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Summary. In this paper we denote by $\pi(x)$ the number of primes $\leq x$, by $\theta(x)$ the logaritm of the product of all primes $\leq x$, and by $\psi(x)$ the logaritm of the least common multiple of positive integers $\leq x$. We also denote by x a real number, by n a natural number and by p a prime number.

Under Riemann hypothesis are known results of the following type:

$$|\theta(x)-x| = O(x^{1/2}\log^2 x) \tag{1}$$

$$|\pi(x)-\text{li } x| = O(x^{1/2}\log x) \tag{2}$$

If this hypothesis is not assumed, poorer asymtotic results can be obtained, but Rosser and Schoerfeld [3] had gives formulas which are true both for small and for great values of x. The aim of this paper is to give explicit formulas for the functions $\pi(x)$ and $\theta(x)$ under Riemann hypothesis, that is, to determine the values of the constants from (1) and (2). We also determine approximate formulas for some functions related to prime numbers.

The most important results are:

$$|\theta(x)-x| < 1,5x^{1/2}\log^2 x \qquad \text{for } x \geq 3 \tag{3}$$

$$|\pi(x)-\text{li } x| < 1,5x^{1/2}(2+\log x) \qquad \text{for } x \geq 3 \tag{4}$$

$$\left|\sum_{p\leq x}\frac{1}{p} - \log\log x - c_1\right| < \frac{5\log x}{x} \qquad \text{for } x \geq 2 \tag{5}$$

$$\left|\sum_{p\leq x}\frac{\log p}{p} - \log x - c_2\right| < 5,23\frac{\log^2 x}{x} \qquad \text{for } x \geq 2 \tag{6}$$

where $c_1=0.26149\ 72128\ 47643$ and $c_2=1.33258\ 22757\ 33221$ are the approximate values of the constants B and E from [3].

For small values of x we need the computations s of Rosser and Schoerfeld [3].

§1. THE FUNCTIONS $\psi(x)$ AND $\theta(x)$

From now on the Riemann hypothesis is assumed to be true. $\rho=\frac{1}{2}+i\gamma$ will denote as usual the zeros of the Riemann zeta function from the critic bord. The inequality (3) for $\theta(x)$ can be infered from a similar one for $\psi(x)$. Such an inequality can be obtained from the basic formula:

$$\psi_1(x)=\int_2^x \psi(u)\,du=\frac{x^2}{2}-\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - x\frac{\zeta'(0)}{\zeta(0)} + \frac{\zeta'(-1)}{\zeta(-1)} - \sum_{r=1} \frac{x^{1-2r}}{2r(2r-1)} \tag{7}$$

which can be found in Inghan [2] pag. 73. Knowing the fact that $\sum_\rho \frac{1}{|\rho|^a}$ is convergent for $a>1$, from (7) follows that:

$$\psi(x)-\psi_1(x+1)-\psi_1(x)=x-\sum_\rho \frac{(x+1)^{\rho+1}-x^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)} - \sum_{r=1} \frac{x^{1-2r}-(x+1)^{1-2r}}{2r(2r-1)} \tag{8}$$

It is known that $\frac{\zeta'(0)}{\zeta(0)} = \log 2\pi < 1,8379$ end the last sum is much smaller than the increasing we'll going to do, so that, our aim is to give an upper bound for the first sum.

According to Inghan [2] pag. 83 we have:

$$\left|\frac{(x+1)^{\rho+1}-x^{\rho+1}}{\rho(\rho+1)}\right| = \frac{1}{\rho}\int_x^{x+1} t^{\rho}\,dt < \frac{(x+1)^{1/2}}{|\gamma|} \tag{9}$$

and also

$$\left|\frac{(x+1)^{\rho+1}-x^{\rho+1}}{\rho(\rho+1)}\right| < \frac{(x+1)^{3/2}+x^{3/2}}{\gamma^2} \tag{10}$$

We need now information about the zeros of $\zeta(s)$. Let $N(T)$ denote the number of ρ's for which $0<\gamma\leq T$ and $F(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}-\frac{7}{8}$. It is known that

$$|N(T)-F(T)|<0.137\log T+0.443\log\log T+4.350 \tag{11}$$

for all $T\geq 2$ (see Backlund [1]). From (11) we can infer the following two lemmas:

LEMMA 1. $N(T+1)-N(T)<1{,}04\log T$ for $T\geq 10^8$ (12)

Proof. $N(T+1)-N(T)<F'(T)+0{,}137(\log(T+1)+\log T)+$ $+0{,}443(\log\log(T+1)+\log\log T)+8{,}7<\frac{1}{2}\log\frac{(T+1)}{2\pi}+0{,}275\log T+0{,}9\log\log T+$ $+8{,}7<1{,}04\log T$.

LEMMA 2. $\sum_{0<\gamma\leq 10^8}\frac{1}{\gamma}<40$ (13)

Proof. The computation was performed using the explicit values of γ for $T<500$ and than adding the products between the number of roots in the intervals $[500\cdot 2^k,\ 500\cdot 2^{k+1}]$, $0\leq k\leq 17$, given by the relation (11), and $\frac{1}{500\,2^k}$ which is larger than the greatest component of the sum in this interval.

THEOREM 1. $|\psi(x)-x|<1{,}5x^{1/2}\log^2 x$ for $x\geq 10^8$ (14)

Proof. We increase the first sum in the formula (8) in the following way:

$$\left|\sum_{\rho}\frac{x^{\ +1}}{(\ +1)}\right| < 2\sum_{0<\gamma\leq 10^8}\left|\frac{x^{\rho+1}}{\rho(\rho+1)}\right| +2\sum_{10^8\leq\gamma\leq[x+2]}\left|\frac{x^{\rho+1}}{\rho(\rho+1)}\right| +$$

$$+2\sum_{[x+2]<\gamma}\frac{x^{\rho+1}}{\rho(\rho+1)} < 2,0002x^{1/2}\sum_{0<\gamma<10^8}\frac{1}{\gamma} +$$

$$+2,0002x^{1/2}\sum_{10^8\leq\gamma<[x+2]}\frac{1}{\gamma}+4,0004x^{3/2}\sum_{[x+2]\leq\gamma}\frac{1}{\gamma^2}$$

because $(x+1)^{1/2}<1,0001x^{1/2}$ and $(x+1)^{3/2}\ 1,0001x^{3/2}$ for $x\geq 10^8$.

Lemma 1 enable us to determine the values of the last two sums:

$$\sum_{10^8\leq\gamma<[x+2]}\frac{1}{\gamma}\leq\sum_{10^8\leq\gamma<[x+2]}\frac{1,04\log n}{n}\leq$$

$$1,04\int_{10^8-1}^{x+2}\frac{\log t}{t}dt=1,04\cdot\frac{1}{2}\log^2 t\Big|_{10^8-1}^{x+2}$$

$$\sum_{[x+2]\leq\gamma}\frac{1}{\gamma^{-2}}\leq\sum_{[x+2]\leq\gamma}\frac{1,04\log n}{n^2}\leq 1,04\int_{x+1}^{\infty}\frac{\log t}{t^2}dt$$

$$\leq 1,04\left[-\frac{\log t}{t}\right]_{x+1}^{\infty}+\frac{1}{x}\quad .$$

From (8), the last three relations and Lemma 2 follows that:

$$\psi(x)<x+80.0081x^{1/2}+1,04x^{1/2}\log^2 x+4,0004x^{1/2}\log x<$$

$$< x+1,5x^{1/2}\log^2 x \qquad \text{for} \quad x\geq 10^8.$$

An analogous proof for the lower bound enable us to conclude the proof of the theorem.

THEOREM 2. $\left|\theta(x)-x\right|<1,5x^{1/2}\log^2 x \quad \text{for } x\geq 3$

<u>Proof</u>. For $x\geq 10^8$ Theorem 2 is a consequence of (14) combined with relations $\theta(x)<\psi(x)$ for every x and $0,98x^{1/2}<\psi(x)-\theta(x)$ for $x\geq 121$ which is Theorem 14 from [3]. For $x\leq 10^8$ Theorem 2 is a poorer result than $x-2,05282x^{1/2}<\theta(x)<x$ for $0<x\leq 10^8$ computed in Theorem 18, [3].

§2. THE FUNCTIONS $\pi(x)$, $\sum_{p\leq x}\frac{1}{p}$ AND $\sum_{p\leq x}\frac{\log p}{p}$

Theorem 2 can be used to determine estimations for other functions related to prime numbers. Using the properties of Stieljes integral for the function f which later became precise we have:

$$\sum_{p\leq x} f(p) = \int_2^x \frac{f(t)}{\log t}\,d\theta(t) \tag{15}$$

and

$$\sum_{p\leq x} f(p) = \frac{f(x)\theta(x)}{\log x} - \int_2^x \theta(t)\left(\frac{f(t)}{\log t}\right)' dt \tag{16}$$

which follows after an integration by parts.

Taking $f(t)=1$, (16) gives:

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{(t)}{t\log^2 t}\,dt \tag{17}$$

Now from Theorem 2 and (17) we can obtaine inequalities for (x):

$$\pi(x) < \frac{x+1,5x^{1/2}\log^2 x}{\log x} + \int_2^x \frac{\log 2}{t\log^2 t}dt + \int_3^x \frac{t+1,5t^{1/2}\log^2 t}{t\log^2 t}dt$$

$$\text{li } x + 1,5x^{1/2}(2+\log x) \qquad \text{for } x \geq 3$$

and analogously for the lower bound.

It is usefull to remark for reference Theorem 16, [3] which is more precise for smaller values of x; and was computed without the assumption of Riemann hypothesis

$$\text{li } x - \text{li } x^{1/2} < \pi(x) \qquad \text{for } 11 \leq x \leq 10^8$$

$$\pi(x) < \text{li } x \qquad \text{for } 2 \leq x \leq 10^8 .$$

Ofen it is hard to work with approximations in terms of li x, so we'll transform (16) to obtain estimation for the other functions in the following way:

$$\sum_{p \leq x} f(p) = \frac{f(x)(\theta(x)-x)}{\log x} - \int_2^x (\theta(t)-t)\left(\frac{f(t)}{\log t}\right)' dt + \int_2^x \frac{f(t)}{\log t}dt + \frac{2f(2)}{\log 2}$$

which can be write also

$$\sum_{p\ x} f(p) = \frac{f(x)(\theta(x)-x)}{\log x} + \int_x^{\infty} (\theta(t)-t)\left(\frac{f(t)}{\log t}\right)' dt + \int_2^x \frac{f(t)}{\log t}dt + C_f \qquad (18)$$

where $C_f = \frac{2f(2)}{\log 2} - \int_2^x (\theta(t)-t)\left(\frac{f(t)}{\log t}\right)' dt$ is constant.

In [3] was computed the approximate values for $C_{1/x} = C_1$ and $C_{\log x/x} = C_2$ as we announced in the first part of the paper.

For $f(x) = \frac{1}{x}$ and $f(x) = \frac{\log x}{x}$ (18) gives:

$$\sum_{p\leq x}\frac{1}{p} = \log\log x+C_1+\frac{\theta(x)-x}{x\log x} - \int_x^{\infty}\frac{(\theta(t)-t)(1+\log t)}{t^2\log^2 t} \tag{19}$$

and respectively

$$\sum_{p\leq x}\frac{\log p}{p}=\log x+C_2+\frac{\theta(x)-x}{x} - \int_x^{\infty}\frac{\theta(t)-t}{t^2}dt \tag{20}$$

From these last two relations and Theorem 2 follows that

$$\left|\sum_{p\leq x}\frac{1}{p} - \log\log x-C_1\right| < \frac{9+4{,}5\log x}{\sqrt{x}} \qquad \text{for } x\geqslant 3 \tag{21}$$

and

$$\left|\sum_{p\leq x}\frac{\log p}{p} - \log x-C_2\right| < \frac{4{,}5\log^2 x+12\log x+24}{\sqrt{x}} \qquad \text{for } x\geqslant 3 \tag{22}$$

which can be combined with Theorems 20 and 21 from [3]:

$$\log\log x+C_1<\sum_{p\leq x}\frac{1}{p}<\log\log x + C_1+\frac{2}{x^{1/2}\log x} \qquad \text{for } 2\leq x\leq 10^8$$

$$\log x+C_2<\sum_{p\leq x}\frac{\log p}{p}<\log x + C_2+\frac{2{,}06123}{x^{1/2}} \qquad \text{for } 2\leq x\leq 10^8$$

to conclude the proof of (5) and (6).

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