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INTRODUCTION

This paper is concerned with the following question raised by Graham [2]:

It is true that $\max\limits_{1\leq i,j\leq n} \frac{a_i}{(a_i,a_j)} \geq n$ for every chain of integers $0 < a_1 < \ldots < a_n$?

In [10] it has been proved that this statement is true for sufficiently large n. There is given no effective bound n_o for n especially because in the proof the following result of Huxley ([4]) is used:

For every $\varepsilon > 0$ there exists n_ε such that for every $n > n_\varepsilon$ there exists a prime number between n and $n + n^{7/12+\varepsilon}$. Here n_ε is not given effectivelly. This difficulty can be overcome if we assume Riemann Hypothesis to be true. The aim of this paper is to obtain, under Riemann Hypothesis, an effective value for n_o. This value is $n_o = 10^{70}$.

1. REDUCTION TO SOME INEQUALITIES

We shall use the notation and the proofs from [10]. We denote: $p = p(n)$ the greatest prime number less than $2n$, $f(n) = 2n - p$, $\tau(i)$ = the number of divisors of i, $m^{\theta_m} = \max\limits_{1<i<m} \tau(i)$. To obtain an effective value for n_o it is sufficient to give a bound satisfying the inequalities from [10], Prop. 2. This six inequalities are:

(1) $n - 1 \leq \frac{p+1}{2} \cdot \sqrt{2}$ (p. 33)

(2) There exists $c_o > 0$ and $m_o \in N^*$ such that for any $m \geq m_o$ and any $K \leq m$, the interval $(K, K+m)$ contains at least $\frac{c_o m}{\log m}$ prime numbers (p. 33)

(3) $\frac{\sqrt{n}}{2} > m_o$ (p. 37)

(4) $2\log n < \frac{c_o\sqrt{n}}{4\log n}$ (p. 37)

(5) $n^{4\theta_n} \cdot 6f(n) \cdot n^{(-1/4+3\varepsilon)} \leq \frac{c_o f(n)}{8\log n}$ (p. 38)

In (5) the factor $6 \cdot f(n) \cdot n^{(-1/4+3\varepsilon)}$ from (p. 36 II) resulting from the inequalities (a), (b) and (c) has to be replaced by $\frac{6(f(n))^4}{n^2}$, which is obtained from (a), (b) and (c) by neglecting the inequality $f(n) < n^{7/12+\varepsilon}$. Thus we have:

(5') $\frac{6(f(n))^4 \cdot n^{4\theta_n}}{n^2} \leq \frac{c_o f(n)}{8\log n}$

(6) $\frac{c_1}{6n^{4\theta_n}\log n} \geq \frac{7}{n^{1/3}}$, where $c_1 = c_o/8$ (p. 39).

Neglecting the last inequality in ((**), p. 38) we may replace the factor $1/n^{1/3}$ in (6) by $f(n)/n$, obtaining thus:

(6') $\frac{c_1}{6n^{4\theta_n}\log n} > \frac{7f(n)}{n}$.

If we apply the results of the following section to the above inequalities we obtain an n_o of the order of magnitude of e^{10^6}. In order to obtain a better result, the proof of (Prop. 2, [10]) has to be improved, for obtaining more advantageous inequalities (special attention must be paid to the factors $n^{4\theta_n}$ in (5') and (6') which are large).

We shall show briefly how this can be done.

To improve (5') we treat simultaneously (I) and (II) (p. 36) by counting the number A of couples of the form

$$\begin{pmatrix} (p-\beta)d_1 \\ \beta d_1 \end{pmatrix}, \quad \begin{pmatrix} (p-\beta)d_2 \\ \beta d_2 \end{pmatrix}$$

where β varies in $\mathcal{J}$ and for fixed β, keeping fixed d_1 (such that, for example, its "$(p-\beta)$ - component" is maximum), and varying d_2 (i.e. varying $\frac{d_2}{d_1} = \frac{X_2Y_2}{X_1Y_1}$).

Applying (a), (b) and (c) (p. 36-37) to x_1, x_2 and y_1, y_2 we have:

$$d_o = \frac{p-\beta}{x_1x_2} < \frac{(f(n))^2}{n} \quad ; d_1 = \frac{\beta}{y_1y_2} < \frac{(f(n))^2}{n}$$

$$\delta_o = x_1 - x_2 < \frac{2f(n)}{n}; \quad |\delta_1| = |y_1 - y_2| < \frac{2f(n)}{n}$$

$$x_2 \in \left(\frac{\sqrt{n}}{\sqrt{d_o}} - \frac{f(n)}{\sqrt{nd_o}} - \delta_o, \frac{\sqrt{n}}{\sqrt{d_o}}\right) \text{ (the length of this interval being } < \frac{3f(n)}{\sqrt{n}}\text{)}.$$

The absolute value has been introduced because it is not known that the "β-component" of d_1 is maxim, too. Noticing that the system (x_1,x_2,y_1,y_2,β) is uniquelly determined by the system $(d_o, \delta_o, x_2, \delta_1, d_1)$ (p being fixed) we have:

$$A < 24\left(\frac{f(n)}{\sqrt{n}}\right)^7.$$

Thus:

$$\text{(5'')} \qquad 24\left(\frac{f(n)}{\sqrt{n}}\right)^7 \leq \frac{c_o f(n)}{8\log n}.$$

To improve (6') observe first that the system of statements (1)-(5) of (b) (p. 39) can be applied to an arbitrary quadruple $(y^i_1, y^i_2, \bar{y}^i_1, \bar{y}^i_2)$ $(i \in \overline{1,m})$ having the property $\{y^i_1, y^i_2, \bar{y}^i_1, \bar{y}^i_2\} \cap \{2,3,\dots,f(n)\} = \emptyset$. More exactly, (1), (2), (3) and (5) remain true, the fact that $m_o \geq 2$ being used only in (4). From (1), (2), (3) and (5) two possibilities result:

(1°) $\bar{y}^i_1 = \bar{y}^i_2 = 1$, $y^i_1 > 1$, $y^i_2 > 1$, and in that case (4) implies

$$\begin{cases} \alpha(p-\alpha)x = \beta_i(p-\beta_i)y^i \\ \bar{x} = y^i \end{cases}$$

thus β_i is uniquelly determined and hence i is uniquelly determined in $\{1,\dots,m\}$.

(2°) $y_1^i = y_2^i = 1$, $\bar{y}_1^i > 1$, $\bar{y}_2^i > 1$ and this implies analogously that i is uniquelly determined in $\{1,\dots,m\}$.

Thus, there are at most two indices $i \in \{1,\dots,m\}$ such that

$\{y_1^i, y_2^i, \bar{y}_1^i, \bar{y}_2^i\} \cap \{2,3,\dots,f(n)\} = \emptyset$.

Suppose (for working a choice) that for $m_1 \geq \frac{m-2}{4}$ indices i, we have $1 < y_1^i \leq f(n)$.

Hence, for $m_o \geq \frac{m}{4n^{\theta_n}}$ indices, we have the same value $1 < y_1 \leq f(n)$ for y_1^i.

Then, we can proceed as in (a) (p. 39) and we obtain:

$$\frac{c_1}{24n^{\theta_n}\log n} \geq \frac{7f(n)}{n} \tag{6''}$$

2. INEQUALITIES FOR ARITHMETIC FUNCTIONS UNDER RIEMANN HYPOTHESIS

The exponent "$7/12 + \varepsilon$" from Huxley's result can be replaced by "$11/20 + \varepsilon$" (cf. [6]) and using Riemann Hypothesis it can be replaced by "$\frac{1}{2} + \varepsilon$" (cf. [5]) (in fact in this case we have $|\pi(x) - \mathrm{li}\, x| = O(x^{\frac{1}{2}}\log x)$ [7]). But we need an inequality valid for all x, and this will be done in this section.

As usually, we note by (x) the logarithm of the product of all primes $\leq x$ and by (x) the logarithm of the least common multiple of positive integers $\leq x$.

We summarize the results of this section in the following:

PROPOSITION 1. Under Riemann Hypothesis the following estimations occur:

(a) $|\Psi(x) - X| < 1,493X^{\frac{1}{2}}\log^2 x$ for $x \geq 10^8$

(b) $|\theta(x) - X| < 1,5X^{\frac{1}{2}}\log^2 x$ for $x \geq 3$

(c) $\left|\pi(x) - \mathrm{li}\,x\right| < 1{,}5x^{\frac{1}{2}}(2 + \log x)$ for $x \geq 3$

(d) $f(n) < 4{,}5\sqrt{n}\log^2 n$ for $n \geq 10^{25}$

(e) For any $m \geq 10^9$ and any $K \leq m$, the interval $(K, K + m)$ contains at least $\frac{8m}{9\log m}$ prime numbers.

Proof. (a) We start from the basic formula ([5], p. 73)

$$(7)\qquad \psi_1(x) = \int_2^x \psi(u)du = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - x\frac{\zeta'(0)}{\zeta(0)} + \frac{\zeta'(-1)}{\zeta(0)} - \sum_{r=1}^{\infty} \frac{x^{1-2r}}{2r(2r-1)}$$

where $\rho = \frac{1}{2} + i\gamma$ denotes the zeros of the zeta function from the critic band. From (7) it follows (note that $\sum_\rho \frac{1}{|\rho|^a}$ is convergent for $a > 1$):

$$(8)\qquad \psi(x) < \psi_1(x+1) - \psi_1(x) = x - \sum_\rho \frac{(x+1)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)} - \sum_{r=1}^{\infty} \frac{x^{1-2r} - (x+1)^{1-2r}}{2r(2r-1)}.$$

It is known that $\frac{\zeta'(0)}{\zeta(0)} = \log 2\pi < 1{,}8379$ and the last sum is much smaller than the increasing we shall make, so that, our aim is now to give an upper bound for the first sum.

We have

$$(9)\qquad \left|\frac{(x+1)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)}\right| = \left|\frac{1}{\rho}\int_x^{x+1} t^\rho\, dt\right| < \frac{(x+1)^{\frac{1}{2}}}{|\gamma|}$$

and also

$$(10)\qquad \left|\frac{(x+1)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)}\right| \leq \frac{(x+1)^{3/2} + x^{3/2}}{\gamma^2}$$

We need now information about the zeros of $\zeta(s)$. Noting by $N(T)$ the number of ρ's for which $0 < \gamma < T$ and defining $F(T) = (T/2\pi)\log(2T/2\pi) - T/2\pi - 7/8$, then:

$$(11)\qquad \left|N(T) - F(T)\right| < 0{,}137\log T + 0{,}443\log\log T + 4{,}350$$

for all $T \geq 2$ (see [1]). From (11) the following two results can be inferred:

(12) $\quad N(T+1) - N(T) < 1{,}04 \log T \qquad \text{for } T \geq 10^8$

(13) $\quad \sum_{0<\gamma<10^8} 1/\gamma < 40$

(the computation for (13) was performed using the explicit values of γ for $T < 500$ and the inequality (11) for the intervals $[500 \cdot 2^k, 500 \cdot 2^{k+1}]$, $0 \leq k \leq 17$).

To increase the first sum in (8) we use (9) and (13) for $0 < \gamma < 10^8$, (9) and (12) for $10^8 \leq \gamma < [x+2]$, (10) and (12) for $[x+2] \leq \gamma$, obtaining thus:

$$\psi(x) < x + 80{,}008x^{\frac{1}{2}} + 1{,}04x^{\frac{1}{2}}\log^2 x + 4{,}0004x^{\frac{1}{2}}\log x < x + 1{,}493x^{\frac{1}{2}}\log^2 x$$

for $x \geq 10^8$, and analogously for the lower bound.

(b) For $x \geq 10^8$ we can combine (a) with the relations $\theta(x) < \psi(x) < \theta(x) + 1{,}4262x^{\frac{1}{2}}$ ([9], Theorem 13).

For $x \leq 10^8$, evenmore the inequalities $x - 2{,}05282x^{\frac{1}{2}} < \theta(x) < x$ are true (cf. [9] Theorem 18).

(c) We have:

$$\pi(x) = \theta(x)/\log x + \int_2^x (\theta(t)/t\log^2 t)dt$$

From (b) we obtain

$$(x) < \frac{x + 1{,}5x^{\frac{1}{2}}\log^2 x}{\log x} + \int_2^x (\log^2/t\log^2 t)dt + \int_3^x \frac{t + 1{,}5t^{\frac{1}{2}}\log^2 t}{t\log^2 t}dt < \mathrm{li}x + 1{,}5x^{\frac{1}{2}}(2 + \log x),$$

and analogously for the lower bound.

(d) We have $\pi(2n) = \pi(p)$; thus (c) implies:

$$f(n)/\log 2n < \int_p^{2n} dt/\log t = \mathrm{li}(2n) - \mathrm{li}(p) < 3(2n)^{\frac{1}{2}}(2 + \log 2n) < \frac{4.5\sqrt{n}\log^2 n}{\log 2n}$$

for $n \geq 10^{25}$.

(e) We have:

$$\pi(k+m) - \pi(k) > \int_k^{k+m} (dt/\log t) - 3(k+m)^{\frac{1}{2}}(2 + \log(k+m)) > (m/\log 2m) -$$

$$- 3\sqrt{2m}(2 + \log 2m) > 8m/9\log m \qquad \text{for } m \geq 10^9.$$

3. EFFECTIVE DETERMINATION OF n_o

We now return to the inequalities of § 1. We choose $c_o = 8/9$ and $m_o = 10^9$ as in Proposition 1 (e). The inequality (5") is implied (using Proposition 1 (d)) by the inequality:

$$\log n \geq 2\log(9 \cdot 24 \cdot (4,5)^6) + 26\log\log n$$

which is verified to be true for $n \geq 10^{70}$ (in fact, for $n \geq 10^{69,878}$). For such an n it is easy to verify (1), (3) and (4) (for (1) we use Proposition 1 (e)). The only problem now is to verify (6") where a factor n^{θ_n} is still present. It is known (see [8] or [3]) that $\theta_n = O(1/\log\log n)$, but if we apply this method and the estimation for $\pi(x)$ given in Proposition 1 (c) we get an n_o greater than 10^{70} (but not so much more greater). So, we prefer to verify more directly that (6") is true for all $n \geq 10^{70}$. The inequality (6") is implied by:

$$(14) \qquad \theta_n \leq \tfrac{1}{2} - \frac{3\log\log n + \log(7 \cdot 4,5 \cdot 9 \cdot 24)}{\log n}$$

To verify (14), it is sufficient to prove that $\theta_n \leq 0,35041864$ for $n \geq 10^{70}$ (this number being obtained by introducing the value $n = 10^{70}$ in the right side of (14)).

Suppose there exists $n \geq 10^{70}$ such that $\theta_n < 0,35041864$ and let $m \leq n$ be maxim such that $\tau(m) = n^{\theta_n}$. Then $m > 10^{67}$. For, if $m \leq 10^{67}$, then there exists $q < 1000$ such that $q \nmid m$, and thus $mq < n$ and $\tau(mq) > \tau(m) = n^{\theta_n}$ which is impossible. Therefore $m > 10^{67}$ and $\tau(m) \geq m^{0,35041868}$ (i.e. $\tau(m)^{2,85383} \geq m$).

Let $m = 2^{\alpha_2} \cdot 3^{\alpha_3} \cdot \ldots$ be the decomposition of m into primes. Then

$$(15) \qquad \prod_q \frac{(\alpha_q + 1)^{2,85373}}{q^{\alpha_q}} = \frac{\tau(m)^{2,85373}}{m} \geq 1$$

For q prime and $\alpha \in N^*$ let $g(\alpha, q) = \dfrac{(\alpha + 1)^{2,85373}}{q^{\alpha}}$ and $G(q) = \max\limits_{\alpha} g(\alpha, q)$

Then:

$$G(2) = g(3,2) < 6,53171$$

$$G(3) = g(2,3) < 2,55466$$

$$G(5) = g(1,5) < 1,44574$$

$$G(7) = g(1,7) < 1,03267$$

$$G(q) = g(1,q) < 7,22867/q < 1 \quad \text{for } q > 7$$

$M = G(2) \cdot G(3) \cdot G(5) < 24,91173$; $M \cdot G(q) < 180,0785/q$, thus if $\alpha_q \geq 1$ then $q \leq 179$.

Moreover $M \cdot G(11) \cdot G(13) \cdot G(17) \cdot G(19) \cdot G(23) < 1$, thus there are at most four factors $q > 7$ for which $\alpha_q \geq 1$ (say q_1, q_2, q_3 and q_4). Since the products $M \cdot g(2,11) \cdot G(13) \cdot G(17) \cdot G(19)$, $M \cdot g(2,11) \cdot g(2,13)$, $M \cdot g(3,11)$ are all less than 1, it follows that $\sum_{i=1}^{4} \alpha_{q_i} \leq 4$, hence

$$\prod_{i=1}^{4} q_i^{\alpha_{q_i}} < 179^4 < 10^{10}.$$

It follows that $2^{\alpha_2} \cdot 3^{\alpha_3} \cdot 5^{\alpha_5} \cdot 7^{\alpha_7} > 10^{57}$ and $g(\alpha_2, 2) \cdot g(\alpha_3, 3) \cdot g(\alpha_5, 5) \cdot$ $\cdot\, g(\alpha_7, 7) \geq 1$.

But $\quad G(2) \cdot G(3) \cdot G(5) \cdot g(4, 7) < 0,9925 < 1$

$$G(2) \cdot G(3) \cdot g(5,5) \cdot G(7) < 0,91636 < 1$$

$$G(2) \cdot g(8,3) \cdot G(5) \cdot G(7) < 0,7857 < 1$$

$$g(13, 2) \cdot G(3) \cdot G(5) \cdot G(7) < 0,8685 < 1.$$

Thus $\alpha_2 \leq 12$, $\alpha_3 \leq 7$, $\alpha_5 \leq 4$, $\alpha_7 \leq 3$ and this would imply that $2^{12} \cdot 3^7 \cdot 5^4 \cdot 7^3 >$ $> 10^{57}$, which is not the case.

We have obtained the following:

PROPOSITION 2. Under Riemann Hypothesis, Graham's Conjecture is true for any $n \geq 10^{70}$.

REMARK. Of course our aim was to prove that under Riemann Hypothesis Graham's statement is true for every n, but our attempt has failed. We do not want to increase the length of this note by showing how (by similar methods) n_0 can be still decreased. It suffices to say that we have made all the computations again with various values for n and we have made the exponent 70 in Proposition 2 less than 60 (but no less than 50).

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