INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

INFINITESIMAL DEFORMATIONS OF NEGATIVE WEIGHTS AND HYPERPLANE SECTIONS

by

Lucian BADESCU

PREPRINT SERIES IN MATHEMATICS
No. 13/1988

BUCURESTI

INFINITESIMAL DEFORMATIONS OF NEGATIVE

WEIGHTS AND HYPERPLANE SECTIONS

by

Lucian BADESCU*

February 1988

*) *Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.*

INFINITESIMAL DEFORMATIONS OF NEGATIVE WEIGHTS AND HYPERPLANE SECTIONS

Lucian Bădescu

Introduction

Consider the following:

Problem. Let (Y,L) be a normal polarized variety over an algebraically closed field k, i.e. a normal projective variety Y over k together with an ample line bundle L on Y. Then one may ask under which conditions the following statement holds:

(+) Every normal projective variety X containing Y as an ample Cartier divisor such that the normal bundle of Y in X is L, is isomorphic to the projective cone over (Y,L), and Y is embedded in X as the infinite section.

Recall that the projective cone over (Y,L) is by definition the projective variety $C(Y,L) = \mathrm{Proj}(S[T])$, where S is the graded k-algebra $S(Y,L) = \bigoplus_{i=1}^{\infty} H^{o}(Y,L^{i})$ associated to (Y,L), and the polynomial S-algebra $S[T]$ (with T an indeterminate) is graded by $\deg(sT^{i}) = \deg(s) + i$ whenever $s \in S$ is homogeneous. The infinite section of $C(Y,L)$ is by definition $V_{+}(T)$, and it is isomorphic to Y.

This problem has classical roots (see [3] for some historical hints). In [1], [2], [3] and [4], among other things, we produced several examples of polarized varieties (Y,L) satisfying (+). If Y is smooth of dimension $\geqslant 2$, and if T_{Y} is the tangent bundle of Y, Fujita subsequently proved in [6] the following general criterion: (Y,L) satisfies (+) if $H^{1}(Y,T_{Y}\otimes L^{i}) = o$ for every $i < o$.

In this paper we prove two main results. The first one (which is in the spirit of [4]) considers the case where Y has singularities, and is a criterion for (Y,L) to satisfy (+). This criterion (see theorem 1 in §1) improves a result of [4] and involves the space of first order infinitesimal deformations of the k-algebra $S(Y,L)$. In §2 we apply it to check that the singular Kummer varieties of dimension ≥ 3 and the symmetric products of certain varieties satisfy (+) with respect to any line bundle. In §3 we make a few remarks when Y is smooth and state an open question. It should be noted that in the first two sections the Schlessinger's deformations theory (see [18], [19]) plays an essential role.

The second main result (see theorem 6 in §4) shows that if Y is a P^n-bundle $(n \geq 1)$ over a smooth projective curve B of positive genus and if X is a normal singular projective variety containing Y as an ample Cartier divisor, then X is isomorphic to the cone C(Y,L). The case $B = P^1$ was discussed in [3], while the case when X is smooth, in [1] and [2]. Putting these results together, we get a complete description of all normal projective varieties containing a P^n-bundle $(n \geq 1)$ over a curve as an ample Cartier divisor (see theorem 7 in §4).

Unless otherwise specified, the terminology and the notations used are standard.

§1. The first main result

In the set-up and notations of the above problem, the graded k-algebra $S = S(Y,L)$ is finitely generated because L is ample (see e.g. [8], chap. III). Let $a_1, \dots, a_n$ be a minimal system of homogeneous generators of S/k, and denote by $k[T_1, \dots, T_n]$ the polynomial k-algebra in n indeterminates $T_1, \dots, T_n$, graded by the conditions that $\deg(T_i) = \deg(a_i) = q_i$ for every $i = 1, \dots, n$. Then S is isomorphic as a graded k-algebra to $k[T_1, \dots, T_n]/I$ in such a way that a_i corresponds to $T_i \bmod I$ for every $i = 1, \dots, n$ (I is the kernel of the homomorphism mapping T_i to a_i). Let $f_1, \dots, f_r \in I$ be a minimal system of homogeneous generators of I, and set:

$$(1) \qquad d = \max(d_1, \dots, d_r), \text{ where } d_i = \deg(f_i).$$

Then we have:

Theorem 1. In the above notations assume the following:

i) $H^1(Y, L^i) = o$ for every $i \in \mathbb{Z}$, or equivalently, $\operatorname{depth}(S_{S_+}) \geq 3$, where S_+ is the irrelevant maximal ideal of S.

ii) $T^1_S(-i) = o$ for every $1 \leq i \leq d$, where d is given by (1), $T^1_S = T^1(S/k, S)$

is the space of first order infinitesimal deformations of the k-algebra S, and $T_S^1 = \bigoplus_{t\in\mathbb{Z}} T_S^1(i)$ is the decomposition arising from the G_m-action of the graded k-algebra S (see [18], or also [14] for the definition of T_S^1).

Then the property (+) holds for (Y,L).

Proof. Let X be a normal projective variety containing Y as an ample Cartier divisor such that $O_X(Y)\otimes O_Y = L$. Let $t \in H^0(X,O_X(Y))$ be a global equation of Y in X, i.e. $\mathrm{div}_X(t) = Y$. Denote by S' the graded k-algebra $S(X,O_X(Y)) = \bigoplus_{t=o}^{\infty} H^0(X,O_X(iY))$. Then using the standard exact sequence

$$o \longrightarrow O_X((i-1)Y) \xrightarrow{\ t\ } O_X(iY) \longrightarrow L^i \longrightarrow o,$$

the hypothesis i), and a theorem of Serre saying that $H^1(X,O_X(iY)) = o$ for every $i \ll o$, one immediately sees that $S'/tS' \cong S$ (isomorphism of graded k-algebras, where deg(t) = 1).

Then choose $b_1,\ldots,b_n \in S'$ homogeneous elements of degrees $q_1,\ldots,q_n$ respectively, such that $b_i \bmod tS' = a_i$, $i = 1,\ldots,n$. Then $S' = k[b_1,\ldots,b_n,t]$. Denote by P the polynomial k-algebra $k[T_1,\ldots,T_n,T]$ in n+1 indeterminates $T_1,\ldots,T_n,T$, graded by $\deg(T_i) = q_i$, $i = 1,\ldots,n$, and $\deg(T) = 1$. For every $m \geq 1$ set $S^m = S'/t^mS'$, and consider the surjective homomorphism $\varphi_m : P \longrightarrow S^m$ such that $\varphi_m(T_i) = b_i'$, $i = 1,\ldots,n$, and $\varphi_m(T) = t'$, where for every $b \in S'$ we have denoted by b' the element $b \bmod t^mS'$. Let $F_1,\ldots,F_s$ be a system of homogeneous generators of the ideal $J = \mathrm{Ker}(\varphi_m)$, and put $e_i = \deg(F_i)$, $i = 1,\ldots,s$.

Now, according to [18], §1 (or also [14]), we can consider:

- The S^m-module $\mathrm{Ex}(S^m/k,S)$ of all isomorphism classes of extensions of S^m over k by the S^m-module $S = S^m/t'S^m$ (recall that an extension of S^m/k by S is a k-algebra E together with a surjective homomorphism of k-algebras $E \longrightarrow S^m$ whose kernel is a square-zero ideal of E, isomorphic as an S^m-module to S).

- The S^m-module $T^1(S^m/k,S)$ defined by the following exact sequence

(2) $$\mathrm{Der}_k(P,S) \xrightarrow{\ \psi\ } \mathrm{Hom}_{S^m}(J/J^2,S) \longrightarrow T^1(S^m/k,S) \longrightarrow o ,$$

where $\mathrm{Der}_k(P,S)$ is the S^m-module of all k-derivations of P in S, and ψ is defined in the following way: if $D \in \mathrm{Der}_k(P,S)$ then $\psi(D)$ is the element of $\mathrm{Hom}_{S^m}(J/J^2,S)$ defined by the restriction D/J (which necessarily vanishes on J^2). It turns out that $T^1(S^m/k,S)$ is independent of the choice of the presentation

P/J of S^m.

Now, the point is that there is a canonical isomorphism of S^m-modules (see [18], theorem 1, page 12, or also [14], page 410):

$$(3) \qquad \alpha : \mathrm{Ex}(S^m/k,S) \longrightarrow T^1(S^m/k,S).$$

Since S^m is a graded k-algebra, $T^1(S^m/k,S)$ has a natural gradation $T^1(S^m/k,S) = \bigoplus_{i\in\mathbb{Z}} T^1(S^m/k,S)(i)$ arising from the G_m-action of S^m (see [17], page 19).

Returning to our situation, consider the element of $\mathrm{Ex}(S^m/k,S)$ given by the exact sequence

$$(a_m) \qquad o \longrightarrow S \cong t^m S'/t^{m+1}S' \longrightarrow S^{m+1} \longrightarrow S^m \longrightarrow o\ .$$

We need to compute $\alpha((a_m)) \in T^1(S^m/k,S)$ explicitly. By the definition of the isomorphism α (see [18]) we need to consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
o & \longrightarrow & J/J^2 & \longrightarrow & P/J^2 & \longrightarrow & P/J & \longrightarrow & o \\
 & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} & & \downarrow & & \\
o & \longrightarrow & t^m S'/t^{m+1}S' \overset{w}{=} S & \longrightarrow & S^{m+1} & \longrightarrow & S^m & \longrightarrow & o
\end{array}$$

where u is the map deduced from φ_m. Thus $v(F_i \bmod J^2) = t^m G_i(b_1,\dots,b_n,t) \bmod t^{m+1}S'$, with $G_i(b_1,\dots,b_n,t) \in S'_{e_i-m}$ homogeneous of degree e_i-m. Then $w \circ v \in \mathrm{Hom}_{S^m}(J/J^2,S)$ corresponds to the vector $(G'_1,\dots,G'_s)$, with $G'_i = w(t^m G_i(b_1,\dots,b_n,t) \bmod t^{m+1}S') = G_i(a_1,\dots,a_n,o)$, and recalling the exact sequence (2) we have

$$((a_m)) = \text{class of } w \circ v \in T^1(S^m/k,S).$$

According to the explicit description of the gradation of $T^1(S^m/k,S)$ given in [17], page 19, the elements of $T^1(S^m/k,S)(j)$ of degree j correspond to those elements of $\mathrm{Hom}_{S^m}(J/J^2,S)$ given by vectors $(h_1,\dots,h_s)$ with $h_i \in S_{e_i+j}$ homogeneous of degree e_i+j, $i = 1,\dots,s$. Since $\deg(G'_i) = e_i-m$, the foregoing discussion implies:

$$(4) \qquad \alpha((a_m)) \in T^1(S^m/k,S)(-m) \quad \text{for every } m \geqslant 1.$$

Now take $m = 1$. Since $S^1 = S$, it follows that $\alpha((a_1)) \in T^1(S/k,S)(-1)$, and hence $\alpha((a_1)) = o$ by hypothesis ii). But the trivial extension of $\mathrm{Ex}(S/k,S)$ is

$$o \longrightarrow S \cong TS[T]/(T^2) \longrightarrow S[T]/(T^2) \longrightarrow S[T]/(T) \cong S \longrightarrow o,$$

and therefore there is an isomorphism of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S \cong TS[T]/(T^2) & \longrightarrow & S[T]/(T^2) & \longrightarrow & S[T]/(T) \cong S & \longrightarrow & 0 \\
 & & \Vert & & \downarrow \wr & & \Vert & & \\
0 & \longrightarrow & S \cong t'S^2 & \longrightarrow & S^2 & \longrightarrow & S & \longrightarrow & 0
\end{array}$$

such that the vertical isomorphism in the middle maps $T\mathrm{mod}(T^2)$ into t'.

Assume now that we know that for some m, $2 \leqslant m \leqslant d$, there is an isomorphism $S[T]/(T^i) \cong S^i$ for every $1 \leqslant i \leqslant m$, which maps $T\mathrm{mod}(T^i)$ into $t' = t\mathrm{mod}t^i S'$. Then recall that there is a general exact sequence (see [18])

$$T^1(S^m/S,S) \longrightarrow T^1(S^m/k,S) \longrightarrow T^1(S/k,S),$$

where the maps are homogeneous and the second one corresponds to the inclusion $S \hookrightarrow S^m$ obtained by composing the natural inclusion $S \hookrightarrow S[T]/(T^m)$ with the isomorphism $S[T]/(T^m) \cong S^m$. Using this and hypothesis ii) we infer that the map $T^1(S^m/S,S)(-m) \longrightarrow T^1(S^m/k,S)(-m)$ is surjective, which together with (4) implies that the extension (a_m) comes from $Ex(S^m/S,S) \cong T^1(S^m/k,S)$. In other words, S^{m+1} is an S-algebra and the canonical surjective map $S^{m+1} \longrightarrow S^m$ is a map of S-algebras. Then we can easily define an isomorphism of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S \cong T^m S[T]/(T^{m+1}) & \longrightarrow & S[T]/(T^{m+1}) & \longrightarrow & S[T]/(T^m) & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow \wr & & \\
0 & \longrightarrow & S \cong t^m S'/t^{m+1}S' & \longrightarrow & S^{m+1} & \longrightarrow & S^m & \longrightarrow & 0
\end{array}$$

where the middle vertical isomorphism is the homomorphism of S-algebras mapping $T\mathrm{mod}(T^{m+1})$ into $t' = t\mathrm{mod}t^{m+1}S'$.

Summing up, we have proved by induction on m that there is an isomorphism of graded k-algebras $S[T]/(T^{d+1}) \cong S^{d+1}$ such that $T\mathrm{mod}(T^{d+1})$ corresponds to $t\mathrm{mod}t^{d+1}S'$. In particular, there is a commutative diagram

$$\begin{array}{ccc}
S & \xrightarrow{\ h\ } & S^{d+1} = S'/t^{d+1}S' \\
 & {}_{\mathrm{id}} \searrow \quad \swarrow {}_{\text{canonical surjection}} & \\
 & S = S^1 &
\end{array}$$

Choose homogeneous elements $c_i \in S'_{q_i}$ such that $h(a_i) = c_i \bmod t^{d+1}S'$, $i = 1,\ldots,n$. Then we claim that

$$(5) \qquad f_i(c_1,\ldots,c_n) = o \text{ for every } i = 1,\ldots,r.$$

Indeed, since $f_i(c_1,\ldots,c_n) \bmod t^{d+1}S' = f_i(h(a_1),\ldots,h(a_n)) = h(o) = o$, it follows that $f_i(c_1,\ldots,c_n) \in t^{d+1}S'$ for every $i = 1,\ldots,r$. If for some i we would have $f_i(c_1,\ldots,c_n) \neq o$, it would follow that $d_i = \deg(f_i(c_1,\ldots,c_n)) \geq d+1$, a contradiction because $d = \max(d_1,\ldots,d_r)$.

Finally, using (5) we can construct a homomorphism of graded k-algebras $f: S \longrightarrow S'$ by putting $f(a_i) = c_i$. The equations (5) show that this definition is correct. Then we get a unique homomorphism of graded k-algebras $g: S[T] \longrightarrow S'$ such that $g/S = f$ and $g(T) = t$. Then it is clear that g is surjective, and hence an isomorphism, because both $S[T]$ and S' are domains of the same dimension. In other words, we have proved that X is isomorphic to the projective cone $C(Y,L)$. Q.E.D.

Remarks. 1) Theorem 1 had beed proved in [4] in the stronger hypothesis that $T^1_S = o$, where we had in mind an application to weighted projective spaces.

2) Unfortunately, the hypothesis i) of theorem 1 is quite restrictive. We do not know whether theorem 1 still remains valid if one drops hypothesis i), even if one assumes for example that $\operatorname{char}(k) = o$ and $T^1_S(-i) = o$ for every $i \geq 1$.

Corollary 1. In the notations of theorem 1, assume that ii) holds. Let X be a normal projective variety containing Y as an ample Cartier divisor such that the normal bundle of Y in X is L. If $H^1(X, O_X(iY)) = o$ for every $i \geq o$, then X is isomorphic to the projective cone $C(Y,L)$ and Y is embedded in X as the infinite section.

Indeed, the exact sequence from the beginning of the proof of theorem 1 together with the hypothesis that $H^1(X, O_X(iY)) = o$ for every $i \geq o$ imply that $S'/tS' \cong S$ (in the proof of theorem 1 the hypothesis i) was used only to deduce this isomorphism).

Another immediate consequence of the proof of theorem 1 is the following purely algebraic result:

Corollary 2. *Let $S = k[T_1,\dots,T_n]/I$ be an $\mathbb{N}$-graded k-algebra, where the polynomial k-algebra $k[T_1,\dots,T_n]$ in the indeterminates $T_1,\dots,T_n$ is graded by $\deg(T_i) = q_i > 0$, $i = 1,\dots,n$, for some fixed system of weights $(q_1,\dots,q_n)$, and I is the ideal generated by some homogeneous polynomials $f_1,\dots,f_r$ of positive degrees. Let S' be an $\mathbb{N}$-graded k-algebra such that S'/tS' is isomorphic to S as a graded k-algebra, for some homogeneous element $t \in S'$ of degree 1. If $T^1_S(-i) = 0$ for every $1 \leq i \leq \max(\deg(f_1),\dots,\deg(f_r))$, then S' is isomorphic (as a graded k-algebra) to the polynomial S-algebra S[T] in such a way that t is mapped into T.*

§2. Applications of theorem 1

The tools for verifying hypotheses of type ii) of theorem 1 have been developed by Schlessinger in [19]. The lemma 1 below (which is essentially due to Schlessinger) provides examples of singular normal polarized varieties (Y,L) satisfying the condition ii) of theorem 1.

Start with a smooth projective variety V and with a finite group G acting on V. Denote by Y the quotient variety V/G and by $f: V \longrightarrow Y$ the canonical morphism. Let L be an ample line bundle on Y and set $M = f^*(L)$. Since f is a finite morphism, M is also ample. Let $S = S(Y,L)$ and $A = S(V,M)$ be the graded k-algebras associated to (Y,L) and (V,M) respectively.

Lemma 1. *In the above notations assume the following:*

i) *$\mathrm{Dim}(V) \geq 3$ and char(k) is either zero, or prime to the order |G| of G.*

ii) *G acts on V freely outside some closed G-invariant subset of V of codimension ≥ 3.*

iii) *$H^1(V,M^{-i}) = 0$ for every $i \geq 1$ (in characteristic zero this is always fulfilled by Kodaira's vanishing theorem).*

iv) *$H^1(V,T_V \otimes M^{-i}) = 0$ for every $i \geq 1$, where T_V is the tangent bundle of V.*

Then $T^1_S(-i) = 0$ for every $i \geq 1$.

Proof. Since lemma 1 is not given in [19] in this form, we include its proof for the convenience of the reader. From ii) we infer that the singular locus of Y, Sing(Y), is of codimension ≥ 3, and that f is étale outside Sing(Y). Using this, the normality of Y and [16], §7, it follows that $f_*(M^i)^G = L^i$ for every

$i \geqslant o$. This shows that G acts on A by automorphisms of graded k-algebras and that the invariant k-algebra A^G coincides with S. Consider the cartesian diagram

$$\begin{array}{ccc} \mathrm{Spec}(A)-(A_+) = W & \xrightarrow{\quad g \quad} & U = \mathrm{Spec}(S)-(S_+) = W/G \\ \Big\downarrow q & & \Big\downarrow p \\ V & \xrightarrow{\quad f \quad} & Y = V/G \end{array}$$

with q and p the canonical projections of the G_m-bundles W and U respectively (see [8], chap. II, § 8). If F is the ramification locus of f, then $q^{-1}(F)$ is the ramification locus of g, and hence g acts freely on W outside a closed G-invariant subset of W. In particular, the singular locus Z of U is of codimension $\geqslant 3$ in U. Then by [19] and [20] we get that $T_U = g_*(T_W)^G$, where T_U is the tangent sheaf of U. Since $\mathrm{char}(k) = o$ or $\mathrm{char}(k)$ is prime to $|G|$, it follows that T_U is a direct summand of $g_*(T_W)$, and in particular

(6) $H^1(U,T_U)$ is a direct summand of $H^1(U,g_*(T_W)) = H^1(W,T_W)$.

On the other hand, it is well known that there is a canonical exact sequence (see e.g. [14] or [21])

$$o \longrightarrow O_W \longrightarrow T_W \longrightarrow q^*(T_V) \longrightarrow o$$

which yields the exact sequence

$$(7)\qquad \begin{array}{ccccc} H^1(W,O_W) & \longrightarrow & H^1(W,T_W) & \longrightarrow & H^1(W,q^*(T_V)) \\ \| & & & & \| \\ \bigoplus_{i\in\mathbb{Z}} H^1(V,M^i) & & & & \bigoplus_{i\in\mathbb{Z}} H^1(V,T_V\otimes L^i) \end{array}$$

The vertical isomorphisms in (7) give the natural gradings on $H^1(W,O_W)$ and on $H^1(W,q^*(T_V))$ respectively. But the middle space in (7) also has a natural gradation $H^1(W,T_W) = \bigoplus_{i\in\mathbb{Z}} H^1(W,T_W)(i)$ arising from the G_m-action on W, and all these three gradations are compatible with the maps in (7). Therefore, using hypotheses iii) and iv) we get that $H^1(W,T_W)(i) = o$ for every $i < o$. There is also a natural gradation $H^1(U,T_U) = \bigoplus_{i\in\mathbb{Z}} H^1(U,T_U)(i)$ arising from the G_m-action on U, and this gradation is compatible via (6) with the gradation of $H^1(W,T_W)$, and consequently we get

(8) $\qquad H^1(U,T_U)(i) = o$ for every $i < o$.

Since U has only quotient singularities in codimension $\geqslant 3$, by [19] and [20] we infer that all the singularities of U are rigid, and in particular,

$\text{depth}_Z(T_U) \geq 3$. Then the exact sequence of local cohomology shows that the restriction map $H^1(U,T_U) \longrightarrow H^1(U-Z,T_U)$ is an isomorphism.

Finally, since U has only quotient (and hence Cohen-Macaulay) singularities and $\text{codim}_U(Z) \geq 3$, by [19] and [20] we get $T^1_S \cong H^1(U-Z,T_U)$. Recalling (8) and the isomorphism $H^1(U-Z,T_U) \cong H^1(U,T_U)$ we get the conclusion of lemma 1. Q.E.D.

Now we illustrate how theorem 1 can be applied - via lemma 1 - on some examples. First we apply theorem 1 to the singular Kummer varieties of dimension ≥ 3. Recall that a singular Kummer variety Y is a variety of the form V/G, where V is an abelian variety of dimension $d \geq 2$ and $G \subset \text{Aut}(V)$ is the subgroup of order 2 generated by the involution $u: V \longrightarrow V$ defined by $u(x) = -x$ for every $x \in V$ ($-x$ is the inverse of x in the group-law of V). Since for $\text{char}(k) \neq 2$ there are exactly 2^{2d} points of order 2 on V (see [16]), $Y = V/G$ has exactly 2^{2d} singularities (which are all quotient singularities). Now we have:

<u>Theorem 2.</u> <u>Let Y be a singular Kummer variety of dimension $d \geq 3$, and let L be an arbitrary ample line bundle on Y. If $\text{char}(k) \neq 2$ then the property (+) holds for (Y,L).</u>

<u>Proof.</u> We first show that lemma 1 implies that $T^1_S(-i) = o$ for every $i \geq 1$, with $S = S(Y,L)$. Indeed, the hypotheses i) and ii) of lemma 1 are clearly satisfied, while iii) and iv) follow using the fact that the tangent bundle of an abelian variety is trivial, together with the fact that the Kodaira's vanishing theorem for an abelian variety holds in arbitrary characteristic (see [16], §16).

It remains to check that $H^1(Y,L^i) = o$ for every $i \in \mathbb{Z}$ (which is the first hypothesis of theorem 1). If $f: V \longrightarrow Y$ is the canonical morphism, then by [19], L^i is a direct summand of $f_*f^*(L^i)$ because $\text{char}(k) \neq 2 = |G|$, and hence $H^1(Y,L^i)$ is a direct summand of $H^1(Y,f_*f^*(L^i)) = H^1(V,f^*(L^i))$. By [16], §16 the latter space is zero for $i \neq o$ because $f^*(L)$ is ample. On the other hand, if $i = o$, according to Schlessinger [19], page 24, we infer that $H^1(Y,O_Y) = H^1(V,O_V)^G$, and G acts on $H^1(V,O_V)$ by $t \longrightarrow -t$. It follows that $H^1(Y,O_Y) = o$. Applying theorem 1 we get the conclusion. Q.E.D.

Further examples of singular normal varieties satisfying (+) with respect to any ample line bundle are the symmetric products of certain smooth projective varieties. Let Z be a smooth projective variety of dimension $d \geq 3$, and let Y be the symmetric product $Z^{(n)} = V/G$, where: $n \geq 2$ is a fixed integer, $V = Z^n$ (the direct product of Z with itself n times), and G is the symmetric group of degree n acting on V by $g.(z_1,\dots,z_n) = (z_{g(1)},\dots,z_{g(n)})$ for every $g \in G$ and $(z_1,\dots,z_n) \in V$. Then the ramification locus of the canonical morphism $f: V \longrightarrow Z^{(n)}$ has codimension in V equal to $d = \dim(V) \geq 3$.

Theorem 3. Let Z be a smooth projective variety of dimension $d \geq 3$ such that $H^1(Z,M) = o$ for every line bundle M on Z, and let $n \geq 2$ be an integer such that either $\mathrm{char}(k) = o$, or $n > \mathrm{char}(k)$. Then for every ample line bundle L on $Y = Z^{(n)}$ the property (+) holds for (Y,L).

Note. The simplest examples of varieties Z satisfying the hypotheses of theorem 3 are all smooth hypersurfaces in P^{d+1} with $d \geq 3$.

Proof of theorem 3. The hypotheses imply in particular that $H^1(Z,O_Z) = o$, and then the see-saw principle (see [16], §5) immediately implies that $f^*(L) \cong p_1^*(L_1) \otimes \dots \otimes p_n^*(L_n)$, with $L_1,\dots,L_n \in \mathrm{Pic}(Z)$ and $p_i: V \longrightarrow Z$ the projection of V onto the i-th factor. Since L is ample on Y and f is finite, $f^*(L)$ is ample on V, and hence L_i is ample on Z for every $i = 1,\dots,n$. As in the proof of theorem 2, it will be sufficient to check the following:

$$H^1(V,f^*(L^i)) = o \quad \text{for every } i \in \mathbb{Z}, \text{ and}$$

$$H^1(V,T_V \otimes f^*(L^i)) = o \text{ for every } i < o ,$$

in order to deduce (via lemma 1) that the hypotheses of theorem 1 are satisfied. But these vanishings are easily checked using the Künneth's formulae, the fact that $T_V = p_1^*(T_Z) \oplus \dots \oplus p_n^*(T_Z)$, the hypotheses of the theorem and the fact that L_i is ample for $i = 1,\dots,n$ (which implies that $H^o(Z,L_i^j) = o$ for every $j < o$ and $i = 1,\dots,n$). Then the conclusion of the theorem follows from theorem 1. Q.E.D.

§3. A few remarks when Y is smooth

In this section we shall assume that Y is smooth and $\mathrm{char}(k) = o$. Then it is known that the space $T_S^1(i)$ can be computed in the following way (see [23], page 337 and theorem 3.7). First, there is an exact sequence of vector bundles

$$o \longrightarrow O_Y \longrightarrow M \longrightarrow T_Y \longrightarrow o$$

which is the dual of the exact sequence

$$o \longrightarrow \Omega_Y^1 \longrightarrow F \longrightarrow O_Y \longrightarrow o$$

corresponding to the image of L in $H^1(Y,\Omega^1_Y)$ via the canonical map $H^1(Y,O^*_Y) \cong$ $\cong \mathrm{Pic}(Y) \longrightarrow H^1(Y,\Omega^1_Y)$ induced by the map $O^*_Y \longrightarrow \Omega^1_Y$ given by $f \longmapsto df/f$. Then it is proved in loc. cit. that

$$(9) \qquad T^1_S(i) = \mathrm{Ker}(H^1(Y,M\otimes L^i) \longrightarrow H^1(Y,\bigoplus_{j=1}^{n} L^{i+q_j})) \quad \text{for every } i\in\mathbb{Z},$$

where $S = S(Y,L)$ and $q_1,\dots,q_n$ have the same meaning as at the beginning of § 1.

Using (9), the first exact sequence and the Kodaira's vanishing theorem, it follows that the condition " $T^1_S(-i) = o$ for every $i\geq 1$ " is a consequence of the condition " $H^1(Y,T_Y\otimes L^{-i}) = o$ for every $i\geq 1$ ". If Y is smooth and $\mathrm{char}(k) = o$, one can get rid of the unpleasant hypothesis i) of theorem 1 because of the following:

<u>Theorem 4</u> (See [6]). <u>Let (Y,L) be a smooth polarized variety of dimension ≥ 2 such that</u> $H^1(Y,T_Y\otimes L^{-i}) = o$ <u>for every $i\geq 1$ and $\mathrm{char}(k) = o$. Then the property (+) holds for (Y,L).</u>

Theorem 4 is proved in [6]; it is also a quick consequence of theorem 2 in [22]. Using theorem 4 and the main result of [22] we prove the following:

<u>Theorem 5. Let (Y,L) be a smooth polarized variety such that: $\mathrm{char}(k) = o$, $\dim(Y)\geq 2$,</u> $H^1(Y,T_Y\otimes L^{-i}) = o$ <u>for $i = 1$ and $i = 2$, and the linear system |L| contains a smooth divisor. Then the property (+) holds for (Y,L).</u>

<u>Proof</u>. By theorem 4 it will be sufficient to show that $H^1(Y,T_Y\otimes L^{-i}) = o$ for every $i\geq 1$. Let $H\in|L|$ be a smooth divisor of |L|. Since $\dim(Y)\geq 2$, H is also connected. If we denote by L_H the restriction $L\otimes O_H$ and by T_H the tangent bundle of H, we have the canonical exact sequence

$$o \longrightarrow T_H\otimes L_H^{-i} \longrightarrow (T_Y\otimes L^{-i})/H \longrightarrow L_H^{1-i} \longrightarrow o,$$

which yields the exact sequence

$$(10_i) \qquad H^o(H,T_H\otimes L_H^{-i}) \longrightarrow H^o(H,(T_Y\otimes L^{-i})/H) \longrightarrow H^o(H,L_H^{1-i}).$$

For every $i\geq 2$ the last space is zero. On the other hand, by the main result of [22] (which extends a theorem of Mori-Sumihiro), the first space could be $\neq o$ only if $(H,L_H) \cong (P^1,O(1))$ (and then $i = 2$), in which case it follows easily that $(Y,L) \cong (P^2,O(1))$, and whence (Y,L) has the property (+). Thus we may assume that $H^o(H,T_H\otimes L^{-i}) = o$ for every $i\geq 2$. Then by (10_i) we get $H^o(H,(T_Y\otimes L^{-i})/H) = o$ for every $i\geq 2$. Finally, using this and the exact sequence

$$(11_i)\qquad o \longrightarrow T_Y\otimes L^{-i-1} \longrightarrow T_Y\otimes L^{-i} \longrightarrow (T_Y\otimes L^{-i})/H \longrightarrow o$$

we infer that the map $H^1(Y,T_Y\otimes L^{-i-1}) \longrightarrow H^1(Y,T_Y\otimes L^{-i})$ is injective for every $i\geq 2$. Therefore $H^1(Y,T_Y\otimes L^{-i}) = o$ for every $i\geq 1$. Q.E.D.

Corollary. Let (Y,L) be a smooth polarized variety of dimension $d\geq 2$ such that there is a smooth divisor $H\in |L|$ for which the exact sequence

$$(12)\qquad o \longrightarrow T_H \longrightarrow T_Y/H \longrightarrow L_H \longrightarrow o$$

is not split (in particular, $H^1(H,T_H\otimes L_H^{-1}) \neq o$). Assume moreover that $char(k) = o$ and $H^1(Y,T_Y\otimes L^{-1}) = o$. Then the property (+) holds for (Y,L).

Proof. According to the proof of theorem 5, the exact sequence (11_1) shows that it is sufficient to prove that $H^o(H,(T_Y\otimes L^{-1})/H) = o$.

The exact sequence (10_1) yields the exact sequence

$$(13)\qquad H^o(H,T_H\otimes L_H^{-1}) \longrightarrow H^o(H,(T_Y\otimes L^{-1})/H) \longrightarrow H^o(H,O_H) \xrightarrow{\partial} H^1(H,T_H\otimes L_H^{-1}).$$

By [22], the first space could be $\neq o$ only in one of the following cases: either $(H,L_H)\cong (P^{d-1},O(1))$, or $(H,L_H)\cong (P^1,O(2))$. In the first case $(Y,L)\cong (P^d,O(1))$, and hence (Y,L) has the property (+); the second case is ruled out because then $H^1(H,T_H\otimes L_H^{-1}) = o$, and hence (12) splits. Therefore we may assume $H^o(H,T_H\otimes L_H^{-1}) = o$, and then (13) shows that $H^o(H,(T_Y\otimes L^{-1})/H) = o$ if and only if $\partial(1)\neq o$. Since $\partial(1)$ is the obstruction in $H^1(H,T_H\otimes L_H^{-1})$ such that (12) be split, we get the result. Q.E.D.

Remark. In a more special situation, L'vovskii proved in [15] a better result than theorem 5 or its corollary. More precisely, assume that $Y\subset P^n$ is a smooth non-degenerate projective subvariety of P^n of dimension ≥ 2 and degree ≥ 3, such that $H^1(Y,T_Y(-1)) = o$ and $char(k) = o$. Let $X\subset P^{n+1}$ be an irreducible subvariety of P^{n+1} such that $X\cap P^n = Y$, and X is smooth along Y and transveral to P^n, where P^n is embedded in P^{n+1} as a hyperplane. Then X is a cone over Y. In fact, L'vovskii has an even weaker assumption than $H^1(Y,T_Y(-1)) = o$ (loc. cit.). His proof uses completely different ideas.

Coming back to the above corollary, we may ask the following:

Question. Let (Y,L) be a smooth polarized variety of dimension $d\geq 2$ such that L is generated by its global sections. Find sufficient conditions ensuring that there is a smooth member $H\in |L|$ such that the corresponding exact sequence

(12) is not split. Or, enumerate the situations when (12) is split for H general.

A necessary condition such that this question has a positive answer is that $H^1(H, T_H \otimes L_H^{-1}) \neq o$ for $H \in |L|$ general. Is it also sufficient ? In the case of surfaces, the pairs (Y,L) for which $H^1(H, T_H \otimes L_H^{-1}) = o$ for $H \in |L|$ general, can be easily enumerated. Indeed, by duality and Riemann-Roch on the curve H one gets that this happens if and only if $(H, L_H) \cong (P^1, O(i))$ with i = 1, 2, or 3. And by a well known classical result, (Y,L) is isomorphic to one of the following: $(P^2, O(1))$, $(P^1 \times P^1, O(1,1))$, or any smooth hyperplane section of $P^1 \times P^2 \subset P^5$ via the Segre embedding (the latter surfaces are all isomorphic to the projective plane blown up at a point).

§4. P^n-bundles over an irrational curve as hyperplane sections

Let B be a smooth projective curve, and let E be a vector bundle of rank n+1 on B, with $n \geq 1$. Denote by Y = P(E) the projective bundle associated to E, and by $p: Y \longrightarrow B$ the canonical projection. The main result of this section is the following:

Theorem 6. In the above notations, assume that the genus of B is positive and char(k) = o. Let X be a singular normal projective variety containing Y = = P(E) as an ample Cartier divisor. Then X is isomorphic to the projective cone C(Y,L) and Y is embedded in X as the infinite section, where L is the normal bundle of Y in X.

The motivation of theorem 6 lies in the fact that, combining it with some results from [1], [2], and [3], we get the following complete description of all normal projective varieties whose hyperplane sections are P^n-bundles over a curve:

Theorem 7. Assume that B is a smooth projective curve of arbitrary genus, and let Y = P(E) be a P^n-bundle over B ($n \geq 1$). Assume furthermore char(k) = o. Let X be a normal projective variety containing Y as an ample Cartier divisor. Then one has one of the following possibilities:

a) $X = P^3$, $Y = P^1 \times P^1$, and Y is embedded in X as a quadric.

b) X is isomorphic to a smooth hyperquadric in P^4, $Y = P^1 \times P^1$, and Y is the intersection of X with a hyperplane of P^4.

c) *There is an exact sequence of vector bundles on B of the form*

$$0 \longrightarrow O_B \longrightarrow F \xrightarrow{\varphi} E' \longrightarrow 0$$

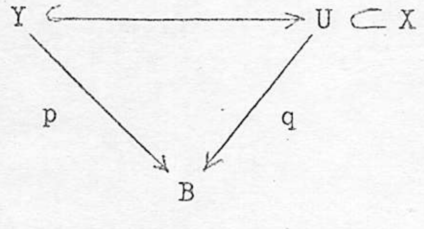
such that F is an ample vector bundle in the sense of [10], $E' = E \otimes L'$ for some $L' \in \mathrm{Pic}(B)$, $X \cong P(F)$, and $Y \cong P(E')$ is embedded in X via surjection φ.

d) *X is isomorphic to the cone C(Y,L), with L the normal bundle of Y in X, and Y is embedded in X as the infinite section.*

Remarks. 1) In certain cases (but not in all) theorem 6 was proved in [3], theorem 6.

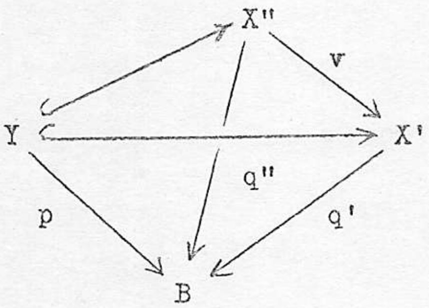
2) Theorem 7 is obtained as the result of a long case-by-case discussion (see [1], theorem 5, [2], theorems 1, 2 and 3, and [3], theorems 3, 4 and 5, and theorem 6 above). The most difficult case is when Y is a surface, i.e. E is a rank two vector bundle. Note that the proof of the result in case $Y = P(O_{P^1} \oplus O_{P^1}(-1))$ and X is smooth is completely given in our short note, L. Badescu, The projective plane blown up at a point as an ample divisor, Atti Accad. Ligure Sci. Lettere, 38 (1981), 3-7 (cf. also lemma 2 in [3] and its proof, for the case X is singular). Another proof of theorem 7 in case X is smooth, was subsequently given by P. Ionescu in [12], as an application of the general adjunction mapping, using Mori's theory of extremal rays and Kawamata-Shokurov contraction theorem.

Proof of theorem 6. According to [2] and [3], the Lefschetz theorem and the Albanese mapping yield the commutative diagram



where U is an open neighbourhood of Y in X (in fact, we can take $U = X_{reg}$). Then X has finitely many singularities, and by Hironaka [11], there is a desingularization $f: X'' \longrightarrow X$ with the following properties: f induces an isomorphism $f^{-1}(U) = U$, the rational map $q'' = q \circ f : X'' \longrightarrow B$ is in fact a morphism, and the exceptional fibres of f are divisors of normal crossings (i.e. with smooth 1-codimensional components intersecting transversely). Then the normal bundle of Y in X'' is L, and since L is ample, L is in particular, p-ample. One of the main point in the proof of theorem 6 is the following lemma, which is essentially the relativization of theorem 4.2, chap. III of [9].

Lemma 2. Let $q'':X'' \longrightarrow B$ be a surjective morphism between the normal projective varieties X" and B, and let Y be an effective Cartier divisor on X" such that the restriction $p:Y \longrightarrow B$ of q" is surjective. Assume that the normal bundle L of Y in X is p-ample. Then there is a canonical commutative diagram



with X' a normal projective variety, $q':X' \longrightarrow B$ a morphism, v a birational morphism such that v is an isomorphism in a neighbourhood of Y, and v(Y) is a q'-ample effective Cartier divisor on X'.

Proof of lemma 2. First we are going to show that for $i \gg o$ the following three conditions are satisfied:

i) L^i is p-very ample,

ii) The canonical map $q''^* q''_*(O_{X''}(iY)) \longrightarrow O_{X''}(iY)$ is surjective,

iii) The canonical map $q''_*(O_{X''}(iY)) \longrightarrow p_*(L^i)$ is surjective.

Indeed, since L is p-ample, i) holds. Now we prove iii). Consider the exact sequence ($i \geq 1$)

$$q''_*(O_{X''}(iY)) \longrightarrow p_*(L^i) \longrightarrow R^1 q''_*(O_{X''}((i-1)Y)) \xrightarrow{\varepsilon_i} R^1 q''_*(O_{X''}(iY)) \longrightarrow R^1 p_*(L^i)$$

induced by the exact sequence $o \longrightarrow O_{X''}((i-1)Y) \longrightarrow O_{X''}(iY) \longrightarrow L^i \longrightarrow o$. The last sheaf is zero for $i \gg o$ because L is p-ample (see [8], chap. III, (2.2.1)). Hence the map ε_i is surjective for every $i \geq j$ (say). Since q" is a projective morphism, $R^1 q''_*(O_{X''}(jY))$ is coherent on B, and therefore ε_i becomes an isomorphism for $i \gg o$, i.e. iii) holds.

To prove ii), observe that by [8], chap. II (3.4.7), ii) is equivalent to the fact that for every affine open subset D = Spec(A) of B, the sheaf $O_{X''}(iY)/q''^{-1}(D)$ is generated by its global sections for $i \gg o$. But by iii) the natural map $H^o(D, q''_*(O_{X''}(iY))) \cong H^o(q''^{-1}(D), O_{X''}(iY)) \longrightarrow H^o(D, p_*(L^i)) \cong$

$\cong H^0(p^{-1}(D),L^i)$ is surjective for $i \gg o$ because D is affine. Using the fact that L^i is p-very ample, it follows that $L^i/p^{-1}(D)$ is generated by its global sections, and hence (by the above surjectivity), $O_{X''}(iY)/q''^{-1}(D)$ is generated by its global sections.

Now fix an $i \gg o$ such that i), ii) and iii) are fulfilled. From ii) it follows that there is a unique B-morphism $v_1 : X \longrightarrow P := P(q''_*(O_{X''}(iY)))$ such that $v_1^*(O_P(1)) \cong O_{X''}(iY)$. Set $X_1 = v_1(X'')$ and $Y_1 = v_1(Y)$. Since L^i is p-very ample, we know that $v_1/Y : Y \longrightarrow Y_1$ is an isomorphism and that iY_1 is a B-very ample Cartier divisor on X_1. Furthermore $Y = v_1^{-1}(Y_1)$ because a global equation of the Cartier divisor iY on X" separates points x, x' such that $x \in Y$ and $x' \in X''-Y$. Then consider the Stein factorization of v_1

$$\begin{array}{ccc} X'' & \xrightarrow{\quad v \quad} & X' := \mathrm{Spec}(v_{1*}(O_{X''})) \\ & {\scriptstyle v_1} \searrow \quad \swarrow {\scriptstyle w} & \\ & X_1 & \end{array}$$

Since $v_*(O_{X''}) \cong O_{X'}$ and X" is normal, X' is also normal. Notice that $v/Y : Y \longrightarrow Y' = v(Y)$ is an isomorphism and $Y = v^{-1}(Y')$, so by Zariski's main theorem (see [8], chap. III, (4.4.1)), v is an isomorphism in a neighbourhood of Y in X". Since w is a finite morphism and Y_1 is B-ample, $Y' = w^*(Y_1)$ is q'-ample on X', where q' is the composition $X' \xrightarrow{w} X_1 \hookrightarrow P \longrightarrow B$.

Lemma 2 is proved.

<u>Note</u>. The above proof of lemma 2 is an adaptation of the proof of theorem 4.2, chap. III in [9] to the relative case.

<u>Proof of theorem 6, continued</u>. We apply lemma 2 to the desingularization X" of X such that $q'' = q \circ f$ is a morphism , and get the normal projective variety X' as in lemma 2 (in particular, Y becomes an effective Cartier divisor on X' which is q'-ample). Notice that v blows down to points only subvarieties of X" that are contained in the exceptional locus of f, and since X' is normal, by [8], chap. II, (8.11.1) we infer that there is a unique morphism $u : X' \longrightarrow X$ such that $q \circ u = q'$ and $f = u \circ v$. Notice also that the construction of u and X' is canonical and depends only on X and the rational map q, and not of the choice of the desingularization $f : X'' \longrightarrow X$.

With this construction in hand, we can proceed further. Since $Y = P(E)$ and

L is a p-ample line bundle, there is an $M \in Pic(B)$ and a positive integer $s \geq 1$ such that

$$(14) \qquad L \cong O_Y(s) \otimes p^*(M^{-1}), \text{ where } O_Y(s) = O_{P(E)}(s).$$

Replacing E by $E \otimes N$, with $N \in Pic(B)$ of sufficiently high degree, we get that $L = O_{P(E \otimes N)}(s) \otimes p^*(N^{-s} \otimes M^{-1})$. In other words, we may assume that in (14) M has sufficiently high degree.

According to the Lefschetz theorem, there is an $F \in Pic(U)$ such that $F \otimes O_Y \cong$ $\cong O_Y(1)$ (cf. e.g. [2], the proof of theorem 2). Set $U'' = f^{-1}(U)$ and $U' = u^{-1}(U)$. Since $U'' \cong U \cong U'$, we may consider the sheaf F on U", and since X" is smooth, F extends (non-uniquely) to a line bundle on X", still denoted by F. Since the map $Pic(U) \longrightarrow Pic(Y)$ is injective, (14) can be translated into $F^s/U'' \cong$ $\cong (O_{X''}(Y) \otimes q''^*(M))/U''$. Therefore there is a divisor D supported by the exceptional fibres of f, such that $F^s \cong O_{X''}(Y) \otimes q''^*(M) \otimes O_{X''}(D)$. If $D = D_+ - D_-$, with D_+ and $D_- \geq o$, after replacing F by $F \otimes O_{X''}(D_-)$ (which still has the restriction $O_Y(1)$ to Y), we may assume that $D \geq o$. Furthermore, since M is of sufficiently high degree, for a general divisor $b_1 + \dots + b_m \in |M|$ (with $b_i \neq b_j$ for $i \neq j$), the fibres $X''_i = q''^{-1}(b_i)$ are all smooth and transverse to all components of D as well as to all their possible intersections. Thus, replacing D by $D'' = D + D'$, with $D' = \sum_{t=1}^{m} X_i$, we get

$$(15) \qquad F^s \cong O_{X''}(Y) \otimes q''^*(D''),$$

where D" is a normal crossing positive divisor on X" such that $D'' = D + D'$, with Supp(D) contained in the exceptional fibres of f and D' a sum of distinct fibres of q" (and hence D' is reduced). Then, according to Kawamata and Viehweg, (see [13], or [21]), for every $i \in \mathbb{Z}$ we put $F^{(i)} = F^i \otimes O_{X''}(-[iD''/s])$, where if $\Delta = \sum_j a_j \Delta_j$ is a $\mathbb{Q}$-divisor on X" (with $\Delta_j \neq \Delta_{j'}$ if $j \neq j'$), $[\Delta]$ denotes the integral divisor $\sum_j [a_j] \Delta_j$, where $[a]$ denotes the largest integer $\leq a$. Notice that if $i = js + r$ is an arbitrary integer such that $o \leq r \leq s-1$, then by (15) we get:

$$(16) \qquad F^{(i)} \cong O_{X''}(jY) \otimes F^{(r)}.$$

Now, the second main point in the proof of theorem 6 is the following:

<u>Lemma 3.</u> $R^b q''_*((F^{(i)})^{-1}) = o$ for every $i \geq 1$ and $b = o,1$.

<u>Proof of lemma 3.</u> This proof follows the well known philosophy consisting in using global vanishing theorems to get local ones (cf. e.g. [5], appendix 1)

Let N be a sufficiently ample line bundle on B such that $N \otimes R^b q''_*((F^{(i)})^{-1})$ is generated by its global sections and such that $H^a(B, N \otimes R^b q''_*((F^{(i)})^{-1})) = o$ for $a \geq 1$, $b = o,1$ and $i \geq 1$ (i fixed). Consider the Leray spectral sequence

$$E_2^{a,b} = H^a(B, N \otimes R^b q''_*((F^{(i)})^{-1})) \Longrightarrow H^{a+b}(X'', q''^*(N) \otimes (F^{(i)})^{-1}).$$

By the choice of N, we have $E_2^{a,b} = o$ for $a > o$, which implies that

$$H^o(B, N \otimes R^b q''_*((F^{(i)})^{-1})) \cong H^b(X'', q''^*(N) \otimes (F^{(i)})^{-1}).$$

Since $N \otimes R^b q''((F^{(i)})^{-1})$ is generated by its global sections, it is sufficient to show that the left-hand side is zero, or by the above isomorphism, that the right-hand-side is zero. To this end, using the fact that N is sufficiently ample, by Bertini we can choose a divisor $c_1 + \dots + c_e \in |N|$ (with $c_i \neq c_j$ if $i \neq j$) such that $X''_i = q''^{-1}(c_i)$ is smooth, not included in Supp(D"), and transverse to D. Then we have the exact sequence

$$H^b(X'', (F^{(i)})^{-1}) \longrightarrow H^b(X'', q''^*(N) \otimes (F^{(i)})^{-1}) \longrightarrow H^b(Z, (F^{(i)})^{-1}),$$

with $Z = \sum_{i=1}^{e} X''_i$. Notice that since $f^*(O_X(Y)) \cong O_{X''}(Y)$ and Y is ample on X, $O_{X''}(iY)$ is generated by its global sections for $i \gg o$ and the self-intersection number $(O_{X''}(Y)^{\cdot \dim(X'')}) > o$ (and hence $O_{X''}(Y)$ is nef and big in the terminology of [21]). Therefore, recalling (15) and the definition of $F^{(i)}$, the first cohomology space is zero by the Kawamata-Viehweg vanishing theorem ([13],[21]). The third cohomology space is also zero by the same vanishing theorem applied on the smooth (but possibly disconnected) variety Z, and hence the middle space is zero. Q.E.D.

<u>Corollary (to lemma 3)</u>. <u>For every $i \in \mathbb{Z}$ set</u> $G_i = v_*(F^{(i)})$. <u>Then</u> $R^1 q'_*(G_{-i}) =$ $= o$ <u>for every $i \geq 1$.</u>

<u>Proof of the corollary</u>. From the definitions we easily get that $F^{(-i)} \cong$ $\cong (F^{(i)})^{-1} \otimes O_{X''}(D_1 + D_2)$, with D_1 and D_2 reduced effective divisors on X" such that $D_1 \leq D'$ and $D_2 \leq D$. Since $O_{X''}(D_1) = q''^*(N)$ for some $N \in \mathrm{Pic}(B)$, we get $G_{-i} = q'^*(N) \otimes v_*((F^{(i)})^{-1} \otimes O_{X''}(D_2))$ (by projection's formula). Thus, it will be sufficient to show that

$$(17) \qquad R^1 q'_*(v_*((F^{(i)})^{-1} \otimes O_{X''}(D_2)) = o \quad \text{for every } i \geq 1.$$

But we have an exact sequence

$$o \longrightarrow v_*((F^{(i)})^{-1}) \longrightarrow v_*((F^{(i)})^{-1} \otimes O_{X''}(D_2)) \longrightarrow R \longrightarrow o$$

where the support of R is contained in the image B' of the union of the exceptional fibres of f under the morphism v. Let $r:B' \longrightarrow B$ denote the restriction of q' to B'. We get the exact sequence

$$R^1q'_*(v_*((F^{(i)})^{-1})) \longrightarrow R^1q'_*(v_*((F^{(i)})^{-1}\otimes O_{X''}(D_2)) \longrightarrow R^1r_*(R).$$

Since $B'\cap Y = \emptyset$ and Y is q'-ample (lemma 2), the fibres of r are all finite, and hence r is finite, and in particular, the third sheaf is zero. By considering the spectral sequence

$$E_2^{b,a} = R^bq'_*(R^av_*((F^{(i)})^{-1})) \Longrightarrow R^{a+b}q''_*((F^{(i)})^{-1})$$

we get that that $R^1q'_*(v_*((F^{(i)})^{-1})) \subseteq R^1q''_*(F^{(i)})^{-1})$, and from lemma 3 it follows that the first sheaf in the above exact sequence is also zero, whence (17) holds.

<u>Proof of theorem 6, continued</u>. Having lemma 3 and its corollary (which is the second main ingredient of the proof), we can finally conclude the proof as follows. Recalling (14), we distinguish two cases.

<u>Case s = 1</u>. Replacing E by $E\otimes M^{-1}$, we may assume that $L = O_Y(1)$. Then by lemma 3 and its corollary, $R^bq'_*(O_{X'}(-Y)) = o$ for b = o,1. Now, the exact sequence

$$o \longrightarrow O_{X'}((i-1)Y) \xrightarrow{t} O_{X'}(iY) \longrightarrow O_Y(i) \longrightarrow o$$

(where $t\in H^o(X',O_{X'}(Y))$ is a global equation of Y on X') yields the exact sequence

$$R^1q'_*(O_{X'}((i-1)Y)) \longrightarrow R^1q'_*(O_{X'}(iY)) \longrightarrow R^1p_*(O_Y(i)) \quad (i\geqslant o).$$

Since by [8], chap. III (2.1.15), $R^1p_*(O_Y(i)) = o$ for every $i\geqslant o$, and since we know that $R^1q'_*(O_{X'}(-Y)) = o$, by induction on i we get that $R^1q'_*(O_{X'}(iY)) = o$ for every $i\geqslant o$. In particular, the above exact sequence yields for every $i\geqslant o$ the exact sequence

$$(18_i) \quad o \longrightarrow q'_*(O_{X'}((i-1)Y)) \xrightarrow{t} q'_*(O_{X'}(iY)) \longrightarrow p_*(O_Y(i)) \longrightarrow o.$$

By [8], chap. III (2.1.15) again, $\bigoplus_{i=o}^{\infty} p_*(O_Y(i)) \cong S(E)$, where S(E) is the symmetric O_B-algebra of E. Denoting by $S = \bigoplus_{i=o}^{\infty} q'_*(O_{X'}(iY))$, from (18_i) we get $S/tS \cong S(E)$. Since S(E) is generated by its homogeneous part $S^1(E) = E$ of degree one and since deg(t) = 1, it follows that the graded O_B-algebra S is generated by $S_1 = q'_*O_{X'}(Y)$. In particular, the natural homomorphism $S(F) \longrightarrow S$ is surjective, where $F = q'_*O_{X'}(Y)$. On the other hand, since $q'_*(O_{X'}(-Y)) = o$ (lemma 3), and $\bigoplus_{i=o}^{\infty} p_*(O_Y(i)) \cong S(E)$, by induction on i in (18_i) we infer that S_i is a locally free O_B-module of rank $\binom{n+i+1}{i}$ for every $i\geqslant o$. It follows that the surjective maps $S^i(F) \longrightarrow S_i$ are all isomorphisms because $S^i(F)$ and S_i are vector bundles of the same rank. Thus $S \cong S(F)$, and recalling that Y is q'-ample (lemma 2), we get that $X' \cong P(F)$ is the projective bundle associated to F. The exact sequence (18_1) becomes

$$(19) \qquad o \longrightarrow O_B \longrightarrow F \longrightarrow E \longrightarrow o.$$

a) The exact sequence (19) does not split. Then a result of Gieseker (see [7], theorem 2.2, or also [5], (4.16)) together with the fact that $O_Y(1)$ is ample (which implies E ample), show that F is ample, or equivalently, $O_{X'}(Y) = O_{P(F)}(1)$ is ample (and not only q'-ample). Since $X' = X''$ is a desingularization of X whose exceptional locus does not intersect Y, it must be zero-dimensional. In other words, $f:X'' = X' \longrightarrow X$ has finite fibres, and hence, by Zariski's main theorem, f is an isomorphism. In particular, X is non singular, and this contradicts the hypotheses of theorem 6. Therefore case a) is impossible.

b) The exact sequence (19) is split. Then $F = E \oplus O_B$, and the surjection $E \oplus O_B \longrightarrow O_B$ yields the zero section $B \overset{i}{\hookrightarrow} V(E) = \mathrm{Spec}(S(E)) \hookrightarrow X' \cong P(F)$, where the second map is the natural open immersion whose complement is $Y = P(E)$ (see [8], chap. II, §8). Since E is an ample vector bundle on B, by Grauert's criterion of ampleness for vector bundles (see [10], (3.5)), the zero section $i(B) \subset X'$ can be blown down to get the projective variety $\mathrm{Proj}(\bigoplus_{i=0}^{\infty} H^o(B, S^i(E))[T])$ (with T an indeterminate of degree 1). Since $S^i(E) = p_*(O_Y(i))$ for every $i \geqslant o$, the latter variety is nothing but the cone $C(Y,L)$. Now, the morphism $f:X' = P(F) \longrightarrow X$ has to contract the curve $i(B)$ to a point (since Y is ample on X), and hence one gets a morphism $C(Y,L) \longrightarrow X$. Since Y is ample on both $C(Y,L)$ and X, as in case a) we infer that this morphism is in fact an isomorphism, and theorem 6 is proved in case $s = 1$.

<u>Case $s \geqslant 2$.</u> Let $i \in \mathbb{Z}$ be an arbitrary integer, and set $i = js + r$, with $o \leqslant r \leqslant s-1$. Since $v^*(O_{X'}(jY)) = O_{X''}(jY)$, by (16) and the projection's formula

$$(20) \qquad G_i \cong O_{X'}(jY) \otimes G_r, \quad \text{with } G_o = O_{X'}.$$

Furthermore, by (14), (15) and the definition of the G_i's we have

$$(21) \qquad G_i \otimes O_Y \cong O_Y(i) \otimes p^*(M^{-j} \otimes M_r),$$

where $M_o, \ldots, M_{s-1}$ are line bundles on B ($M_o = O_B$). Then by (20) and (21) for every $i \geqslant o$ we have the exact sequence

$$o \longrightarrow G_{i-s} \xrightarrow{t} G_i \longrightarrow O_Y(i) \otimes p^*(M^{-j} \otimes M_r) \longrightarrow o .$$

Let $D = Spec(A)$ be an arbitrary affine open subset of B such that $E/D \cong O_D^{n+1}$, $M/D \cong O_D$ and $M_r/D \cong O_D$ for every $o \leqslant r \leqslant s-1$. Set $X'_D = q'^{-1}(D)$ and $Y_D = p^{-1}(D)$. Then the above exact sequence restricted to X'_D becomes

$$o \longrightarrow G_{i-s} \xrightarrow{t} G_i \longrightarrow O_{Y_D}(i) \longrightarrow o$$

Since by the corollary of lemma 3 we have $H^1(X'_D, G_{i-s}) = R^1q'_*(G_{i-s})/D = o$ for every $i < s$, exactly as in case $s = 1$ we get

$$H^1(X'_D, G_i) = o \quad \text{for every } i \in \mathbb{Z} .$$

and hence the exact sequence $(i \geqslant o)$

$$(22_i) \quad o \longrightarrow H^o(X'_D, G_{i-s}) \xrightarrow{t} H^o(X'_D, G_i) \longrightarrow H^o(Y_D, O_Y(i)) \longrightarrow o.$$

Denote again by S the graded A-algebra $S = \bigoplus_{i=0}^{\infty} H^o(X'_D, G_i)$. Then $t \in S_s = H^o(X'_D, O_{X'}(Y))$ is an homogeneous element of degree s. Since $H^o(Y_D, O_Y) = A$ and by lemma 3, $H^o(X'_D, O_{X'}(-Y)) = q''_*(O_{X''}(-Y))/D = q''_*((F^{(s)})^{-1})/D = o$, it follows that $S_o = A$. Then $\bigoplus_{i=0}^{\infty} H^o(Y_D, O_Y(i)) \cong A[T_o, \ldots, T_n]$, where $T_o, \ldots, T_n$ are n+1 indeterminates (all of degree 1). By (22_1) there are n+1 elements of $S_1 = H^o(X'_D, G_1)$ such that $t_i/Y_D = T_i$ for $i = o, 1, \ldots, n$. Let S' denote the graded A-subalgebra of S generated by $t_o, \ldots, t_n$ and t. Then one has a surjective map of graded A-algebras $A[T, T_o, \ldots, T_{n+1}] \longrightarrow S$ mapping T into t and T_i into t_i, $i = o, \ldots, n$, where the polynomial A-algebra $A[T, T_o, \ldots, T_n]$ in n+2 variables is graded by $deg(T) = s$ and $deg(T_i) = 1$ for $i = o, \ldots, n$. That it is easy to see that this map is in fact an isomorphism of graded A-algebras.

On the other hand, by considering the exact sequences (22_{is}) $(i \geqslant o)$ and using the fact that $H^o(X'_D, O_{X'}(-Y)) = o$, an easy induction on i implies that $S'^{(s)} = S^{(s)}$, where according to [8], chap. II, $S^{(s)}$ denotes the graded A-algebra such that $(S^{(s)})_i = S_{is}$ for every $i \geqslant o$. Recalling also that Y is a q'-ample Cartier divisor on X', we infer that

$$X'_D \cong Proj(S) \cong Proj(S^{(s)}) = Proj(S'^{(s)}) \cong Proj(S'),$$

or else, that X'_D is isomorphic to the (n+2)-dimensional weighted projective space $P_A(1, \ldots, 1, s)$ over A of weights $(1, \ldots, 1, s)$. Furthermore, the restriction $q' : X'_D \longrightarrow D$ coincides to the canonical projection of $P_A(1, \ldots, 1, s)$ onto $D = Spec(A)$. In particular, for every $b \in B$, $X'_b = q'^{-1}(b)$ is isomorphic to the weighted projective space $P(1, \ldots, 1, s)$ over k and $Y_b = p^{-1}(b)$ is contained in

X'_b as the infinite section (i.e. the subvariety $V_+(T)$ of $P(1,\dots,1,s)$).

Summing up, we showed that there is a closed subset B' of X' such that q' defines an isomorphism of B' on B, $B' \cap X'_D = V_+(T_o,\dots,T_n) = \mathrm{Proj}(A[T])$, and $B' \cap X_b$ is precisely the vertex x_b of the cone $X_b = P(1,\dots,1,s)$ $(s \geqslant 2)$. Let $y \in Y$ be an arbitrary point and let L_y be the generating line of the cone $X_{p(y)}$ joining y and $x_{p(y)}$. Then $X'-B'$ is the disjoint union of all $L_y - x_{p(y)}$ (when $y \in Y$), and hence we get a well-defined function $\pi : X'-B' \longrightarrow Y$ by putting $\pi(x) = y$ if $x \in L_y$. The above discussion shows that π is in fact an algebraic morphism defined in a neighbourhood V of Y in X' (or in X) which is a retraction of $Y \subset V$. Then using lemma 3 in [3] (cf. also [6], (3.1)), we infer that $X \cong C(Y,L)$ also if $s \geqslant 2$.

Theorem 6 is completely proved. Q.E.D.

R e f e r e n c e s

1. L. Bădescu, On ample divisors, Nagoya Math. J., 86 (1982), 155-171.

2. L. Bădescu, On ample divisors: II, Proceedings Week of Algebraic Geometry, Bucharest 1980, Teubner-Texte Math. Band 40, Leipzig 1981, 12-32.

3. L. Bădescu, Hyperplane sections and deformations, Proceedings Algebraic Geometry, Bucharest 1982, Springer Lect. Notes Math. 1056, 1-33.

4. L. Bădescu, On a criterion for hyperplane sections, Math. Proc. Camb. Phil. Soc., 103 (1988), 59-67.

5. T. Fujita, On the hyperplane section principle of Lefschetz, J. Math. Soc. Japan, 32 (1980), 153-169.

6. T. Fujita, Rational retractions onto ample divisors, Scient. Papers Coll. Art. Sci? Univ. Tokyo, 33 (1983), 33-39.

7. D. Gieseker, P-ample bundles and their Chern classes, Nagoya Math. J., 43 (1971), 91-116.

8. A. Grothendieck & J. Dieudonné, Eléments de Géométrie Algébrique, chap. II, III, Publ. Math. IHES, 8, 11 (1961).

9. R. Hartshorne, Ample subvarieties of algebraic varieties, Springer Lect. Notes Math. 156 (1970).

10. R. Hartshorne, Ample vector bundles, Publ. Math. IHES, 29 (1966), 63-94.

11. H. Hironaka, Resolution of singularities of an algebraic variety over a field of char. zero, Annals Math., 79 (1964), 109-326.

12. P. Ionescu, Generalized adjunction and applications, Math. Proc. Camb. Phil. Soc., 99 (1986), 457-472.

13. Y. Kawamata, A generalization of Kodaira-Ramanujam's vanishing theorem, Math. Ann., 261 (1982), 43-46.

14. S. Kleiman & J. Landolfi, Geometry and deformations of special Schubert varieties, Compos. Math., 23 (1971), 407-434.

15. S.M. L'vovskii, Prolongation of projective manifolds and deformations, VINITI Preprint, Moskow State University, 1987 (in Russian).

16. D. Mumford, Abelian varieties, TATA Lecture Notes Math., Bombay 1968.

17. H. Pinkham, Deformations of algebraic varieties with G_m-action, Astérisque, 20 (1974), Soc. Math. France.

18. M. Schlessinger, Infinitesimal deformations of singularities, Thesis, Harvard Univ., 1964.

19. M. Schlessinger, Rigidity of quotient singularities, Invent. Math., 14 (1971), 17-26.

20. M. Schlessinger, On rigid singularities, Rice Univ. Studies, 51 (1973), 147-162.

21. E. Viehweg, Vanishing theorems, Journ. reine angew. Math., 355 (1982), 1-8.

22. J. Wahl, A cohomological characterization of P^n, Invent. Math., 72 (1983), 315-322.

23. J. Wahl, Equisingular deformations of normal surface singularities:I, Annals Math., 104 (1976), 325-356.

INCREST Bucharest, Dept. of Mathematics
B-dul Păcii 220, 79622 Bucharest,
Rumania