\ b\in P_n^{\times}\})_{n\geq 2}$ is a bijection from the set of $\underline{\underline{K}}$-domains with underlying domain A onto the set of $\underline{\underline{K}}$-fields with underlying field F=Q(A), whose inverse is given by $(P_n)_{n\geq 2} \mapsto (P_n \cap A)_{n\geq 2}$.

The proof is easy.

With $\underline{\underline{K}}=(K, v)$ as above, let $(\underline{\underline{L}}_\omega)_K$ be the augmentation of the language $\underline{\underline{L}}$ of rings with constants standing for the elements of K and one-place relation symbols P_n for $n\geq 2$. Denote by $\underline{\underline{D}}_{\underline{\underline{K}}}$ the universal $(\underline{\underline{L}}_\omega)_K$ - theory of $\underline{\underline{K}}$-domains and by $\underline{\underline{pCF}}_{\underline{\underline{K}}}$ the inductive $(\underline{\underline{L}}_\omega)_K$ - theory obtained from the $\underline{\underline{L}}_K$-theory of p-adically closed p-valued fields of type (e, f) extending $\underline{\underline{K}}$, by adding the axioms - definitioms $P_n(x_1) \leftrightarrow (\exists x_2)x_1=x_2^n$ for $n\geq 2$. The following theorem is an analogue of Theorems 1.1 and 1.4.

<u>Theorem</u> 1.14. Assume $\underline{\underline{K}}=(K, v)$ is a p-adically closed p-valued field of type (e, f).

i) $\underline{\underline{pCF}}_{\underline{\underline{K}}}$ admits elimination of quantifiers.

ii) The complete $(\underline{\underline{L}}_\omega)_K$-theory $p\underline{\underline{CF}}_K$ is the model completion of $\underline{\underline{D}}_{\underline{\underline{K}}}$.

Proof. i) is immediate by [28] Theorem 5.6. In order to prove ii), a model theoretic reformulation of i) according to [33] Theorem 13.2, it suffices to show that $\underline{\underline{D}}_{\underline{\underline{K}}}=(p\underline{\underline{CF}}_{\underline{\underline{K}}})_\forall$, the set of all universal $(\underline{\underline{L}}_\omega)_K$-sentences which follow from $p\underline{\underline{CF}}_{\underline{\underline{K}}}$. In semantic terms, we have to show that, given an integral domain A extending K and a family $(P_n)_{n\geq 2}$ of subsets of A, $\underline{\underline{A}}=(A, P_n{:}n\geq 2)$ is a $\underline{\underline{K}}$-domain iff $\underline{\underline{A}}$ is an $(\underline{\underline{L}}_\omega)_K$-substructure of some model of $p\underline{\underline{CF}}_{\underline{\underline{K}}}$. By Lemma 1.13, we may assume that A=F is a field.

Assume $\underline{\underline{F}}=(F, P_n{:}n\geq 2)$ is a substructure of some model $\underline{\underline{L}}=(L, L^n{:}n\geq 2)$ of $p\underline{\underline{CF}}_{\underline{\underline{K}}}$, i.e., F is a subfield of L and $P_n=F\cap L^n$ for $n\geq 2$. Then the conditions i), ii), iv), v) from Lemma 1.11 are trivially satisfied. As the extension L/K is elementary and the groups $K^\times/K^{\times n}$ are finite for $n\geq 2$ it follows that the canonic morphism $K^\times/K^{\times n}\to L^\times/L^{\times n}$ is an isomorphism for $n\geq 2$, and hence the condition iii) from Lemma 1.11 is also verified. In order to verify the condition vi) from the afore mentioned lemma, let $a\in\underline{\underline{m}}_w^{2v(n)+1}$, $n\geq 2$, and consider the polynomial $f(X)=X^n-1-a\in\mathcal{O}_w[X]$. As $w(f(1))>2v(n)=2w(f'(1))$ it follows by Newton's lemma that f(X) admits a root b in L and hence $1+a=b^n\in F\cap L^n=P_n$. Consequently, $\underline{\underline{F}}$ is a $\underline{\underline{K}}$-field.

Conversely, assume that $\underline{\underline{F}}$ is a $\underline{\underline{K}}$-field. Since the class of $\underline{\underline{K}}$-fields is inductive it follows by Zorn's lemma that there exists a maximal algebraic $\underline{\underline{K}}$-field extension of $\underline{\underline{F}}$, so we may assume from the beginning that the $\underline{\underline{K}}$-field $\underline{\underline{F}}$ is algebraically maximal. It remains to show that $\underline{\underline{F}}$ is a model of $p\underline{\underline{CF}}_{\underline{\underline{K}}}'$.

Let $\mathcal{O}_w$ be the valuation ring of F given by the condition v) of Lemma 1.12. First let us show that the valued field (F, w) is Henselian. Let (F', w') be the Henselization of (F, w) and consider the family $(P_n')_{n\geqslant 2}$ of subsets of F' given by $P_n'=P_n\cdot F'^n$. Let us show that $\underline{\underline{F}}'=(F',P_n':n\geqslant 2)$ is a $\underline{\underline{K}}$-field extension of $\underline{\underline{F}}$. The conditions i), ii), iv) of Lemma 1.12 are obviously verified. According to Lemma 1.12, iii), $K^X/_{K^{Xn}} \simeq F^X/_{P_n^X}$ for $n\geqslant 2$, so we have to show that the canonic morphism $F^X/_{P_n^X} \to F'^X/_{P_n'^X}$ is an isomorphism for $n\geqslant 2$. First let us check that $F\cap P_n'=P_n$ for $n\geqslant 2$. As $F\cap P_n'=P_n\cdot(F\cap F'^n)$, it suffices to show that $F\cap F'^n\subset P_n$. Since wF=w'F', we get $F\cap F'^n=F\cap F^n\mathcal{O}_{v'}^{Xn}= =F^n\cdot(F\cap\mathcal{O}_{v'}^X)$, so we have to show that $F\cap\mathcal{O}_{v'}^{Xn}\subset P_n$. Let $x\in\mathcal{O}_{v'}^X$ be subh that $x^n\in F$. As (F', w') is a p-valued field extension of (F, W) of type (e, f), it follows that the canonic morphisms $\mathcal{O}_v/_{\underline{\underline{m}}_v^k}\to\mathcal{O}_w/_{\underline{\underline{m}}_w^k}$ and $\mathcal{O}_w/_{\underline{\underline{m}}_w^k}\to\mathcal{O}_{w'}/_{\underline{\underline{m}}_{w'}^k}$ are isomorphisms for $k\geqslant 1$. Let $y\in\mathcal{O}_v^X$ be such that $w'(x-y)>2v(n)$. Consequently, $xy^{-1}\in 1+ +\underline{\underline{m}}_{w'}^{2v(n)+1}$, and hence $(xy^{-1})^n\in F\cap(1+\underline{\underline{m}}_{w'}^{2v(n)+1})=1+\underline{\underline{m}}_w^{2v(n)+1}$. It follows $x^n\in F^n\cdot(1+\underline{\underline{m}}_w^{2v(n)+1})\subset P_n$, according to Lemma 1.12, i) and vi) applied to the $\underline{\underline{K}}$-field $\underline{\underline{F}}$. Next let us show that $F'^X=F^X\cdot P_n'^X$. Since wF=w'F', we get $F'^X=F^X\mathcal{O}_{w'}^X$, so it suffices to show that $\mathcal{O}_{w'}^X\subset F^X\cdot P_n'^X$. Let $x\in\mathcal{O}_{w'}^X$ and $y\in\mathcal{O}_v^X$ be such that $w'(x-y)>2v(n)$. As (F', w') is Henselian, it follows by Newton's lemma that $xy^{-1}\in\mathcal{O}_{w'}^{Xn}$, and hence $x\in\mathcal{O}_v^X\cdot\mathcal{O}_{w'}^{Xn}\subset F^X\cdot P_n'^X$, as required. Finally, the condition v) from Lemma 1.12 follows by Lemma 1.7, while the condition vi) is immediate by Newton's lemma. Thus $\underline{\underline{F}}'$ is a $\underline{\underline{K}}$-field extension of $\underline{\underline{F}}$. As F'/F is algebraic and $\underline{\underline{F}}$ is assumed to be algebraically maximal we conclude that $\underline{\underline{F}}'=\underline{\underline{F}}$, and hence (F, w) is Henselian.

Now it remains to show that $P_n=F^n$ for $n\geqslant 2$. Indeed, if so, we get in particular that $wF=\mathbb{Z}+n.wF$ for $n\geqslant 2$, i.e., wF is a $\mathbb{Z}$-group, and hence $\underline{\underline{F}}$ is a model of $pCF_{\underline{\underline{K}}}$.

Assuming the contrary, let $m\geqslant 2$ be such that $P_m\neq F^m$. Since $F^{kl}=F^k\cap F^l$ and $P_{kl}=P_k\cap P_l$ if $(k, l)=1$, we may assume that m is a prime's power. Moreover we may assume that m is a prime. Indeed, if $P_{m^k}=F^{m^k}$ for some $m\geqslant 2$, $k\geqslant 1$, then $F^{m^{k+1}}\subset P_{m^{k+1}}=$ $=F^{m^k}\cap P_{m^{k+1}}=(K.F^{m^k})^{m^k}\cap P_{m^{k+1}}\subset K^{m^{k+1}}.F^{m^{2k}}\subset F^{m^{k+1}}$, i.e., $P_{m^{k+1}}=$ $=F^{m^{k+1}}$.

Let $a\in P_m\setminus F^m$ and $\dot{w}$ be the valuation $F^\times \xrightarrow{w} wF \to wF/\mathbb{Z}$. First let us show that $\dot{w}(a)\notin m.\dot{w}F$. Assuming the contrary, we get $a=\pi^i bc^m$ with $0\leqslant i<m$, $b\in \mathcal{O}_w^\times$, $c\in F^\times$, so we may assume without loss of generality that $a=\pi^i b$, with $0\leqslant i<m$, $b\in \mathcal{O}_w^\times$. As $\mathcal{O}_v/\underline{\underline{m}}_v^k\cong \mathcal{O}_w/\underline{\underline{m}}_w^k$ for $k\geqslant 1$, there exists $d\in \mathcal{O}_v^\times$ such that $w(b-d)>2v(m)$. By Newton's lemma, $db^{-1}\in F^m$, and hence $\pi^i d=a.(db^{-1})\in P_m\cap K=K^m$. It follows $a\in F^m$, contrary to the assumption.

Now let t be an m'th root of a in the algebraic closure of F, and let us put $F'=F(t)$. As (F, w) is Henselian, w extends uniquely to a valuation w' of F'. Using the same argument as in [28] Theorem 3.1, it follows that $[F':F]=(w'F':wF)=m$ and (F', w') is a p-valued field extension of (F, w) of type (e, f). Let $\sqrt{F}=\{x\in F'^\times : x^n\in F \text{ for some } n\geqslant 1\}$ be the group of radical elements of F' over F. According to [28] Theorem 3.8, $w'F'=$ $=w'(\sqrt{F})$ and the canonic morphism $\sqrt{F}/_{F^X} \to w'F'/_{wF} \simeq Z/_{mZ}$ is an isomorphism, so $\sqrt{F}=\bigcup_{0\leq i<m} t^i F^X$, i.e., tF^X is a generator of the cyclic group $\sqrt{F}/_{F^X}$.

First let us show that $F'=F'^n.\sqrt{F}=F.F'^n.t^Z$ for $n\geq 2$. As $w'F'=wF+Zw'(t)$, we get $F'=F.t^Z.\mathcal{O}^X_{w'}$ so it suffices to show that $\mathcal{O}^X_{w'}=\mathcal{O}^X_w.\mathcal{O}^{Xn}_{w'}$. Let $x\in\mathcal{O}^X_{w'}$. Since (F', w') is a p-valued field extension of (F, w) of type (e, f), there is $y\in\mathcal{O}^X_w$ such that $w'(x-y)>2v(n)$, and hence $x=y.(xy^{-1})\in\mathcal{O}^X_{w'}.\mathcal{O}^{Xn}_{w'}$ by Newton's lemma applied to the Henselian valued field (F', w').

Now we define a family $(P'_n)_{n\geq 2}$ of subsets of F' in such a way that $\underline{\underline{F}}'=(F', P'_n: n\geq 2)$ is a $\underline{\underline{K}}$-field extension of F. Let $n\geq 2$. If $m\nmid n$ let us put $P'_n=P_n.F'^n$. As we must have $P'_{m^i.n}=P'_{m^i}\cap P'_n$ for $i\geq 1$ and $m\nmid n$, it remains to define P'_{m^i} for $i\geq 1$. In order to do this we define inductively a sequence $(b_i)_{i\geq 0}$ of elements in K^X such that $a_i:=a.\prod_{j=0}^{i} b_j^{m^j}\in P_{m^{i+1}}$ for $i\geq 0$. For $i=0$, set $b_o=1$. Then $a_o=$ $=a\in P_m$ by assumption. Let $i\geq 0$ and $(b_j)_{0\leq j\leq i}$ be a sequence of elements in K^X such that $a_j\in P_{m^{j+1}}$ for $0\leq j\leq i$. As $K^{X m^{i+1}}/K^{X m^{i+2}} \simeq$

$P^X_{m^{i+1}}/P^X_{m^{i+2}}$, we may choose some $b_{i+1}\in K^X$ such that $a_{i+1}=a_i b_{i+1}^{m^{i+1}}\in$

$P_{m^{i+2}}$. Note that the b_i's are uniquely determined modulo

$K^{Xm}\mu_{m^i}$, where $\mu_{m^i}=\{x\in K: x^{m^i}=1\}$, thanks to the isomorphism

$K^X/_{K^{Xm}\mu_{m^i}} \tilde{\to} K^{X m^i}/_{K^{X m^{i+1}}}$ for $i\geq 0$, induced by the map $x\mapsto x^{m^i}$.

Now let us put $t_i=t.\prod_{j=1}^{i} b_j^{m^{j-1}}$ for $i\geq 0$. We get $t_o=t$, $t_i^m=a_i$ and

$t_{i+1} \cdot t_i^{-1} \in K^{X^{m^i}}$ for $i \geq 0$. Let us define $P'_{m^i} = P_{m^i} F'^{m^i} t_i^{Z}$ for $i \geq 1$.

We have to show that $\underline{\underline{F}}'$ is a $\underline{\underline{K}}$-field extension of $\underline{\underline{F}}$. The conditions i), ii) and iv) of Lemma 1.12 are trivially satisfied. In order to check iii) and the fact that $\underline{\underline{F}}'$ extends $\underline{\underline{F}}$, we have to show that the canonic morphism $F^X/_{P_n^X} \to F'^X/_{P'^X_n}$ is an isomorphism for each $n \geq 2$.

We distinguish two cases:

Case 1: $m \nmid n$. Let us show that $F' = F.P'_n = F.F'^n$. As $F' = FF'^n t^{Z}$ it suffices to show that $t \in F.F'^n$. Writing $1 = km + ln$ with $k, l \in \underline{Z}$, we get $t = a^k t^{ln} \in F.F'^n$, as required. It remains to show that $F \cap P'_n = P_n$. Since $P'_n = P_n . F'^n$, it suffices to show that $F \cap F'^n = F^n$. Let $x \in F'^X$ be such that $x^n \in F$. Thus $x \in \sqrt{F}$ and its order modulo F^X divides n and m, the order of $\sqrt{F}/_{F^X}$, so $x \in F$ and $x^n \in F^n$.

Case 2: $n = m^i$, $i \geq 1$. As $F' = FF'^n t^{Z}$ and $t \equiv t_i \mod F^X$, we get $t \in F.t_i^{Z} \subset F.P'_n$, and hence $F' = F.P'_n$. It remains to show that $F \cap P'_n = P_n$.

It suffices to verify that $F \cap F'^n t_i^{Z} \subset P_n$. Let $y \in F'^X$, $j \in \underline{Z}$ be such that $x = y^n t_i^j \in F$. Thus $y^n \in F^X . t_i^{Z} = F^X . t^{Z} = \sqrt{F}$, and hence $y \in \sqrt{F}$, i.e., $y = z t_i^k$ with $z \in F^X$, $k \in \underline{Z}$. Consequently, $x = z^n t_i^{kn+j} \in F^X$, and $m \mid kn+j$. As $n = m^i$, $i \geq 1$, we get $m \mid j$, and hence $x = z^n a_i^{km^{i-1} + \frac{j}{m}} \in P_n$ since $a_i \in P_{nm}$.

Finally, the condition v) from Lemma 1.12 is a consequence of Lemma 1.7, while the condition vi) follows by Newton's lemma. Thus $\underline{\underline{F}}'$ is a proper algebraic $\underline{\underline{K}}$-field extension of $\underline{\underline{F}}$, contrary to the assumption that $\underline{\underline{F}}$ is algebraically maximal. ∎

Given a $\underline{\underline{K}}$-field $\underline{\underline{F}}$, there exists by Zorn's lemma a maximal algebraic $\underline{\underline{K}}$-field extension $\tilde{\underline{\underline{F}}}$ of $\underline{\underline{F}}$. According to the proof of Theorem 1.14, $\tilde{\underline{\underline{F}}}$ is a model of $p\underline{\underline{C}}\underline{\underline{F}}_{\underline{\underline{K}}}$; call it a p-adic closure of the $\underline{\underline{K}}$-field $\underline{\underline{F}}$. The following statement shows that the concept above is a suitable analogue of the concepts of algebraic closure of a field and real closure of an ordered field.

Theorem 1.15. Let $\underline{\underline{K}}$ be a p-adically closed p-valued field of type (e, f). Given a $\underline{\underline{K}}$-field $\underline{\underline{F}}=(F, P_n : n\geqslant 2)$, its p-adic closure is unique up to an $\underline{\underline{F}}$-isomorphism.

Proof. Let $\tilde{\underline{\underline{F}}}_i$, i=1,2, be p-adic closures of $\underline{\underline{F}}$, and let $Z_i=\{f\in F[X]: f \text{ has a root in } \tilde{F}_i\}$ for i=1,2. According to Theorem 1.14, the $\underline{\underline{K}}$-fields $\tilde{\underline{\underline{F}}}_1$ and $\tilde{\underline{\underline{F}}}_2$ are elementarily equivalent over $\underline{\underline{F}}$, and hence $Z_1=Z_2$. As the field extensions $\tilde{\underline{\underline{F}}}_1/F$ and $\tilde{F}_2/F$ are algebraic separable we may apply [1] Lemma 5 and conclude that $\tilde{F}_1$ and $\tilde{F}_2$ are isomorphic over F as fields. Then, obviously, the $\underline{\underline{K}}$-fields $\tilde{\underline{\underline{F}}}_1$ and $\tilde{\underline{\underline{F}}}_2$ are isomorphic over the $\underline{\underline{K}}$-field $\underline{\underline{F}}$. ∎

Remark. The theorem above is an immediate consequence of [28] Corollary 3.11. A general criterion for two algebraic Henselian valued field extensions of a given valued field of characteristic zero to be isomorphic over the given valued field is proved by the author in [6].

2. Spectral spaces induced by theories of fields: Affine varieties over arbitrary fields

A key role in the following two sections is played by the celebrated Stone's duality theorem. Consider the category DL whose objects are the distributive lattices $L=(L,\vee,\wedge,0,1)$ with the customary binary operations $\vee,\wedge$, a smallest element 0 and a greatest element 1 with respect to the partial order $a\leq b \Leftrightarrow a=a\wedge b \Leftrightarrow b=a\vee b$. If L, L' are objects of DL then the set $DL(L, L')$ of morphisms from L to L' consists of the maps $f:L\to L'$ satisfying $f(a\vee b)=f(a)\vee f(b)$, $f(a\wedge b)=f(a)\wedge f(b)$ for $a,b\in L$, and $f(0)=0$, $f(1)=1$.

On the other hand let us consider the category SS whose objects are usually called spectral spaces by ring theorists, respectively coherent spaces by category and lattice theorists. Thus the objects of SS are the topological spaces X satisfying i) X is sober, i.e., every irreducible closed subset of X is the closure of a unique point of X, and ii) the family L(X) of quasi-compact open subsets of X is closed under finite intersection (in particular, X itself is quasi-compact) and forms a base for the topology of X. A morphism $f:X\to Y$ in SS, called a coherent map, is a (continuous) map subject to $f^{-1}(U)\in L(X)$ for each $U\in L(Y)$.

Theorem 2.1 (Stone's representation theorem for distributive lattices). The category DL is dual to the category SS. This duality induces a duality between the category BA of Boolean algebras and the category BS of Boolean spaces (i.e., compact totally disconnected spaces).

<u>Proof</u>. The duality sends a distributive lattice L to the prime spectrum S(L) of proper prime filters of L; its open sets may be identified with arbitrary ideals of L, a point $F \in S(L)$ being in an open set $I \in Id(L)$ iff $F \cap I$ is non-empty. Conversely, the duality sends a spectral space X to the distributive lattice L(X) of all quasi-compact open subsets of X. For details, see [19] Ch.II. ∎

In particular, if X and Y are spectral spaces, and $f: X \to Y$ is a coherent epi, L(Y) is identified with the sublattice $\{f^{-1}(U): U \in L(Y)\}$ of L(X), and Y is canonically isomorphic over X with the quotient space $Y' = X/_{\sim}$, where $x \sim y \Leftrightarrow \{U \in L(Y): f(x) \in U\} = \{U \in L(Y): f(y) \in U\}$, whose base is the family of sets $\{x \bmod \sim : f(x) \in U\}$ for all $U \in L(Y)$. Thus, Y is completely determined up to a canonic isomorphism over X by a sublattice of L(X). Let us apply this general scheme to the following more concrete situation: Consider a first order language $\underline{\underline{L}}$, an $\underline{\underline{L}}$-theory T and let B=B(T) be the Lindenbaum Boolean algebra of $\underline{\underline{L}}$-sentences up to equivalence modulo T; two L-sentences φ, ψ are identified iff $T \vdash (\varphi \leftrightarrow \psi)$, i.e., the $\underline{\underline{L}}$-sentence $\varphi \leftrightarrow \psi$ follows from T. Let X= =S(B) be the Boolean space assigned by Stone's duality to the Boolean algebra B. The underlying set of X is the set of all complete $\underline{\underline{L}}$-theories extending T, which is identified to the class Mod(T) of the models of T up to the elementary equivalence $\equiv$: if A_1, A_2 are models of T (write $A_i \models T$, i=1,2) then $A_1 \equiv A_2$ iff for each $\underline{\underline{L}}$-sentence φ, $A_1 \models \varphi \Leftrightarrow A_2 \models \varphi$. The family of sets

$D_\varphi=\{T':T'$ is a complete $\underline{\underline{L}}$-theory such that $T\subset T'$ and $T'\vdash\varphi\}$ with $\varphi\in B$ is a base for the topology of X. Let L be a sublattice of B. Then the spectral space Y=S(L) assigned by Stone's duality to L is identified with the quotient space $X/_{\equiv_L}$, where the equivalence relation $\equiv_L$ is given by: $T_1'\equiv_L T_2'\Leftrightarrow\{\varphi\in L:T_1'\vdash\varphi\}=\{\varphi\in L:T_2'\vdash\varphi\}$ for $T_i'\in X$, i=1,2, or in semantic terms, $A_1\equiv_L A_2\Leftrightarrow\{\varphi\in L:A_1\models\varphi\}=\{\varphi\in L:A_2\models\varphi\}$ for $A_i\in Mod(T)$, i=1,2. In other words, Y is identified with the set of $\underline{\underline{L}}$-theories $T_F:=T\cup F\cup\{\neg\varphi:\varphi\in L\setminus F\}$ for all prime filters F of L, with the base given by the family $D_{L,\varphi}:=\{T_F:\varphi\in F\}=\{T_F:T_F\vdash\varphi\}$ for $\varphi\in L$.

Now let us apply the previous scheme to a still more concrete situation: Assume that $\underline{\underline{L}}$ is an augmentation of the customary first order language of rings and T is an $\underline{\underline{L}}$-theory of fields. Let $\underline{\underline{F}}=\underline{\underline{F}}(T)$ be the Boolean algebra of all $\underline{\underline{L}}$-formulas up to equivalence modulo T.

<u>Defintion</u> 2.2. A subset M of $\underline{\underline{F}}$ is <u>closed under polynomial substitution</u> (abbreviated cps) if for each $\underline{\underline{L}}$-formula $\varphi(x_1,\dots,x_m)$, such that φ modulo T belongs to M, and for arbitrary polynomials $f_i\in Z[x_1,\dots,x_n;a_1,\dots,a_k]$, where $n,k\in\underline{N}$, $1\le i\le m$, and the parameters $a_1,\dots,a_k$ are constants of $\underline{\underline{L}}$, the class modulo T of the $\underline{\underline{L}}$-formula $\psi(x_1,\dots,x_n):=\varphi(f_1,\dots,f_m)$ belongs to M too.

Obviously, given a family $(M_i)_{i\in I}$ of cps subsets of $\underline{\underline{F}}$, the intersection $\bigcap_{i\in I} M_i$ is cps too, so we may speak on the cps subset of $\underline{\underline{F}}$ generated by some subset of $\underline{\underline{F}}$. Similarly, we may speak on the cps sublattice of $\underline{\underline{F}}$ generated by some subset of $\underline{\underline{F}}$.

Fix a cps sublattice L of $\underline{\underline{F}}$. Given an $\underline{\underline{L}}$-structure A which is also a commutative ring, call it an $\underline{\underline{L}}$-ring, let $\underline{\underline{L}}_A$ be the augmentation of the language $\underline{\underline{L}}$ with constants which are names for the elements of A, $D(A)_+$ be the <u>positive diagram</u> of A, i.e., the set of all atomic $\underline{\underline{L}}_A$ - sentences which are true on A, and D(A) be the <u>diagram</u> of A, i.e., the union of $D(A)_+$ with the set of all negated atomic $\underline{\underline{L}}_A$-sentences which are true on A. Denote by T_A the $\underline{\underline{L}}_A$ - theory $T \cup D(A)_+$, whose models are identified with the $\underline{\underline{L}}$-morphisms $A \to F$, with $F \models T$, and let B(A) be the Boolean algebra of all $\underline{\underline{L}}_A$ - sentences up to equivalence modulo T_A, and L(A) be the sublattice of B(A) induced by L, consisting of the classes modulo T_A of the $\underline{\underline{L}}_A$ - sentences $\varphi(f_1,\ldots,f_m)$, where $\varphi(x_1,\ldots,x_m)$ is an $\underline{\underline{L}}$-formula, such that φ modulo $T \in L$ and $f_1,\ldots,f_m \in A$. Denote by $Spec_{T,L}(A)=S(L(A))$ the spectral space assigned by Stone duality to the distributive lattice L(A). One gets in this way a contravariant functor $Spec_{T,L}$ from the category of $\underline{\underline{L}}$-rings into the category $\underline{\underline{SS}}$ of spectral spaces, assigned to the pair (T, L).

A particularly important (for algebraic - geometric applications) cps lattice will be the object of the rest of this section, while other relevant cps lattices will be investigated in Section 3.

Given an $\underline{\underline{L}}$-theory T of fields, let us denote by ZL=ZL(T) the cps sublattice of the Boolean algebra $\underline{\underline{F}}=\underline{\underline{F}}(T)$, generated by the class modulo T of the $\underline{\underline{L}}$-formula $x_1 \neq 0$, and call it the <u>Zariski lattice</u> assigned to T. If A is an $\underline{\underline{L}}$-ring we abbreviate $Spec_{T,ZL}(A)$ by $Spec_T(A)$.

In particular, if $\underline{L}$ is the language of rings and T is the theory $\underline{Fd}$ of fields or the theory $\underline{ACF}$ of algebraically closed fields, the spectral space $Spec_T(A)$ is identified with the Zariski prime spectrum Spec(A) of the commutative ring A. In general, if T is arbitrary, then $Spec_T(A)$ is identified with the subspace of Spec(A) consisting of the prime ideals $\underline{p}$ of A for which the field $k(\underline{p})=Q(A/\underline{p})$ is embeddable in some model of T. In particular, we get $Spec_T(A)=Spec_{T'}(A)$, where $T'=\underline{Fd}\vee T_{\forall}$. For instance, if T the theory $\underline{RCF}$ of real closed fields, in which case T' is the theory of formally real fields, then the underlying set of $Spec_T(A)$ consists of those $\underline{p}\in Spec(A)$ for which $k(\underline{p})$ is formally real.

Let us fix for the rest of this section a base field K. Let $\underline{L}_K$ be the language $\underline{L}$ of rings augmented with constants which are names for the elements of K and $T=Th(K, \langle a\rangle a\in K)$ be the complete $\underline{L}_K$ - theory of K. The models of T are the elementary extensions of the base field K. If A is a K-algebra, we write $Spec_K(A)$ instead of $Spec_T(A)$. $Spec_K(A)$ is the subspace of Spec(A) consisting of those $\underline{p}\in Spec(A)$ for which K is existentially complete in $k(\underline{p})$. Thus, $Spec_K(A)$ is Spec(A) if K is algebraically closed (by Theorem 1.1), $\{\underline{p}\in Spec(A): A/\underline{p}\cong K\}$ if K is finite, $\{\underline{p}\in Spec(A): k(\underline{p})$ is formally real$\}$ if $\underline{K}$ is real closed (by Theorem 1.4), and $\{\underline{p}\in Spec(A): k(\underline{p})$ is formally p-adic over $\underline{K}$, i.e., there is a valuation w of $k(\underline{p})$ such that $(k(\underline{p}),w)$ is a p-valued field extension of $\underline{K}$ of type (e, f)$\}$ if $\underline{K}=(K, v)$ is a p-adically closed p-valued field of type (e, f) (by Theorem 1.14).

Given an ideal I of A, let $V_K(I)$ be the closed subset $\{\underline{p}\in Spec_K(A): I\subset p\}$ of $Spec_K(A)$, and let $rad_K(I)= \bigcap\, p$ be the so called K-radical of I. If I is the null ideal of A, let us put $\mathrm{Rad}_K(A):=\mathrm{rad}_K(I)$. I is called a K-radical ideal if $I=$ $=\mathrm{rad}_K(I)$. In some concrete situations $\mathrm{rad}_K(I)$ admits more explicit descriptions. For instance, $\mathrm{rad}_K(I)$ is the milradical of I, if K is algebraically closed; $\mathrm{rad}_K(I)=\{f\in A: -f^{2n}\in\sum A^2+I$ for some $n\geqslant 1\}$, if K is real closed (see [35]); if $\underline{K}=(K, v)$ is a p-adically closed p-valued field of type (e, f), $A=K[\underline{X}]$, $\underline{X}=(X_1,\dots,X_n)$ and $I=(f_1,\dots,f_m)$, then $\mathrm{rad}_K(I)=\{g\in A: g^l=\sum_{i=1}^{m}\lambda_i f_i$ with $l\geqslant 1$, $\lambda_i=\frac{s_i}{1-\pi t_i}$, $s_i\in A[\gamma F]$, $t_i\in Z[\gamma F]\}$, where $\pi\in K$, $v(\pi)=1$, $F=K(\underline{X})$, $\gamma F=\{\gamma(g): g\in F\}$ and $\gamma(X)=\frac{1}{\pi}\,\frac{X^q-X}{(X^q-X)^2-1}$, $q=p^f$ (see [18] Theorem 1.1).

Obviously, $V_K(I)=V_K(\mathrm{rad}_K(I))$ and the map $I\mapsto V_K(I)$ induces a galoisian correspondence between the K-radical ideals of A and the closed subsets of $\mathrm{Spec}_K(A)$. As $\mathrm{Spec}_K(A)$ is sober, $V_K(I)$ is irreducible iff $\mathrm{rad}_K(I)\in\mathrm{Spec}_K(A)$. In particular, $\mathrm{Spec}_K(A)$ is irreducible iff $\mathrm{Rad}_K(A)\in\mathrm{Spec}_K(A)$. Since $\mathrm{Spec}_K(A)$ is a spectral space, the basic open sets $D_K(f)=\{\underline{p}\in\mathrm{Spec}_K(A): f\notin\underline{p}\}$, $f\in A$, are quasi-compact and hence the set of K-radical ideals of A is inductive with respect to the partial order given by inclusion, i.e., the union of a chain of K-radical ideals is a K-radical ideal. Consequently, we get by standard arguments (see [22] Ch.6, §1):

Lemma 2.3. Given a K-algebra A, the next statements are equivalent:

i) For every K-radical ideal I there exists a finitely generated ideal J such that $I=\mathrm{rad}_K(J)$;

ii) Every ascending chain $I_1 \subset I_2 \subset \dots$ of K-radical ideals is stationary.

iii) Every non-empty set of K-radical ideals has a maximal element.

Definition 2.4. A K-algebra A satisfying the equivalent conditions above is called K-noetherian.

Obviously, A is K-noetherian iff the space $\mathrm{Spec}_K(A)$ is noetherian. The next lemma is immediate by standard arguments (see [34] Ch.I, §3, Theorem 1.2).

Lemma 2.5. If the K-algebra A is K-noetherian then $\mathrm{Spec}_K(A)$ has a unique decomposition in finitely many irreducible components.

Now let $\mathrm{Max}_K(A)=\{\underline{p}\in \mathrm{Spec}_K(A): V_K(\underline{p})=\{\underline{p}\}\}$ be the set of closed points of $\mathrm{Spec}_K(A)$. The closed subsets of $\mathrm{Max}_K(A)$ with respect to the topology induced from $\mathrm{Spec}_K(A)$ have the form $\underline{V}_K(I)=V_K(I)\cap \mathrm{Max}_K(A)$, where I ranges over the ideals of A. If I is an ideal of A, let $\mathrm{Jrad}_K(I)$ be the ideal $\bigcap_{\underline{p}\in \underline{V}_K(I)} \underline{p}$; call it the Jacobson K-radical of I. Obviously, $\mathrm{rad}_K(I)\subset \mathrm{Jrad}_K(I)$.

Theorem 2.6. (Nullstellensatz). Assume that the K-algebra A is finitely generated. Then

i) $\mathrm{Max}_K(A)=\{\underline{p}\in \mathrm{Spec}(A): A/\underline{p}\simeq K\}$ and $\mathrm{Max}_K(A)$ is dense in $\mathrm{Spec}_K(A)$;

ii) the map $V_K(I)\mapsto \underline{V}_K(I)$ is an isomorphism of inf-complete lattices from the lattice of closed subsets of $\mathrm{Spec}_K(A)$ onto the lattice of closed subsets of $\mathrm{Max}_K(A)$;

iii) $Jrad_K(I)=rad_K(I)$ for every ideal I of A.

<u>Proof</u>. We may assume without loss of generality that A is a polynomial algebra $K[\underline{X}]$, $\underline{X}=(X_1,\dots,X_n)$.

In order to prove the theorem, it suffices to show that for every prime ideal $\underline{p}\in Spec_K(A)$ and every $f\in A\setminus\underline{p}$, there exists $\underline{q}\in Spec(A)$ such that $A/\underline{q}\not\cong K$, $\underline{p}\subset\underline{q}$ and $f\notin\underline{q}$.

Let $\underline{p}$ and f be as above, and $g_1,\dots,g_m$ be generators of $\underline{p}$. Consider the existential $\underline{L}_K$-sentence $\varphi:=(\exists\underline{x})(\bigwedge_{i=1}^{m} g_i(\underline{x})=0)\wedge(f(\underline{x})\neq 0)$ which is obviously true on $k(\underline{p})=Q(A/\underline{p})$. As K is existentially complete in $k(\underline{p})$, φ is true on K too, and hence there is $\underline{a}\in K^m$ such that $g_i(\underline{a})=0$ for $1\leq i\leq m$ and $f(\underline{a})\neq 0$. Thus the substitution $\underline{X}\mapsto\underline{a}$ induces a K-morphism $u:A/\underline{p}\to K$. The prime ideal $\underline{q}=Ker(u)$ satisfies the required conditions. ∎

The theorem above identifies $Max_K(A)$, where $A=K[\underline{X}]$, $\underline{X}=(X_1,\dots,X_n)$, $n\in\underline{N}$, with the affine space K^n, and the closed subsets $\underline{V}_K(I)$ of $Max_K(A)$ with the subsets $\{\underline{a}\in K^n: f(\underline{a})=0 \text{ for each } f\in I\}$ of K^n; call the latter ones affine K-<u>varieties</u> (abbreviated K-<u>varieties</u>). Call <u>Zariski</u> K-<u>topology</u> on K^n the topology whose closed sets are exactly the K-varieties in K^n. If $Y=\underline{V}_K(I)$ is a K-variety in K^n, let $J_K(Y)$ be the ideal $\{f\in A: f(\underline{a})=0 \text{ for each } \underline{a}\in Y\}$. According to Theorem 2.6, $J_K(Y)=rad_K(I)$, and the map $Y\mapsto J_K(Y)$ is a galoisian correspondence between the K-varieties in K^n and the K-radical ideals of A.

For $Y=\underline{V}_K(I)$, denote by $K[Y]$ the <u>coordinate</u> K-<u>algebra</u> $A/J_K(Y)=A/rad_K(I)$ of the K-variety Y. Y is irreducible with respect to the Zariski K-topology iff $J_K(Y)\in Spec_K(A)$. For an irreducible K-variety Y, let us denote by $K(Y)=Q(K[Y])$ the <u>field of rational functions</u> on Y.

<u>Definition</u> 2.7. Let $Y\subset K^n$, $Z\subset K^m$ be K-varieties. A map $f:Y\to Z$ is called <u>regular</u> if there exist $f_1,\dots,f_m\in K[Y]$ such that $f(\underline{a})=(f_1(\underline{a}),\dots,f_m(\underline{a}))$ for every $\underline{a}\in Y$.

The correspondence $Y\mapsto K[Y]$ extends to an equivalence from the category of K-varieties, with regular maps as morphisms onto the category of finitely generated K-algebras A, which are K-<u>reduced</u>, i.e., $\mathrm{Rad}_K(A)=0$. This equivalence induces an equivalence between the category of irreducible K-varieties and the category of finitely generated K-algebras A for which the null ideal belongs to $\mathrm{Spec}_K(A)$. A finitely generated field extension F of K is isomorphic over K to K(Y) for some irreducible K-variety Y iff K is existentially complete in F.

<u>Definition</u> 2.8. The K-variety $Y\subset K^n$ is <u>defined</u> (<u>rational</u>) over a subfield k of K if its defining ideal $J_K(Y)$ is generated by polynomials in $k[\underline{X}]$, $\underline{X}=(X_1,\dots,X_n)$.

If $Y\subset K^n$ is a K-variety and k is a subfield of K, let us denote by $Y(k)=Y\cap k^n$ the set of k-<u>points</u> of Y. If F is a field extension of K, let $Y\otimes_K F$ be the F-variety $\underline{V}_F(J_K(Y)F[\underline{X}])$.

The following lemmas are immediate.

<u>Lemma</u> 2.9. Let $Y\subset K^n$ be a K-variety, k be a subfield of K and F be a field extension of K.

i) $J_F(Y\otimes_K F)=J_K(Y)\,F[\underline{X}]$ and $F[Y\otimes_K F]=K[Y]\otimes_K F$; in particular the F-variety $Y \otimes_K F$ is defined over K.

ii) $Y \otimes_K F$ is the smallest F-variety $Z \subset F^n$ satisfying $Y \subset Z(K)$.

iii) If Y is defined over k, then $Y(k) \subset k^n$ is a k-variety, $J_k(Y(k)) = \mathrm{rad}_k(J_K(Y) \cap k[\underline{X}])$ and $Y(k) \otimes_k K \subset Y$, with equality if k is existentially complete in K.

<u>Lemma</u> 2.10. Let F/K be a field extension.

i) The map $Y \subset F^n \mapsto K[\underline{X}]/J_F(Y) \cap K[\underline{X}]$, $\underline{X} = (X_1, \ldots, X_n)$, extends to an equivalence from the category of F-varieties defined over K, with regular maps defined over K as morphisms, onto the category of finitely generated K-algebras A for which $A \otimes_K F$ is F-reduced.

ii) The functor $Y \mapsto Y(K)$ from the category of F-varieties defined over K into the category of K-varieties is the left adjoint of the functor $Z \mapsto Z \otimes_K F$.

In the following, let us fix the base field K and let $F = \widetilde{K}$ be the algebraic closure of K, or an arbitrary algebraically closed field extension of K. The K-varieties are identified with the $\widetilde{K}$-varieties Y defined over K satisfying $Y = Y(K) \otimes_K \widetilde{K}$.

<u>Lemma</u> 2.11. The necessary and sufficient condition for the K-variety Y to be irreducible is that the $\widetilde{K}$-variety $Y \otimes_K \widetilde{K}$ is irreducible.

<u>Proof</u>. Let $A = K[Y]$, $B = F[Y \otimes_K \widetilde{K}] = A \otimes_K \widetilde{K}$. By Lemma 2.9, the canonic K-morphism $u: A \to B$ is injective. Thus if $Y \otimes_K \widetilde{K}$ is irreducible, i.e., B is an integral domain, then A is an integral domain too. On the other hand, A is K-reduced, i.e., the null ideal of

exist $\underline{\underline{p}}_1,\ldots,\underline{\underline{p}}_m \in \mathrm{Spec}_K(A)$ such that $\bigcap_{i=1}^{m} \underline{\underline{p}}_i=0$. As $0\in\mathrm{Spec}(A)$, it follows immediately that moreover $0\in\mathrm{Spec}_K(A)$, i.e., Y is irreducible.

Conversely, if Y is irreducible, then K is existentially complete in the field $F=K(Y)=Q(A)$, and hence the field extension F/K is regular. Consequently, $B=A\otimes_K \widetilde{K}$ is an integral domain, i.e., $Y\otimes_K \widetilde{K}$ is irreducible. ∎

If Y is an irreducible $\widetilde{K}$-variety, let us denote by Y_{reg} the Zariski open set of regular (simple, smooth) points of Y. If Z is an irreducible K-variety, let $Z_{reg}:=Z\cap(Z\otimes_K \widetilde{K})_{reg}$.

<u>Lemma</u> 2.12. If Z is an irreducible K-variety then Z_{reg} is dense in Z with respect to the Zariski K-topology. In particular Z_{reg} is non-empty.

<u>Proof</u>. Let $Z\subset K^n$ and $g\in K[\underline{\underline{X}}]\setminus J_K(Z)$. We have to show that $\{a\in Z_{reg}: g(\underline{\underline{a}})\neq 0\}$ is non-empty. Let $f_1,\ldots,f_m\in K[\underline{\underline{X}}]$ be generators of $J_K(Z)$ and hence of $J_K(Y)$, where $Y=Z\otimes_K \widetilde{K}$. Given a field extension F of $\widetilde{K}$, the necessary and sufficient condition for a point $\underline{\underline{b}}$ of $Y\otimes_{\widetilde{K}} F=Z\otimes_K F$ to be regular is that there exists a minor $h\in K[\underline{\underline{X}}]$ of order n-dim Y of the Jacobian matrix $(\frac{\partial f_i}{\partial X_j})_{1\leq i\leq m,\ 1\leq j\leq n}$ such that $h(\underline{\underline{b}})\neq 0$, where $\dim Y=\mathrm{trdeg}(\widetilde{K}(Y)/K)=\mathrm{trdeg}(K(Z)/K)$. Let $\underline{\underline{y}}=\underline{\underline{X}} \bmod J_K(Z)=\underline{\underline{X}} \bmod J_K(Y)$ and take $F=\widetilde{K}(Y)=K(Z)\otimes_K \widetilde{K}$. Since $\underline{\underline{y}}$ is generic, there exists a minor $h\in K[\underline{\underline{X}}]$ as above such that $h(\underline{\underline{y}})\neq 0$. As K is existentially complete in K(Z), there exists $\underline{\underline{a}}\in K^n$ such that $f_i(\underline{\underline{a}})=0$ for $1\leq i\leq m$ and $h(\underline{\underline{a}})g(\underline{\underline{a}})\neq 0$. Thus $\underline{\underline{a}}\in Z_{reg}$ and $g(\underline{\underline{a}})\neq 0$,

as required. ∎

The next stratification lemma follows from Lemmas 2.5 and 2.12 by standard arguments (see [27] Ch.1, §1A).

Lemma 2.13. If Y is a K-variety then there exists finitely many irreducible K-varieties $Y^i \subset Y$, $1 \leq i \leq m$, such that

$$Y = \bigcup_{i=1}^{m} Y^i_{reg}.$$

3. Spectral spaces induced by theories of fields: Regularity and finiteness properties.

K being a fixed base field, let $\underline{\underline{L}}_K$ be the language $\underline{\underline{L}}$ of rings augmented with constants standing for the elements of K, T be the $\underline{\underline{L}}_K$ theory $Th(K, \langle a \rangle_{a \in K})$ and $\underline{\underline{F}}$ be the Boolean algebra of the $\underline{\underline{L}}_K$ - formulas up to equivalence modulo T.

Given an $\underline{\underline{L}}_K$ - formula $\varphi(x_1, \ldots, x_m)$, let $\tilde{\varphi}(x_1, \ldots, x_m, x_{m+1}) := (x_{m+1} \neq 0) \wedge \left[(\exists z_1) \ldots (\exists z_m) \varphi(z_1, \ldots, z_m) \wedge \bigwedge_{i=1}^{m} x_i = x_{m+1} \cdot z_i\right]$; call $\tilde{\varphi}$ the homogenization of φ.

Definition 3.1. A subset M of $\underline{\underline{F}}$ is called homogeneous if whenever $\varphi(x_1, \ldots, x_m)$ is an $\underline{\underline{L}}_K$-formula such that $\varphi \bmod T \in M$, then $\tilde{\varphi} \bmod T$ belongs to M too.

One checks easily that the Zariski lattice $ZL = ZL(T)$ is homogeneous.

Given a cps sublattice L of $\underline{\underline{F}}$ and a K-algebra A, let $D(A)_+$, $D(A)$, T_A, $B(A)$ and $Spec_{K,L}(A) := Spec_{T,L}(A)$ be as defined in Section 2. In the following we assume always that <u>the cps sublattice</u> L <u>contains</u> ZL, so we get a coherent epi $\mathrm{Spec}_{K,L}(A) \to \mathrm{Spec}_K(A)$. Thus, the points of the spectral space $\mathrm{Spec}_{K,L}(A)$ are identified with pairs $(\underline{\underline{p}}, f)$, where $\underline{\underline{p}} \in \mathrm{Spec}_K(A)$ and f is a K-embedding of the integral domain $A/_{\underline{\underline{p}}}$ into some model F of T, up to the equivalence relation given by $(\underline{\underline{p}}_1, f_1 : A/_{\underline{\underline{p}}_1} \to F_1) \sim (\underline{\underline{p}}_2, f_2 : A/_{\underline{\underline{p}}_2} \to F_2)$ iff $\underline{\underline{p}}_1 = \underline{\underline{p}}_2$ and $F_1 \equiv_{L(A/\underline{\underline{p}}_1)} F_2$. One checks easily that the underlying set of $\mathrm{Spec}_{K,L}(A)$ is the disjoint union $\bigcup_{\underline{\underline{p}} \in \mathrm{Spec}_K(A)} \mathrm{Spec}_{K,L}(k(\underline{\underline{p}}))$ if L is supposed to be homogeneous.

Now let Y be a K-variety and $A = K[Y]$ be its coordinate K-algebra. As the set Y is identified with $\mathrm{Hom}_K(A, K)$ and $\mathrm{Spec}_{K,L}(K)$ is a singleton, we get a canonic embedding of Y into $\mathrm{Spec}_{K,L}(A)$, inducing on Y a topology, called L-<u>topology</u>, with the family $\gamma_{Y,L} := \{\underline{\underline{D}}_\varphi := D_{L,\varphi} \cap Y\}_{\varphi \in L(A)}$ as base of open sets. Since, by assumption, $ZL \subset L$, it follows that the L-topology on Y is finer as the Zariski K-topology on Y. Obviously, $\gamma_{Y,L}$ is a sublattice of the power set P(Y) with respect to finite unions and intersections.

<u>Lemma</u> 3.2. The map $\phi_L : L(A) \to \gamma_{Y,L} : \varphi \mapsto \underline{\underline{D}}_\varphi$ is an isomorphism of lattices.

<u>Proof</u>. Let $\psi_1, \psi_2 \in L(A)$. We have to show that $\underline{\underline{D}}_{\psi_1} \subset \underline{\underline{D}}_{\psi_2} \Rightarrow D_{L,\psi_1} \subset D_{L,\psi_2}$. Let $\widehat{\psi}_i(x_1, \ldots, x_m)$, i=1,2, be $\underline{\underline{L}}_K$-formulas, and $\underline{\underline{f}}_i \in A^m$, i=1,2, be such that $\psi_i = \widehat{\psi}_i(\underline{\underline{f}}_i) \bmod T$ for i=1,2. We have to show that $T_A = T \cup D(A)_+ \vdash \widehat{\psi}_1(\underline{\underline{f}}_1) \to \widehat{\psi}_2(\underline{\underline{f}}_2)$. Write $A = K[\underline{\underline{y}}]$, $\underline{\underline{y}} = (y_1, \ldots$

$\ldots,y_n)$, and assume that $J_K(Y)=\mathrm{Ker}(K[\underline{\underline{X}}]\to A:\underline{\underline{X}}\mapsto \underline{\underline{y}}\}$ is generated by $g_1,\ldots,g_l$. Denote by "$\underline{\underline{x}}\in Y$" the $\underline{\underline{L}}_K$-formula $\bigwedge_{i=1}^{l} g_i(\underline{\underline{x}})=0$, $\underline{\underline{x}}=(x_1,\ldots,x_n)$. Then $T_A=T\cup\{\text{"}\underline{\underline{y}}\in Y\text{"}\}$. Since, by assumption, $K\models \text{"}\underline{\underline{x}}\in Y\text{"}\to(\widehat{\Psi}_1(\underline{\underline{f}}_1(\underline{\underline{x}}))\to\widehat{\Psi}_2(\underline{\underline{f}}_2(\underline{\underline{x}})))$ and $K\models T$, we get $T_A\vdash\widehat{\Psi}_1(\underline{\underline{f}}_1)\to\widehat{\Psi}_2(\underline{\underline{f}}_2)$, as required. ∎

Let $\widehat{\gamma}_{Y,L}$ bethe spectral space assigned by Stone duality to the distributive lattice $\gamma_{Y,L}$. The points of $\widehat{\gamma}_{Y,L}$ are the prime filters of $\gamma_{Y,L}$ and the sets $[\underline{\underline{D}}]=\{\underline{\underline{F}}\in\widehat{\gamma}_{Y,L}: \underline{\underline{D}}\in\underline{\underline{F}}\}$ with $\underline{\underline{D}}\in\gamma_{Y,L}$ is a base of $\widehat{\gamma}_{Y,L}$. The isomorphism ϕ_L above induces a homeomorphism $\widehat{\phi}_L: \widehat{\gamma}_{Y,L}\to \mathrm{Spec}_{K,L}(A)$:

$$\underline{\underline{F}}\mapsto T_A\cup\{\varphi\in L(A):\phi_L(\varphi)\in\underline{\underline{F}}\}\cup\{\neg\varphi:\varphi\in L(A),\phi_L(\varphi)\notin\underline{\underline{F}}\}.$$

In particular, Y is dense in $\mathrm{Spec}_{K,L}(A)$.

Now let $\overline{L}$ be the cps sublattice of $\underline{\underline{F}}$ generated by $L\cup\{(x_1=0)\bmod T\}$. Thus the members of $\overline{L}$ are the classes modulo T of the $\underline{\underline{L}}_K$ - formulas having the form $\bigvee_{i=1}^{l}[\varphi_i(\underline{\underline{x}})\wedge\bigwedge_{j=1}^{s_i}(f_{ij}(\underline{\underline{x}})=0)]$, or equivalently, having the form $\bigwedge_{i=1}^{l}[\varphi_i(\underline{\underline{x}})\vee(f_i(\underline{\underline{x}})=0)]$, where f_{ij}, $f_i\in K[\underline{\underline{x}}]$, $\underline{\underline{x}}=(x_1,\ldots,x_n)$, and $\varphi_i \bmod T\in L$ for $1\leq i\leq l$. Call the members of the distributive lattice $\gamma_{Y,\overline{L}}$ (i.e., the basic $\overline{L}$-open subsets of Y) the L-<u>constructible</u> subsets of Y. The embedding $L(A)\subset\overline{L}(A)$ induces the coherent epis $\lambda_L:\mathrm{Spec}_{K,\overline{L}}(A)\to\mathrm{Spec}_{K,L}(A)$, $\lambda'_L:\widehat{\gamma}_{Y,\overline{L}}\to\widehat{\gamma}_{Y,L}$ such that $\lambda_L\circ\widehat{\phi}_{\overline{L}}=\widehat{\phi}_L\,\lambda'_L$.

Since $ZL\subset L$, it follows:

<u>Lemma</u> 3.3. The map $\lambda_L:\mathrm{Spec}_{K,\overline{L}}(A)\to\mathrm{Spec}_{K,L}(A)$ is bijective

Thus, the underlying sets of $\mathrm{Spec}_{K,L}(A)$ and $\mathrm{Spec}_{K,\overline{L}}(A)$ can be identified, and so, the topology on $\mathrm{Spec}_{K,\bar{\bar{L}}}(A)$ is a rafination of the topology on $\mathrm{Spec}_{K,L}(A)$.

Lemma 3.4. Let $U,V\subset Y$ be L-constructible sets such that $U\subset V$. The necessary and sufficient condition for $\phi_{\bar{\bar{L}}}^{-1}(U)$ to be open in $\phi_{\bar{\bar{L}}}^{-1}(V)$ with respect to the topology of $\mathrm{Spec}_{K,L}(A)$ is that $U=V\cap \underline{\underline{D}}$ for some $\underline{\underline{D}}\in\gamma_{Y,L}$.

Proof. The if part is trivial. Conversely, assuming that $\phi_{\bar{\bar{L}}}^{-1}(U)$ is L-open in $\phi_{\bar{\bar{L}}}^{-1}(V)$, there exists a family $(\varphi_i)_{i\in I}$ of elements of L(A) such that $\phi_{\bar{\bar{L}}}^{-1}(U)=\phi_{\bar{\bar{L}}}^{-1}(V)\cap\bigcup_{i\in I}D_{L,\varphi_i}$. As $\phi_{\bar{\bar{L}}}^{-1}(U)$ is open quasi-compact in $\mathrm{Spec}_{K,\bar{\bar{L}}}(A)$, there exists a finite subset J of I such that $\phi_{\bar{\bar{L}}}^{-1}(U)=\phi_{\bar{\bar{L}}}^{-1}(V)\cap\bigcup_{i\in J}D_{L,\varphi_i}$. Consequently, $U=V\cap\underline{\underline{D}}$, where $\underline{\underline{D}}=\bigcup_{i\in J}\underline{\underline{D}}_{\varphi_i}\in\gamma_{Y,L}$. ∎

Corollary 3.5. Let $U,V\subset Y$ be L-constructible sets such that $\phi_{\bar{\bar{L}}}^{-1}(U)$ is L-open in $\phi_{\bar{\bar{L}}}^{-1}(V)$. Then U is L-open in V.

Definition 3.6. We say that the lattice L has <u>finiteness property</u> if the converse of Corollary 3.5. holds, i.e., for every K-variety Y and arbitrary $U,V\in\gamma_{Y,\bar{\bar{L}}}$, if U is L-open in V then $U=V\cap\underline{\underline{D}}$ for some $\underline{\underline{D}}\in\gamma_{Y,L}$.

Now let us assume that Y is an irreducible K-variety, and let $A=K[Y]$, $F=K(Y)$. Assuming L homogeneous, the canonic morphism of lattices $\psi_L:L(A)\to L(F)$ is onto and so we get an embedding $\hat{\psi}_L:\mathrm{Spec}_{K,L}(F)=\mathrm{Spec}_{K,\bar{\bar{L}}}(F)\hookrightarrow\mathrm{Spec}_{K,L}(A)$. Note that $\bar{\bar{L}}$ is homogeneous too. Thus we have the following commutative diagram

$$\begin{array}{ccccc} \hat{\gamma}_{Y,\bar{L}} & \xrightarrow[\sim]{\hat{\phi}_{\bar{L}}} & \mathrm{Spec}_{K,\bar{L}}(A) & \xleftarrow{\hat{\psi}_{\bar{L}}} & \\ \lambda'_L \downarrow & & \lambda_L \downarrow & & \\ \hat{\gamma}_{Y,L} & \xrightarrow[\sim]{\hat{\phi}_L} & \mathrm{Spec}_{K,L}(A) & \xhookleftarrow{\hat{\psi}_L} & \mathrm{Spec}_{K,L}(F) \end{array}$$

The following result is a generalization of Brumfiel's ultrafilter theorem for orders [11] p.232, [8] I.7.

Lemma 3.7. With the notations and hypothesis above,

i) $\mathrm{Im}(\hat{\phi}_L^{-1}\circ\hat{\psi}_L)=\{\mathcal{F}\in\hat{\gamma}_{Y,L}: \underline{\underline{D}}_f\subset\mathcal{F}$ for every $0\neq f\in A\}$, where $\underline{\underline{D}}_f=\{\underline{\underline{a}}\in Y: f(\underline{\underline{a}})\neq 0\}$;

ii) $\mathrm{Im}(\hat{\phi}_{\bar{L}}^{-1}\circ\hat{\psi}_{\bar{L}})=\{\mathcal{F}\in\hat{\gamma}_{Y,\bar{L}}$: U is dense in Y with respect to the Zariski K-topology on Y for every $U\in\mathcal{F}\}$.

Proof i) is trivial.

ii) Let $\mathcal{F}\in\mathrm{Im}(\hat{\phi}_{\bar{L}}^{-1}\circ\hat{\psi}_{\bar{L}})$. Then there exists a field extension N of F such that the extension N/K is elementary and $\mathcal{F}=\{\underline{\underline{D}}_\varphi : \varphi\in\bar{L}(A),\ N\models\varphi\}$. Let $0\neq f\in A$. Then $\underline{\underline{D}}_f\in\mathcal{F}$ and hence $U\cap\underline{\underline{D}}_f$ is non-empty for each $U\in\mathcal{F}$.

Conversely, let $\mathcal{F}\in\hat{\gamma}_{Y,\bar{L}}$ be such that U is Zariski-dense in Y for every $U\in\mathcal{F}$. Let N be a model of $T_A=T\cup D(A)_+$ such that $\hat{\phi}_{\bar{L}}^{-1}(N)=\mathcal{F}$. We have to show that N is a field extension of F. Assuming the contrary, let $0\neq f\in A$ be such that $N\models(f=0)$. Then $U=\{\underline{\underline{a}}\in Y: f(\underline{\underline{a}})=0\}\in\mathcal{F}$, and hence $U\cap\underline{\underline{D}}_f$ is non-empty since U is Zariski dense in Y, a contradiction. ∎

Lemma 3.8. Assume that L is homogeneous and Y is an irreducible K-variety. Let us denote by $\overline{Y_{reg}}$ the closure of Y_{reg} with respect to the L-topology on Y, and by $\overline{\mathrm{Im}(\hat{\psi}_L)}$ the closure of $\mathrm{Im}(\hat{\psi}_L)$ in $\mathrm{Spec}_{K,L}(A)$, $A=K[Y]$. Identifying Y with a subset of $\mathrm{Spec}_{K,L}(A)$, we have $Y\cap\overline{\mathrm{Im}(\hat{\psi}_L)}\subset\overline{Y_{reg}}$.

<u>Proof</u>. Let $\underline{\underline{a}}\in Y\cap\overline{\mathrm{Im}(\hat{\psi}_L)}$ and $\varphi\in L(A)$ be such that $\underline{\underline{a}}\in\underline{\underline{D}}_\varphi$. We have to show that $Y_{reg}\cap\underline{\underline{D}}_\varphi$ is non-empty. As $D_{L,\varphi}\cap\overline{\mathrm{Im}(\hat{\psi}_L)}$ is non-empty by assumption, there exists a field extension N of $F=K(Y)$ such that the extension N/K is elementary and φ is true on N. Thus, N satisfies the $\underline{\underline{L}}_K$-sentence $\Theta:=(\exists\underline{\underline{x}})$ "$\underline{\underline{x}}\in Y$"$_{reg}\wedge\varphi(\underline{\underline{x}})$, (set $\underline{\underline{x}}=\underline{\underline{y}}$, the generic point of Y), and hence Θ is true on K too, i.e., $Y_{reg}\cap\underline{\underline{D}}_\varphi$ is non-empty, as required. ∎

<u>Definition</u> 3.9. We say that the homogeneous cps lattice L has <u>regularity property</u> if the opposite inclusion of Lemma 3.8 holds for every irreducible K-variety Y, i.e., for each $\varphi\in L(K[Y])$ $D_{L,\varphi}\cap\mathrm{Im}(\hat{\psi}_L)$ is non-empty (equivalently $T\cup D(K[Y])\cup\{\varphi\}$ is consistent) whenever $Y_{reg}\cap\underline{\underline{D}}_\varphi$ is non-empty.

<u>Lemma</u> 3.10. Assume that L has regularity property, Y is an irreducible K-variety and U is an L-open subset of Y such that $U\cap Y_{reg}$ is non-empty. Then U is Zariski dense in Y.

<u>Proof</u>. Let $\underline{\underline{a}}\in U\cap Y_{reg}$ and $\varphi\in L(A)$, $A=K[Y]$ be such that $\underline{\underline{a}}\in\underline{\underline{D}}_\varphi\subset U$. As L has regularity proerty and $\underline{\underline{D}}_\varphi\cap Y_{reg}$ is non-empty, $T\cup D(A)\cup\{\varphi\}$ is consistent, and hence $\underline{\underline{D}}_\varphi\cap\underline{\underline{D}}_f$ is non-empty for every $0\neq f\in A$, i.e., $\underline{\underline{D}}_\varphi$ (and hence U) is Zariski-dense in Y. ∎

In the rest of this section we apply the general theory above to the particular case when the base field K is algebraically closed, or real closed, or p-adically closed, and L is the cps sublattice of $\underline{\underline{F}}$ generated by the classes modulo T of the $\underline{\underline{L}}$-formulas $\varphi_n(x_1) := (x_1 \neq 0) \wedge ((\exists x_2) x_1 = x_2^n)$ for all $n \geq 2$.

<u>Lemma</u> 3.11. i) If K is algebraically closed then L=ZL.
ii) If K is real closed then L is the cps sublattice of $\underline{\underline{F}}$ generated by the class modulo T of the L-formula φ_2. In particular, ZL ⊂ L.
iii) If K is p-adically closed then ZL ⊂ L.

<u>Proof</u>. i) If K is algebraically closed then $T \vdash \varphi_n(x_1) \leftrightarrow (x_1 \neq 0)$ for $n \geq 2$.
ii) If K is real closed then $T \vdash \varphi_n(x_1) \leftrightarrow \varphi_2(x_1)$ if n is even, and $T \vdash \varphi_n(x_1) \leftrightarrow (x_1 \neq 0)$ if n is odd. We get also $T \vdash (x_1 \neq 0) \leftrightarrow (\varphi_2(x_1) \vee \varphi_2(-x_1))$.
iii) If K is p-adically closed then $K^{\times}/K^{\times 2}$ is finite and

$$T \vdash (x_1 \neq 0) \leftrightarrow \bigvee_{a \in K^{\times}/K^{\times 2}} \varphi_2(a x_1). \qquad \blacksquare$$

<u>Lemma</u> 3.12. Let K, L be as above. Then L is homogeneous and $\overline{L}$ is the whole Boolean algebra $\underline{\underline{F}}$.

<u>Proof.</u> The homogeneity of L is immediate. For the convenience of the reader let us prove this in the case when K is real closed. Then each member of L is the class modulo T of some $\underline{\underline{L}}_K$-formula $\psi(x_1, \ldots, x_m)$ of the form $\psi(\underline{\underline{x}}) := \bigvee_{i=1}^{\ell} \bigwedge_{j=1}^{s_i} \varphi_2(f_{ij}(\underline{\underline{x}}))$, with $f_{ij} \in K[\underline{\underline{x}}]$. Thus, $T \vdash \tilde{\psi}(\underline{\underline{x}}, x_{m+1}) \leftrightarrow \{[\varphi_2(x_{m+1}) \wedge \bigvee_{i=1}^{\ell} \bigwedge_{j=1}^{s_i} \varphi_2(\tilde{f}_{ij}(\underline{\underline{x}},

$$x_{m+1}))\Big]\vee\Big[\varphi_2(-x_{m+1})\wedge\bigvee_{i=1}^{\ell}\bigwedge_{j=1}^{s_i}\varphi_2(\varepsilon_{ij}\tilde{f}_{ij}(\underline{\underline{x}}, x_{m+1}))\Big]\Big\}, \text{ where}$$

$$\tilde{f}_{ij}(\underline{\underline{x}}, x_{m+1})=x_{m+1}^{\deg f_{ij}} f_{ij}\Big(\frac{x_1}{x_{m+1}}, \dots, \frac{x_m}{x_{m+1}}\Big),$$

$$\varepsilon_{ij}=\begin{cases} 1 & \text{if } \deg f_{ij} \text{ is even,} \\ -1 & \text{otherwise.}\end{cases}$$

The last part of the statement is immediate by elimination of quantifiers, according to Theorems 1.1, 1.4 and 1.14. ∎

Lemma 3.13. Let A be a K-algebra.

i) If K is algebraically closed then $\mathrm{Spec}_{K,L}(A)$ is the Zariski prime spectrum Spec(A).

ii) If K is real closed then $\mathrm{Spec}_{K,L}(A)$ is the real spectrum Specr(A) of A consisting of the pairs $(\underline{\underline{p}}, P)$, where $\underline{\underline{p}}\in\mathrm{Spec}(A)$ and P is an order on $k(\underline{\underline{p}})=Q(A/\underline{\underline{p}})$, with the Coste-Roy topology [13, 14] given by the base of open sets $D(a_1,\dots,a_n)$: $:=\{(\underline{\underline{p}}, P): a_i \bmod \underline{\underline{p}}\in P^{\times}=P\setminus\{0\} \text{ for } 1\le i\le n\}$ with $a_1,\dots,a_n\in A$. In particular, if A=F is a field then $\mathrm{Spec}_{K,L}(F)$ is the Boolean space of orders of F with respect to the Harrison topology.

iii) If $\underline{\underline{K}}=(K, v)$ is a p-adically closed p-valued field of type (e, f) then $\mathrm{Spec}_{K,L}(A)$ consists of the pairs $(\underline{\underline{p}}, (P_n)_{n\geq 2})$, where $\underline{\underline{p}}\in\mathrm{Spec}(A)$ and $P_n\subset k(\underline{\underline{p}})$ such that $(k(\underline{\underline{p}}), P_n{:}n\geq 2)$ is a $\underline{\underline{K}}$-field, with the topology given by the subbase of open sets $D(f,m):=\{(\underline{\underline{p}}, (P_n)_{n\geq 2}): f \bmod \underline{\underline{p}}\in P_m^{\times}\}$, for $f\in A$, $m\geq 2$. In particular, if A=F is a field then $\mathrm{Spec}_{K,L}(F)$ is the Boolean space of all $\underline{\underline{K}}$-field structures on F.

Proof. i) is trivial, while ii) and iii) follow by Theorems 1.4 and 1.14. ∎

Set $\mathrm{Specp}(A):=\mathrm{Spec}_{K,L}(A)$ in the p-adic case and call it the p-adic spectrum of the K-algebra A.

Lemma 3.14. Let Y be a K-variety, $A=K[Y]$, and identify Y with a dense subset of $\mathrm{Spec}_{K,L}(A)$.

i) If K is real closed then the L-topology on Y is the strong topology induced by the unique order of K.

ii) If $\underline{K}=(K, v)$ is p-adically closed then the L-topology on Y is the topology induced by the valuation v of K.

Proof. i) is immediate.

ii) We may assume without loss of generality that Y is the affine space K^n. Let τ be the product topology on K^n, and $V_\alpha=\{\underline{a}=(a_1,\ldots,a_n)\in Y: v(a_i)>\alpha \text{ for } 1\leq i\leq n\}$, $\alpha\in vK$, be the fundamental system of open neighbourhoods of $\underline{0}$. On the other hand, let $\underline{D}(f, m) = D(f, m)\cap Y= =\{\underline{a}\in Y: f(\underline{a})\in K^{\times m}\}$, $f\in K[\underline{X}]$, $\underline{X}=(X_1,\ldots,X_n)$, $m\geq 2$, be the subbase of the L-topology on Y. First let us show that the L-topology is finer as τ, i.e., $\underline{a}+V_\alpha$ is L-open for $\underline{a}\in Y$, $\alpha\in vK$. Let $\pi\in K$ be such that $v(\pi)=1$, $b\in K^{\times}$ be such that $v(b)=\alpha$, and $f_{i,p}(\underline{X})=1+\frac{(X_i-a_i)^2}{\pi b^2}$ if $p\neq 2$, respectively $f_{i,2}(X)=1+\frac{(X_i-a_i)^3}{\pi^2 b^3}$ if $p=2$, for $1\leq i\leq n$. Since $\mathcal{O}_v=\{c\in K: 1+\pi c^2\in K^{\times 2}\}$ if $p\neq 2$, respectively $\mathcal{O}_v=\{c\in K: 1+\pi c^3\in K^{\times 3}\}$ if $p=2$, it follows $\underline{a}+V_\alpha\subset\bigcap_{i=1}^{n}\underline{D}(f_{i,p}, m_p)$, where $m_p=\begin{cases}2 & \text{if } p\neq 2\\ 3 & \text{if } p=2\end{cases}$,

i.e., $\underline{a}+V_\alpha$ is L-open. Conversely, let us show that τ is finer as the L-topology. Let $f\in K[\underline{X}]$, $m\geq 2$. We have to show that $\underline{D}(f, m)$ is τ-open. For each $b\in K^{\times m}$, consider the τ-open set

$U_b=\{\underline{a}\in Y: v(f(\underline{a})-b)>v(b)+2v(m)\}$. Since, by Newton's lemma, $1+\underline{m}_v^{2v(m)+1}\subset K^{\times m}$, it follows that $\underline{D}(f, m)=\bigcup_{b\in K^{\times m}} U_b$ is τ-open. ∎

<u>Theorem</u> 3.15. If K, L are as above then L has regularity property.

<u>Proof</u>. Let Y be an irreducible K-variety, $A=K[Y]=K[\underline{y}]$ and $\psi \in L(A)$ be such that $Y_{reg} \cap \underline{D}_{\psi}$ is non-empty. We have to show that $T \cup D(A) \cup \{\psi\}$ is consistent. We distinguish three cases:

i) K is algebraically closed: As L=ZL, we may assume that ψ is the class modulo T_A of the $\underline{L}_A$-sentence $\bigwedge_{i=1}^{l} f_i(x) \neq 0$, with $f_i \in A$, $1 \leq i \leq l$. Since $Y_{reg} \cap \underline{D}_{\psi}$ is non-empty, we get $f_i \neq 0$ for $1 \leq i \leq l$, and hence the required conclusion is immediate.

ii) K is real closed: By Lemma 3.11, we may assume that ψ is the class modulo T_A of the $\underline{L}_A$-sentence $\bigwedge_{i=1}^{l} \varphi_2(f_i(\underline{y}))$, with $f_i \in A$, $1 \leq i \leq l$. By hypothesis, there exists $\underline{a} \in Y_{reg}$ such that $\underline{f}_i(a) > 0$ for $1 \leq i \leq l$. The conclusion, i.e., there exists an order P on F=K(Y) such that $f_i \in P^X$ for $1 \leq i \leq l$, follows by Artin-Lang theorem [7] Theorem 1.3.

iii) $\underline{K}=(K, v)$ is p-adically closed: We may assume that ψ is the class modulo T_A of the $\underline{L}_A$-sentence $\bigwedge_{i=1}^{l} \varphi_m(f_i(\underline{y}))$ with $m \geq 2$, $f_i \in A$, $1 \leq i \leq l$. Let $\underline{a} \in Y_{reg} \cap \underline{D}_{\psi} = Y_{reg} \cap \bigcap_{i=1}^{l} \underline{D}(f_i, m)$. According to [18] Corollara A.3, the K-morphism $A \to K: \underline{y} \mapsto \underline{a}$ extends to a place Q of F=K(Y) over K such that FQ=K. Let v_Q be the valuation of F assigned to Q and w be the composite valuation $v \circ v_Q$ of F. Then w extends v, $O_w = Q^{-1}(O_v)$, $F_w \cong K_v$ and vK is a convex subgroup of wF. In particular, $w(\pi)=1$, where π is an element of K such that $v(\pi)=1$, and the ordered group vK is existentially complete in wF, by [36] Theorem 2.6. According to the AX-Kochen-Ershov transfer principle [3], [16], [2], [20], [38], [41], [4], the Henselian valued field (K, v) is existentially complete in the valued field (F, w), i.e., (F, w) is embeddable over (K, v) into some p-adically closed p-valued field (F', w') of the same type as (K, v). In order to conclude that $T \cup D(A) \cup \{\psi\}$ is consistent, it remains to show that $Q^{-1}(K^{\times n}) \subset F'^{\times n}$ for $n \geq 2$. Let Q' be the place of F' over K, extending Q, whose valuation $v_{Q'}$ is the composite map $F'^{\times} \xrightarrow{w'} w'F' \to w'F'/\langle vK \rangle$, where $\langle vK \rangle$ is the convex hull of vK in w'F'. Then K=FQ is identified with a subfield of the residue field F'Q'. Let $a \in Q^{-1}(K^{\times n}) \subset Q'^{-1}((F'Q')^{\times n})$, $n \geq 2$. Since w' is Henselian and $O_{w'} \subset O_{v_{Q'}}$ it follows that $(F', v_{Q'})$ is Henselian. As the residue characteristic of $(F', v_{Q'})$ is zero, we may apply Hensel's lemma to the polynomial $X^n - a$ and conclude that $a \in F'^{\times n}$, as required. ∎

<u>Theorem</u> 3.16. If K, L are as above then L has finiteness property.

<u>Proof</u>. Let Y be a K-variety, and $U, V \in \gamma_{Y, \underline{\underline{F}}}$ be such that U is L-open in V. We have to show that $U = V \cap \underline{\underline{D}}$ for some $\underline{\underline{D}} \in \gamma_{Y,L}$.

We distinguish three cases:

i) K is algebraically closed: Trivial, since L=ZL and thus $\gamma_{Y,L}$ is the whole set of Zariski open subsets of Y.

ii) K is real closed: Then the statement above is nothing else than the fundamental finiteness theorem for open semi-algebraic sets, conjectured by Brumfiel [11] "Unproved Proposition" 8.1.2, and proved by various techniques in [10], [13], [15], [37].

iii) $\underline{\underline{K}}$=(K, v) is p-adically closd: Let $(\underline{\underline{L}}_{\omega})_K$ be the language $\underline{\underline{L}}_K$ augmented with the one-place relation symbols P_n, $n \geq 2$, and $pCF_{\underline{\underline{K}}}$

be the $(\underline{\underline{L}}_{\omega})_{\underline{\underline{K}}}$ - theory obtained from the $\underline{\underline{L}}_K$-theory T of p-adically closed field extensions of $\underline{\underline{K}}$ of the same type as $\underline{\underline{K}}$, by adding the defining axioms $P_n(x_1) \leftrightarrow (\exists x_2) x_1 = x_2^n$, $n \geqslant 2$. As $\overline{L}$ is the whole Boolean algebra $\underline{\underline{F}}$, the statement we have to prove is equivalent with the following one: given the $\underline{\underline{L}}_K$ - formulas $\Psi_i(\underline{\underline{x}})$, $\underline{\underline{x}} = (x_1, \ldots, x_m)$, i=1,2, if $T \vdash \Psi_1 \rightarrow \Psi_2$ and $[\Psi_1]$ is closed in $[\Psi_2]$ with respect to the product topology on K^m induced by the valuation v, where $[\Psi_i] := \{\underline{\underline{a}} \in K^m : K \models \Psi_i(\underline{\underline{a}})\}$, i=1,2, then there exists a positive quantifier free $(\underline{\underline{L}}_{\omega})_K$ - formula $\theta(\underline{\underline{x}})$ such that $p\underline{\underline{CF}}_{\underline{\underline{K}}} \vdash \Psi_1 \leftrightarrow (\Psi_2 \wedge \theta)$. In order to prove the latter statement, it suffices, according to van den Dries's Lyndon-Robinson-type lemma [37], to prove that the formulas Ψ_1, Ψ_2 as above satisfy the following lifting property: given two models F,N of T, an intermediate ring between K and F, a K-morphism $f: A \rightarrow N$ and a point $\underline{\underline{a}} = (a_1, \ldots, a_m) \in A^m$, if $f(A \cap F^n) \subset N^n$ for $n \geqslant 2$ and $F \models \Psi_1(\underline{\underline{a}})$, then $N \models \Psi_2(f(\underline{\underline{a}})) \rightarrow \Psi_1(f(\underline{\underline{a}}))$.

First let us note that $f(A \cap F^n) = f(A) \cap N^n$, $n \geqslant 2$, i.e., $\underline{\underline{B}} = (f(A),\ f(A \cap F^n) : n \geqslant 2)$ is a sub-$\underline{\underline{K}}$-domain of $\underline{\underline{N}} = (N,\ N^n : n \geqslant 2)$. Indeed, let $x \in A$ be such that $f(x) \in N^{\times n}$. As $F^{\times} = K^{\times} F^{\times n}$, there is $y \in K^{\times}$ such that $xy \in F^{\times n}$, and hence $f(x)y = f(xy) \in N^{\times n}$. Thus, $y \in K^{\times} \cap N^n = K^{\times n}$ and $x \in A \cap F^n$.

Let v_F, v_N be the p-valuations of F, respectively N, extending v, and let I=Ker f. For $x, y \in A$, $v_N(f(x)) \leqslant v_N(f(y))$ iff <u>either</u> $v_F(x) \leqslant v_F(y)$ or $x \in I$ and $y \in I$. Indeed, if $v_F(x) \leqslant v_F(y)$, then $x^2 + \pi y^2 \in A \cap F^2$ if $p \neq 2$, respectively $x^3 + \pi y^3 \in A \cap F^3$ if p=2, where π is an element of K such that $v(\pi) = 1$. Applying f we get $f(x)^2 + \pi f(y)^2 \in N^2$ if $p \neq 2$, respectively $f(x)^3 + \pi f(y)^3 \in N^3$, i.e., $v_N(f(x)) \leqslant v_N(f(y))$. Conversely, assume

$v_N(f(x)) \leqslant v_N(f(y))$ and $v_F(x) > v_F(y)$, i.e., $v_F(x) \geqslant v_F(\pi y)$. Consequently, $v_N(f(y)) \geqslant v_N(f(x)) \geqslant v_N(\pi f(y))$, i.e., $x \in I$ and $y \in I$. Thus, the prime ideal I of A is convex with respect to the valuation v_F, i.e., given $x \in I$, $y \in A$, if $v_F(x) \leqslant v_F(y)$ then $y \in I$. As we shall show in Proposition 3.20, there exists a valuation w of F such that $A \cup O_{v_F} \subset O_w$ and $\mathfrak{m}_w \cap A = I$. Thus, the epi $A \twoheadrightarrow f(A)$ induced by f is identified with the restriction of the canonic epi $p: O_w \twoheadrightarrow F_w$.

Let $\mathbf{F}_w = (F_w, F_w^n : n \geqslant 2)$. Let us show that $\mathbf{B}$ is a sub-$\mathbf{K}$-domain of $\mathbf{F}_w$, i.e., $p(O_w \cap F^n)$ ~~$= p(O_w^n) = F_w^n$ for $n \geqslant 2$, and $(\mathbf{F}_w$, i.e., $p(O_w^n)$~~ (is a model of $p\mathbf{CF}_{\mathbf{K}}$) $= p(O_w^n) = F_w^n$ for $n \geqslant 2$, and $\mathbf{F}_w$ is a model of $p\mathbf{CF}_{\mathbf{K}}$, i.e., F_w is a model of T. The first requirement is obviously fulfiled since O_w is Henselian of residue characteristic zero, as O_w contains the Henselian valuation ring O_{v_F}. Moreover, v_F induces a Henselian valuation $\bar{w}$ on F_w such that $\bar{w}$ extends v, $(F_w)_{\bar{w}} \simeq F_{v_F} \simeq K_v$ and the value group $\bar{w}F_w$ is identified with an intermediate convex subgroup between vK and $v_F F$. In particular $\bar{w}(\pi) = v(\pi)$ is the smallest positive element of $\bar{w}F_w$ and $\bar{w}F_w$ is a Z-group. Thus, $(F_w, \bar{w})$ is a p-adically closed p-valued field extending $\mathbf{K}$ of the same type as $\mathbf{K}$, i.e., F_w is a model of T, as required.

So $\mathbf{B}$ is a common sub-$\mathbf{K}$-domain of the models $\mathbf{N}$ and $\mathbf{F}_w$ of $p\mathbf{CF}_{\mathbf{K}}$. According to Theorem 1.14, $\mathbf{N}$ and $\mathbf{F}_w$ are elementarily equivalent over $\mathbf{B}$. In particular, the sentence $\psi_2(f(\underline{a})) \rightarrow \psi_1(f(\underline{a}))$ is true on N iff it is true on F_w, and hence we may assume from the beginning that $A = O_w$, $N = F_w$ and f is the canonic epi $p: O_w \twoheadrightarrow F_w$.

As (F, w) is Henselian of residue characteristic zero, we may identify F_w with an intermediate model of T between K and F, according to [2] Proposition 16. By assumption $[\psi_1]$ is closed in $[\psi_2]$ with respect to the topology on K^m induced by v. Since the property to be closed with respect to v is $\underline{L}_K$ - definable and F_w is an elementary extension of K it follows that $[\psi_1]_{F_w}$ is closed in $[\psi_2]_{F_w}$, where $[\psi_i]_{F_w} = \{\underline{b} \in F_w^m : F_w \models \psi_i(\underline{b})\}$, i=1,2. On the other hand, $\psi_1(\underline{a})$ is true on F by hypothesis. We have to show that $F_w \models \psi_2(p(\underline{a})) \rightarrow \psi_1(p(\underline{a}))$. Assuming the contrary, $p(\underline{a}) \in [\psi_2]_{F_w}$ and there exists $\alpha \in \bar{w}F_w = v_F(F_w^X)$ such that $\underline{z} = (z_1, \ldots, z_m) \notin [\psi_1]_{F_w}$ whenever $\underline{z} \in F_w^m$ and $\bar{w}(z_i - p(a_i)) > \alpha$ for $1 \leq i \leq m$. As F is an elementary field extension of F_w and the p-valuations $\bar{w}$ and v_F are L_K-definable, it follows that (F, v_F) is an elementary valued field extension of $(F_w, \bar{w})$. On the other hand, $v_F(a_i - p(a_i)) > \alpha$ since $w(a_i - p(a_i)) > 0$, $1 \leq i \leq m$, and $\bar{w}F_w = v_F(F_w^X)$ is a convex subgroup of $v_F F$. Consequently, $F \models \neg\psi_1(\underline{a})$, contrary to the hypothesis. ∎

The rest of this section is devoted to the unproved statement used in the proof of the theorem above.

<u>Definition</u> 3.17. Given a valued field (F, v) and a subring A of F, an ideal I of A is called <u>convex</u> (with respect to the valuation v) if for arbitrary $a \in I$ and $b \in A$, $b \in I$ whenever $v(a) \leq v(b)$.

Using Definition 1.9, one gets easily:

<u>Lemma</u> 3.18. Let (F, v) be a valued field, A be a subring of F and I be a prime ideal of A. The necessary and sufficient condition for I to be convex is that (A/I, R) is a valued domain,

where $R=\{(a \bmod I,\ b \bmod I): a,b\in A,\ v(a)\leq v(b)\}$.

Definition 3.19. Given (F, w) and $A\subset F$, the convex radical of the ideal I of A is the set $\mathrm{crad}(I)=\{a\in A: v(b)\leq nv(a)$ for some $b\in I,\ n\geq 1\}$.

Obviously, crad(I) is a convex ideal, containing the nilradical of I, and crad (crad(I))=crad(I). The necessary and sufficient condition for I=crad(I) is that I is convex and I is a nilradical. $\mathrm{crad}(I)\neq A$ iff $I\subset \underline{\underline{m}}_v$. One checks easily that crad(I) is the intersection of all convex prime ideals of A containing I.

Proposition 3.20. (Place extension theorem for valuations). Let (F, v) be a valued field, A be a subring of F and I be a convex prime ideal of A. Then there exists a valuation ring O_w such that $A\cup O_v\subset O_w$ and $A\cap \underline{\underline{m}}_w=I$.

Proof. Let M be the set of all pairs (B, J), where B is an intermediate ring between A and F, J is a convex prime ideal of B and $A\cap J=I$, partially ordered by the relation $(B, J)\leq(B', J')$ iff $B\subset B'$ and $B\cap J'=J$. As $(A, I)\in M$ and $(M,\leq)$ is inductive, it follows by Zorn's lemma that M has a maximal element, so we may assume that $M=\{(A, I)\}$. Then obviously A is a local ring and I is its maximal ideal. Let us show that A is a valuation ring. Let $x\in F\setminus A$, $B=A[x]$ and $J=IB$. Assuming that $J\subset \underline{\underline{m}}_v$, $\mathrm{crad}(J)\neq B$, and hence there exists a convex prime ideal $\underline{\underline{p}}$ of B such that $J\subset\underline{\underline{p}}$. Since I is a maximal ideal of A, it follows $(A, I)\lneq(B, \underline{\underline{p}})$, a contradiction.

Thus, $J\not\subset \underline{\underline{m}}_v$, and hence $v(\sum_{i=0}^{n} a_i x^i)\leq 0$ for some $a_i \in I$, $0\leq i\leq n$. As $I\subset \underline{\underline{m}}_v$, one gets $v(x)<0$. Assuming $x^{-1}\notin A$, we get similarly $v(x)>0$, a contradiction. We conclude that A is a valuation ring. ∎

4. The Riemann space of a commutative ring

In the first part of this section we show that natural generalizations of the concept of Riemann space of a field can be given in the frame developed in Section 2.

Let $\underline{\underline{L}}_{val}$ be the language of valued domains, i.e., the language $\underline{\underline{L}}$ of rings augmented with a two-place relation symbol R. The valued fields (K, v) are particular $\underline{\underline{L}}_{val}$-structures on which the relation R is interpreted as follows: aRb iff $v(a)\leq v(b)$. Let T=ACVF be the $\underline{\underline{L}}_{val}$-theory of non-trivial valued algebraically closed fields, $\underline{\underline{F}}(T)$ be the Boolean algebra of $\underline{\underline{L}}_{val}$-formulas up to equivalence modulo T, and ZL_{val} be the cps sublattice of $\underline{\underline{F}}(T)$ generated by the class modulo T of the $\underline{\underline{L}}_{val}$-formula $x_1\neq 0 \wedge x_1 R x_2$.

As $T\vdash (x_1\neq 0)\leftrightarrow(x_1\neq 0 \wedge x_1 R x_1)$, the Zariski lattice ZL is contained in ZL_{val} , and ZL_{val} is homogeneous. The following statement is a reformulation of a basic model-theoretic result due to Robinson [30] .

Theorem 4.1. $\underline{\underline{F}}(T)$ is generated as Boolean algebra by ZL_{val}.

Proof. As $T\vdash (x_1=0)\leftrightarrow(0Rx_1)$, the Boolean subalgebra B of $\underline{\underline{F}}(T)$ generated by ZL_{val} is the cps Boolean subalgebra generated by the class modulo T of the formula $x_1 R x_2$. The equality $B=\underline{\underline{F}}(T)$ is equivalent to the fact that the $\underline{\underline{L}}_{val}$ - theory T admits elimination of quantifiers, i.e., T is the model completion of the universal theory $T_{\forall}$ of valued domains [30], [25]. ∎

Given a commutative ring A with 1, let $D(A)_+$ be the positive $\underline{\underline{L}}$ - diagram of A, $(\underline{\underline{L}}_{val})_A$ be the augmentation of $\underline{\underline{L}}_{val}$ with constants standing for elements of A and T_A be the $(\underline{\underline{L}}_{val})_A$ - theory $T \cup D(A)_+$. The models of T_A are the pairs $((K,v),f)$ where (K,v) is a non - trivial valued algebraically closed field and $f : A \to K$ is a non - null morphism. Let B(A) be the Boolean algebra of $(\underline{\underline{L}}_{val})_A$ - sentences up to equivalence modulo T_A, and $ZL_{val}(A)$ be the sublattice of B(A) induced by ZL_{val}. $ZL_{val}(A)$ is generated by the classes modulo T_A of the sentences $a \neq o \wedge aRb$ for $a,b \in A$. Let R(A) be the spectral space assigned by Stone duality to the distributive lattice $ZL_{val}(A)$, and call it the <u>Riemann space</u> of A. The underlying set of R(A) is identified with the set of pairs $(\underline{\underline{p}}, v)$, where $\underline{\underline{p}} \in Spec(A)$ and v is a valuation, may be the trivial one, on $k(\underline{\underline{p}}) = Q(A/\underline{\underline{p}})$, while the topology is given by the subbase of open sets $D(a,b) = \{(\underline{\underline{p}},v) \in R(A) : a \notin \underline{\underline{p}}$ and $v(b \bmod \underline{\underline{p}}) \geqslant v(a \bmod \underline{\underline{p}})\}$, for $a,b \in A$. In particular, if A = K is a field then R(K) is the customary Riemann space of valuations of K ([40], Ch.6, §17), with the <u>Zariski topology</u> given by the subbase of open sets $D(f) = \{v \in R(K) : v(f) \geqslant 0\}$, $f \in K$. Thus, we get a contravariant functor R from the category of commutative rings into the category of spectral spaces. Note also that the canonic projection $R(A) \to Spec(A)$: $(\underline{\underline{p}},v) \mapsto \underline{\underline{p}}$ is a coherent epi, with a coherent canonic section $i : \underline{\underline{p}} \mapsto (\underline{\underline{p}}$, the trivial valuation on $k(\underline{\underline{p}}))$.

On the other hand, we may consider the Boolean space BR(A) assigned by Stone duality to the Boolean algebra B(A).

According to Theorem 4.1, its underlying set is identified with that of R(A), while its topology is finer than that of R(A) and is given by basic clopen sets $D(a_1,\dots,a_n, b_1,\dots,b_n; c_1,\dots,c_m, d_1,\dots,d_m) = \{(\underline{\underline{p}},v) : v(a_i \bmod \underline{\underline{p}}) \leqslant v(b_i \bmod \underline{\underline{p}}), 1 \leqslant i \leqslant n$, and $v(c_j \bmod \underline{\underline{p}}) < v(d_j \bmod \underline{\underline{p}})$, $1 \leqslant j \leqslant m\}$. In particular, if A = K is a field then the basic open sets of BR(K) have the form $D(f_1,\dots,f_n; g_1,\dots,g_m) = \{v : v(f_i) \geqslant 0, 1 \leqslant i \leqslant n$ and $v(g_j) > 0$, $1 \leqslant j \leqslant m\}$. The canonic projection of BR(A) onto its Boolean space BSpec(A), the constructible spectrum of A, is continuous, and its canonic section i is continuous too, identifying B Spec(A) with a closed subspace of BR(A).

Given a field K and a K-algebra A, we may consider the closed subspaces R(A/K), BR(A/K) of R(A), BR(A), having as underlying set the set of pairs $(\underline{\underline{p}}, v)$, where $\underline{\underline{p}} \in Spec(A)$ and v is a valuation of $k(\underline{\underline{p}})$ lying over K. Clearly, R(A/K) is a spectral space, while BR(A/K) is a Boolean one.

Now we introduce real and p-adic versions for the Riemann space of a ring defined above. First let us consider the real case.

Definition 4.2. A valued-ordere domain is a triple (A,R,P) where (A, R) is a valued domain and (A, P) is an ordered domain satisfying the following compatibility condition: $(a+b)b \in P$, whenever bRa but not aRb, for $a,b \in A$. A valued-ordered field is a valued-ordered domain (F, R, P), where F is a field. Identifying R with its corresponding valuation v, the compatibility condition above reads as follows: $1+\underline{\underline{m}}_v \subset P$, i.e., the valuation ring O_v is convex in F with respect to the order P.

Given a valued-ordered domain (A, R, P), there exists a unique structure (F, v, P') of valued-ordered field on F=Q(A) such that $R=\{(a, b) \in A \times A : v(a) \leq v(b)\}$ and $P=A \cap P'$. If (F, v, P) is a valued-ordered field then $\bar{P}=\{a \bmod \underline{\underline{m}}_v : a \in P \cap O_v\}$ is an order on the residue field F_v. Moreover, it is well known that the function $P \mapsto \bar{P}$ maps the set of orders of F which are compatible with v onto the set of all orders of F_v.

Let $\underline{\underline{L}}_{val-ord}$ be the language of valued-ordered domains and T=RCVOF be the $\underline{\underline{L}}_{val-ord}$ - theory of non-trivial-valued-ordered real closed fieds.

Theorem 4.3. T=RCVOF is complete and admits elimination of quantifiers. It is the model completion of the $\underline{\underline{L}}_{val-ord}$-theory of valued-ordered domains.

Proof. The models of T are identified with the Henselian non-trivial valued fields (K, v) for which the residue field K_v is real closed and the value group vK is divisible. According to Ax-Kochen-Ershov transfer principle for Henselian valued fields of residue characteristic zero[3], [16], T is complete and model complete, since the theories of real closed fields and of divisible Abelian ordered groups are so. It remains to show that every valued-ordered domain $\underline{\underline{A}}$ has an up to isomorphism unique minimal extension to a model of T. Obviously we may assume that $\underline{\underline{A}}$=(K, v, P) is a valued-ordered field. First assume that v is non-trivial. Let $\tilde{K}$ be the real closure of the ordered field (K, P); thus $P=K\cap \tilde{K}^2$. By Lang's place extension theorem for orders [11], 7.7.4, p.152, v extends to a valuation w of $\tilde{K}$ such that $1+\underline{\underline{m}}_w \subset \tilde{K}^2$, i.e., $(\tilde{K}, w, \tilde{K}^2)$ is a model of T extending $\underline{\underline{A}}$. The unicity up to isomorphism is immediate since the real closures of the ordered fied (K, P) are conjugate in the algebraic closure K^a of K and the valuations of K^a extending the valuation v of K are conjugate too.

Now assume that v is trivial and let K'=K(x) be a pure transcendental extension of K. There exists a unique pair (v', P') such that (K', v', P') is a valued-ordered field extension of (K, v, P), $v'(x)>0$ and $x\in P'$, given by $O_{v'}=K[x]_{(x)}$ and $P'\cap K[x]= =\{x^n f(x): n\geq 0, f\in K[x], f(0)\in P^X\}\cup\{0\}$. So it remains to apply to (K', v', P') the procedure above. ∎

Using the previous result, we may proceed as in the first part of the section to get two contravariant functors Rr and BRr defined on the category of commutative rings with values in the category of spectral spaces, respectively Boolean spaces. Given a commutative ring A, the underlying set of Rr(A) and BRr(A) is identified with the set of triples $(\underline{\underline{p}}, v, P)$, where $\underline{\underline{p}}\in Spec(A)$ such that $(k(\underline{\underline{p}}), v, P)$ is a valued-ordered field; the valuation v may be the trivial one. The topology on the spectral space Rr(A), called the <u>real Riemann space</u> of A, is given by the subbase of open sets $D(a, b)=\{(\underline{\underline{p}}, v, P): a \bmod \underline{\underline{p}}\in P^X, v(a \bmod \underline{\underline{p}}) \leq v(b \bmod \underline{\underline{p}})\}$ for $a,b\in A$, while the topology on the Boolean space BRr(A) is generated by the clopen sets D(a, b) as above and the clopen sets $D'(a, b)=\{(\underline{\underline{p}}, v, P): v(a \bmod \underline{\underline{p}})<v(b \bmod \underline{\underline{p}})\}$, $a,b\in A$. Note that the canonic epis $Rr(A)\to Specr(A)$, $BRr(A)\to BSpecr(A)$, their canonic sections and the canonic maps $Rr(A)\to R(A)$, $BRr(A)\to BR(A)$ are coherent. The constructible (semialgebraic) real spectrum BSpecr(A) is identified with a closed subspace of BRr(A). In particular, if A=F is a field, the underlying set of Rr(F) and BRr(F) is the set of pairs (v, P) for which (F, v, P) is a valued ordered field. The topology of Rr(F) is given by the subbase of open sets $[a>0]$, $[v(a)\geq 0]$, $a\in F$, while the topology of BRr(F) is given by the subbase above extended with the sets $[v(a)>0]$, $a\in F$. Given an ordered field K=(K, P) and a K-algebra A, we may consider th spectral space $Rr(A/\underline{\underline{K}})$ and the Boolean space $BRr(A/\underline{\underline{K}})$, having as underlying set, the set of triples (p, v, Q) in $Rr(\underline{\underline{A}})$ subject to $K\subset O_v$ and $P\subset Q$.

Now let us consider the p-adic case. Let $\underline{\underline{K}}=(K, v)$ be a p-adically closed p-valued field of type (e, f). Let $\pi\in K$ be such that $v(\pi)=1$.

Definition 4.4. A valued $\underline{\underline{K}}$-domain is a structure $\underline{\underline{A}}=(A, R, P_n: n\geq 2)$, such that (A, R) is a valued domain, $(A, P_n: n\geq 2)$ is a $\underline{\underline{K}}$-domain (see Definition 1.11) and the following compatibility conditions are satisfied:

i) $a^2+\pi b^2\in P_2 \Rightarrow aRb$ if $p\neq 2$, respectively $a^3+\pi b^3\in P_3 \Rightarrow aRb$ if $p=2$, for $a, b\in A$;

ii) $1Ra$ for each $a\in K$.

A valued $\underline{\underline{K}}$-field is a valued $\underline{\underline{K}}$-domain $\underline{\underline{F}}=(F, R, P_n: n\geq 2)$, where F is a field. Identifying R with its corresponding valuation w, the compatibility conditions above read as follows:

i)' $\{a\in F: 1+\pi a^2\in P_2\}\subset O_w$ if $p\neq 2$, respectively $\{a\in F: 1+\pi a^3\in P_3\}\subset O_w$ if $p=2$; in other words, $O_{v_F}\subset O_w$, where v_F is the p-valuation of F;

ii)' $K\subset O_w$.

There exists a canonic bijection between the $\underline{\underline{K}}$-domain structures on an integral domain and the $\underline{\underline{K}}$-field structures on its quotient field. The next lemma puts in evidence a lifting property for valued $\underline{\underline{K}}$-fields.

Lemma 4.5. Let F be a field extension of K and w be a valuation of F/K. The function $(P_n)_{n\geq 2}\mapsto (\bar{P}_n)_{n\geq 2}$, where $\bar{P}_n=\{a \bmod \underline{\underline{m}}_w:$

$a \in O_w \cap P_n\}$, maps the set of the $\underline{\underline{K}}$-field structures on F which are compatible with w onto the set of $\underline{\underline{K}}$-field structures on F_w.

<u>Proof</u>. One checks easily that the map above is well defined. Note also that for every $(P_n)_{n \geq 2}$ as above, $P_n \cap O_w^{\times}$ is the preimage of $\bar{P}_n^{\times}$ through the canonic epi $q: O_w \to F_w$. It remains to prove the surjectivity. Let $(Q_n)_{n \geq 2}$ be a $\underline{\underline{K}}$-field structure on F_w. Let $Q = \bigcap_{n \geq 2} Q_n^{\times}$, Q'_n be the preimage of $Q_n^{\times}$ through the epi q, and $Q' = \bigcap_{n \geq 2} Q'_n$. Denote by S the set of the $\underline{\underline{K}}$-field structures $\underline{\underline{P}} = (P_n)_{n \geq 2}$ on F which are compatible with w and satisfy $\bar{P}_n = Q_n$ for $n \geq 2$, up to the equivalence relation: $\underline{\underline{P}} \sim \underline{\underline{P}}'$ iff there exists an isomorphism of valued $\underline{\underline{K}}$-fields $(F, w, \underline{\underline{P}}) \to (F, w, \underline{\underline{P}}')$ inducing the identity on F_w. We have to show that S in non-empty. The set S can be described in cohomological terms as follows.

Denote by $\hat{F}_w^{\times}$ the Abelian profinite group $\varprojlim_{n \geq 2} F_w^{\times}/Q_n^{\times} \cong \varprojlim_{n \geq 2} K^{\times}/K^{\times n} \cong \varprojlim_{n \geq 2} O_w^{\times}/Q'_n$, and let G be the group defined by the exact sequence

$$1 \to F_w^{\times}/Q \to \hat{F}_w^{\times} \to G \to 1.$$

The exact sequence above induces the exact sequence

$$\mathrm{Hom}(wF,\ G) \xrightarrow{\lambda} \mathrm{Ext}(wF,\ F_w^{\times}/Q) \to \mathrm{Ext}(wF,\ F_w^{\times}),$$

where Hom stands for group morphisms and $\mathrm{Ext} = \mathrm{Ext}^1_{\mathbb{Z}}$. As $\hat{F}_w^{\times} \cong \varprojlim_{n \geq 2} K^{\times}/K^{\times n}$ is $\mathbb{Z}$-complete and wF is torsion free, it follows by [17] §39, 54 that $\mathrm{Ext}(wF, \hat{F}^X_w)=0$, i.e., λ is onto. Denote by S' the preimage through λ of the class of the canonic exact sequence

$$1 \to F^X_w/Q \simeq O^X_w/Q' \to F^X/Q' \to wF \to 1.$$

We define a canonic bijection $\mu : S \to S'$, proving in particular that S is non-empty. If P is a representative of some $s \in S$, the isomorphisms $K^X/K^{Xn} \xrightarrow{\sim} F^X/P^X_n$, $n \geq 2$ determine a canonic morphism $F^X/Q' \to \varprojlim_{n\geq 2} F^X/P^X_n \simeq \hat{F}^X_w$, inducing the identity on F^X_w/Q.

The morphism above induces a morphism $\mu(s): wF \to G$ which does not depend on the choice of the representative $\underline{\underline{P}}$ of s. By construction, $\mu(s) \in S'$. Conversely, given $f \in S'$, there exists $h: F^X/Q' \to \hat{F}^X_w$ such that $(F^X/Q'\ ;\ h, w: F^X/Q' \to wF)$ is the pullback of the pair $(\hat{F}^X_w \to G,\ f: wF \to G)$. Let $P_n = h^{-1}(\hat{F}^{Xn}_w) \cup \{0\}$, $n \geq 2$. By the universality of the pullback, the family $\underline{\underline{P}} = (P_n)_{n\geq 2}$ is uniquely determined by f up to an automorphism of the valued field (F, w) inducing the identity on F_w. One checks easily that $\underline{\underline{P}}$ is a $\underline{\underline{K}}$-field structure on F which is compatible with w and $\bar{P}_n = Q_n$. Thus we get a map $\nu : S' \to S$. It follows immediately that μ and ν are inverse each to other. ∎

<u>Remark</u>. If $\mathrm{Ext}(wF, F^X_w/Q)=0$ then $S \simeq \mathrm{Hom}(wF, G)$. In particular, if w is discrete then $S \simeq G$. In this case, the lifting of Q is unique up to isomorphism iff the canonic morphism $F^X_w \to \hat{F}^X_w$ is onto.

Let $\underline{\underline{L}}_{val,K}$ be the language of valued $\underline{\underline{K}}$-domains and T be the $\underline{\underline{L}}_{val,K}$-theory of the valued K-fields $(F, w, P_n : n \geq 2)$ for which w is non-trivial and F is p-adically closed; in particular $P_n = F^n$, $n \geq 2$. The p-adic analogue of Theorem 4.3 reads as follows.

Theorem 4.6. T is complete and admits elimination of quantifiers. It is the model completion of the $L_{val,K}$ - theory of valued K-domains.

Proof. The models of T are identified with the Henselian non-trivial valued fields (F, w) subject to $K \subset O_w$, F_w is a p-adically closed field extension of K of type (e, f) and wF is divisible. The completeness and model completeness of T is a consequence of the Ax-Kochen-Ershov transfer principle for Henselian valued fields of residue characteristic zero, Theorem 1.14 and the completeness, model completeness of the theory of divisibile Abelian ordered groups. In order to finish the proof, it suffices to show that each valued K-field $F = (F, w, P_n : n \geq 2)$ extends to a model $\tilde{F} = (\tilde{F}, \tilde{w}, \tilde{F}^n : n \geq 2)$ of T, which is almost minimal in the following sense: for every model N of T extending F there exists an elementary extension N' of N such that $\tilde{F}$ is embeddable over F into N'.

First suppose that w is non-trivial and let $\tilde{F}$ be the p-adic closure of the K-field $(F, P_n : n \geq 2)$. By Theorem 1.15, $\tilde{F}$ is unique up to an isomorphism. Applying Proposition 3.20 to the valued field $(\tilde{F}, \tilde{v})$, where $\tilde{v}$ is the p-valuation of $\tilde{F}$, the subring O_w of $\tilde{F}$ and the convex (with respect to $\tilde{v}$) prime ideal m_w, we may extend w to a valuation $\tilde{w}$ of $\tilde{F}$ such that $O_{\tilde{v}} \subset O_{\tilde{w}}$, geting the required model $\tilde{F}$ of T. In fact, $\tilde{F}$ is the up to isomorphism unique minimal extension of F to a model of T.

Now assume that w is trivial, and let $F'=F(x)$ be a pure transcendental extension of F of transcendency degree one. Let w' be the unique valuation of F' subject to $F\subset O_{w'}$ and $w'(x)>0$. $O_{w'}$ is the localization of $F[x]$ with respect to the maximal ideal $xF[x]$, $\underline{\underline{m}}_{w'}=xO_{w'}$, and $F'_{w'}=F$. Let $P'_n=x^{Z}\cdot P_n^{X}\cdot(1+\underline{\underline{m}}_{w'})\cup\{0\}$, $n\geq 2$. Then $\underline{\underline{F}}'=(F',w',P'_n:n\geq 2)$ is a valued $\underline{\underline{K}}$-field extension of $\underline{\underline{F}}$. Let $\tilde{\underline{\underline{F}}}$ be the minimal extension of $\underline{\underline{F}}'$ to a model of T constructed as above. In order to show that $\tilde{\underline{\underline{F}}}$ has the required property, it suffices to show that, given a non-trivial-valued $\underline{\underline{K}}$-field extension $\underline{\underline{F}}''$ of $\underline{\underline{F}}$, $\underline{\underline{F}}'$ is embeddable over $\underline{\underline{F}}$ into some elementary extension of $\underline{\underline{F}}''$. Given $\underline{\underline{F}}''$ as above, there exists an $\aleph_1$-saturated elementary extension of it; take for instance an ultrapower with respect to a non-principal ultrafilter on the set of natural numbers Thus we may assume from the beginning that $\underline{\underline{F}}''=(F'', w'', P''_n : n\geq 2)$ is $\aleph_1$-saturated. As w'' is non-trivial and $\underline{\underline{F}}''$ is $\aleph_1$-saturated, there exists $y\in\bigcap_{n\geq 2}F''^{Xn}$ such that $w''(y)>0$. Since w is trivial, y is transcendental over F. The substitution $x\mapsto y$ induces an $\underline{\underline{F}}$-embedding of $\underline{\underline{F}}'$ into $\underline{\underline{F}}''$, as required. ∎

<u>Remark</u>. According to Lemma 4.5, the obstruction to minimality in the trivial valuation case is given by the group morphism $\Theta:F^{X}\to\hat{F}^{X}=\varprojlim_{n\geq 2} F^{X}/P_n^{X}$. More precisely, the necessary and sufficient condition for the trivial-valued $\underline{\underline{K}}$-domain $\underline{\underline{F}}$ to have an up to isomorphism unique minimal extension to a model of T is that Θ is onto.

Using Theorem 4.6, we get two contravariant functors Rp and BRp defined on the category of K-algebras with values in the category of spectral, respectively Bollean spaces. Given a K-algebra A, the underlying set of Rp(A) and BRp(A) is identified with the set of systems ($\underline{\underline{p}}$, w, P_n:n$\geq$2), where $\underline{\underline{p}}\in$Spec(A) and (w,$P_n$:n$\geq$2) is a valued $\underline{\underline{K}}$-field structure on k($\underline{\underline{p}}$). The topology on the spectral space Rp(A), called the p-adic Riemann space of A, is given by the subbase of open sets D(a,b,m)={($\underline{\underline{p}}$,w,$P_n$:n$\geq$2): :a mod $\underline{\underline{p}}\in P_m^X$, w(a mod $\underline{\underline{p}}$)$\leq$w(b mod $\underline{\underline{p}}$)} for a,b$\in$A, m$\geq$2, while the topology of the Boolean space BRp(A) is generated by the clopen sets D(a,b,m) (as above and the clopen) sets D'(a,b)={($\underline{\underline{p}}$, w, P_n:n$\geq$2):w(a mod $\underline{\underline{p}}$)<w(b mod $\underline{\underline{p}}$)}, a,b$\in$A. The canonic epis Rp(A)$\rightarrow$ Specp(A), Brp(A)$\rightarrow$ BSpec p(A), their canonic sections and the canonic maps Rp(A)$\rightarrow$R(A), BRp(A)$\rightarrow$ BR(A) are coherent, and BSpecp(A), the constructible p-adic spectrum, is identified with a closed subspace of BRp(A). If A=F is a field extension of K, the underlying set of Rp(F) and BRp(F) is the set of valued $\underline{\underline{K}}$-fields with universe F. The topology on Rp(F) is given by the subbase of open sets [a$\in P_m^X$], [w(a)$\geq$0], m$\geq$2, a$\in$F, while the topology on BRp(F) is given by the subbase above extended with the sets [w(a)>0], a$\in$F.

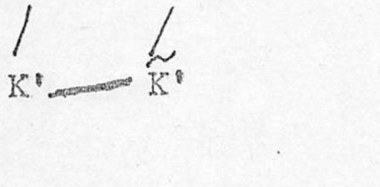
The rest of this section is devoted to a density theorem on Riemann spaces which plays a basic role in the following sections.

Fix a base field K, assumed to be either algebraically closed, or real closed, or p-adically closed of type (e, f). Given a K-algebra A, let us consider the Boolean space X(A) and its closed subspace X'(A), where X(A):=BR(A/K), X'(A):=BSpec(A) if K is algebraically closed, BRr(A/K), BSpecr(A) if K is real closed, BRp(A), BSpecp(A) if K is p-adically closed. Let Z(A) be the pre-image through the canonic map X(A)$\rightarrow$R(A/K) of the set of pairs ($\underline{\underline{p}}$, w) for which w is discrete with residue field k($\underline{\underline{p}}$)$_w$=K.

Theorem 4.7. With the notations above, assume that for every $p\in X'(A)$, the field $k(p)$ is finitely generated over K. Then Z(A) is dense in the open subspace $X(A)\setminus X'(A)$ of the Boolean space X(A), i.e., X(A) is the union of X'(A) and the closure of Z(A).

Proof. Without loss of generality we may assume that A=F is a finitely generated field extension of K. The present proof is inspired by the proof of the main theorem of [21]. We distingusih three cases:

Case 1: K is algebraically closed. We have to show that $D\cap Z(F)$ is non-empty for every basic non-empty open set $D=D(f_1,\dots,f_n;g_1,\dots,g_m)=\{w: w(f_i)\geq 0,\ 1\leq i\leq n,\ w(g_j)>0,\ 1\leq j\leq m\}$, $m\geq 1$, $g=g_1\neq 0$. Choose some $w\in D$ and denote also by w some extension to a valuation of the algebraic closure $\tilde{F}$ of F. w induces on $K'=K(g)$ a discrete valuation with valuation ring $K[g]_{(g)}$ and residue field K. Let $\tilde{K}'\subset\tilde{F}$ be the algebraic closure of K' with the valuation induced by the fixed valuation w of $\tilde{F}$. Thus we get a diagram of valued field extensions of K



Write $F.\tilde{K}'=\tilde{K}'(t_1,\dots,t_k,y)$, where $t_1,\dots,t_k$ are algebraically independent over K', so trdeg(F/K)=k+1, and y is algebraic separable over $\tilde{K}'(t)$. Let $H\in\tilde{K}'[T,Y]$ be irreducible, monic in Y, such that $H(\underline{t}, y)=0$. Then $f_i = \dfrac{F_i(\underline{t}, y)}{G(\underline{t})}$, $g_j = \dfrac{G_j(\underline{t}, y)}{G(\underline{t})}$ for some polynomials G, F_i, G_j with coefficients in $\tilde{K}'$, $1\leq i\leq n$, $2\leq j\leq m$.

As the extension $\tilde{F}/\tilde{K}'$ of non-trivial valued algebraically closed fields is elementary, there exists $(\underline{t}', y')\in\tilde{K}'^{k+1}$ satisfying i) $H(\underline{t}', y')=0$, $\frac{\partial H}{\partial Y}(\underline{t}', y')\neq 0$, and ii) $w(F_i(\underline{t}', y'))\geq w(G(\underline{t}'))$, $G(\underline{t}')\neq 0$, $1\leq i\leq n$, and $w(G_j(\underline{t}', y'))> w(G(\underline{t}'))$, $2\leq j\leq m$.

Now let $L\subset\tilde{K}'$ be a finite extension of K' such that $(\underline{t}', y')\in L^{k+1}$, $F\subset L(\underline{t}, y)$ and the coefficients of the polynomials H, G, F_i $(1\leq i\leq n)$, G_j $(2\leq j\leq m)$ belong to L. Obviously the induced valuation $w|L$ is discrete and $L_w=K$. Let $(\hat{L}, \hat{w})$ be the completion of the valued field $(L, w|L)$. As $w|L$ is discrete, $(\hat{L}, \hat{w})$ is Henselian. On the other hand, $\hat{L}$ is of infinite transcendency degree over L, since the completion of the discrete valued field $(K', w|K')$, isomorphic to the valued field $K((X))$ of formal power series in one indeterminate X, is embeddable in $\hat{L}$.

As the set S of the points $(\underline{t}'', y'')\in\hat{L}^{k+1}$ satisfying the condition ii) above, with $\hat{w}$ instead of w and $(\underline{t}'', y'')$ instead of $(\underline{t}', y')$, is open with respect to the topology induced by $\hat{w}$, there exists an open neighbourhood $U\subset\hat{L}$ of 0 such that $\prod_{i=1}^{k}(t_i'+U)\times(y'+U)\subset S$. Applying the implicit function theorem [29] Theorem 7.4 to the Henselian valued field $(\hat{L}, \hat{w})$, we get some open neighbourhoods $V, V'\subset\hat{L}$ of 0 and a continuous map $\lambda:\prod_{i=1}^{k}(t_i'+V)\to y'+V'$, such that $\lambda(\underline{t}'')$ is the unique root of $H(\underline{t}'', Y)\in\hat{L}[Y]$ in $y'+V'$ for each $\underline{t}''$ in the domain of λ. In particular, $\lambda(\underline{t}')=y'$. Obviously, we may assume that $V\cup V'\subset U$. Thus, if $\underline{t}''\in\mathrm{dom}(\lambda)$ then $H(\underline{t}'', \lambda(\underline{t}''))=0$ and $(\underline{t}'', \lambda(\underline{t}''))\in S$.

As trdeg $(\hat{L}/L)$ is infinite, we may proceed as in [21] to find $\underline{\underline{t}}''\in\mathrm{dom}(\lambda)$ such that $t''_1,\ldots,t''_k$ are algebraically independent over L. Let N be the algebraic closure of $L(\underline{\underline{t}}'')$ in $\hat{L}$. Then the substitution $\underline{\underline{t}}\mapsto\underline{\underline{t}}''$, $y\mapsto\lambda(\underline{\underline{t}}'')$ defines an L-embedding of the field $L(\underline{\underline{t}}, y)$ into N, inducing a K'-embedding $\varphi: F\to N$. It remains to observe that the valuation $\varphi^{-1}(\hat{w}|N)\in D\cap Z(F)$, as required.

Case 2: K is real closed. As the proof is quite similar, we point out only the specific facts. In this case, the basic open set D has the form $\{(w, P): w(f_i)\geq 0,\ 1\leq i\leq n,\ w(g_j)>0,\ 1\leq j\leq m,\ r_k\in P^X,\ 1\leq k\leq l\}$ with $m\geq 1$, $g=g_1\neq 0$. Choosing $(w, P)\in D$ we take $\tilde{F}$ to be the up to isomorphism unique minimal extension of (F, w, P) to a valued real closed field. The minimal extension of $(K'=K(g),\ w|K',\ P\cap K')$ is identified with the algebraic closure of K' in $\tilde{F}$. With the notations from the case 1 suitably modified, we get, by model completeness of the theory of non-trivial-valued real closed fields, some $(\underline{\underline{t}}', y')\in\tilde{K}'^{k+1}$ such that i) and ii) above, completed with the specific condition $G(\underline{\underline{t}}')R_i(\underline{\underline{t}}', y')\in P^X$, $1\leq i\leq l$, with $r_i=\dfrac{R_i(\underline{\underline{t}}, y)}{G(\underline{\underline{t}})}$, are satisfied. As above take $L\subset\tilde{K}'$ to be a large enough finite extension of K'. As the fixed valuation w of $\tilde{F}$ and the unique order of $\tilde{F}$ induce the same topology on L, the respective completions of L are identified, and hence $(\hat{L},\hat{w},\hat{P})$ is a Henselian discrete valued-ordered field extension of L, with $\hat{L}_{\hat{w}}=K$. The last part of the proof is quite identical with that from the case 1.

Case 3: $\underline{\underline{K}}=(K, v)$ is p-adically closed. Now we may take a basic open set D of the form $\{(w, P_n: n\geq 2): w(f_i)\geq 0,\ 1\leq i\leq n,$

$w(g_j)>0,\ 1\leq j\leq m,\ r_k\in P_s^{\times},\ 1\leq k\leq 1\},\ m\geq 1,\ s\geq 2,\ g=g_1\neq 0$. Choosing $(w, P_n:n\geq 2)\in D$, we take $\tilde{F}$ to be the up to isomorphism unique minimal extension of the non-trivial-valued $\underline{\underline{K}}$-field $(F, w, P_n:n\geq 2)$ to a valued p-adically closed $\underline{\underline{K}}$-field, and we identify the minimal extension of the non-trivial-valued $\underline{\underline{K}}$-field $(K'=K(g), w/K', P_n\cap K':n\geq 2)$ with the algebraic closure $\tilde{K}'$ of K' in $\tilde{F}$. The proof continues as in the case I. We have only to use the following facts:

a) the model completeness of the theory of non-trivial-valued p-adically closed $\underline{\underline{K}}$-fields, applied to the extension $\tilde{F}/\tilde{K}'$;

b) the p-valuation v of $\tilde{F}$ and the valuation w of $\tilde{F}$ induce the same topology on the convenient finite extension $L\subset\tilde{K}'$ of K', and thus, the completion $\hat{L}$ of L with respect to v has a canonic structure of Henselian discrete-valued $\underline{\underline{K}}$-field. ∎

5. Some natural continuous maps on the Riemann space of a field

Given a totally ordered Abelian group Γ, one defines a map $\Gamma\times\Gamma\setminus\{0\}\to R\cup\{\pm\infty\}:(\alpha,\beta)\mapsto\alpha:\beta$, as follows: Let Δ_β be the convex subgroup of Γ generated by $\beta\neq 0$, and Δ'_β be the maximal convex subgroup of Γ properly contained in Δ_β, i.e., $\beta\notin\Delta'_\beta$. Then the ordered group $\Delta_\beta/\Delta'_\beta$ is embeddable in the ordered group R of reals. Fix a morphism of ordered groups $\lambda:\Delta_\beta\to R$ with $\mathrm{Ker}\lambda=\Delta'_\beta$. If $\alpha\notin\Delta_\beta$,

$$\text{set } \alpha:\beta=\begin{cases}+\infty, & \text{if } \mathrm{sign}(\alpha)=\mathrm{sign}(\beta)\\ -\infty, & \text{otherwise.}\end{cases}$$

If $\alpha\in\Delta_\beta$, set $\alpha:\beta=\frac{\lambda(\alpha)}{\lambda(\beta)}\in\mathbb{R}$. As λ is unique, up to multiplication by a positive constant, the definition does not depend on the choice of λ. If $f:\Gamma\to\Gamma'$ is a morphism of ordered Abelian groups, $\alpha\in\Gamma$, $\beta\in\Gamma\setminus\operatorname{Ker} f$, then $\alpha:\beta=f(\alpha):f(\beta)$. If $\alpha\in\Gamma$, $0\neq\beta\in\Gamma$ and $\gamma\in\Delta_\beta\setminus\Delta'_\beta$, then $\alpha:\beta=(\alpha:\gamma)(\gamma:\beta)$, with the usual convention:

$$(\pm\infty)\cdot r=\begin{cases}\pm\infty & \text{if } r>0\\ \mp\infty & \text{if } r<0.\end{cases}$$

The next lemma is immediate.

<u>Lemma</u> 5.1. Let F/K be a field extension, v be a valuation of F/K and A be an intermediate ring between K and F, such that A is finitely generated over K. Define the map $\alpha_v:F\to vF$ by $\alpha_v(f)=\max(0,-v(f))$ for $f\in F$, and let $\Gamma_{v,A}$ be the convex subgroup of vF generated by $\alpha_v(A)$. For a system $\underline{\underline{g}}=(g_1,\dots,g_n)$ of generators of A over K, let us put $\beta_{v,\underline{\underline{g}}}=\max_{1\leq i\leq n}\alpha_v(g_i)$. Then $\Gamma_{v,A}$ is the convex subgroup of vF generated by $\beta_{v,\underline{\underline{g}}}$. Moreover, for each $f\in A$, there is a bound $N_{f,\underline{\underline{g}}}\in\mathbb{N}$, independent on v, such that $\alpha_v(f)\leq N_{f,\underline{\underline{g}}}\beta_{v,\underline{\underline{g}}}$.

With the notations above, denote by $R(F/K)_A$ the closed sub space of the Riemann space R(F/K) consisting of the valuations v for which $A\not\subset O_v$. For a system $\underline{\underline{g}}=(g_1,\dots,g_n)$ of generators of the K-algebra A and some $f\in A$, define the map $u_{\underline{\underline{g}},f}:R(F/K)_A\to[0,\infty)\subset\mathbb{R}$ according to the rule: $u_{\underline{\underline{g}},f}(v)=\alpha_v(f):\beta_{v,\underline{\underline{g}}}$. One checks easily that the map $u_{\underline{\underline{g}},f}$ is continuous.

Now let $\underline{\underline{f}}=(f_j)_{j\in J}$ be an arbitrary family of elements of A such that $\underline{\underline{f}}$ generates the K-algebra A. In particular, $\underline{\underline{f}}$ may be the whole A. Then the family of continuous maps $(u_{\underline{\underline{g}},f_j})_{j\in J}$ induces a continuous map $u_{\underline{\underline{g}},\underline{\underline{f}}}:R(F/K)_A\to[0,\infty)^J\setminus\{\underline{\underline{0}}\}$, where $\underline{\underline{0}}=(0)_{j\in J}$.

Denote by $\underline{P}_J$ the "projectivized" version of the Hausdorff space $[0,\infty)^J$ as defined in [26] I.3. $\underline{P}_J$ is the quotient of the Hausdorff space $[0,\infty)^J\setminus\{\underline{\underline{0}}\}$ by the closed equivalence relation identifying the points $(t_j)_{j\in J}$ and $(at_j)_{j\in J}$ for $a>0$. If $(t_j)_{j\in J}$ is an element of $[0,\infty)^J\setminus\{\underline{\underline{0}}\}$, the point of $\underline{P}_J$ determined by (t_j) is denoted by $[t_j]$(" homogeneous coordinates"). As the canonic map $[0,\infty)^J\setminus\{\underline{\underline{0}}\}\to\underline{P}_J$ is open, $\underline{P}_J$ is a Hausdorff space.

Taking the composite of $u_{\underline{\underline{g}},\underline{\underline{f}}}$ with the canonic map above we get a continuous map $u_{\underline{\underline{f}}}:R(F/K)_A\to\underline{P}_J$ which does not depend on the choice of $\underline{\underline{g}}$. As the domain of $u_{\underline{\underline{f}}}$ is quasi-compact, its image $Im(u_{\underline{\underline{f}}})$ is a compact subspace of $\underline{P}_J$.

Let us call a point of $\underline{P}_J$ <u>integral</u> if it has the form $[t_j]_{j\in J}$, where $t_j\in\mathbb{Z}$ for every $j\in J$. Denote by $\underline{P}_J^{int}$ the set of integral points of $\underline{P}_J$. The next result is an immediate consequence of Theorem 4.7.

<u>Proposition</u> 5.2. $Im(u_{\underline{\underline{f}}})\cap\underline{P}_J^{int}$ is dense in $Im(u_{\underline{\underline{f}}})$.

If K is real closed, let $Rr(F/K)_A$ be the preimage of $R(F/K)_A$ through the continuous canonic map $Rr(F/K)\to R(F/K)$, and $u_{\underline{\underline{f}},r}:Rr(F/K)_A\to\underline{P}_J$ be the continuous map induced by $u_{\underline{\underline{f}}}$. Its image is a compact subspace of $\underline{P}_J$. Similarly, if K is a p-adically closed field, we may consider the preimage $Rp(F)_A$ of $R(F/K)_A$ through the continuous map $Rp(F)\to R(F/K)$ and the corresponding continuous map $u_{\underline{\underline{f}},p}:Rp(F)_A\to\underline{P}_J$. Using Theorem 4.7, we get real and p-adic versions of Proposition 5.2.

6. Compactification of affine varieties over locally compact fields: A non-standard approach

Given a locally compact Hausdorff space X, a compactification of X is a pair $(\hat{X}, \eta)$, where $\hat{X}$ is a compact Hausdorff space and $\eta : X \to \hat{X}$ is a continuous embedding mapping homeomorphically X onto an open dense subset of $\hat{X}$. The following general procedure of compactification is described in [26] I.3. Assume that $f: X \to Y$ is a continuous map and Y is a compact Hausdorff space. Let X^+ denote the one-point compactification of X, in which X is identified with the complement of a point +. Consider the continuous embedding $i: X \to X^+ \times Y : x \mapsto (x, f(x))$, and let $\hat{X}$ be the closure of i(X) in $X^+ \times Y$, and η be regarded as a map of X into $\hat{X}$. Set $B = p^{-1}(+)$, where $p: \hat{X} \to X^+$ is the projection to the first factor. One gets $\hat{X} = \eta(X) \cup B$, $\eta(X)$ is open dense in $\hat{X}$ and η maps X homeomorphically onto $\eta(X)$. The pair $(\hat{X}, \eta)$ is called the compactification of X determined by f. The points of B are called the ideal points of the compactification. The projection $q: \hat{X} \to Y$ maps B homeomorphically onto a closed subset of Y; we may identify B with q(B), and also X with $\eta(X)$. In the following we give a description of the set B of ideal points in terms of the non-standard Analysis. The basic notions of the non-standard Analysis can be found in [31], [32], [23].

Consider the higher order structure (X, Y, f) and let $(X^*, Y^*, f^* = f)$ be an enough saturated enlargement of (X, Y, f). For every $x \in X$ (similarly for the points of Y), we denote by $\mu(x)$ the monad of x with respect to the topology of X, i.e., $\mu(x) = \bigcap D^*$, where D ranges over all open neighbourhoods of x. More generally, if $x \in X^*$, let $\mu(x) = \bigcap D^*$, where D ranges over the open subsets of X for which $x \in D^*$. As X and Y are Hausdorff spaces, $\mu(x) \cap \mu(x')$ is empty for every pair of <u>standard</u> points $x, x' \in X (\in Y)$ such that $x \neq x'$. Since $f: X \to Y$ is continuous, $f(\mu(x)) \subset \mu(f(x))$ for each $x \in X^*$. A point $x \in X^*$ is called <u>near-standard</u> if $x \in \mu(a)$ for some $a \in X$. As X is Hausdorff, for every near-standard point $x \in X^*$ there is a unique $a \in X$ such that $x \in \mu(a)$; call a the <u>standard part</u> of x and denote it by st(x). Denote by $nst(X^*)$ the set of all near-standard points of X^*. Thus we get canonic retracts st: $:nst(X^*) \to X$, $st: nst(Y^*) \to Y$, of the canonic embeddings $X \to nst(X^*)$, $Y \to nst(Y^*)$. As Y is compact, we get $nst(Y^*) = Y^*$.

<u>Lemma</u> 6.1. The set $B \subset Y$ of ideal points of the compactification determined by $f: X \to Y$ is the image of $X^* \setminus nst(X^*)$ through the composite map $X^* \xrightarrow{f} Y^* = nst(Y^*) \xrightarrow{st} Y$.

<u>Proof.</u> Consider the commutative diagram

$$\begin{array}{ccc} X^* & \longrightarrow & (X^+ \times Y)^* = nst((X^+ \times Y)^*) = (X^* \cup \{+\}) \times Y^* \\ \uparrow & & \swarrow \, st \\ X & \xrightarrow{i} & X^+ \times Y \end{array}$$

Let $(x, y) \in X^+ \times Y$ be such that $(x, y) \notin i(X)$, i.e., either $x = +$, or $x \in X$ and $y \neq f(x)$. We have to show that the necessary and sufficient condition for (x, y) to belong to $\hat{X} = \overline{i(X)}$ is that $x = +$ and $y = st(f(z))$ for some $z \in X^* \setminus nst(X^*)$. Assuming $x \in X$, we get $(x, y) \in \hat{X}$ iff the intersection $\mu((x, y)) \cap i(X)^*$ is non-empty. As $\mu((x, y)) = \mu(x) \times \mu(y)$ and $i(X)^* = \{(z, f(z)) : z \in X^*\}$, it follows $(x, y) \in \hat{X} \Leftrightarrow$ there is $z \in X^*$ such that $z \in \mu(x)$ and $f(z) \in \mu(y) \Leftrightarrow f(x) = y$, contrary to the assumption

$(x, y) \notin i(X)$. Assume $x=+$. Then the monad $\mu(+)$ in $(X^+)^* = X^* \cup \{+\}$ is $(X^* \setminus nst(X^*)) \cup \{+\}$, and hence $(+, y) \in \hat{X} \Leftrightarrow$ there is $z \in X^* \setminus nst(X^*)$ such that $f(z) \in \mu(y)$, i.e., $y = st(f(z))$. ∎

Now let us apply the procedure above to the following algebraic-geometric situation. Let L be a non-discrete locally compact field. Thus L is either $\mathbb{C}$, $\mathbb{R}$, or a local field, i.e., a Cauchy complete valued field with a discrete valuation v(set $vL = \mathbb{Z}$) and finite residue field $L_v \simeq \mathbb{F}_q$, $q=p^f$; if char L=0 then L is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, while $L \simeq \mathbb{F}_q((T))$ if char L=p. Let $| \ | : L \to [0, \infty)$ be the corresponding absolute value; if L is a local field then $|x| = q^{-v(x)}$ for $x \in L$, and so $v(x) = -\log_q(|x|)$; set also $v(x) := -\log_e(|x|)$ in the case $L = \mathbb{C}, \mathbb{R}$.

Let $Y \subset L^n$ be an affine L-variety. Y is closed in the affine space L^n with respect to the topology induced from L. Thus Y has a canonic structure of Hausdorff locally compact space. Given a family $\underline{f} = (f_j)_{j \in J}$ of regular functions in the coordinate L-algebra $L[Y] = L[\underline{y}]$, such that $\underline{f}$ generates $L[Y]$ over L, one defines as in [26] I.3, a continuous map $\Theta_{\underline{f}} : Y \to P_{\mathcal{J}}$, according to the rule: $\Theta_{\underline{f}}(\underline{a}) = [\log(|f_j(\underline{a})| + c)]_{j \in J}$, where $c > 1$ is a real constant, whose only role is to assure that the logarithms are well defined and strictly positive, so that $\Theta_{\underline{f}}$ is well-defined. According to [26] Proposition I.3.1, the closure $\overline{\Theta_{\underline{f}}(Y)}$ of $\Theta_{\underline{f}}(Y)$ in $P_{\mathcal{J}}$ is compact; it is also metrizable if J is countable. Thus we may consider the compactification $\hat{Y}$ of Y determined by $\Theta_{\underline{f}}$ and apply Lemma 6.1 in order to describe in non-standard terms the set B(Y) of the ideal points of the compactification.

Thus we have to consider an enlargement L^* of L, assumed to be $\aleph_1$ - saturated. For instance, we may consider an ultrapower with respect to a non-principal ultrafilter on the set of natural numbers. Let L^*_{fin} be the subring of L^* consisting of the <u>finite</u> elements of L^*/L with respect to the canonic extension $|\ |: L^* \to R^*$ of the absolute value of L, i.e., $L^*_{fin}=\{x\in L^*: |x|\leq a$ for some $a\in R$, $a>0\}$. It is well known that L^*_{fin} is a Henselian valuation ring of L^*, whose maximal ideal L^*_{inf} consists of the <u>infinitesimals</u> of L^*/L, i.e., $L^*_{inf}=\{x\in L^*: |x|<a$ for every $a\in R$, $a>0\}$ and whose residue field is identified with L. The corresponding valuation $\mathring{v}$ is the composite map $(L^*)^X \xrightarrow{v} R^* \to \mathring{R}$, where $\mathring{R}=R^*/R^*_{fin} \simeq Z^*/Z$ is a divisible ordered group, and v is the canonic extension to L^* of the map $v: L^X \to R$; in the local field case, the extension of v to L^* is the composite valuation of the valuation $v: L^X \to Z$ and the valuation $\mathring{v}$. The enlargement τ^* of the topology τ of L generates a topology in the usual sense on L^* which coincides with the topology induced by the valuation $\mathring{v}$. Note that $nst(L^*)=L^*_{fin}=O_{\mathring{v}}$, $\mu(0)=L^*_{inf}=m_{\mathring{v}}$ and the map $st: nst(L^*) \to L$ is the canonic epi $O_{\mathring{v}} \to L^*_{\mathring{v}} = L$.

Given the L-variety $Y\subset L^n$, theinternal set Y^* is identified with the L^* -variety $Y\otimes_L L^*$ and the internal ring $L[Y]^*$ is $*$-generated by the coordinate L^*-algebra $L^*[Y\otimes_L L^*]= =L[Y]\otimes_L L^*$. It follows easily that $nst(Y^*)=\{\underline{\underline{a}}\in Y^*: L[\underline{\underline{a}}]\subset O_{\mathring{v}}\}$ and the composite map $Y^*\setminus nst(Y^*) \xrightarrow{\theta^*_{\underline{\underline{f}}}} \overline{\theta_{\underline{\underline{f}}}(Y)}^*=nst(\overline{\theta_{\underline{\underline{f}}}(Y)}^*) \xrightarrow{st} \overline{\theta_{\underline{\underline{f}}}(Y)}$ $\hookrightarrow P_J$ is nothing else than the map $\underline{\underline{a}}\mapsto u_{\underline{\underline{f}}(\underline{\underline{a}})}(\mathring{v})$ for $\underline{\underline{a}}\in Y^*$ such that $L[\underline{\underline{a}}]\not\subset O_{\mathring{v}}$, where $u_{\underline{\underline{f}}(\underline{\underline{a}})}: R(L^*/L)_{L[\underline{\underline{a}}]} \to P_J$ is the continuous map assigned to the family $\underline{\underline{f}}(\underline{\underline{a}})=(f_j(\underline{\underline{a}}))_{j\in J}$ of generators of $L[\underline{\underline{a}}]$, as defined in Section 5. As a consequence of Lemma 6.1, we get the required non-standard description of B(Y).

Proposition 6.2. $B(Y)=\{u_{\underline{\underline{f}}(\underline{\underline{a}})}(\mathring{v}) : \underline{\underline{a}}\in Y^{*},\ L[\underline{\underline{a}}]\not\subset O_{\mathring{v}}\}$.

7. Compactification through valuations: The main results

Let us assume that the locally compact base field L is of characteristic zero, $Y\subset L^n$ is an irreducible L-variety and $\underline{\underline{f}}=(f_j)_{j\in J}$ is a countable family of generators of the L-algebra of coordinates $L[Y]$. By Lemma 2.9, there exists a countable subfield K of L such that K is existentially complete in L, Y is defined over K, $Y=Y(K)\otimes_K L$ and $\underline{\underline{f}}$ generates the K-algebra of coordinates $K[Y(K)]=K[\underline{\underline{y}}]$ of the irreducible K-variety $Y(K)\subset K^n$. According to Theorems 1.1, 1.4, 1.14, the condition upon K to be existentially complete in L is equivalent to the stronger one: L is an elementary extension of K, and also to the weaker one: K is algebraically closed in L. Thus K is algebraiccally closed, or real closed, or p-adically closed of a suitable type (e, f).

Consider the non-empty set Y_{reg} of regular points on Y, and let $\overline{Y_{reg}}$ be the closure of Y_{reg} in Y with respect to the Hausdorff locally compact topology on Y. If $L=\underline{C}$ then $\overline{Y_{reg}}=Y$, according to [34] Ch.7 . For $L\neq\underline{C}$ it is possible to have $\overline{Y_{reg}}\neq Y$. For a description of $\overline{Y_{reg}}$ in the case $L=\underline{R}$ or a local field of characteristic zero (more generally for real closed and p-adically close fields) see [8] Theorem 1.10, and Lemma 3.8, Theorem 3.15 of Section 3. Note that $\overline{Y_{reg}}$ is definable by a first order formula in the language of K. Now let C be a clopen subset of $\overline{Y_{reg}}$. If $L=\underline{C}$ then C=Y since Y is connected [34] Ch.7, §2. If $L=\underline{R}$ then the semi-algebraic set $\overline{Y_{reg}}$ has finitely many connected components which are semialgebraic too [14], and hence C is a union of such components; in particular C is semialgebraic definable over the countable real closed subfield K of R. If L is a local field then Y is totally disconnected, and hence there exists a lot of clopen subsets of $\bar{Y}_{reg}$. We shall assume only that C is definable in the language of fields with parameters from the field L. Of course, we may assume that the countable subfield K of L is large enough such that C is definable over K. Let $\hat{C}$ be the compactification of C determined by the restriction to C of the continuous map $\theta_f: Y \to P_{\mathcal{Y}}$ defined in Section 6. According to Proposition 6.2, the compact space of the ideal points of the compactification $\hat{C}$ admits the non-standard description $B(C) = \{u_{f(a)}(\mathring{v}) : a \in C^*, L[a] \not\subset O_{\mathring{v}}\}$. The main goal of this section is to give an alternative description of B(C) in valuation-theoretic terms.

Let $A=K[Y(K)]=K[y]$ and $F=K(Y(K))=K(y)$ be the coordinate K-algebra and the field of rational functions of the irreducible K-variety $Y(K) \subset K^n$.

First let us consider the complex case, and let $u_f : R(F/K)_A \to P_{\mathcal{Y}}$ be the continuous function defined in Section 5.

<u>Theorem</u> 7.1 (compare with [26] Theorem I.3.6). $B(Y)=\mathrm{Im}(u_f)$.

<u>Proof</u>. Let us show that $B(Y) \subset \mathrm{Im}(u_f)$. Let $a=(a_1,\dots,a_n) \in Y^*$ be such that $C[a] \not\subset O_{\mathring{v}}$. We have to find $w \in R(F/K)_A$ satisfying $u_f(w)=u_{f(a)}(\mathring{v})$. By the definition given in Section 5, $u_{f(a)}(\mathring{v}) = [\alpha_{\mathring{v}}(f_j(a)) : \beta_{\mathring{v},a}]_{j \in J}$, where $\alpha_{\mathring{v}}(f_j(a))=\max(0,-\mathring{v}(f_j(a)))$,

$\beta_{\dot{v},\underline{a}} = \max_{1\le i\le n} \alpha_{\dot{v}}(a_i) = -\min_{1\le i\le n} \dot{v}(a_i)$. By continuity of the polynomial functions with respect to the valuation $\dot{v}$, there exists a family $(\gamma_j)_{j\in J}$ of non-negative elements in $\dot{v}C^* = \dot{R}$ such that $\beta_{\dot{v},\underline{a}} = \beta_{\dot{v},\underline{b}}$ and $\alpha_{\dot{v}}(f_j(\underline{a})) = \alpha_{\dot{v}}(f_j(\underline{b}))$ for every $\underline{b} = (b_1,\ldots,b_n) \in Y^*$ subject to $\dot{v}(\underline{b}-\underline{a}) = \min_{1\le i\le n} \dot{v}(b_i - a_i) > \gamma_j$. Consider the countable system of formulas in the language of valued fields with parameters in C, in variables $\underline{z} = (z_1,\ldots,z_n)$: "$\underline{z} \in Y^*_{reg}$", $\dot{v}(\underline{z}-\underline{a}) > \gamma_j$ for $j\in J$. Since $\overline{Y_{reg}} = Y$ it follows that the system above is finitely satisfiable. On the other hand, the enlargement $(C^*, |\,|)$ of $(C, |\,|)$ is assumed to be $\aleph_1$-saturated, and hence the valued field $(C^*, \dot{v})$ is $\aleph_1$-saturated too. Consequently there exists $\underline{b} \in Y^*_{reg}$ such that $u_{\underline{f}(\underline{b})}(\dot{v}) = u_{\underline{f}(\underline{a})}(\dot{v})$. As $\underline{b} \in Y^*_{reg} = (Y(K) \otimes_K C^*)_{reg}$, it follows by [18] Corollary A.2 that there exists a valuation $\dot{w}$ of F/K such that $\underline{y} \bmod \underline{m}_{\dot{w}} = \underline{b}$ and $F_{\dot{w}} = K(\underline{b})$. Denote by w the composite of the valuations $\dot{w}$ and $\dot{v}|K(\underline{b})$. Then $w \in R(F/K)_A$ and $\dot{v}K(\underline{b})$ is identified with a convex subgroup of wF. Finally, we get $u_{\underline{f}}(w) = u_{\underline{f}(\underline{b})}(\dot{v}) = u_{\underline{f}(\underline{a})}(\dot{v})$, as required.

Conversely, let $w \in R(F/K)_A$. We have to find some $\underline{a} \in Y^*$ such that $C[\underline{a}] \not\subset O_{\dot{v}}$ and $u_{\underline{f}(\underline{a})}(\dot{v}) = u_{\underline{f}}(w)$. Since the theory of non-trivial valued algebraically closed fields admits elimination of quantifiers, (F, w) is a countable non-trivial valued field and $(C^*, \dot{v})$ is an $\aleph_1$-saturated non-trivial valued algebraically closed field with the common trivial valued subfield K, it follows that there exists a K-embedding $\lambda : (F, w) \to (C^*, \dot{v})$. Let $\underline{a} = \lambda(\underline{y})$. Then $\underline{a} \in Y^*_{reg}$, $C[\underline{a}] \not\subset O_{\dot{v}}$ as $A \not\subset O_w$ and $u_{\underline{f}(\underline{a})}(\dot{v}) = u_{\underline{f}}(w)$ since wF is identified with an ordered subgroup of $\dot{v}C^* = \dot{R}$. ∎

Next let us consider the real case, and let $u_{...}$: $Rr(F/K) \to R$ ...

be the continuous map defined in Section 5. Given a clopen subset C of $\overline{Y}_{reg}$, it follows, by the finiteness theorem for open semialgebraic sets, that there exist some polynomial functions $h_{ij}\in A=K[Y(K)]$, $1\leq i\leq m$, $1\leq j\leq l_i$, such that $C=\bigcup_{i=1}^{m}\bigcap_{j=1}^{l_i}\{\underline{a}\in\overline{Y}_{reg}: h_{ij}(\underline{a})>0\}$. Denote by $\tilde{C}$ the clopen subset of $Rr(F/K)_A$ given by $\tilde{C}=\bigcup_{i=1}^{m}\bigcap_{j=1}^{l_i}\{(w,P)\in Rr(F/K)_A: h_{ij}\in P^{x}\}$.

Theorem 7.2 (compare with [26] Proposition I.4.2)

$B(C)=u_{\underline{f},r}(\tilde{C})$.

Proof. First let us show that $B(C)\subset u_{\underline{f},r}(\tilde{C})$. Let $\underline{a}\in C^{*}$ be such that $R[\underline{a}]\not\subset O_{\dot{v}}$. We look for some $(w, P)\in\tilde{C}$ such that $u_{\underline{f},r}(w, P)=u_{\underline{f}(\underline{a})}(\dot{v})$. Proceeding as in the proof of Theorem 7.1, we find some $\underline{b}\in Y^{*}_{reg}\cap C^{*}$ such that $u_{\underline{f}(\underline{b})}(\dot{v})=u_{\underline{f}(\underline{a})}(\dot{v})$. Applying [18] Corollary A.2, we get a valuation $\dot{w}$ of F/K such that $\underline{y} \bmod \underline{m}_{\dot{w}}=\underline{b}$ and $F_{\dot{w}}=K(\underline{b})$. Let w be the composite of the valuations $\dot{w}$ and $\dot{v}|_{K(\underline{b})}$. As $F_{\dot{w}}$ inherits from R^{*} a structure $(\dot{v}|_{K(\underline{b})}, K(\underline{b})\cap R^{*2})$ of valued-ordered field, there is at least one order P of F such that $(P\cap O_{\dot{w}}) \bmod \underline{m}_{\dot{w}}=K(\underline{b})\cap R^{*2}$ and $1+\underline{m}_{w}\subset P$. It follows $(w, P)\in\tilde{C}$ and $u_{\underline{f},r}(w, P)=u_{\underline{f}(\underline{b})}(\dot{v})=u_{\underline{f}(\underline{a})}(\dot{v})$, as required.

Conversely, let $(w, P)\in\tilde{C}$. We look for some $\underline{a}\in C^{*}$ such that $R[\underline{a}]\not\subset O_{\dot{v}}$ and $u_{\underline{f}(\underline{a})}(\dot{v})=u_{\underline{f},r}(w, P)$. Applying Theorem 4.3 to the $\aleph_1$ - saturated non-trivial valued-ordered real closed field $(R^{*},\dot{v}, R^{*2})$, the countable non-trivial valued-ordered field (F, w, P) and their common trivial valued-ordered subfield K, we get a K-embedding $\lambda:(F, w, P)\rightarrow(R^{*}, \dot{v}, R^{*2})$. Then

$\underline{\underline{a}}=\lambda(\underline{\underline{y}})\in Y^*_{reg}\cap C^*$, $R[\underline{\underline{a}}]\not\subset O_{\dot{v}}$ and $u_{\underline{\underline{f}}(\underline{\underline{a}})}(\dot{v})=u_{\underline{\underline{f}},r}(w, P)$. ∎

Finally, let us consider the p-adic case. Denote also by v the restriction to K of the discrete valuation v of L. Thus $\underline{\underline{K}}=(K, v)$ is p-adically closed of the same type as (L, v). Let $u_{\underline{\underline{f}},p}:Rp(F)_A\to P_{\underline{\underline{y}}}$ be the continuous map defined in Section 5. Consider a clopen subset C of $\overline{Y}_{reg}$ which is definable by a first order formula in the language of K. According to the p-adic analogue of the finiteness theorem for open semialgebraic sets (see Theorem 3.16), there exist some polynomial functions $h_{ij}\in A=K[Y(K)]$, $1\leq i\leq m$, $1\leq j\leq l_i$, and a natural number $s\geq 2$ such that $C=\bigcup_{i=1}^{m}\bigcap_{j=1}^{l_i}\{\underline{\underline{a}}\in\overline{Y}_{reg}:h_{ij}(\underline{\underline{a}})\in L^{\times s}\}$. Denote by $\tilde{C}$ the clopen subset of $Rp(F)_A$ given by $\tilde{C}=\bigcup_{i=1}^{m}\bigcap_{j=1}^{l_i}\{(w, P_n: n\geq 2)\in Rp(F)_A:h_{ij}\in P_s^{\times}\}$.

Theorem 7.3. $B(C)=u_{\underline{\underline{f}},p}(\tilde{C})$.

Proof. Let $\underline{\underline{a}}\in C^*$ be such that $L[\underline{\underline{a}}]\not\subset O_{\dot{v}}$. As in the proof of Theorem 7.1, we get $\underline{\underline{b}}\in Y^*_{reg}\cap C^*$ and a valuation $\dot{w}$ of F/K such that $u_{\underline{\underline{f}}(\underline{\underline{b}})}(\dot{v})=u_{\underline{\underline{f}}(\underline{\underline{a}})}(\dot{v})$, $\underline{\underline{y}}$ mod $\underline{\underline{m}}_{\dot{w}}=\underline{\underline{b}}$ and $F_{\dot{w}}=K(\underline{\underline{b}})$. The residue field $F_{\dot{w}}$ inherits from L^* a valued $\underline{\underline{K}}$-field structure $(\dot{v}|_{K(\underline{\underline{b}})}, K(\underline{\underline{b}})\cap L^{*n}: n\geq 2)$. According to Lemma 4.5, this structure can be lifted to a valued $\underline{\underline{K}}$-field structure $(w, P_n:n\geq 2)$ on F such that $O_w=\{a\in O_{\dot{w}}: a \text{ mod } \underline{\underline{m}}_{\dot{w}}\in O_{\dot{v}}\}$ and $(P_n\cap O_{\dot{w}})$ mod $\underline{\underline{m}}_{\dot{w}}=K(\underline{\underline{b}})\cap L^{*n}$ for $n\geq 2$. Thus $(w, P_n:n\geq 2)\in\tilde{C}$ and $u_{\underline{\underline{f}},p}(w, P_n:n\geq 2)=u_{\underline{\underline{f}}(\underline{\underline{b}})}(\dot{v})=u_{\underline{\underline{f}}(\underline{\underline{a}})}(\dot{v})$. Therefore $B(C)\subset u_{\underline{\underline{f}},p}(\tilde{C})$.

Conversely, let $(w, P_n:n\geq 2)\in\tilde{C}$. Applying Theorem 4.6 to the $\aleph_1$-saturated non-trivial-valued p-adically closed $\underline{\underline{K}}$-field $(L^*, \dot{v}, L^{*n}:n\geq 2)$ and the countable non-trivial-valued $\underline{\underline{K}}$-field

$(F, w, P_n : n \geq 2)$, we get an embedding $\lambda : (F, w, P_n : n \geq 2) \to (L^*, \dot{v}, L^{*n} : n \geq 2)$. Then $\underline{\underline{a}} = \lambda(\underline{\underline{y}}) \in Y^*_{reg} \cap C^*$, $L[\underline{\underline{a}}] \not\subseteq O_{\dot{v}}$ and $u_{\underline{\underline{f}}(\underline{\underline{a}})}(\dot{v}) = u_{\underline{\underline{f}},p}(w, P_n : n \geq 2)$. Thus, $u_{\underline{\underline{f}},p}(\tilde{C}) \in B(C)$. ∎

We now drop the assumption that the L-variety Y is irreducible. The final result is an immediate consequence of Lemma 2.13, Theorem 4.7 and Theorems 7.1, 7.2, 7.3.

Corollary 7.4. Assume Y is an arbitrary L-variety.

a) ([26] Corollary I.3.8). If $L = \underline{C}$ then $B(Y) \cap \underline{P}_J^{int}$ is dense in $B(Y)$.

b) ([26] Corollary I.4.5). If $L = \underline{R}$ and C is a clopen subset of Y, i.e., a union of connected components of Y, then $B(C) \cap \underline{P}_J^{int}$ is dense in $B(C)$.

c) If L is a local field of characteristic zero and C is a first order definable clopen subset of Y, then $B(C) \cap \underline{P}_J^{int}$ is dense in $B(C)$.

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