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ON THE MILNOR FIBRATIONS OF WEIGHTED HOMOGENEOUS POLYNOMIALS

Alexandru Dimca

Let $\mathbf{w} = (w_o, \dots, w_n)$ be a set of integer positive weights and denote by S the polynomial ring $\mathbf{C}[x_o, \dots, x_n]$ graded by the conditions $\deg(x_i) = w_i$. For any graded object M we denote by M_k the homogeneous component of M of degree k. Let $f \in S_N$ be a weighted homogeneous polynomial of degree N.

The Milnor fibration of f is the locally trivial fibration $f: \mathbf{C}^{n+1} \setminus f^{-1}(0) \to \mathbf{C} \setminus \{0\}$, with tipical fiber $F = f^{-1}(1)$ and geometric monodromy $h: F \longrightarrow F$, $h(x) = (t^{w_o}x_o, \dots, t^{w_n}x_n)$ for $t = \exp(2\pi i/N)$. Since $h^N = 1$, it follows that the (complex) monodromy operator $h^* : H^{\cdot}(F) \longrightarrow H^{\cdot}(F)$ is diagonalizable and has eigenvalues in the group $G = \{t^a; a = 0, \dots, N-1\}$ of the N-roots of unity.

We denote by $H^{\cdot}(F)_a$ the eigenspace corresponding to the eigenvalue t^{-a}, for $a = 0, \dots, N-1$.

When f has an isolated singularity at the origin, the only nontrivial cohomology group $H^k(F)$ is for $k = n$ and the dimensions $\dim H^n(F)_a$ are known by the work of Brieskorn [2]. But as soon as f has a nonisolated singularity, it seems that even the Betti numbers $b_k(F)$ are known only in some special cases, see for instance [9], [14], [17], [20], [23].

The first main result of our paper is an explicit formula for the cohomology groups $H^k(F)$ and for the eigenspaces $H^k(F)_a$. Let $\Omega^{\cdot}$ be the complex of global algebraic differential forms on $\mathbf{C}^{n+1}$, graded by the convention $\deg(u dx_{i_1} \wedge \dots \wedge dx_{i_k}) = p + w_{i_1} + \dots + w_{i_k}$ for $u \in S_p$. We introduce a new differential on $\Omega^{\cdot}$, namely $D_f(\omega) = d\omega - (|\omega|/N) df \wedge \omega$, for $\omega \in \Omega^k_p$ with $|\omega| = p$ the degree of ω and d the usual exterior differential, similar to Dolgachev [8], p. 61.

For $a = 0, \dots, N-1$ we denote by $\Omega^{\cdot}_{(a)}$ the subcomplex in $\Omega^{\cdot}$ given by $\bigoplus_{s \geq 0} \Omega^{\cdot}_{-a+sN}$.

To a D_f-closed form $\omega \in \Omega^{k+1}$ we can associate the element $\delta(\omega) = [i^* \Delta(\omega)]$ in the de Rham cohomology group $H^k(F)$, where Δ is the contraction with the Euler vector field (as in [12], p.467 in the homogeneous case and [8], p. 43 in the weighted homogeneous case) and $i : F \longrightarrow \mathbf{C}^{n+1}$ denotes the inclusion.

Theorem A

The maps $\delta : H^{k+1}(\Omega^{\cdot}, D_f) \longrightarrow \widetilde{H}^k(F)$ and $\delta : H^{k+1}(\Omega^{\cdot}_{(a)}, D_f) \longrightarrow \widetilde{H}^k(F)_a$ are isomorphisms for any $k \geq 0$, $a = 0, \dots, N-1$, with $\widetilde{H}$ denoting reduced cohomology.

In fact, using homotheties inside $\Omega^{\cdot}$, it is easy to see that there is a similar result taking instead of D_f the differential $D'_f \omega = d\omega + df \wedge \omega$.

However, the apparently more complicated differential D_f is more natural (e.g. $(\Omega^{\cdot}, D_f)$ is a differential graded algebra while $(\Omega^{\cdot}, D'_f)$ is not!).

The proof of this Theorem depends on a comparision between spectral sequences naturally associated to the two sides of these equalities (1.8).

Our second main theme is that these spectral sequences can be used to perform explicit computations, in spite of the fact that the E_1-term has infinitely many nonzero entries and that degeneration at the E_2-term happens only in special cases (see Remark (3.11) below).

The eigenspaces $H^{\cdot}(F)_0$ are particularly interesting. If $\mathbf{P} = \mathbf{P}(\mathbf{w})$ denotes the weighted projective space Proj(S), V the hypersurface $f = 0$ in $\mathbf{P}$ and $U = \mathbf{P} \setminus V$ the complement, then there is a natural identification $H^{\cdot}(F)_0 = H^{\cdot}(U)$. We establish a relation between the filtration on $H^{\cdot}(F)_0$ induced by the spectral sequence mentioned above and the (mixed) Hodge filtration on $H^{\cdot}(U)$, having a substantial consequence for explicit computations.

Note that the Betti numbers $b_k(V)$ are completely determined by $b_k(U)$ and hence one can get by our method at least upper bounds for all $b_k(V)$ as well as the exact value of the top interesting one (i.e. $b_{n+m-1}(V)$ where $m = \dim f^{-1}(0)_{sing}$) in a finite number of steps (2.8).

Then we specialize to the usual projective space $\mathbf{P}^n$ and to the case when V has only isolated singularities and relate the global spectral sequences used until now to some local spectral sequences associated to each singularity.

In the end we compute two numerical examples to give a precise idea about how one has to proceed in practice. The example (4.3) may seem rather tedious, but as long as one misses a better approach, it is a nice illustration of our method.

1. Some spectral sequences

In this section we shall use many notations and results from Dolgachev [8] without explicit reference.

Let $\Delta : \Omega^k \longrightarrow \Omega^{k-1}$ denote the contraction with the Euler vector field $\sum w_i x_i \partial / \partial x_i$. For $k \geq 1$ we put $\overline{\Omega}^k = \ker(\Delta : \Omega^k \to \Omega^{k-1}) = \mathrm{im}(\Delta : \Omega^{k+1} \to \Omega^k)$ and let $\Omega^k_{\mathbf{P}}$ denote the associated sheaf on $\mathbf{P}$. One has also the twisted sheaves $\Omega^k_{\mathbf{P}}(s)$, for any $s \in \mathbb{Z}$.

Let $i : U \longrightarrow \mathbf{P}$ denote the inclusion and put $\Omega^k_U(s) = i^* \Omega^k_P(s)$.

The Milnor fiber F is an affine smooth variety and according to Grothendieck [13] one has $H^\cdot(F) = H^\cdot(\Gamma(F, \Omega^\cdot_F))$. Let $p : F \longrightarrow U$ denote the canonical projection and note that

(1.1) $$p_* \Omega^\cdot_F = \bigoplus_{a=0}^{N-1} \Omega^\cdot_U(-a)$$

If we let $A^\cdot_a = \Gamma(U, \Omega^\cdot_U(-a))$ and $A^\cdot = \bigoplus_{a=0}^{N-1} A^\cdot_a$, then we clearly have

(1.2) $$H^\cdot(F) = H^\cdot(A), \quad H^\cdot(F)_a = H^\cdot(A^\cdot_a)$$

There is a natural increasing filtration F_s on $A^\cdot_a$, related to the order of the pole a form in $A^\cdot_a$ has along V, namely

(1.3) $F_sA^j_a = 0$ for $s < 0$ and $F_sA^j_a = \{\omega/f^s;\ \omega \in \overline{\Omega}^j_{sN-a}\}$ for $s \geq 0$ similar to [12].

But for obvious technical reasons it is more convenient to consider the decreasing filtration

(1.4) $$F^sA^j_a = F_{j-s}A^j_a$$

The filtration F^s is compatible with d, exhaustive (i.e. $A^\cdot_a = \bigcup F^sA^\cdot_a$) and bounded above ($F^{n+1}A^\cdot = 0$). Here d denotes the differential of the complex $A^\cdot_a$ which is induced by the exterior differential d in $\Omega^\cdot_F$ via (1.1) and which is given explicitly by the formula

(1.5) $d(\omega/f^s) = d_f(\omega)\cdot f^{-s-1}$, where $d_f(\omega) = fd\omega - (|\omega|/N)df\wedge\omega$.

By the general theory of spectral sequences e.g. [16], p. 44 we get the next <u>geometric</u> spectral sequence.

(1.6) Proposition

<u>There is an E_1-spectral sequence $(E_r(f)_a, d_r)$ with</u>

$$E_1^{s,t} = H^{s+t}(F^sA^\cdot_a/F^{s+1}A^\cdot_a)$$

<u>and converging to the cohomology eigenspace $H^\cdot(F)_a$</u>.

Moreover one can sum these spectral sequences for $a = 0, \ldots, N-1$ and get a spectral sequence $(E_r(f), d_r)$ converging to $H^\cdot(F)$. And $(E_r(f)_0, d_r)$ and $(E_r(f), d_r)$ are in fact <u>spectral sequences of algebras converging to their limits as algebras.</u> Note that $H^\cdot(F)_0 \simeq H^\cdot(U)$, either using the fact that $U = F/G$, G acting on F via the geometric monodromy or the fact that $\Omega^\cdot_U$ is a resolution of $\mathbb{C}$ [22].

We pass now to the construction of some purely <u>algebraic</u> spectral sequences. Let (B_a, d', d'') be the double complex $B_a^{s,t} = \Omega^{s+t+1}_{tN-a}$, $d' = d$ and $d''(\omega) = -|\omega|/N df\wedge\omega$ for a homogeneous differential form ω. Note that the associated total complex $B^\cdot_a$, with $B_a^k = \bigoplus_{s+t=k} B_a^{s,t}$, $D = d' + d''$ is precisely the complex $(\Omega^{\cdot -1}_{(a)}, D_f)$.

Similarly $B^\cdot = \oplus B^\cdot_{(a)} = (\Omega^{\cdot -1}, D_f)$.

Consider the decreasing filtration F^p on $B_a^\cdot$ given by $F^p B_a^k = \bigoplus_{s \geq p} B_a^{s,k-s}$ and similarly on $B^\cdot$. Using the contraction operator Δ, we define the next complex morphisms, compatible with the filtrations:

$$\bar{\delta} : B_a^\cdot \longrightarrow A_a^\cdot \quad \text{and} \quad \bar{\delta} : B^\cdot \longrightarrow A^\cdot$$

$$\bar{\delta}(\omega) = \Delta(\omega) f^{-t} \quad \text{for} \quad \omega \in B_a^{s,t}.$$

Note that $B^\cdot$ and $A^\cdot$ are in fact differential graded algebras, but $\bar{\delta}$ is not compatible with the products.

(1.7) Proposition

There is an E_1-spectral sequence $('E_r(f)_a, d_r)$ with

$$E_1^{s,t} = H^{s+t}(F^s B_a^\cdot / F^{s+1} B_a^\cdot)$$

and converging to the cohomology $H^\cdot(B_a^\cdot)$. The operator $\bar{\delta}$ induces a morphism $\delta_r : ('E_r(f)_a, d_r) \longrightarrow (E_r(f)_a, d_r)$ of spectral sequences.

Moreover one can sum these spectral sequences $'E_r(f)_a$ and get a spectral sequence $('E_r(f), d_r)$ converging to $H^\cdot(B^\cdot)$ and a morphism $('E_r(f), d_r) \longrightarrow (E_r(f), d_r)$. The proof of these facts is standard e.g. [16], p. 49. Let $\widetilde{E}_r(f)_o$ (resp. $\widetilde{E}_r(f)$) denote the reduced spectral sequence associated to $E_r(f)_o$ (resp. $E_r(f)$) which is obtained by replacing the term at the origin $E_1^{o,o} = E_\infty^{o,o} = \mathbb{C}$ by zero. For $a \neq 0$, we put $\widetilde{E}_r(f)_a = E_r(f)_a$.

We clearly have natural morphisms $\widetilde{\delta}_r : 'E_r(f)_a \longrightarrow \widetilde{E}_r(f)_a$, $\widetilde{\delta}_r : 'E_r(f) \longrightarrow \widetilde{E}_r(f)$ induced by δ_r. We can state now a basic result.

(1. 8) Theorem

The morphisms $\widetilde{\delta}_r$ are isomorphisms for $r \geq 1$ and they induce isomorphisms $H^\cdot(B_a^\cdot) = \widetilde{H}^\cdot(F)_a$ and $H^\cdot(B^\cdot) = \widetilde{H}^\cdot(F)$.

Proof

Since $F^{n+1}B^\cdot = F^{n+1}A^\cdot = 0$, the filtrations F are strongly convergent [16], p. 50 and hence it is enough to show that $\widetilde{\delta}_1$ is an isomorphism. The vertical columns in $'E_1(f)$ correspond to certain homogeneous components in the Koszul complex $K^\cdot$.

(1.9) $\quad K^\cdot : 0 \rightarrow \Omega^o \xrightarrow{df} \Omega^1 \xrightarrow{df} \dots \xrightarrow{df} \Omega^{n+1} \longrightarrow 0$

of the partial derivatives $f_i = (\partial f)/(\partial x_i)$, $i = 0, \dots n$ in S. To describe the vertical columns in $E_1(f)$ is more subtle. Note that $fK^\cdot$ is a subcomplex in $K^\cdot$ and let $\overline{K}^\cdot$ denote the quotient complex $K^\cdot/fK^\cdot$. There is a map $\overline{\Delta} : \overline{K}^\cdot \longrightarrow \overline{K}^{\cdot -1}$ induced by Δ which is a complex morphism and hence $\widetilde{K}^\cdot = \ker \overline{\Delta}$ is a subcomplex in $\overline{K}^\cdot$.

Let $\widetilde{\Delta}$ denote the composition $K^\cdot \longrightarrow \overline{K}^\cdot \xrightarrow{\overline{\Delta}} \overline{K}^{\cdot -1}$.

Then the vertical lines in $E_1(f)$ correspond to certain homogeneous components in the cohomology groups $H^\cdot(\widetilde{K}^\cdot)$. The morphism $\widetilde{\delta_1}$ corresponds to $\widetilde{\Delta}^* : H^\cdot(K^\cdot) \longrightarrow H^\cdot(\widetilde{K}^{\cdot -1})$ and a well-defined inverse for $\widetilde{\Delta}^*$ is given by the map

(1.10) $\nabla : H^\cdot(\widetilde{K}^{\cdot -1}) \longrightarrow H^\cdot(K^\cdot)$, $\nabla[\Delta(\omega)] = [df \wedge \Delta(\omega)/(Nf)]$.

To check this, use that $df \wedge \omega = 0$ implies $0 = \Delta(df \wedge \omega) = Nf\omega - df \wedge \Delta(\omega)$.

(1.11) Example

Assume that f has an isolated singularity at the origin. Then $f_o, \dots, f_n$ form a regular sequence in S and we get $'E_1^{s,t}(f)_a = 0$ for $s + t \neq n$ and

$'E_1^{n-t,t}(f)_a \simeq H^{n+1}(K^\cdot)_{tN-a} \simeq Q(f)_{tN-a-w}$

where $Q(f) = S/(f_o, \dots, f_n)$, $w = w_o + \dots + w_n$. Moreover, the Poincaré series for $Q(f)$ (see for instance [7], p. 109) implies that $Q(f)_k = 0$ for $k > (n+1)N - 2w$. Hence in this case all our spectral sequences are finite and degenerate at the E_1-term (the degeneracy of the component $a = 0$ being equivalent to Griffiths' Theorem 4.3 in [12]). Note that one can have $'E_1^{-1,n+1}(f)_a \neq 0$. In general, one has the next result about the size of the spectral sequence $'E_r(f)$.

(1. 12) Proposition

$'E_r^{s,t}(f) = 0$ for any $r \geq 1$ and $s + t < n - m$, where $m = \dim f^{-1}(0)_{sing}$.

Proof

The result follows using the description of $'E_1^{s,t}(f)$ in terms of the Koszul complex and Greuel generalized version of the de Rham-Lemma, see [11], (1.7).

(1.13) Corollary

$\widetilde{H}^k(F) = 0$ for $k < n - m$.

This result is implied also by [15], but (1.12) will be used below in (2.8) in a crucial way.

(1.14) Remark (Sebastiani-Thom construction)

Let $w' = (w'_o, \dots, w'_{n'})$ be a new set of weights and $f' \in \mathbb{C}[y_o, \dots, y_{n'}]$ a homogeneous polynomial of degree N with respect to these weights.

Let $C_f^{\cdot}$ be the complex $(\Omega^{\cdot}, D_f)$ associated to f in the Introduction and let $C_{f'}^{\cdot}$, $C_{f+f'}^{\cdot}$ denote the similar complexes associated to f' and f + f'. Then it is easy to check that

$$(1.15) \qquad C_{f+f'}^{\cdot} = C_f^{\cdot} \otimes C_{f'}^{\cdot}$$

Combined with Theorem A, this gives

$$(1.16) \qquad H^k(F'') = \bigoplus_{s+t=k-1} H^s(F) \otimes H^t(F')$$

where F', F'' denote the Milnor fibers of f' and f + f' respectively, compare to [17]. Keeping trace of the homogeneous components in the formula (1.15) gives

$$H^{\cdot}(F'')_o = \bigoplus_c H^{\cdot}(F)_c \otimes H^{\cdot}(F')_{N-c}$$

with $c = 0, \dots, N-1$ and $H^{\cdot}(F')_N = H^{\cdot}(F')_o$.

When $f' = y_o^N$, (1.11) shows that $H^{\cdot}(F')_o = 0$ and $H^{\cdot}(F')_c = \langle y_o^{N-c-1} dy_o \rangle$, a 1-dimensional vector space, for $c = 1, \dots, N-1$.

It follows that at vector space level

$$H^{\cdot}(F'')_o \simeq \bigoplus_{c=1,N-1} H^{\cdot}(F)_c$$

This isomorphism can be identified already at the E_1-terms of the corresponding spectral sequences. In this sense $'E_r(f)$ is built from two pieces: $'E_r(f)_o$ and $'E_r(f + y_o^N)_o$. Hence in order to understand the behaviour of $'E_r(f)$ it is enough to concentrate on the piece $'E_r(f)_o$. And this is precisely what we do in the next two sections.

2. The relation with the Hodge filtration

Let us consider the decreasing filtration F^s on $H^{\cdot}(U)$ defined by the filtration F^s on $A_o^{\cdot}$, namely

$$(2.1) \qquad F^s H^{\cdot}(U) = \operatorname{im} \left\{ H^{\cdot}(F^s A_o^{\cdot}) \longrightarrow H^{\cdot}(A_o^{\cdot}) = H^{\cdot}(U) \right\}$$

On the other hand there is on $H^{\cdot}(U)$ the decreasing <u>Hodge filtration</u> F_H^s introduced by Deligne [5].

(2.2) Theorem

<u>One has</u> $F^s H^{\cdot}(U) \supset F_H^{s+1} H^{\cdot}(U)$ <u>for any s and</u> $F^o H^{\cdot}(U) = F_H^1 H^{\cdot}(U) =$
$= F_H^o H^{\cdot}(U) = H^{\cdot}(U)$.

Proof

Let $p : \mathbf{P}^n \longrightarrow \mathbf{P}$ be the projection presenting $\mathbf{P}$ as the quotient of $\mathbf{P}^n$ under the group $G(w)$, the product of cyclic groups of orders w_i.

Then $\widetilde{f} = p^*(f) = f(x_o^{w_o}, \dots, x_n^{w_n})$ is a homogeneous polynomial of degree N and let $\widetilde{U}$ be the complement of the hypersurface $\widetilde{f} = 0$ in $\mathbf{P}^n$.

Since $H^{\cdot}(U)$ can be identified to the fixed part in $H^{\cdot}(\widetilde{U})$ under the group $G(w)$ and since the monomorphism $p^* : H^{\cdot}(U) \longrightarrow H^{\cdot}(\widetilde{U})$ is clearly compatible with the filtrations F^s and F^s_H, it is enough to prove (2.2) for $\widetilde{U}$.

To simply the notation, we assume that $w = (1, \dots, 1)$ from the beginning. Then U is smooth and it is easier to describe the construction of the Hodge filtration [22].

Let $p : X \longrightarrow \mathbf{P}^n$ be a proper modification with X smooth, $D = p^{-1}(V)$ a divisor with normal crossings in X and $\overline{U} = X \setminus D$ isomorphic to U via p.

From this point on it is more suitable to work with holomorphic differential forms on our algebraic varieties. If $\Omega^{\cdot}_U$ is this holomorphic sheaves complex, ${}^a\Omega^{\cdot}_U$ the algebraic version of it and $i : U \longrightarrow \mathbf{P}^n$ is the inclusion, then one has inclusions $i_*({}^a\Omega^{\cdot}_U) \subset \Omega^{\cdot}_{P^n}(*V) \subset i_*\Omega^{\cdot}_U$, where $\Omega^{\cdot}_{P^n}(*V)$ denotes the sheaves of meromorphic differential forms on $\mathbf{P}^n$ with polar singularities along V. By Grothendieck [13], the inclusion $i_*({}^a\Omega^{\cdot}_U) \subset \Omega^{\cdot}_{P^n}(*V)$ induces isomorphisms at the hypercohomology groups. And the same is true for the inclusions $\Omega^{\cdot}_X(\log D) \subset \Omega^{\cdot}_X(*D) \subset j_*\Omega^{\cdot}_{\overline{U}}$ where $j : \overline{U} \longrightarrow X$ is the inclusion, $\Omega^{\cdot}_X(*D)$ is defined similarly to $\Omega^{\cdot}_{P^n}(*V)$ and $\Omega^{\cdot}_X(\log D)$ is the complex of holomorphic differential forms with logarithmic poles along D [22].

Recall that there is a trivial filtration $\sigma_{\geq}$ on any complex $K^{\cdot}$, by defining $\sigma_{\geq s}K^{\cdot}$ to be the subcomplex of $K^{\cdot}$ obtained by replacing the first s terms in $K^{\cdot}$ by 0. The Hodge filtration is given by

(2.3) $$F^s_H H^j(U) = \operatorname{im}\left\{ \mathbf{H}^j(\sigma_{\geq s}\Omega^{\cdot}_X(\log D)) \longrightarrow \mathbf{H}^j(\Omega^{\cdot}_X(\log D)) \right\}$$

via the identifications

$$\mathbf{H}^{\cdot}(\Omega^{\cdot}_X(\log D)) = \mathbf{H}^{\cdot}(j_*\Omega^{\cdot}_{\overline{U}}) = \mathbf{H}^{\cdot}(\Omega^{\cdot}_{\overline{U}}) = H^{\cdot}(\overline{U}) = H^{\cdot}(U).$$

The filtration F^s on the complex $A^{\cdot}_o$ is related to a filtration F^s on the complex $\Omega^{\cdot}_{P^n}(*V)$ defined in the following way: $F^s\Omega^j_{P^n}(*V)$ is the sheaf of meromorphic j-forms on $\mathbf{P}^n$ having poles of order at most $j - s$ along V for $j \geq s$ and $F^s\Omega^j_{P^n}(*V) = 0$ for $j < s$. Note that $F^s\Omega^j_{P^n}(*V) \simeq \Omega^j_{P^n}((j-s)N)$ for $j \geq s$. We get next a filtration on the complex $\Omega^{\cdot}_X(*D) \simeq p^*(\Omega^{\cdot}_{P^n}(*V))$ by defining $F^s\Omega^{\cdot}_X(*D) = p^*(F^s\Omega^{\cdot}_{P^n}(*V))$.

At stalks level, a germ $\omega \in \Omega^j_X(*D)_x$ belongs to $F^s\Omega^j_X(*D)_x$ if and only if

$p^*(u)^{j-s}\cdot\omega\in\Omega^j_{X,x}$, where $u = 0$ is a local equation for V around the point $y = p(x)$. If $v_1,\dots,v_n$ are local coordinates on X around x such that $v_1\dots v_k = 0$ is a local equation for D, then $p^*(u)$ vanishes on D and hence $p^*(u) = v_1^{a_1}\dots v_k^{a_k}w$ for some germ $w\in\mathcal{O}_{X,x}$ and integers $a_i\geq 1$.

Using the definitions, it follows that $\Omega^j_X(\log D)\subset F^s\Omega^j_X(*D)$ for $j > s$ and $j > 0$. And $\Omega^o_X(\log D) = \Omega^o_X\subset F^s\Omega^o_X(*D)$ for $s \not> 0$. We can state this as follows.

(2. 4) Lemma

(i) $\sigma_{\geq s+1}\Omega^{\cdot}_X(\log D)\subset F^s\Omega^{\cdot}_X(*D)$ for $s\geq 0$;

(ii) $\Omega^{\cdot}_X(\log D)\subset F^o\Omega^{\cdot}_X(*D)$.

We can hence write the next commutative diagram

$$\begin{array}{ccccc} H^{\cdot}(\sigma_{\geq s+1}\Omega^{\cdot}_X(\log D)) & \longrightarrow & H^{\cdot}(F^s\Omega^{\cdot}_X(*D)) & \xleftarrow[\sim]{p^*} & H^{\cdot}(F^s\Omega^{\cdot}_{P^n}(*V)) \\ \downarrow & & \downarrow & & \downarrow\alpha \\ H^{\cdot}(\Omega^{\cdot}_X(\log D)) & \longrightarrow & H^{\cdot}(\Omega^{\cdot}_X(*D)) & \xleftarrow[\sim]{p^*} & H^{\cdot}(\Omega^{\cdot}_{P^n}(*V)) \end{array}$$

Now $H^{\cdot}(\Omega^{\cdot}_{P^n}(*V)) = H^{\cdot}({}^a\Omega^{\cdot}_U) = H^{\cdot}(A^{\cdot}_o) = H^{\cdot}(U)$. To compute $H^{\cdot}(F^s\Omega^{\cdot}_{P^n}(*V))$ we use the E_2-spectral sequence $E_2^{p,q} = H^p(H^q(P^n, K^{\cdot}))$ converging to $H^{\cdot}(K^{\cdot})$, where $K^{\cdot} = F^s\Omega^{\cdot}_{P^n}(*V)$ and Bott's vanishing theorem [8].

It follows that $E_2^{p,o} = H^p(F^sA^{\cdot}_o)$, $E_2^{s,s} = H^s(P^n,\Omega^s_{P^n})$ and $E_2^{p,q} = 0$ in the other cases. This spectral sequence degenerates at E_2 since one can represent the generator of $E_2^{s,s}$ by a $\bar\partial$-harmonic form γ and hence $d\gamma = 0$. On the other hand $\alpha(\gamma) = 0$, since γ belongs to the kernel of the map $H^{2s}(P^n)\xrightarrow{i^*}H^{2s}(U)$. In fact this map is zero for $s > 0$. To see this, it is enough to show that $i^*(c) = 0$, where $c = c_1(\mathcal{O}(1))$ is the first Chern class of the line bundle $\mathcal{O}(1)$ (in cohomology with complex coefficients!). But $Ni^*(c) = 0$, since it corresponds to the Chern class of $\mathcal{O}(N)|_U$ and this line bundle has a section (induced by f) without any zeros.

It follows that $\mathrm{im}(\alpha) = F^sH^{\cdot}(U)$ and this gives the first part in (2.2).

The similar diagram associated to the inclusion (2.4. ii) gives $F^oH^{\cdot}(U) = F^o_HH^{\cdot}(U) = H^{\cdot}(U)$.

To see that $F^o_H = F^1_H$ we relate the mixed Hodge structure on $H^{\cdot}(U)$ to the mixed Hodge structure on $H^{\cdot}(V)$. Consider the exact sequence in cohomology with compact supports of the pair (P^n, V)

(2.5) $\quad \dots\to H^k_c(U)\longrightarrow H^k(P^n)\longrightarrow H^k(V)\longrightarrow H^{k+1}_c(U)\longrightarrow\dots$

This is an exact sequence of MHS (mixed Hodge structures) and it gives an isomorphism of MHS $H_c^{k+1}(U) \simeq H_o^k(V)$, the primitive cohomology of V [10]. Poincaré duality gives a natural identification (U is a $\mathbb{Q}$-homology manifold):

$$H^s(U) = \mathrm{Hom}(H_c^{2n-s}(U), H_c^{2n}(U))$$

Since $H_c^{2n}(U) \simeq H^{2n}(P^n) \simeq \mathbb{C}(-n)$, we get the following relations among mixed Hodge numbers

$$h^{p,q}(H^s(U)) = h^{n-p,\, n-q}(H_o^{2n-s-1}(V))$$

This gives $h^{o,q}(H^s(U)) = 0$ for any q and s, which shows that $F_H^o H^{\cdot}(U) = F_H^1 H^{\cdot}(U)$, ending the proof of (2.2).

(2.6) Remark. It is an interesting open question to decide whether one has equality $F^s H^{\cdot}(U) = F_H^{s+1} H^{\cdot}(U)$ for any s.

This is true when V is a quasismooth hypersurface (use (1.11) and the computation of $h^{p,q}(H_o^{\cdot}(V))$ given in [21]).

We also note that there is a similar inclusion $F^s H^{\cdot}(F) \supset F_H^{s+1} H^{\cdot}(F)$ among filtrations on the cohomology of the Milnor fiber F, but we will not prove this here. And using the formulas for $h^{p,q}(H^{\cdot}(F))$ given in [21] it follows that $F^s = F_H^{s+1}$ in the case when V is quasismooth (i.e. when f = 0 has an isolated singularity at the origin).

In spite of these important relations with the Hodge filtration, we resist the temptation to change our filtration F^s to $\tilde{F}^s = F^{s-1}$ (in order to have $\tilde{F}^s = F_H^s$) since we want $E_r(f)$ and $E_r(f)_o$ to remain spectral sequences of algebras.

Note also that the filtration F^s on $H^{\cdot}(F)$ is very close to the filtrations considered by Scherk and Steenbrink in the isolated singularity case in [19].

(2.7) Corollary

(i) $E_\infty^{s,t}(f)_o = 0$ for $s < 0$;

(ii) Any element in $H^k(U)$ can be represented by a differential k-form with a pole along V of order at most k.

We note that this can be regarded as an extension of Griffith's Theorem 4.2 in [12]. On the side of numerical computations of Betti numbers we get the following important consequence. Recall that $m = \dim f^{-1}(0)_{sing}$.

(2.8) Theorem

Let $b_j^o(V) = \dim H_o^j(V)$ denote the primitive Betti numbers of V. Then

(i) $b_j^o(V) = 0$ for $j < n - 1$ or $j > n - 1 + m$;

(ii) For $k \in [0,m]$ and $r \geq 1$ one has

$$b^o_{n-1+k}(V) = b_{n-k}(U) \leq \sum_{s=0}^{n-k-1} \dim E_r^{s,n-k-s}(f)_o$$

When $k = m$ and $r \geq n - m$ the above inequality is an equality.

Proof. Use (1.6), (1.7), (1.8), (1.12) and (2.7).

Using the end of Remark (1.14) or (2.6) we get similarly:

$E^{s,t}(f)_a = 0$ for $s < -1$ and $a = 1, \dots, N - 1$. There is also an analog of (2.8) for $\dim H^j(F)_a$ but we leave the details for the reader.

3. The isolated singularities case

In this section we restrict to the homogeneous case, $w_i = 1$ for $i = 0, \dots, n$.

Let $g : (\mathbb{C}^n, o) \longrightarrow (\mathbb{C}, o)$ be an analytic function germ and let $(Y,o) = (g^{-1}(0), o)$ be the hypersurface singularity defined by g. Let $\Omega^{\cdot}_{g,o}$ denote the localization of the stalk at the origin of the holomorphic de Rham complex $\Omega^{\cdot}_{\mathbb{C}^n}$ with respect to the multiplicative system $\{ g^s ; s \geq o \}$.

Choose $\varepsilon > 0$ small enough such that Y has a conic structure in the closed ball $B_\varepsilon = \{ y \in \mathbb{C}^n ; |y| \leq \varepsilon \}$ [4]. Let $S_\varepsilon = \partial B_\varepsilon$ and $K = S_\varepsilon \cap Y$ be the link of the singularity (Y,o). Then Thm. 2 in [13] implies the following.

(3.1) Proposition

$H^{\cdot}(S_\varepsilon \setminus K) \simeq H^{\cdot}(\Omega^{\cdot}_{g,o})$.

One can construct a filtration F^s on $\Omega^{\cdot}_{g,o}$ in analogy to (1.4), namely

$F^s \Omega^j_{g,o} = \{ \omega / g^{j-s} ; \omega \in \Omega^j_{\mathbb{C}^n,o} \}$ for $j \geq s$ and $F^s \Omega^j_{g,o} = 0$ for $j < s$.

(3. 2) Proposition

There is an E_1-spectral sequence of algebras $(E_r(g,o), d_r)$ with

$E_1^{s,t} = H^{s+t}(Gr^s_F \Omega^{\cdot}_{g,o})$

and converging to $H^{\cdot}(S_\varepsilon \setminus K)$ as an algebra.

Assume from now on that (Y,o) is an isolated singularity and let $L^{\cdot} = (\Omega^{\cdot}_{\mathbb{C}^n,o}, dg)$ denote the Koszul complex of the partial derivatives of g. In our case these derivatives form a regular sequence and hence $H^j(L^{\cdot}) = 0$ for $j < n$ and $H^n(L^{\cdot}) = M(g)$, the Milnor algebra of the singularity (Y,o), see for instance [7], p. 90. Let $I^{\cdot}$ denote the quotient complex $L^{\cdot}/gL^{\cdot}$. If $g : M(g) \longrightarrow M(g)$ denotes the multiplication by g, it follows that

$H^j(I^\cdot) = 0$ for $j < n-1$, $H^{n-1}(I^\cdot) = \ker(g)$ and $H^n(I^\cdot) = \operatorname{coker}(g) = T(g)$, the Tjurina algebra of (Y,o), see [7], p. 90.

There is the next analog of (1.8), computing $E_1(g,o)$ in terms of $H^\cdot(I^\cdot)$.

(3. 3) Lemma

The nonzero terms in $E_1(g,o)$ are the following.

(i) $E_1^{s,o}(g,o) = \Omega^s_{\mathbb{C}^n,o}$ for $s \in [o,n]$

(ii) $E_1^{s,1}(g,o) \simeq \Omega^s_Y$ for $s \in [0,n-3]$, there is an exact sequence
$0 \longrightarrow \Omega^{n-2}_Y \xrightarrow{u} E_1^{n-2,1}(g,o) \xrightarrow{v} \ker(g) \longrightarrow 0$ and $E_1^{n-1,1}(g,o) = \Omega^n_{\mathbb{C}^n,o}/g\,\Omega^n_{\mathbb{C}^n,o}$, where

$$\Omega^\cdot_Y = (\Omega^\cdot_{\mathbb{C}^n,o})/(g\,\Omega^\cdot_{\mathbb{C}^n,o} + dg \wedge \Omega^\cdot_{\mathbb{C}^n,o}).$$

(iii) $E_1^{n-t-1,t}(g,o) = \ker(g)$, $E_1^{n-t,t}(g,o) = T(g)$ for $t \geq 2$.

Proof. To get the more subtle point (ii), one uses the well-defined maps

$$u : \Omega^s_Y \longrightarrow E_1^{s,1}(g,o), \quad u(\alpha) = [(dg \wedge \alpha)/g]$$

$$v : E_1^{s,1}(g,o) \longrightarrow H^{s+2}(L^\cdot), v[\beta/g] = [(dg \wedge \beta)/g]$$

and note that $\operatorname{im}(v) \subset \ker(g)$ for $s = n-2$.

(3. 4) Corollary

The only (possibly) nonzero terms in $E_2(g,o)$ are $E_2^{o,o} = E_2^{o,1} = \mathbb{C}$ and $E_2^{n-1-t,t}$, $E_2^{n-t,t}$ for $t \geq 1$.

Proof. Use the exactness of the de Rham complexes [11]:

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^o_{\mathbb{C}^n,o} \longrightarrow \quad \dots \quad \longrightarrow \Omega^n_{\mathbb{C}^n,o} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^o_Y \longrightarrow \quad \dots \longrightarrow \Omega^{n-1}_Y.$$

We can also describe the differentials

$$d_1^t : \ker(g) = E_1^{n-1-t,t} \xrightarrow{d_1} E_1^{n-t,t} = T(g).$$

An $(n-1)$ form α induces an element in $\ker(g)$ if $dg \wedge \alpha = g\,\beta$ and then

(3.5) $\quad d_1^t[\alpha] = [d\alpha - t\beta]$

(3. 6) Example

Assume that (Y,o) is a weighted homogeneous singularity of type

$(w_1, \dots, w_n; N)$, i.e. (Y,o) is defined in suitable coordinates by a weighted homogeneous polynomial g of degree N with respect to the weights $\mathbf{w}$.

Then $M(g) = T(g) = \ker(g)$ and they are all graded $\mathbb{C}$-algebras.

Let $\alpha = \sum_{i=1,n} (-1)^{i+1} w_i x_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$ and note that $dg \wedge \alpha = N \cdot g \omega_n$, with $\omega_n = dx_1 \wedge \dots \wedge dx_n$. It follows that the class of α generates $\ker(g)$. For a monomial $x^a = x_1^{a_1} \dots x_n^{a_n}$ of degree $|x^a| = a_1 w_1 + \dots + a_n w_n$ one has by (3.5)

$$d_1^t(x^a \alpha) = [(w + |x^a| - tN) x^a \omega_n]$$

with $w = w_1 + \dots w_n$.

It follows that $\ker d_1^t \simeq \operatorname{coker} d_1^t \simeq M(g)_{tN-w}$. Hence the E_2-term $E_2(g,o)$ has finitely many nonzero entries and the spectral sequence $E_r(g,o)$ degenerates at E_2 (compare to (1.11)).

(3. 7) Example

For $g = x^2y^2 + x^5 + y^5$, one can find a detailed computation of $M(g)$ and $T(g)$ in [7], p. 96.

It follows easily that $[x^2y^2] \in \ker(g)$ and $d_1^t[x^2y^2] = 0$ for any t. Hence the E_2-term $E_2(g,o)$ has infinitely many nonzero entries and the spectral sequence $E_r(g,o)$ does not degenerate at E_2.

Now we come back to our global setting and let Z denote the singular locus of V.

Consider the restriction morphism

$$(3.8) \qquad \rho : \Omega^{\cdot}_{P^n}(*V) \longrightarrow \Omega^{\cdot}_{P^n}(*V)\big|_Z$$

and the associated morphisms

$$Gr_F^s \rho : Gr_F^s(\Omega^{\cdot}_{P^n}(*V)) \longrightarrow Gr_F^s(\Omega^{\cdot}_{P^n}(*V)\big|_Z)$$

A moment thought shows that $Gr_F^s \rho$ is a quasi-isomorphism for $s < 0$. A computation using an E_2-spectral sequence as in the proof of (2.2) shows that

$$H^{\cdot}(Gr_F^s(\Omega^{\cdot}_{P^n}(*V)) = H^{\cdot}(Gr_F^s A^{\cdot}_o)$$

Assume from now on that Z is a finite set, namely $Z = \{a_1, \dots, a_p\}$. Choose the coordinates on P^n such that $H : x_o = 0$ is transversal to V and $Z \subset P^n \setminus H \simeq \mathbb{C}^n$. We denote again by a_i the corresponding points in $\mathbb{C}^n$ and let $g(y) = f(1,y)$.

Then $\Omega^{\cdot}_{P^n}(*V)\big|_Z = \bigoplus_{j=1,p} \Omega^{\cdot}_{g,a_j}$, this identification being compatible with the

F filtrations. Thus we get

$$H^{\cdot}(Gr_F^s(\Omega^{\cdot}_{P^n}(*V)\big|_Z)) = \bigoplus_{j=1,p} H^{\cdot}(Gr_F^s(\Omega^{\cdot}_{g,a_j}))$$

We can restate these considerations in the next form.

(3. 9) Theorem

The restriction map ρ induces a morphism $\rho_r : E_r(f)_o \longrightarrow \bigoplus_{j=1,p} E_r(g,a_j)$ of spectral sequences such that at the E_1-level $\rho_1^{s,t}$ is an isomorphism for $s < 0$.

(3. 10) Corollary

If all the singularities of V are weighted homogeneous, then $E_2^{s,t}(f)_o = 0$ for $s < 0$.

In fact, the examples seem to suggest that in this case the spectral sequence $E_r(f)_o$ degenerates at E_2.

(3. 11) Remark

Let f be a homogeneous polynomial such that V has an isolated singularity of the type considered (3.7). Then $E_r(f)_o$ surely does not degenerate at E_2. Note that $f : (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ is concentrated in the terminology of [23], p. 206 and our spectral sequence $E_r(f)_o$ is a subobject in the huge spectral sequence considered in [23], p. 209. Hence in this case that spectral sequence does not degenerate at E_2 and this gives a negative answer to the question at the top of p. 209 in [23].

By Theorem (2.8) the interesting Betti numbers for V in this case are just $b_{n-1}(V)$, $b_n(V)$ and we can get $b_n(V)$ from $E_{n-1}(f)_o$.

But one has a simple formula for the Euler-Poincaré characteristic in this case [6]:

$$(3.12) \quad \chi(V) = \chi(V_o) + (-1)^n \sum_{i=1,p} \mu(V,a_i)$$

where V_o denotes a smooth hypersurface in $\mathbb{P}^n$ of degree N and $\mu(V,a_i) = \dim M(g,a_i)$ are the corresponding Milnor numbers.

In this way we get $b_{n-1}(V)$ knowing $b_n(V)$. We remark that there is a formula for $\chi(F)$ similar to (3.12) and which appears in the special case $n = 2$ as Theorem 6. A in [9].

(3. 13) Proposition

$$\chi(F) = 1 + (-1)^n[(N-1)^{n+1} - N\sum_{i=1,p} \mu(V,a_i)]$$

Proof.

If $\overline{F}$ denotes the closure of F in P^{n+1}, one has $\chi(F) = \chi(\overline{F}) \setminus \chi(V)$. One then use (3.12) and the remark that the singularities of $\overline{F}$ are just the N-fold suspensions of the singularities of V and hence

$$\mu(\overline{F}, (a_i : 0)) = (N - 1)\mu(V, a_i).$$

For concrete computations it is useful to use the following. Assume that $f_1, \dots, f_n$ is a regular sequence in S (this can be always achived by a linear change of coordinates!). Then the Koszul complex $K^\cdot$ (1.9) is quasi-isomorphic to the complex

$$(3.14) \quad 0 \longrightarrow Q_1(f) \xrightarrow{f_o} Q_1(f) \longrightarrow 0$$

where $Q_1(f) = S/(f_1, \dots, f_n)$ and f_o denotes multiplication by f_o. An indication of the dimensions of $H^{n+1}(K^\cdot)_k \simeq Q(f)_{k-n-1}$ and $H^n(K^\cdot)_k \simeq \ker(f_o)_{k-n}$ can be obtained from the exact sequence

$$(3.15) \quad 0 \longrightarrow \ker(f_o)_k \longrightarrow Q_1(f)_k \xrightarrow{f_o} Q_1(f)_{k+N-1} \longrightarrow Q(f)_{k+N-1} \longrightarrow 0$$

since the Poincaré series of $Q_1(f)$ is known.

Similar results hold in the non isolated singularities case.

4. Explicit computations

(4. 1) Example (Computation of $H^1(U)$)

Let $f = f_1^{a_1} \dots f_k^{a_k}$ be the decomposition of f in distinct irreducible factors. Then it is known that $b_1(U) = b^o_{2n-2}(V) = k - 1$ and it is easy to check that the closed forms

$$\omega_i = (df_i)/(f_i) - (N_i/N)(df)/(f)$$

where $N_i = \deg(f_i)$, $i = 1, \dots, k$ generate $H^1(U)$ with only one relation: $\sum a_i\omega_i = 0$. Compare to (2.7 ii).

(4. 2) Example (with isolated singularities for V).

Let $f = xyz(x + y + z)$, $n = 2$. Then V consists of 4 lines in general position in P^2 and its topology is simple to describe. However, the dimensions of the eigenspaces $H^\cdot(F)_a$ are more subtle invariants, e.g. one cannot derive them by the methods of Oka [17].

First we compute explicit bases for the homogeneous components of Q(f):

$$Q(f)_o = \langle 1\rangle,\ Q(f)_1 = \langle x,y,z\rangle,\ Q(f)_2 = \langle x^2,y^2,z^2,xy,yz,zx\rangle$$

$Q(f)_3 = \langle x^3, y^3, z^3, x^2y, y^2z, z^2x, xyz\rangle$ and

$Q(f)_k = \langle x^k, y^k, z^k, x^{k-1}y, y^{k-1}z, z^{k-1}x\rangle$ for $k \geq 4$.

Then we look for the elements in $H^2(K^\cdot)$ and define:

$\omega_{xy} = x(x + 2y + z)dy \wedge dz + y(2x + y + z)dx \wedge dz$

and ω_{yz}, ω_{zx} by cyclic symmetry.

Then $df \wedge \omega_{xy} = df \wedge \omega_{yz} = df \wedge \omega_{zx} = 0$ and these three forms give a basis for $H^2(K^\cdot)_4$.

The six forms $x\,\omega_{xy}$, $y\,\omega_{xy}$, $y\,\omega_{yz}$, $z\,\omega_{yz}$, $z\,\omega_{zx}$, $x\,\omega_{zx}$ generate $H^2(K^\cdot)_5$ with one relation among them (their sum is trivial).

And the six forms $x^k\omega_{xy}$, $y^k\omega_{xy}, \dots$ form a basis for $H^2(K^\cdot)_{k+4}$ for any $k \geq 2$.

It is now easy to compute $d_k^1 : H^2(K^\cdot)_k \longrightarrow H^3(K^\cdot)_k$ and the nontrivial kernels and cokernels are listed below together with $E_2^{o,o}(f)_o$:

$E_2^{o,o}(f)_o = E_2^{0,2}(f)_2 = E_2^{o,2}(f)_3 = E_2^{1,1}(f)_1 = \mathbb{C}$

$E_2^{o,1}(f)_o = E_2^{1,1}(f)_o = \mathbb{C}^3$

The computations also show that the spectral sequence degenerates at E_2 and hence we get the complete results. One can restate them by saying that the monodromy operator h^* acts trivially on $H^o(F) = \mathbb{C}$, $H^1(F) = \mathbb{C}^3$ and its action on $H^2(F) = \mathbb{C}^6$ has characteristic polynomial $(t-1)^3(t+1)(t^2+1)$.

(4. 3) Example (with nonisolated singularities for V)

An irreducible cubic surface in P^3 with nonisolated singularities is projectively equivalent to one of the next normal forms [3]

(i) a cone on the nodal cubic curve;

(ii) a cone on the cuspidal cubic curve;

(iii) $S' : x^2z + y^2t = 0$

(iv) $S : x^2z + y^3 + xyt = 0$

The topology of the surfaces (i)-(iii) can be described easier e.g. using [18], so that we concentrate on the last case: $f = x^2z + y^3 + xyt$. The homogeneous components of $Q(f)$ are given by

$Q(f)_o = \langle 1\rangle$ and $Q(f)_k = \langle z,t\rangle_k + \langle z^{k-1}x,\ z^{k-1}y,\ z^{k-2}y^2\rangle$ for $k \geq 1$,

where $\langle z,t\rangle_k$ denotes the vector space of all homogeneous polynomials in z,t of degree k. Hence $\dim Q(f)_k = k + 4$ for $k \geq 2$. Consider now the differential forms:

$$\omega_1 = x dx \wedge dy \wedge dz + y dx \wedge dy \wedge dt$$

$$\omega_2 = x dx \wedge dz \wedge dt + t dx \wedge dy \wedge dt - 3y dx \wedge dy \wedge dz$$

$$\omega_3 = t dx \wedge dy \wedge dz - 2z dx \wedge dy \wedge dt + x dy \wedge dz \wedge dt$$

Then some tedious computations show that:

$$H^3(K^\cdot)_4 = \langle \omega_1, \omega_2, \omega_3 \rangle$$

$$H^3(K^\cdot)_5 = \langle z,t \rangle_1 \omega_2 + \langle z,t \rangle_1 \omega_3 + \langle x\omega_2, y\omega_2, y\omega_3 \rangle$$

$$H^3(K^\cdot)_{k+4} = \langle z,t \rangle_k \omega_2 + \langle z,t \rangle_k \omega_3 + \langle z^{k-1}x, z^{k-1}y, z^{k-2}y^2 \rangle \omega_3$$

for $k \geq 2$. This last vector space has dimension $2k + 5$.
And similarly one gets $H^2(K^\cdot)_{k+4} = \langle z,t \rangle_k \omega$, with $\omega = (6yz - t^2) dx \wedge dy - xt dx \wedge dz - - yt dx \wedge dt - 3xy dy \wedge dz - 3y^2 dy \wedge dt$.

After these complicated formulas it comes as a surprise that the spectral sequence $E_r(f)$ degenerates at E_2 and the only nonzero terms are $E_\infty^{0,0}(f)_0 = E_\infty^{0,2}(f)_1 = E_\infty^{0,2}(f)_2 = \mathbb{C}$.
It follows that $H^\cdot(S) \simeq H^\cdot(\mathbb{P}^2)$ and hence S has the same rational homotopy type as $\mathbb{P}^2$, according to Berceanu [1], who has proved that a projective complete intersection (with arbitrary singularities) is an intrinsically formal space.

Concerning the Milnor fiber one has $H^0(F) = \mathbb{C}$ with trivial action of h^*, $H^2(F) = \mathbb{C}^2$ with the characteristic polynomial of h^* equal to $t^2 + t + 1$ and $H^1(F) = H^3(F) = 0$.

Note that via [17], one can use (4.2) and (4.3) to compute quickly several other examples.

Finally we express our hope that the computers might play an important role in doing concrete computations with the method presented in this paper.

REFERENCES

1. **Berceanu, B.**: Formal algebraic varieties, unpublished manuscript.
2. **Brieskorn, E.**: Die Monodromie der Isolierten Singularitaten von Hyperflachen. Manuscr. Math. 2, 103-161 (1970).
3. **Bruce, J.W., Wall, C.T.C.**: On the classification of cubic surfaces, J. London Math. Soc. (2) 19, 245-256 (1979).
4. **Burghelea, D., Verona, A.**: Local homological properties of analytic sets. Manuscr. Math. 7, 55-66, (1972).
5. **Deligne, P.**: Theorie de Hodge II, III. Publ. Math. IHES 40, 5-58 (1971) and 44, 5-77 (1974).
6. **Dimca, A.**: On the homology and cohomology of complete intersections with isolated singularities. Compositio Math. 58, 321-339 (1986).
7. **Dimca, A.**: Topics on Real and Complex Singularities, Braunschweig-Wiesbaden: Vieweg 1987.
8. **Dolgachev, I.**: Weighted projective varieties. In: Carrell, J.B. (ed) Group Actions and Vector Fields, Proceedings 1981. (Lect. Notes Math., vol. 956, pp. 34-71) Berlin Heidelberg New York: Springer 1982
9. **Esnault, H.**: Fibre de Milnor d'un cone sur une courbe plane singuliere, Invent. Math. 68, 477-496 (1982).
10. **Fujiki, A.**: Duality of mixed Hodge structures of algebraic varieties, Publ. RIMS Kyoto Univ. 16, 635-667 (1980).
11. **Greuel, G.-M.**: Der Gauss-Manin-Zusammenhang isolierter Singularitaten von vollstandigen Durchschnitten. Math. Ann. 214, 235-266 (1975).
12. **Griffiths, P.**: On the periods of certain rational integrals I, II, Ann. Math. 90, 460-541 (1969).
13. **Grothendieck, A.**: On the de Rham cohomology of algebraic varieties, Publ. Math. IHES 29, 351-358, (1966).
14. **Hamm, H.**: Ein Beispiel zur Berechnung der Picard-Lefschetz-Monodromie fur nichtisolierte Hyperflachensingularitaten Math. Ann. 214, 221-234 (1975).
15. **Kato, M., Matsumoto, Y.**: On the connectivity of the Milnor fiber of a holomorphic function at a critical point. In: Manifolds, Proceedings Tokyo 1973, pp. 131-136, Univ. of Tokyo Press 1975.
16. **McClearly, J.**: User's Guide to Spectral Sequences, Publish or Perish: 1985.
17. **Oka, M.**: On the homotopy types of hypersurfaces defined by weighted homogeneous polynomials. Topology 12, 19-32 (1973).
18. **Oka, M.**: On the cohomology structure of projective varieties. In: Manifolds, Proceedings Tokyo 1973, pp. 137-143, Univ. of Tokyo Press 1975.
19. **Scherk, J., Steenbrink, J.H.M.**: On the mixed Hodge structure on the cohomology of the Milnor fiber. Math. Ann. 271, 641-665 (1985).
20. **Siersma, D.**: Quasihomogeneous singularities with transversal type A_1, preprint 452, Utrecht Univ. (1987).
21. **Steenbrink, J.H.M.**: Intersection form for quasihomogeneous singularities. Compositio Math. 34, 211-223 (1977).

22. **Steenbrink, J.H.M.**: Mixed Hodge Structures and Singularities (book to appear).

23. **van Straten, D.**: On the Betti numbers of the Milnor fiber of a certain class of hypersurface singularities. In: Greuel, G.-M., Trautmann, G. (eds) Singularities, Representations of Algebras and Vector Bundles, Proceedings, Lambrecht 1985 (Lect. Notes Math. Vol. 1273, pp. 203-220) Berlin Heidelberg New York: Springer 1987.

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