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SPECTRAL CAPACITIES IN QUOTIENT FRÉCHET SPACES

Florian-Horia Vasilescu

Dedicated to the memory of Constantin Apostol

The definition of a spectral capacity has been proved to be a fruitful one in the theory of spectral decompositions of linear operators. Most of its standard properties still hold in the context of quotient Fréchet spaces.

1. INTRODUCTION

One of the most important concepts introduced by C. Apostol in the theory of spectral decompositions of linear operators is that of *spectral capacity* ([2], Definition 2.1). He defined it as a map E from the family of all closed subsets of the complex plane $\mathbb{C}$, with values closed linear subspaces of a given Banach space X, satisfying the following conditions:

(i) $E(\emptyset) = \{0\}$, $E(\mathbb{C}) = X$;

(1.1) (ii) $E(\bigcap_{m=1}^{\infty} F_m) = \bigcap_{m=1}^{\infty} E(F_m)$ for every sequence of closed subsets $\{F_m\}_{m=1}^{\infty}$ of $\mathbb{C}$;

(iii) $X = E(\bar{G}_1) + \dots + E(\bar{G}_n)$, where $\{G_j\}_{j=1}^{n}$ is an arbitrary finite open cover of $\mathbb{C}$.

C. Apostol showed, in particular, that every decomposable operator T on X (in the sense of C. Foiaş [8]) has a spectral capacity with the property

(1.2) $TE(F) \subset E(F)$ and $\sigma(T,E(F)) \subset F$ for all closed sets $F \subset \mathbb{C}$.

(We denote here by $\sigma(T,Y)$ the *spectrum* of T when acting on Y, where Y is a subspace of X invariant under T.)

This concept of Apostol's has been adapted to various situations, assuming changes of both the domain of definition and the range, assigned to one or several operators; nevertheless, all these versions essentially preserved the requirements (1.1) and (1.2) (see [1], [10], [16], [17], [22], etc.).

Let X be a Fréchet space (in this work all Fréchet spaces are assumed to be locally convex) and let Lat (X) denote the family of all *Fréchet subspaces* ([21]) of X (i.e. those linear subspaces Y of X that have a Fréchet space structure of their own which makes the inclusion $Y \subset X$ continuous). As the notation suggests, $\mathrm{Lat}(X)$ is a lattice with respect to the sum and intersection of subspaces (see, for instance, [18], Lemma 2.1). A *quotient Fréchet space* ([21]) is a linear space of the form X/Y, where X is a Fréchet space and $Y \in \mathrm{Lat}(X)$.

Let X_1/Y_1, X_2/Y_2 be quotient Fréchet spaces and let $T : X_1/Y_1 \to X_2/Y_2$ be a linear map. We define the set

$$G_o(T) = \{(x_1,x_2) \in X_1 \times X_2;\ x_2 \in T(x_1 + Y_1)\},$$

which is obviously a linear space. We say that T is a *linear operator* (or simply *operator*) if $G_o(T) \in \mathrm{Lat}(X_1 \times X_2)$. (Here we adopt the terminology from [19]. A linear operator is called in [18] a *morphism* in the category of quotient Fréchet spaces; it is equivalent to the homonymic concept introduced in [21] in a different manner.) The family of all linear operators from X_1/Y_1 into X_2/Y_2 is a linear space under the usual operations ([18]), denoted by $L(X_1/Y_1, X_2/Y_2)$. When $X_1/Y_1 = X/Y = X_2/Y_2$, we set $L(X/Y) = L(X/Y, X/Y)$, which is, in this case, an algebra ([18]). A *strict operator* $T : X_1/Y_1 \to X_2/Y_2$ is a linear map which is induced by a linear and continuous operator $T_o : X_1 \to X_2$ satisfying $T_o(Y_1) \subset Y_2$; in other words $T(x + Y_1) = T_o x + Y_2$ for each $x \in X_1$. A strict operator is a linear operator ([18]) (Strict operators are called *strict morphisms* in [18] or [21]).

Let X/Y be a fixed quotient Fréchet space. A linear manifold $D = D_o/Y \subset X/Y$ will be called a *(quotient Fréchet) subspace* of X/Y if $D_o \in \mathrm{Lat}(X)$. The family of all subspaces of X/Y, which is easily seen to be a lattice with respect to the sum and intersection of subspaces, will be also denoted by $\mathrm{Lat}(X/Y)$. Direct and inverse images of subspaces via linear operators are subspaces too ([22]; see also [18], Lemma 2.1).

Let $P(X/Y)$ be the family of those linear operators T that are defined on subspaces $D = D(T) \in \mathrm{Lat}(X/Y)$, with values in X/Y. Then $D(T)$, which is called the *domain of definition* of T, has the form $D(T) = D_o(T)/Y$, with $D_o(T) \in \mathrm{Lat}(X)$. This class of operators, which has been studied in [22], is a natural extension of the family of closed operators (and even of the larger family of those linear maps between two Fréchet spaces, whose graph is a Fréchet subspace, which originates in [4]).

If $T \in P(X/T)$, we may also define its iterates. Namely, let $D(T^2) = T^{-1}(D(T))$. Then $D(T^2) \in \mathrm{Lat}(X/Y)$ and $T^2 : D(T^2) \to X/Y$, given by $T^2\xi = T(T\xi)$ $(\xi \in D(T^2))$, is a member of $P(X/Y)$. In general, if T^n has been defined $(n \geq 1)$, we set $D(T^{n+1}) = T^{-1}(D(T^n)) \subset D(T)$. Then $D(T^{n+1}) \in \mathrm{Lat}(X/Y)$ and $T^{n+1}\xi = T(T^n\xi)$ $(\xi \in D(T^{n+1}))$. Note also that $D(T^{n+1}) \in \mathrm{Lat}(D(T^n))$.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $U \subset \mathbb{C}_\infty$ be open. We denote by

$O(U,X)$ the Fréchet space of all holomorphic X-valued functions on U. Let ${}_0O(U,X)$ be equal to $O(U,X)$ if $\infty \notin U$ and equal to $\{f \in O(U,X);\ f(\infty) = 0\}$ if $\infty \in U$. It is known that the assignment $U \to O(U,X)/O(U,Y)$ ($U \subset \mathbb{C}_\infty$ open) is an analytic sheaf $O_{X/Y}$ on $\mathbb{C}_\infty$ whose space of global sections $\Gamma(U,O_{X/Y})$ on U is given by

$$\Gamma(U,O_{X/Y}) = O(U,X)/O(U,Y).$$

The space $\Gamma(U,O_{X/Y})$ will be denoted by $O(U,X/Y)$. The assignment $U \to {}_0O(U,X)/{}_0O(U,Y)$ is a subsheaf ${}_0O_{X/Y}$ of $O_{X/Y}$ whose space of global sections on U equals ${}_0O(U,X)/{}_0O(U,Y)$ and will be denoted by ${}_0O(U,X/Y)$. Note that both X/Y and ${}_0O(U,X/Y)$ are (isomorphic to) subspaces of $O(U,X/Y)$ (see [18], Section 3, for some details).

Let $T \in P(X/Y)$ and let $U \subset \mathbb{C}_\infty$ be open. Then T induces a linear operator

$$T_U : O(U,D(T)) \to O(U,X/Y)$$

which extends T and maps the subspace ${}_0O(U,D(T))$ into ${}_0O(U,X/Y)$ (see [18], Section 2). In other words, $T_U \in P(O(U,X/Y))$ and $D(T_U) = O(U,D(T))$.

Let ζ be the coordinate function on $\mathbb{C}$. Then ζ induces by multiplication a linear operator

$$\zeta_U : {}_0O(U,D(T)) \to O(U,D(T)).$$

Therefore we have a linear operator

$$\zeta_U - T_U : {}_0O(U,D(T)) \to O(U,X/Y) \tag{1.3}$$

for every open $U \subset \mathbb{C}_\infty$. (It is, in fact, a sheaf morphism.)

The *resolvent set* $\rho(T,X/Y)$ of $T \in P(X/Y)$ is the largest open set $V \subset \mathbb{C}_\infty$ such that the linear operator (1.3) is bijective for every open $U \subset V$. The complement $\sigma(T,X/Y)$ of $\rho(T,X/Y)$ in $\mathbb{C}_\infty$ (which is a nonempty closed set) is called the *spectrum* of T. (These concepts have been defined in [18] for $T \in L(X/Y)$ and extended in [22] for $T \in P(X/Y)$.)

A subspace $Z \in \mathrm{Lat}(X/Y)$ is said to be *invariant* under $T \in P(X/Y)$ if $T(Z \cap D(T)) \subset Z$. We denote by $T|Z$ the linear map $T : Z \cap D(T) \to Z$ and call it the *restriction* of T to Z. It is easily seen that $T|Z \in P(Z)$. The family of all invariant subspaces of T will be designated by Inv(T). The spectrum of the operator $T|Z$ will be denoted by $\sigma(T,Z)$.

Let $T \in P(X/Y)$ and let $Z = Z_0/Y \in \mathrm{Inv}(T)$. Then T induces a linear operator $\hat{T} \in P(X/Z_0)$ with $D(\hat{T}) = (D_0(T) + Z_0)/Z_0$, given by $\hat{T}(x + Z_0) = y + Z_0$ ($(x,y) \in G_0(T)$).

From now on by *spectral capacity* we mean a map E defined on the closed subsets of $\mathbb{C}_\infty$, with values in Lat(X/Y), such that (1.1) is fulfilled (with $\mathbb{C}$ replaced by $\mathbb{C}_\infty$ and X by X/Y). A linear operator $T \in P(X/Y)$ is said to be *decomposable* if there exists a spectral capacity E with values in Inv(T) such that $\sigma(T,E(F)) \subset F$ for all closed $F \subset \mathbb{C}_\infty$ (which is essentially (1.2)). Such a spectral capacity is said to be *attached* to T.

This concept of decomposable operator extends the homonymic one due to C. Foiaș (see [8] or [3]), via the characterization from [9] (see also [16], [17], [22] for other extensions). Spectral capacities of this type and decomposable operators from $L(X/Y)$ have been studied in [22]. Unlike [22], we consider in this work "unbounded" decomposable operators, that is, operators from $P(X/Y)$. Our main concern is to recapture, in the present setting, some of the properties of (unbounded) decomposable operators in Fréchet spaces (see [16], Chapter IV). At the same time, we try to prove that the framework of quotient Fréchet spaces allows the development of a sufficiently sophisticated theory of spectral decompositions, in which the contributions of C. Apostol play a central rôle.

2. THE SPECTRUM OF A CLOSED OPERATOR

In this section we shall present a characterization of the resolvent set of a closed operator in a Fréchet space (and therefore of its spectrum) in terms of spaces of holomorphic vector-valued functions (see [18], Proposition 1.2 for continuous operators). Although elementary, it provides, in our opinion, the necessary explanation for the definition of the spectrum of a linear operator in a quotient Fréchet space, as given in the Introduction.

Let X be a fixed Fréchet space (which can be regarded as the quotient Fréchet space $X/\{0\}$), and let $C(X)$ be the family of all closed linear operators, defined on linear subspaces of X, with values in X. It is known ([18], Lemma 2.5) that $L(X)(= L(X/\{0\})$ is precisely the algebra of all linear and continuous operators on X. If $T \in C(X)$, the algebraic isomorphism between D(T) and G(T) (i.e. the graph of T) shows that $D(T) \in \mathrm{Lat}(X)$. Moreover, the operator $T : D(T) \to X$ becomes continuous when D(T) is endowed with this topology.

Following [20] (see also [16]), a point $z_0 \in \mathbb{C}_\infty$ is said to be *regular* for $T \in C(X)$ if z_0 has a neighbourhood V_0 in $\mathbb{C}_\infty$ such that

(1) $(z - T)^{-1} \in L(X)$ for every $z \in V_0 \cap \mathbb{C}$;

(2) the set $\{(z - T)^{-1}x;\ z \in V_0 \cap \mathbb{C}\}$ is bounded in X for each $x \in X$.

The set of all regular points for T, which is obviously an open set in $\mathbb{C}_\infty$, will be denoted by $\rho_W(T,X)$. We shall prove in this section that $\rho_W(T,X) = \rho(T,X)$, where the latter has been defined in the Introduction.

2.1. LEMMA. *Let* $T \in C(X)$ *and let* $z_0 \in \rho_W(T,X) \cap \mathbb{C}$. *Then there exists an open set* $V \subset \mathbb{C}$, $z_0 \in V$, *such that the operator* $\zeta - T : O(V,D(T)) \to O(V,X)$ *is bijective.*

PROOF. Let $V \subset \mathbb{C}$ be an open neighbourhood of z_0 such that $(z - T)^{-1} \in L(X)$ for every $z \in V$ and the set $\{(z - T)^{-1}x;\ z \in V\}$ is bounded in X for every $x \in X$. Then the operator $\zeta - T$ is obviously injective on $O(V,D(T))$. Let us prove that it is surjective too.

Let $g \in O(V,X)$ and set $f(z) = (z - T)^{-1}g(z)$ for $z \in V$. We first show that the function $f : V \to D(T)$ is continuous. For, let $w_0 \in V$ be fixed and let $W \subset V$ be a compact neighbourhood of w_0. It follows from the uniform boundedness principle that the family $\{(z - T)^{-1};\ z \in V\}$ is equally continuous. In particular, the set

(2.1) $$\{(w - T)^{-1}(w_0 - T)^{-1}g(w);\ w \in W\}$$

is bounded in X. Since the set

$$\{T(w - T)^{-1}(w_0 - T)^{-1}g(W);\ w \in W\} \subset$$
$$\subset W \cdot \{(w - T)^{-1}(w_0 - T)^{-1}g(w);\ w \in W\} - \{(w_0 - T)^{-1}g(w);\ w \in W\}$$

is also bounded in X, it results that (2.1) is actually bounded in D(T). Therefore

(2.2) $$\lim_{w \to w_0} (w_0 - w)(w - T)^{-1}(w_0 - T)^{-1}g(w) = 0$$

in D(T). Using the continuity of g, we infer that

$$\lim_{w \to w_0} T(w_0 - T)^{-1}(g(w) - g(w_0)) =$$
$$= \lim_{w \to w_0} w_0(w_0 - T)^{-1}(g(w) - g(w_0)) - \lim_{w \to w_0} (g(w) - g(w_0)) = 0.$$

Therefore

(2.3) $$\lim_{w \to w_0} (w_0 - T)^{-1}(g(w) - g(w_0)) = 0$$

in D(T). From (2.2) and (2.3) we deduce that

$$\lim_{w \to w_0} (f(w) - f(w_0)) = \lim_{w \to w_0} (w_0 - w)(w - T)^{-1}(w_0 - T)^{-1}g(w) +$$
$$+ \lim_{w \to w_0} (w_0 - T)^{-1}(g(w) - g(w_0)) = 0$$

in D(T), which proves the continuity of the function f at w_0. Since $w_0 \in V$ is arbitrary, $f : V \to D(T)$ is continuous.

Note also that

$$\lim_{w\to w_o} ((w - w_o)^{-1}(w_o - T)^{-1}(g(w) - g(w_o)) - (w_o - T)^{-1}f(w)) =$$

$$= (w_o - T)^{-1}g'(w_o) - (w_o - T)^{-1}f(w_o)$$

in X, and that

$$\lim_{w\to w_o} T((w - w_o)^{-1}(w_o - T)^{-1}(g(w) - g(w_o)) - (w_o - T)^{-1}f(w)) =$$

$$= w_o(w_o - T)^{-1}g'(w_o) - g'(w_o) + f(w_o) - w_o(w_o - T)^{-1}f(w_o)$$

in X, where $g'(w_o) = (dg/dw)(w_o)$. Hence the limit

$$\lim_{w\to w_o} (w - w_o)^{-1}(f(w) - f(w_o))$$

exists in D(T). This shows that the function $f : V \to D(T)$ is differentiable at every point $w_o \in V$. By the (vector version of the classical) Looman-Menchoff theorem, it follows that $f \in O(V,D(T))$.

2.2. LEMMA. *Let* $T \in C(X)$ *and let* $V \subset \mathbb{C}_\infty$ *be open. If* $\zeta - T : {}_oO(V,D(T)) \to O(V,X)$ *is bijective, then* $V \subset \rho_W(T,X)$.

PROOF. For every $z \in V \cap \mathbb{C}$ we define the linear map

$$T_z x = ((\zeta - T)^{-1}x)(z), \; x \in X,$$

where x is regarded as as constant function from $O(V,X)$. Since $(\zeta - T)^{-1} : O(V,X) \to {}_oO(V,D(T))$ is continuous (by the closed graph theorem), it is clear that $T_z : X \to D(T)$ is continuous. Notice that

$$(z - T)T_z x = (z - T)((\zeta - T)^{-1}x)(z) = ((\zeta - T)(\zeta - T)^{-1}x)(z) = x$$

for each $x \in X$, and

$$T_z(z - T)y = ((\zeta - T)^{-1}(z - T)y)(z) = ((z - T)(\zeta - T)^{-1}y)(z) =$$

$$= ((\zeta - T)(\zeta - T)^{-1}y)(z) = y$$

for every $y \in D(T)$. This shows that $T_z = (z - T)^{-1}$ for all $z \in V \cap \mathbb{C}$.

Let $z_o \in V$ be fixed and let $V_o \subset V$ be a compact neighbourhood of z_o. Then the set

$$\{(z - T)^{-1}x;\; x \in V_o \cap \mathbb{C}\} \subset ((\zeta - T)^{-1}x)(V_o)$$

is bounded in D(T), and therefore in X. Thus $z_o \in \rho_W(T,X)$, and so $V \subset \rho_W(T,X)$.

2.3. LEMMA. *Let* $T \in C(X)$ *and assume that* $\infty \in \rho_W(T,X)$. *Then there exists an open set* $V_\infty \subset \mathbb{C}_\infty$, $\infty \in V_\infty$, *such the operator* $\zeta - T : {}_0O(V_\infty, D(T)) \to O(V_\infty, X)$ *is bijective.*

PROOF. Since $\infty \in \rho_W(T,X)$, there exists an open neighbourhood V_∞ of ∞ such that $(z-T)^{-1} \in L(X)$ for all $z \in V_\infty \cap \mathbb{C}$ and the set $\{(z-T)^{-1}x;\ z \in V_\infty \cap \mathbb{C}\}$ is bounded for each $x \in X$. Then the operator $\zeta - T$ is clearly injective on ${}_0O(V_\infty, D(T))$. We shall prove that $\zeta - T$ is onto $O(V_\infty, X)$.

Let $g \in O(V_\infty, X)$ and set $f(z) = (z-T)^{-1}g(z)$ for $z \in V_\infty \cap \mathbb{C}$. Since every point $z \in V_\infty \cap \mathbb{C}$ is regular for T, it follows by Lemma 2.1 that $f \in O(V_\infty \cap \mathbb{C}, D(T))$. We shall prove that f is analytic and null at infinity.

Let $W_\infty \subset V_\infty$ be a compact neighbourhood of ∞. Then the set

$$f(W_\infty \cap \mathbb{C}) = \{(z-T)^{-1}g(z);\ z \in W_\infty \cap \mathbb{C}\}$$

is bounded in X (by the uniform boundedness principle). This shows that $f \in O(W_\infty, X)$ (see the proof of Corollary II.4.14 in [16]). Then, from the equation

$$(2.4) \qquad f(z) = z^{-1}Tf(z) + z^{-1}g(z),\ z \in V_\infty \cap \mathbb{C},\ z \neq 0,$$

it follows that $\lim_{z\to\infty} z^{-1}Tf(z)$ exists in X. Since $\lim_{z\to\infty} z^{-1}f(z) = 0$ and T is closed, we must have $\lim_{z\to\infty} z^{-1}Tf(z) = 0$. Using this fact, we obtain, again from (2.4), that $\lim_{z\to\infty} f(z) = 0$. Therefore

$$(2.5) \qquad \lim_{z\to\infty} Tf(z) = \lim_{z\to\infty} (zf(z) - g(z))$$

exists in X. Then we have as above that $\lim_{z\to\infty} Tf(z) = 0$. Consequently both $f(z)$ and $Tf(z) = zf(z) - g(z)$ are analytic in V_∞ and null at ∞. Hence $f \in {}_0O(V_\infty, D(T))$.

The next consequence of Lemma 2.3 is known (see, for instance, [16], Lemma III.3.5) but its proof is seemingly new.

2.4. COROLLARY. *Let* $T \in C(X)$ *be such that* $\infty \in \rho_W(T,X)$. *Then* $T \in L(X)$.

PROOF. We take in the previous proof $g = x$, where $x \in X$ is fixed. Then from (2.5) we infer that $\lim_{z\to\infty} zf(z) = x$. Since $\lim_{z\to\infty} zTf(z)$ exists in X and the operator T is closed, we deduce that $x \in D(T)$. Hence $D(T) = X$, and so $T \in L(X)$.

2.5. THEOREM. *Let* $T \in C(X)$ *and let* $V \subset \mathbb{C}_\infty$ *be open. We have* $V \subset \rho_W(T,X)$ *if and only if the operator*

$$\zeta - T : {}_0O(V, D(T)) \to O(V, X)$$

is bijective.

PROOF. Let $V \subset \rho_W(T,X)$. From Lemmas 2.1 and 2.3 we derive the existence of an open cover $\{V_j\}_{j\in J}$ of V such that $\zeta - T : {}_0O(V_j,D(T)) \to O(V_j,X)$ is bijective for all $j \in J$. Let $g \in O(V,X)$ be given. Then for every $j \in J$ we can find $f_j \in {}_0O(V_j,D(T))$ such that $(\zeta - T)f_j = g|V_j$. Since $(\zeta - T)(f_j - f_k) = 0$ on $V_j \cap V_k$, it follows $f_j = f_k$ on $V_j \cap V_k$. Therefore there is a function $f \in {}_0O(V,D(T))$ such that $f|V_j = f_j$ for all $j \in J$. This shows that $\zeta - T$ is onto $O(V,X)$. As $\zeta - T$ is obviously injective, it must be bijective.

Conversely, the assertion follows from Lemma 2.2.

2.6. COROLLARY. *The set $\rho_W(T,X)$ is the largest open set $V \subset \mathbb{C}_\infty$ with the property that $\zeta - T : {}_0O(V,D(T)) \to O(V,X)$ is bijective. Therefore $\rho_W(T,X) = \rho(T,X)$.*

2.7. REMARK. The family $P(X)(= P(X/\{0\})$ is strictly larger than the family $C(X)$. Indeed, if $Y \in \mathrm{Lat}(X)$ is not closed in X, then the inclusion $i : Y \to X$ is in $P(X)$ but not in $C(X)$. Nevertheless, it is the class $C(X)$ which is the most interesting from the spectral point of view. Specifically, if $T \in P(X)$ and $\rho(T,X) \neq \emptyset$, then $T \in C(X)$.

3. NATURAL SPECTRAL CAPACITIES

The uniqueness of the spectral capacity attached to a decomposable operator, first proved by C. Foiaş [9] (see also [10], [16], [17], [22] for some extensions) makes it a very useful concept in the study of spectral decompositions of linear operators. In this section we shall prove a version of this uniqueness result in our more general setting. Some other properties, extensions of statements from [2], [3], [7], [13], [14], [16], [17], [22], will be also presented.

We shall rely heavily upon the work [18]. We shall also use some assertions from [19] and [22] (generally accompanied by an outline of the proof).

3.1. REMARK. Let

$$0 \to X_1/Y_1 \xrightarrow{S} X_2/Y_2 \xrightarrow{T} X_3/Y_3 \to 0$$

be an exact complex of quotient Fréchet spaces. Then for every open set $U \subset \mathbb{C}_\infty$ the complex

$$0 \to O(U,X_1/Y_1) \xrightarrow{S_U} O(U,X_2/Y_2) \xrightarrow{T_U} O(U,X_3/Y_3) \to 0$$

is also exact. This assertion is proved in [19]. For the convenience of the reader we shall sketch its proof.

If $S \in L(X_1/Y_1,X_2/Y_2)$ is arbitrary, $N(S) = N_0(S)/Y_1$ is the null-space of S, $R(S) = R_0(S)/Y_2$ is the range of S and $U \subset \mathbb{C}_\infty$ is open, then we have the equalities $N_0(S_U) =$

$= O(U,N_o(S))$ and $R_o(S_U) = O(U,R_o(S))$. These equalities follow from the elementary properties of tensor products with nuclear spaces (see, for instance, [5]). Then, if $T \in L(X_2/Y_2, X_3/Y_3)$ and $R(S) \subset N(T)$, we have the equality

$$O(U,N_o(T)/R_o(S)) = N_o(T_U)/R_o(S_U),$$

from which we derive the desired exactness.

From now on X/Y will be a fixed quotient Fréchet space.

3.2. LEMMA. *Let $T \in P(X/Y)$, let $Z = Z_o/Y \in \mathrm{Inv}(T)$ and let $\hat{T} \in P(X/Z_o)$ be the operator induced by T. Then the union of any two of the sets $\sigma(T,X/Y)$, $\sigma(T,Z)$ and $\sigma(\hat{T},X/Z_o)$ contains the third.*

PROOF. *For every open $U \subset \mathbb{C}_\infty$* the following diagram

$$\begin{array}{ccccccccc} 0 \to & {}_oO(U,D(T|Z)) & \xrightarrow{i_U} & {}_oO(U,D(T)) & \xrightarrow{k_U} & {}_oO(U,D(\hat{T})) & \to 0 \\ & \downarrow \zeta_U - (T|Z)_U & & \downarrow \zeta_U - T_U & & \downarrow \zeta_U - \hat{T}_U & \\ 0 \to & O(U,Z) & \xrightarrow{i_U} & O(U,X/Y) & \xrightarrow{k_U} & O(U,X/Z_o) & \to 0 \end{array} \tag{3.1}$$

is commutative, where $i : Z \to X/Y$ is the inclusion and $k : X/Y \to X/Z_o$ is the canonical map. The commutativity of (3.1) follows from the results of [18] (see especially Theorem 2.9). Moreover, the rows of (3.1) are exact, by Remark 3.1. Therefore if any two of the columns of (3.1) are exact (i.e. the corresponding operators are bijective) then the third is exact as well, whence we derive our assertion.

If $T \in P(X/Y)$ and $\xi \in X/Y$, we denote by $\delta_T(\xi)$ the set of those points $z \in \mathbb{C}_\infty$ for which there is an open set V containing z and a section $\phi \in {}_oO(V,D(T))$ such that $(\zeta_V - T_V)\phi = \xi$ (where ξ is regarded as a section in $O(U,X/Y)$). $\delta_T(\xi)$ is an open set which is called the *local resolvent (set)* of T at ξ. The set $\gamma_T(\xi) = \mathbb{C}_\infty \setminus \delta_T(\xi)$ is called the *local spectrum* of T at ξ (see [3], [16], [22] for some stages of these concepts).

A linear operator $T \in P(X/Y)$ is said to have the *single valued extension property* (briefly SVEP) if the operator

$$\zeta_U - T_U : {}_oO(U,D(T)) \to O(U,X/Y)$$

is injective for every open $U \subset \mathbb{C}_\infty$. In this case, for each $\xi \in X/Y$ there exists a uniquely determined section $\xi_T \in {}_oO(W,D(T))$ such that $(\zeta_W - T_W)\xi_T = \xi$, where $W = \delta_T(\xi)$ (see the above references).

3.3. LEMMA. *Let $T \in P(X/Y)$. The local spectrum has the following properties:*

(1) $\gamma_T(0) = \emptyset$;

(2) $\gamma_T(\xi + \gamma) \subset \gamma_T(\xi) \cup \gamma_T(\eta)$ *for all* $\xi, \eta \in X/Y$;

(3) $\gamma_T(z\xi) = \gamma_T(\xi)$ *for all* $\xi \in X/Y$ *and* $z \in \mathbb{C} \setminus \{0\}$;

(4) *if* $\xi \in D(T)$ *and* $\phi \in {}_{o}O(V,D(T))$ *satisfies* $(\zeta_V - T_V)\phi = \xi$ *for some open* $V \subset \delta_T(\xi)$, *then* $\phi \in {}_{o}O(V,D(T^2))$.

PROOF. Properties (1), (2) and (3) are simple exercises.

Let us prove (4). We have:

$$T_V\phi = \phi + \zeta_V\phi \in O(V,D(T)) = D(T_V)\ .$$

In other words,

$$\phi \in D((T_V)^2) = D((T^2)_V) = O(V,D(T^2)).$$

On the other hand, $\phi \in P(X/Y)$, and so $\phi \in {}_{o}O(V,D(T^2))$.

Let $T \in P(X/Y)$, let $W \subset \mathbb{C}_\infty$ be open and let $F = \mathbb{C}_\infty \setminus W$. We define the linear manifold

$$(3.1) \qquad {}_{o}O_c(W,D(T)) = \{\phi \in {}_{o}O(W,D(T)),\ (\zeta_W - T_W)\phi \in X/Y\},$$

which is a subspace of ${}_{o}O(W,D(T))$. Then the image

$$(3.2) \qquad E_T^o(F) = (\zeta_W - T_W)({}_{o}O_c(W,D(T)))$$

is a subspace of X/Y. If $F = \mathbb{C}_\infty$, we set $E_T^o(F) = X/Y$. Since T_W extends T, from the equation

$$T(\zeta_W - T_W)\phi = T_W(\zeta_W - T_W)\phi = (\zeta_W - T_W)T_W\phi,$$

valid for every $\phi \in {}_{o}O(W,D(T^2))$, it follows that T_W maps ${}_{o}O(W,D(T^2))$ into ${}_{o}O_c(W,D(T))$.

3.4. LEMMA. *Let* $T \in P(X/Y)$ *and let* $W \subset \mathbb{C}_\infty$ *be open. Then for every open* $V \subset W$ *the operator*

$$\zeta_V - T_{WV} : {}_{o}O(V,{}_{o}O_c(W,D(T^2))) \to O(V,{}_{o}O_c(W,D(T)))$$

is bijective.

PROOF. There exists a linear and continuous operator

$$\tau_o : O(V,O(W,X)) \to {}_{o}O(V,{}_{o}O(W,X))$$

which is given by the equation

$$(3.3) \qquad (z - w)(\tau_o f)(z,w) = f(z,w) - f(z,z), \qquad z \in \mathbb{C} \cap V, \qquad w \in \mathbb{C} \cap W.$$

We shall also use the linear and continuous operator

$$\delta_o : O(V,O(W,X)) \to O(V,X)$$

given by $(\delta_o f)(z) = f(z,z)$ $(z \in V)$. Let τ (resp. δ) be the strict operator induced by τ_o (resp. δ_o) from $O(V,O(W,X/Y))$ into ${}_oO(V,{}_oO(W,X/Y))$ (resp. from $O(V, O(W,X/Y))$ into $O(V,X/Y)$). Then from (3.3) we derive easily the equality

$$(3.4) \qquad (\zeta_V - \zeta_W)\tau\phi = \phi - \delta\phi,\ \phi \in O(V,O(W,X/Y))$$

(where we use some obvious identifications). We shall show that the operator τ induces a map

$$(3.5) \qquad \tau : O(V,{}_oO_c(W,D(T))) \to {}_oO(V,{}_oO_c(W,D(T^2))),$$

which provides an inverse for $\zeta_V - T_{WV}$.

If $\phi \in O(V,{}_oO_c(W,D(T)))$, then $\zeta_W\phi$ is a section in $O(V,O(W,D(T)))$ and we may write the equalities

$$(3.6) \qquad \begin{aligned}&(\zeta_V - \zeta_W)(\zeta_W - T_{WV})\tau\phi = (\zeta_W - T_{WV})(\zeta_V - \zeta_W)\tau\phi = \\ &= (\zeta_W - T_{WV})(\phi - \delta\phi) = (\zeta_W - T_{WV})\phi + (\zeta_V - \zeta_W)\delta\phi - (\zeta_V - T_{WV})\delta\phi.\end{aligned}$$

Let us prove that

$$(\zeta_W - T_{WV})\phi = (\zeta_V - T_{WV})\delta\phi.$$

Indeed, it is clear that $\delta\zeta_W\phi = \zeta_V\delta\phi$. We also have $\delta T_{WV}\phi = T_V\delta\phi$ since if $(f,g) \in G_o(T_{WV})$, then $\delta_o(f,g) = (\delta_o f,\delta_o g) \in G_o(T_V)$. Therefore

$$\delta(\zeta_W - T_{WV})\phi = (\zeta_V - T_V)\delta\phi.$$

On the other hand, the restriction of δ to $O(V,E_T^o(F))$ is just the identity, where $F = \mathbb{C}_\infty \setminus W$. Hence

$$(\zeta_W - T_{WV})\phi = \delta(\zeta_W - T_{WV})\phi = (\zeta_V - T_V)\delta\phi.$$

If we return to (3.6), we get

$$(\zeta_V - \zeta_W)(\zeta_W - T_{WV})\tau\phi = (\zeta_V - \zeta_W)\delta\phi.$$

Since the map $\zeta_V - \zeta_W$ is injective (which follows from the fact that if the function $(z-w)f(z,w)$ belongs to $O(V,O(W,Y))$, then f itself must be in $O(V,O(W,Y))$), we obtain

$$(\zeta_W - T_{WV})\tau\phi = \delta\phi.$$

Thus

$$(\zeta_V - T_{WV})\tau\phi = (\zeta_V - \zeta_W)\tau\phi + (\zeta_W - T_{WV})\tau\phi = \phi,$$

by (3.4). This shows that τ is a right inverse of $\zeta_V - T_{WV}$. Moreover

$$T_{WV}\tau\phi = \zeta_V\tau\phi - \phi \in O(V,O(W,D(T)),$$

and therefore $\tau\phi \in {}_oO(V,{}_oO_c(W,D(T^2)))$.

Note also that

$$\tau(\zeta_V - T_{WV})\phi = (\zeta_V - T_{WV})\tau\phi = \phi$$

(since obviously $\tau\zeta_V = \zeta_V\tau$ and if $(f,g) \in G_o(T_{WV})$, then $\tau_o(f,g) = (\tau_o f, \tau_o g) \in G_o(T_{WV})$). Consequently (3.5) must be the desired inverse.

3.5. COROLLARY. *Let* $T \in P(X/Y)$, *let* $W \subset \mathbb{C}_\infty$ *be open and let* $F = \mathbb{C}_\infty \setminus W$. *If* $\xi \in E_T^o(F)$ *and* $\phi \in {}_oO_c(W,D(T))$ *satisfies* $(\zeta_W - T_W)\phi = \xi$, *then* $\varepsilon_w\phi \in E_T^o(F)$ *for every* $w \in W$, *where* $\varepsilon_w : O(W,X/Y) \to X/Y$ *is the strict operator induced by the evalution at the point* w.

PROOF. It follows from Lemma 3.4 that we can find a section $\psi \in$ $\in {}_oO(V,{}_oO_c(W,D(T^2)))$ such that $(\zeta_V - T_{WV})\psi = \phi$, where $V \subset W$ is an arbitrary open set. Then we have

$$(\zeta_V - T_V)\varepsilon_{w,V}\psi = \varepsilon_{w,V}(\zeta_V - T_{WV})\psi = \varepsilon_{w,V}\phi = \varepsilon_w\phi,$$

and $\varepsilon_{w,V}\psi \in {}_oO(V,D(T))$. Therefore, for $V = W$, we infer that $\gamma_T(\varepsilon_w\phi) \subset F$.

As a matter of fact, we actually have

$$\gamma_T(\varepsilon_w\phi) = \gamma_T(\xi)$$

for every $w \in \mathbb{C} \cap W$, which can be shown by similar arguments. We omit the details (see [16], Proposition IV.3.4).

3.6. LEMMA. *Let* $T \in P(X/Y)$ *have the* SVEP. *For every closed* $F \subset \mathbb{C}_\infty$ *we set*

$$E_T(F) = \{\xi \in X/Y;\ \gamma_T(\xi) \subset F\}.$$

Then $E_T(F) = E_T^o(F) \in \mathrm{Inv}(T)$.

PROOF. If $\xi \in E_T(F)$, then $\xi = (\zeta_W - T_W)\phi$, where $\phi = \xi_T | W \in {}_oO(W,D(T))$ and $W = \mathbb{C}_\infty \setminus F$. Hence $\xi \in E_T^o(F)$. The inclusion $E_T^o(F) \subset E_T(F)$ is obvious.

That $E_T(F) \in \mathrm{Inv}(T)$ follows from the equality

$$(\zeta_W - T_W)T_W\phi = T(\zeta_W - T_W)\phi,$$

valid for every $\phi \in {}_oO_c(W,D(T^2))$ (which has been already noticed).

The next result extends an assertion which originates in [14] (see also [13], [7], [22]).

3.7. THEOREM. *Let* $T \in P(X/Y)$ *have the* SVEP. *Then for every closed* $F \subset \mathbb{C}_\infty$ *one has the inclusion*

$$\sigma(T, E_T(F)) \subset F \cap \sigma(T, X/Y) .$$

PROOF. If $F = \mathbb{C}_\infty$ the assertion is obvious, so that we may assume $F \neq \mathbb{C}_\infty$. Let $W = \mathbb{C}_\infty \setminus F$ and let $V \subset W$ be open. Then the diagram

$$\begin{tikzcd} E_T(F) \cap D(T) \arrow[r, "T"] & E_T(F) \\ {}_0\mathcal{O}_c(W, D(T^2)) \arrow[u, "\zeta_W - T_W"] \arrow[r, "T_W"] & {}_0\mathcal{O}_c(W, D(T)) \arrow[u, "\zeta_W - T_W"] \end{tikzcd}$$

is easily seen to be commutative. Using the functors ${}_0\mathcal{O}(V, \cdot)$ and $\mathcal{O}(V, \cdot)$, we obtain the commutative diagram

$$\begin{tikzcd} {}_0\mathcal{O}(V, E_T(F) \cap D(T)) \arrow[r, "\zeta_V - T_V"] & \mathcal{O}(V, E_T(F)) \\ {}_0\mathcal{O}(V, {}_0\mathcal{O}_c(W, D(T^2))) \arrow[u, "\zeta_W - T_{WV}"] \arrow[r, "\zeta_V - T_{WV}"] & \mathcal{O}(V, {}_0\mathcal{O}_c(W, D(T))) \arrow[u, "\zeta_W - T_{WV}"] \end{tikzcd} \tag{3.7}$$

(see [18] for some details).

We have to prove that the operator

$$\zeta_V - T_V : {}_0\mathcal{O}(V, E_T(F) \cap D(T)) \to \mathcal{O}(V, E_T(F)) \tag{3.8}$$

is bijective. The space $E_T(F)$ is isomorphic to the space ${}_0\mathcal{O}_c(W, D(T))$, since T has the SVEP. Similarly, the space $E_T(F) \cap D(T)$ is isomorphic to the space ${}_0\mathcal{O}_c(W, D(T^2))$ (by Lemma 3.3 (4)). Therefore, to prove the bijectivity of (3.8) it suffices to prove the bijectivity of $\zeta_V - T_{WV}$, when acting on the lower row of (3.7), which follows from Lemma 3.4. This shows that

$$\sigma(T, E_T(F)) \subset F .$$

As we clearly have

$$E_T(F) = E_T(F \cap \sigma(T, X/Y)) ,$$

it follows from the above result that

$$\sigma(T, E_T(F)) = \sigma(T, E_T(F \cap \sigma(T, X/Y))) \subset F \cap \sigma(T, X/Y) ,$$

which completes the proof of the theorem.

3.8. REMARKS. 1° Theorem 3.7 is connected with another important observation of Apostol's. Namely, he proved directly that if X is a Banach space and $T \in L(X)$ has the SVEP, then there exists a holomorphic functional calculus with functions analytic in neighbourhoods of a given closed set $F \subset \mathbb{C}$, associated to the linear map $T|E_T(F)$ ([2], Theorem 2.10; see also [6], [7], [13], [22] for further development). It follows from Theorem 3.7 that if X is actually a Fréchet space and $T \in C(X)$, then $E_T(F) \in \mathrm{Inv}(T)$ and $\sigma(T,E_T(F)) \subset F$ for each closed $F \subset \mathbb{C}_\infty$. Hence the existence of a holomorphic functional calculus for $T|E_T(F)$ (as well as its consequences) can also be obtained from the general theory of Fréchet space operators (see [16], Section III.3). The Fréchet space structure of $E_T(F)$ and the spectral inclusion $\sigma(T,E_T(F)) \subset F$ (with respect to this structure) has been first noticed in [14] (when $E_T(F)$ is supposed to be closed in X, the assertion goes back to [3]).

2° If $T \in P(X/Y)$ has the SVEP, then the assignment $F \to E_T(F)$ provides a map with the properties (i) and (ii) from (1.1) (with $\mathbb{C}$ replaced by $\mathbb{C}_\infty$). If for every open cover $\{G_j\}_{j=1}^n$ of $\mathbb{C}$ one has

$$X/Y = E_T(\bar{G}_1) + \ldots + E_T(\bar{G}_n),$$

then the operator T is decomposable, via Theorem 3.7.

Conversely, we shall see that every decomposable operator has the SVEP and its spectral capacity is uniquely determined and coincides with the natural one, given by Lemma 3.6.

3° As one might expect (see Corollary 2.4), if $T \in P(x/Y)$ and $\infty \notin \sigma(T,X/Y)$, then $T \in L(X/Y)$. This assertion is obtained in [22]. For the convenience of the reader, we shall sketch its proof. Let $U = \mathbb{C}_\infty \setminus \sigma(T,X/Y)$ and let $\xi = x + Y \in X/Y$. Take $\phi \in {}_0O(U,D(T))$ such that $(\zeta_U - T_U)\phi = \xi$. If $f \in {}_0O(U,D_0(T))$ is in the coset ϕ and $g \in {}_0O(U,X)$ is in the coset $T_U\phi$, then $\zeta f - g - x \in O(U,Y)$. This shows that $x \in D_0(T)$. Therefore $D(T) = X/Y$.

3.9. LEMMA. *Let* $T \in L(X/Y)$ *be such that* $\infty \notin \sigma(T,X/Y)$ *and let* $U \subset \mathbb{C}$ *be open,* $U \supset \sigma(T,X/Y)$. *Then the operator*

$$\zeta_U - T_U : O(U,X/Y) \to O(U,X/Y)$$

is injective.

PROOF. Let $\phi \in O(U,X/Y)$ be such that $(\zeta_U - T_U)\phi = 0$. If $V = U \setminus \sigma(T,X/Y)$, then $\phi|V = 0$. In other words, if $f \in O(U,X)$ is in the coset ϕ, then $f|V \in O(V,Y)$. If

$\Delta \supset \sigma(T,X/Y)$ is a Cauchy domain such that $\bar{\Delta} \subset U$, and Γ is the boundary of Δ, then the Cauchy formula

$$g(z) = (2\pi i)^{-1} \int_\Gamma (w - z)^{-1} f(w)dw, \quad z \in \Delta,$$

defines a function $g \in O(\Delta, Y)$. On the other hand, since $f \in O(U,X)$, we must have $f|\Delta = g|\Delta$. Therefore $f \in O(U,Y)$, that is $\phi = 0$.

3.10. LEMMA. *Let* $T \in P(X/Y)$ *and let* $Z_j = X_j/Y \in \mathrm{Inv}(T)(j = 0,1,2)$ *be such that either* Z_1 *or* Z_2 *is in* $D(T)$ *and* $X/Y = Z_1 + Z_2$. *If* $\sigma(T,Z_0) \cap (\sigma(T,Z_2) \cup \sigma(T,Z_1 \cap Z_2)) = \emptyset$, *then* $Z_0 \subset Z_1$.

PROOF. Let $\theta : Z_0 \to X_2/(X_1 \cap X_2)$ be the operator given by the composite of the canonical map $X/Y \to X/X_1$, restricted to Z_0, and the natural isomorphism from X/X_1 onto $X_2/(X_1 \cap X_2)$ (induced by the decomposition $X/Y = Z_1 + Z_2$). We shall show that $\theta = 0$, which clearly implies our assertion.

Let $\xi \in Z_0$ and let $U = \rho(T,Z_0)$. Then there exists a section $\phi \in {}_0O(U,Z_0 \cap D(T))$ such that $(\zeta_U - T_U)\phi = \xi$. Let

$$\theta_0 : Z_0 \cap D(T) \to (X_2 \cap D_0(T))/(X_1 \cap X_2)$$

be the restriction of θ (note that $X_1 \cap X_2 \subset D_0(T)$ from the hypothesis). Then θ_0 and θ induce, respectively, the operators

$$\theta_1 : {}_0O(U,Z_0 \cap D(T)) \to {}_0O(U,(X_2 \cap D_0(T))/(X_1 \cap X_2)),$$

$$\theta_2 : O(U,Z_0) \to O(U,X_2/(X_1 \cap X_2)).$$

Moreover,

$$\theta_2(\zeta_U - T_U)\phi = (\zeta_U - \hat{T}_U)\theta_1\phi = \theta\xi \tag{3.9}$$

(see [18], Theorem 2.9), where $\hat{T}$ is the operator induced by T in $X_2/(X_1 \cap X_2)$.

Next, let $V = \mathbb{C}_\infty \setminus (\sigma(T,Z_2) \cup \sigma(T,Z_1 \cap Z_2))$. Then

$$\sigma(\hat{T}, X_2/(X_1 \cap X_2)) \subset \sigma(T,X_2/Y) \cup \sigma(T,(X_1 \cap X_2)/Y) =$$
$$= \sigma(T,Z_2) \cup \sigma(T,Z_1 \cap Z_2),$$

by Lemma 3.2. Hence the operator

$$\zeta_V - \hat{T}_V : {}_0O(V,(X_2 \cap D_0(T))/(X_1 \cap X_2)) \to O(V,X_2/(X_1 \cap X_2))$$

is bijective and we can find a section

$$\phi_2 \in {}_0O(V,(X_2 \cap D_0(T))/(X_1 \cap X_2))$$

such that $(\zeta_V - \hat{T}_V)\phi_2 = \theta\xi$. If $\phi_1 = \theta_1\phi$, let us observe that

$$(\zeta_{U \cap V} - \hat{T}_{U \cap V})(\phi_1 | U \cap V - \phi_2 | U \cap V) = 0,$$

by (3.9). Thus there is a section

$$\phi_o \in {}_oO(U \cup V, (X_2 \cap D_o(T))/(X_1 \cap X_2))$$

such that $\phi_o | U = \phi_1$ and $\phi_o | V = \phi_2$. Moreover $(\zeta_{U \cup V} - \hat{T}_{U \cup V})\phi_o = \theta\xi$. But $U \cup V = \mathbb{C}_\infty$. Hence $\phi_o = 0$ (see the proof of Theorem 3.7 from [18]), and so $\theta\xi = 0$. Consequently $Z_o \subset Z_1$.

The next result is a sufficient condition which insures the SVEP (see also [7], Bemerkung I.2.3 for Fréchet space operators with bounded spectrum).

3.11. THEOREM. *Let* $T \in P(X/Y)$ *be such that for every open cover* $\{G_1, G_2\}$ *of* $\mathbb{C}_\infty$ *there are a quotient Fréchet space* Z_o, *an operator* $S \in P(Z_o)$ *and two subspaces* Z_1, $Z_2 \in \mathrm{Inv}(S)$ *such that* $Z_o = Z_1 + Z_2$, $\sigma(S, Z_j \cap Z_k) \subset G_j \cap G_k$ $(j,k = 1,2)$, $X/Y \in \mathrm{Inv}(S)$ *and* $S|(X/Y) = T$. *Then* T *has the* SVEP.

PROOF. It suffices to prove that the operator

$$\zeta_U - T_U : O(U, D(T)) \to O(U, X/Y)$$

is injective for every open disc $U \subset \mathbb{C}$. Let U, V be open discs such that $V \subset \bar{V} \subset U$. Then $G_1 = U$ and $G_2 = \mathbb{C}_\infty \setminus \bar{V}$ provide an open cover of $\mathbb{C}_\infty$. Let $Z_j = X_j/Y$ $(j = 0,1,2)$ and $S \in P(Z_o)$ be given by the hypothesis, with respect to the cover $\{G_1, G_2\}$.

The complex of quotient Fréchet spaces

$$0 \to Z_1 \cap Z_2 \xrightarrow{\alpha} Z_1 \times Z_2 \xrightarrow{\beta} Z_o \to 0$$

is exact, where $\alpha(\eta) = (\eta, -\eta)$ and $\beta(\eta_1, \eta_2) = \eta_1 + \eta_2$. According to Remark 3.1, the complex

$$(3.10) \quad 0 \to O(U, Z_1 \cap Z_2) \xrightarrow{\alpha_U} O(U, Z_1) \times O(U, Z_2) \xrightarrow{\beta_U} O(U, Z_o) \to 0$$

is also exact, where $\alpha_U(\phi) = (\phi, -\phi)$ and $\beta_U(\phi_1, \phi_2) = \phi_1 + \phi_2$. The exactness of (3.10) shows that $O(U, Z_o) = O(U, Z_1) + O(U, Z_2)$ and that $O(U, Z_1) \cap O(U, Z_2) = O(U, Z_1 \cap Z_2)$. Therefore $O(U, Z_o)/O(U, Z_2)$ is isomorphic to $O(U, Z_1)/O(U, Z_1 \cap Z_2)$, which in turn is isomorphic to $O(U, X_1/(X_1 \cap X_2))$. Let

$$\theta : O(U, D(T)) \to O(U, X_1/(X_1 \cap X_2))$$

be the composite of the canonical map

$$O(U, Z_o) \to O(U, Z_o)/O(U, Z_2)$$

and the above isomorphism. We shall prove that the diagram

$$
(3.11)\qquad
\begin{array}{ccc}
O(U,D(S)) & \xrightarrow{\theta} & O(U,X_1/(X_1 \cap X_2)) \\
\Big\downarrow \zeta_U - S_U & & \Big\downarrow \zeta_U - \hat{S}_U \\
O(U,Z_o) & \xrightarrow{\theta} & O(U,X_1/(X_1 \cap X_2))
\end{array}
$$

is commutative, where $\hat{S}$ is induced by S in $X_1/(X_1 \cap X_2)$. First of all note that

$$\sigma(\hat{S},X_1/(X_1 \cap X_2)) \subset \sigma(S,Z_1) \cup \sigma(S,Z_1 \cap Z_2) \subset U$$

by Lemma 3.2 and the hypothesis. Since $\infty \notin U$ we must have $\hat{S} \in L(X_1 \cap X_2))$, by Remark 3.8.3°. If $\phi \in O(U,D(S))$, we can write $\phi = \phi_1 + \phi_2$, with $\phi_j \in O(U,Z_j)$ $(j = 1,2)$. As we have $\infty \notin \sigma(S,Z_1) \subset U$, then $Z_1 \subset D(S)$, as above. Hence $\phi_1 \in O(U,D(S))$, and so $\phi_2 = \phi - \phi_1 \in O(U,D(S))$. Therefore

$$(\zeta_U - S_U)\phi = (\zeta_U - S_U)\phi_1 + (\zeta_U - S_U)\phi_2 ,$$

and $(\zeta_U - S_U)\phi_j \in O(U,Z_j)$ $(j = 1,2)$. Consequently

$$\theta(\zeta_U - S_U)\phi = (\zeta_U - \hat{S}_U)\theta\phi = (\zeta_U - \hat{S}_U)\theta\phi_1 ,$$

showing that (3.11) is commutative.

Now, let $\phi \in O(U,D(T)) \subset O(U,D(S))$ be such that $(\zeta_U - T_U)\phi = 0$, and let $\phi = \phi_1 + \phi_2$ be a decomposition of ϕ as above. Therefore, by the commutativity of (3.11),

$$0 = \theta(\zeta_U - S_U)\phi = (\zeta_U - \hat{S}_U)\theta\phi_1 .$$

According to Lemma 3.9, the operator $\zeta_U - \hat{S}_U$ is injective. Hence $\phi_1 \in O(U,Z_1 \cap Z_2) \subset$ $\subset O(U,Z_2)$, and so $\phi = \phi_1 + \phi_2 \in O(U,Z_2)$. Since $\sigma(S,Z_2) \cap V = \emptyset$, it follows that $\phi|V = 0$. As $V \subset \overline{V} \subset U$ is arbitrary, we must have $\phi = 0$.

3.12. REMARK. When $Y = \{0\}$ and therefore X, Z_j $(j = 0,1,2)$ are Fréchet spaces, then the requirement $\sigma(S,Z_j \cap Z_k) \subset G_j \cap G_k$ $(j,k = 1,2)$ from Theorem 3.11 may be replaced by weaker one $\sigma(S,Z_j) \subset G_j$ $(j = 1,2)$, provided Z_1,Z_2 are *closed* subspaces of Z_o (as stated in [7]). Indeed, in this case, if $U \subset \mathbb{C}$ is an open disc (more generally a simply connected open set) and $\sigma(T,Z_1) \subset U$, then $\sigma(S,Z_1 \cap Z_2) \subset U$, which suffices for the proof of Theorem 3.11. Nevertheless, if $Z_1 \cap Z_2 \in \mathrm{Lat}(Z_1)$ is not closed in Z_1, then the inclusion $\sigma(S,Z_1 \cap Z_2) \subset U$ may not be true, as simple examples show.

For operators with bounded spectrum, the condition from Theorem 3.11 is necessary too, modulo similarities (see also [7]).

3.13. PROPOSITION. *Let* $T \in L(X/Y)$ *have the* SVEP *and assume that*

$\infty \notin \sigma(T,X/Y)$. *Then for every open cover* $\{G_1,G_2\}$ *of* $\sigma(T,X/Y)$ *there are a quotient Fréchet space* Z_o, *an injective operator* $\theta : X/Y \to Z_o$, *an operator* $S \in L(Z_o)$ *and two subspaces* $Z_1, Z_2 \in \mathrm{Inv}(S)$ *such that* $Z_o = Z_1 + Z_2$, $\sigma(S,Z_j) \subset G_j$ $(j = 1,2)$ *and* $S|E_o = T_o$, *where* $T_o = \theta T \theta^{-1}$ *and* $E_o = \theta(X/Y)$.

PROOF. For every open and bounded set $U \subset \mathbb{C}$ we define the quotient Fréchet space

$$F_T(U) = O(U,X/Y)/(\zeta_U - T_U)O(U,X/Y)$$

(see [13] for Fréchet space operators). It is easily seen that T_U and ζ_U induce the same action on $F_T(U)$. Moreover, $\sigma(\zeta_U, F_T(U)) \subset \bar{U}$.

Now, let $\{U_1,U_2\}$ be an open cover of $\sigma(T,X/Y)$ such that $U_j \subset \bar{U}_j \subset G_j$, and with $\bar{U}_j$ compact in $\mathbb{C}$ $(j = 1,2)$. We define the quotient Fréchet space

$$Z_o = F_T(U_1) \times F_T(U_2)$$

and the operator $\theta : X/Y \to Z_o$ given by $\theta\xi = ([\xi]_1,[\xi]_2)$, where $[\xi]_j$ is the coset of ξ in $F_T(U_j)$ $(j = 1,2)$. Since T has the SVEP, the operator θ is clearly injective. We also set $Z_1 = F_T(U_1) \times \{0\}$, $Z_2 = \{0\} \times F_T(U_2)$ and $S \in L(Z_o)$ given by

$$S(\phi_1,\phi_2) = (\zeta_{U_1}\phi_1, \zeta_{U_2}\phi_2), \qquad (\phi_1,\phi_2) \in Z_o .$$

Having these objects defined, our assertions follow easily.

We are now in the position to prove the uniqueness of the spectral capacity attached to a decomposable operator in our sense (see also [9], [10], [16], [17], [22], etc.).

3.14. THEOREM. *Let* $T \in P(X/Y)$ *be decomposable and let* E *be a spectral capacity attached to* T. *Then* T *has the* SVEP *and* $E(F) = E_T(F)$ *for all closed* $F \subset \mathbb{C}_\infty$.

PROOF. That T has the SVEP clearly follows from Theorem 3.11. If $F = \bar{F} \subset \mathbb{C}_\infty$ is fixed and $\xi \in E(F)$, then the section

$$\phi = (\zeta_U - (T|E(F))_U)^{-1}\xi \in {}_oO(U,D(T))$$

satisfies the equation $(\zeta_U - T_U)\phi = \xi$, where $U = \mathbb{C}_\infty \setminus F$. Therefore $\xi \in E_T(F)$.

Conversely, let $\{G_1,G_2\}$ be an open cover of $\mathbb{C}_\infty$ such that $F \subset G_1$ and $\bar{G}_2 \cap F = \emptyset$, and let $Z_j = E(\bar{G}_j)$ $(j = 1,2)$. Since E is a spectral capacity attached to T, we have

$$\sigma(T,Z_2) \cup \sigma(T,Z_1 \cap Z_2) \subset \bar{G}_2 \cup (\bar{G}_1 \cap \bar{G}_2) = \bar{G}_2 .$$

Then it follows from Lemma 3.10 that $E_T(F) \subset E(\bar{G}_1)$.

If $\{G_{1,n}\}_{n=1}^{\infty}$ is a family of open sets such that each $G_{1,n}$ shares the properties of G_1 and $\cap\{\bar{G}_{1,n}; n \geq 1\} = F$, it results from (1.1) that

$$E_T(F) \subset \bigcap_{n=1}^{\infty} E(\bar{G}_{1,n}) = E(F) .$$

Consequently $E_T(F) = E(F)$ for each closed $F \subset \mathbb{C}_\infty$, and the proof of the theorem is completed.

3.15. REMARK. One can also settle in this context the problem solved in [15] for Banach space operators (see also [7], [13], [16], [22], etc.). Namely, let $T \in P(X/Y)$ have the SVEP. Then for every open $U \subset \mathbb{C}_\infty$ one can define the quotient Fréchet space

$$F_T(U) = O(U,X/Y)/(\zeta_U - T_U)_o O(U,D(T)) .$$

The assignment $U \to F_T(U)$, which is a presheaf with respect to the natural restrictions, corresponds to the sheaf model of a Fréchet space operator introduced in [13] (see also [11] for sheaf theoretical results). Since both $U \to O(U,X/Y)$ and $U \to {}_oO(U,D(T))$ (and therefore $U \to (\zeta_U - T_U)_o O(U,D(T))$) are acyclic sheaves (this fact follows from the proof of Proposition 3.5 from [18]), one can derive that $U \to F_T(U)$ is actually a sheaf.

Next, if $F \subset U$ is a closed set, then the natural operator $E_T(F) \to F_T(U)$ is injective. Moreover, the image of $E_T(F)$ via this operator consists of those sections from $F_T(U)$ whose support is in F. If, in addition, T is assumed to be 2-decomposable (i.e. condition (iii) from (1.1) is valid only for $n \leq 2$), then one can show that T is decomposable, via the fact that the sheaf $U \to F_T(U)$ is, in this case, soft, as done in [13], Section 4. We omit the details (the case $T \in L(X/Y)$ with $\infty \notin \sigma(T,X)$ is treated in [22]).

3.16. EXAMPLES. 1° Let K be a compact subset of the real line and let $X = A'(K)$ be the Fréchet space of all analytic functionals carried by K (see, for instance, [12]). If $T \in L(X)$ is the operator induced by the multiplication with the independent variable, then T is decomposable and the spectral capacity of T is given by

$$E_T(F) = A'(F \cap K), \qquad F = \bar{F} \subset \mathbb{C}_\infty$$

(see [17] for details). We note that $E_T(F) \in \mathrm{Lat}(X)$ but, in general, $E_T(F)$ is not a closed subspace of X.

2° Let K_o, K be compact subsets of the real line, $K_o \subset K$, and let $X = A'(K)$, $Y = A'(K_o)$. Then X/Y is a quotient Fréchet space. If $T \in L(X/Y)$ is the strict operator induced by the multiplication with the independent variable, then T is decomposable [22]. The spectral capacity of T is given by

$$E_T(F) = (A'(K_o) + A'(F \cap K))/A'(K_o), \qquad F = \bar{F} \subset \mathbb{C}_\infty .$$

In particular, the strict operator induced by the multiplication with the independent variable in spaces of hyperfunctions on the real line [12] (which are quotient Fréchet spaces) is decomposable.

3° To get a genuine "unbounded" decomposable operator in a quotient Fréchet space, it suffices to take a direct sum of an operator of the previous type and, say, an unbounded selfadjoint operator.

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