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ON SUPERANALYTIC ALGEBRAS. I

by

Paul FLONDOR and Eugen PASCU

During the last years the interest in superalgebra and supergeometry is growing. In this paper we consider a notion of superanalytic algebra and study some of its first properties. By means of some natural functors, we have been able to reduce ourselves to the more or less classical case of analytic algebras. § 1 is mainly devoted to the fixing of notation and terminology.

§ 1. $\mathbb{Z}_2$ - graded structures

For the general notion of a group (resp. ring, module) graded with respect to a commutative monoid Δ and for some fundamental properties of graded algebraic structures see e.g. [3].

In this section, we shall consider some definitions and results for the case $\Delta = \mathbb{Z}_2 := \{0,1\}$ - the (additive) group $\mathbb{Z}/2\mathbb{Z}$.

We shall use the additive notation for abelian groups, the neutral element being denoted by 0.

DEFINITION 1. Let G be an abelian group. A $\mathbb{Z}_2$-grading on G is a family $(G_\lambda)_{\lambda\in\mathbb{Z}_2}$ of subgroups of G such that $G = G_o \oplus G_1$. A supergroup is an abelian group G together with a $\mathbb{Z}_2$-grading on G. If $G = G_o \oplus G_1$ and $H = H_o \oplus H_1$ are supergroups, a morphism between these supergroups is a group morphism $u: G \to H$ such that $u(G_o) \subset H_o$, $u(G_1) \subset H_1$.

Supergroups and their morphisms form in a natural way a category.

DEFINITION 2. If $G = G_0 \oplus G_1$ is a supergroup, $x \in G$ is called homogeneous if $x \in G_0$ or $x \in G_1$. The elements of G_0 are called even and the elements of G_1 are called odd ($0 \in G$ is the only element both even and odd). Each $x \in G$ may be uniquely written $x = x_0 + x_1$, with $x_0 \in G_0$, $x_1 \in G_1$; x_0, x_1 are called the homogeneous components of x.

For each homogeneous element $x \in G$, $x \neq 0$ we define $p(x) \in \mathbb{Z}$ by:

$$p(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd.} \end{cases}$$

In the sequel, when the notation p(x) is used, we are tacitly assumming that $x \neq 0$ is homogeneous.

REMARK 1. When no confusion is possible, we shall denote a supergroup $G = G_0 \oplus G_1$ only by G.

REMARK 2. Let G be an abelian group. By setting $G_0 = G$, $G_1 = \{0\}$, one obtains a $\mathbb{Z}_2$-grading on G. This is called the trivial grading on G. If G, H are supergroups with respect to the trivial grading, every group morphism $u : G \to H$ is a supergroup morphisms. Thus, the category of abelian groups becomes (in a canonical way) a full subcategory of the category of supergroups.

In the following, rings are supposed to be unitary, the neutral element with respect to multiplication is denoted by 1, ring morphisms preseve 1 and a subring has the same unit as the ring itself. Modules are left modules.

DEFINITION 3. Let A be a ring. A $\mathbb{Z}_2$-grading of the additive group A is compatible with the ring structure if $A_\lambda A_\mu \subset A_{\lambda+\mu}$ for each $\lambda, \mu \in \mathbb{Z}_2$. A ring together with a $\mathbb{Z}_2$-grading compatible with the ring structure will be called a superring. If $A = A_0 \oplus A_1$, $B = B_0 \oplus B_1$ are superrings, a morphism between them is a ring morphism $f : A \to B$ such that $f(A_0) \subset B_0$, $f(A_1) \subset B_1$.

Superrings and their morphisms form in a natural way a category.

REMARK 3. If A is a ring, the trivial grading $A_o = A$, $A_1 = \{0\}$ is compatible with the ring structure. If A and B are rings with the trivial grading, each ring morphism $f : A \to B$ is a superring morphism. Thus, the category of rings becomes (in a natural way) a full subcategory of the category of superrings.

REMARK 4. As in the case of supergroups, if no confusion is possible, we shall denote a superring $A = A_o \oplus A_1$ only by A.

REMARK 5. If $A = A_o \oplus A_1$ is a superring, then A_o is a subring of A (in particular $1 \in A_o$); hence A and A_1 are in a natural way A_o-modules.

DEFINITION 4. A superring A is called commutative if $xy = (-1)^{p(x)p(y)}yx$ for each x and y.

If $A = A_o \oplus A_1$ is a commutative superring, then $A_o \subset Z(A)$ (:= the center of the ring A) and if $x \in A_1$, then $x^2 + x^2 = 0$. Moreover, if 1 + 1 is invertible, then $x^2 = 0$ for each $x \in A_1$.

REMARK 6. If A is a superring with respect to the trivial grading, then A is a commutative superring iff it is a commutative ring. If $A = A_o \oplus A_1$ is a commutative superring, then A is in a natural way an A_o-algebra.

DEFINITION 5. Let $A = A_o \oplus A_1$ be a superring, and let M be an A-module. A $\mathbb{Z}_2$-grading of the additive group $M = M_o \oplus M_1$ is compatible with the A-module structure if $A_\lambda M_\mu \subset M_{\lambda+\mu}$ for each $\lambda, \mu \in \mathbb{Z}_2$. An A-module, together with a $\mathbb{Z}_2$--grading compatible with the A-module structure will be called an A-supermodule. If $M = M_o \oplus M_1$, $N = N_o \oplus N_1$ are A-supermodules, a morphism between them is an A-module morphism $u : M \to N$ such that $u(M_o) \subset N_o$, $u(M_1) \subset N_1$.

A-supermodules and their morphisms form in a natural way a category.

REMARK 7.1. If $A = A_o \oplus A_1$ is a superring and $M = M_o \oplus M_1$ is an A-supermodule, then M_o, M_1 are A_o-modules.

2. If A is a superring with respect to the trivial grading, then $M = M_o \oplus M_1$ is an A-supermodule, iff M_o, M_1 are A-submodules of M.

3. If A is a superring, then A is an A-supermodule in a canonical way.

4. If A is a superring with respect to the trivial grading, and M is an A-module, then the trivial grading of M is compatible with the A-module structure of M. If M, N are A-modules with the trivial grading, every A-module morphism is an A-supermodule morphism. Thus, the category of A-modules becomes (in a canonical way) a full subcategory of the category of A-supermodules.

5. Supergroups may be regarded as $\mathbb{Z}$-supermodules (here $\mathbb{Z}$ is with the trivial grading).

DEFINITION 6. Let $A = A_o \oplus A_1$ be a superring and let $M = M_o \oplus M_1$ be an A-supermodule. An A-submodule N of M is called a graded submodule of M if $N = (N \cap M_o) \oplus (N \cap M_1)$. In this case, N becomes in a natural way an A-supermodule.

PROPOSITION 1. Under the notations in Definition 6, the following statements are equivalent

i) N is a graded submodule of M.

ii) Homogeneous components of elements of N belong to N.

iii) N is generated (as an A-module) by homogeneous elements (in M).

For the proof see [3].

DEFINITION 7. If A is a superring, a superideal in A is a graded submodule of the A-supermodule A (see also remark 7.3).

PROPOSITION 2. Let $\mathfrak{a}$ be a superideal in a commutative superring A. Then $\mathfrak{a}$ is a two-sided ideal in A.

Proof. Let $x = x_0 + x_1$ and $t = t_0 + t_1$, $x \in \mathfrak{a}$, $t \in A$. As x_0 and x_1 belong to $\mathfrak{a}$. (by proposition 1), we have:

$$xt = x_0t_0 + x_0t_1 + x_1t_0 + x_1t_1 = t_0x_0 + t_1x_0 + t_0x_1 - t_1x_1 .$$

Hence $xt \in \mathfrak{a}$.

Let now A be a superring, let $M = M_0 \oplus M_1$ be an A supermodule and let N be a graded submodule of M. By means of natural identifications, we obtain $M/N = M_0/N_0 \oplus M_1/N_1$ ($N_0 = M_0 \cap N, N_1 = M_1 \cap N$). One obtains thus a $\mathbb{Z}_2$-grading on M/N, which is compatible with the A-module structure; M/N will be considered an A-supermodule in this manner and this grading will be called the quotient grading.

If M and N are as above, the natural projection $M \xrightarrow{\pi} M/N$ is an A-supermodule morphism.

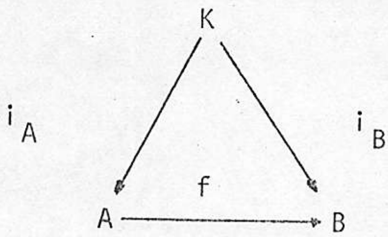
If A is a superring and $\mathfrak{a} \subset A$ is a two sided superideal, then $A/\mathfrak{a}$ with the quotient grading is a superring. Moreover, if A is commutative, so is $A/\mathfrak{a}$.

Let now K be a commutative field, char $K \neq 2$ (fixed for the rest of the section). We shall consider K as a superring by means of the trivial grading.

DEFINITION 8. A K-superalgebra is a superring A, together with a superring morphism $i_A : K \to A$; i_A is called the structure morphism of A.

REMARK 8. If A is a commutative ring, trivially graded, then A is a K-superalgebra iff it is a K-algebra (in the sense of [4]).

DEFINITION 9. If $i_A : K \to A$, and $i_B : K \to B$ are K superalgebras, a morphism between them is a superring morphism $f : A \to B$ such that the diagram



is commutative.

K-superalgebras and their morphisms form in a natural way a category.

DEFINITION 10. A K-superalgebra $i_A : K \to A$ is called commutative iff A is a commutative superring.

The category of commutative K-superalgebras contains as a full subcategory the category of commutative K-algebras (which are considered as K-superalgebras by means of the trivial grading).

PROPOSITION 3. Let $i_A : K \to A = A_o \oplus A_1$ be a commutative K superalgebra. For $S \subset A$ denote by (S) the ideal generated by S in A. Then

i) There exists an isomorphism of commutative rings

$u : A/(A_1) \to A_o/(A_1^2)$

ii) Denote by $\pi : A \to A/(A_1)$ the canonical projection. Let $a = a_o + a_1 \in A$. Then the following statements are equivalent

1. a is invertible in A.

2. a_o is invertible in A_o.

3. $\pi(a)$ is invertible in $A/(A_1)$.

REMARK 3. If $i_A : K \to A$ is a K-superalgebra and $a \subset A$ is a two sided superideal in A, then the composition $K \xrightarrow{i_A} A \xrightarrow{\pi} A/a$ furnishes a canonical K--superalgebra structure on A/a and the canonical projection π becomes a K-superalgebra morphism.

Let $i_A : K \to A = A_o \oplus A_1$ and $i_B : K \to B = B_o \oplus B_1$ be two K-superalgebras. Let $i : K \to A \otimes_K B$ be given by $i = i_A \otimes i_B$.

Let

$$(A \otimes_K B)_o = (A_o \otimes_K B_o) \oplus (A_1 \otimes_K B_1)$$

$$(A \otimes_K B)_1 = (A_o \otimes_K B_1) \oplus (A_1 \otimes_K B_o)$$

We obtain a $\mathbb{Z}_2$-grading on $A \otimes_K B$.

By defining a multiplication on $A \otimes_K B$ by:

$$(x \otimes y) \cdot (z \otimes t) = (-1)^{p(y)p(z)} xz \otimes yt$$

for homogeneous elements $x,z \in A$, $y,t \in B$ and extending by K-linearity to $A \otimes_K B$, $A \otimes_K B$ becomes a superring; by means of $i : K \to A \otimes_K B$, in fact, we have defined a K-superalgebra structure. Moreover if A and B are commutative, so is $A \otimes_K B$.

DEFINITION 11. The K-superalgebra $i : K \to A \otimes_K B$ is called the tensor product of the K-superalgebras $i_A : K \to A$ and $i_B : K \to B$.

REMARK 10. If no confusion is possible, we shall denote a K-superalgebra $i_A : K \to A$ only by A.

DEFINITION 12. A commutative superring A is called local if the set of all non invertible elements is a superideal in A. In this case this superideal is the only maximal ideal of A and will be denoted by m_A.

PROPOSITION 4. If A is a local superring and $1 + 1 \neq 0$, then the quotient grading on A/m_A is the trivial one.

Proof. If $x \in A_1$ and x were invertible, by denoting its inverse by y, it follows $y \in A_1$ and $xy = 1 = yx = -1$ and hence $1 + 1 = 0$ which contradicts our hypotheses. It follows then $A_1 \subset m_A$.

DEFINITION 13. A superring morphism $f: A \to B$ between the local superrings A and B is local if $f(m_A) \subset m_B$.

DEFINITION 14. A K superalgebra $i_A : K \to A$ is local iff A is a local superring.

DEFINITION 15. If $i_A : K \to A$ is a local K-superalgebra, then A/m_A is a commutative field R (see proposition 4 and remark 9). By means of the map $i_R : K \to R$ given by $i_R := \pi \circ i_A$, R becomes a K-superalgebra.

$i_R : K \to R$ is called the residue field of $i_A : K \to A$.

REMARK 11. If $i_A : K \to A$ is a local K-superalgebra, then i_A is injective and it follows that $1 + 1$ is invertible in A. For each local K superalgebra $i_A : K \to A$ we can consider the K-algebra $K \to A_o$, naturally determined by i_A.

PROPOSITION 5. Let $i_A : K \to A = A_o \oplus A_1$ be a K superalgebra. The following statements are equivalent:

i) A is a local K-superalgebra;

ii) A_o is a local K-algebra;

iii) $A/(A_1)$ is a local K-algebra.

Proof. i)$\Longrightarrow$ii) Obvious.

ii)$\Longrightarrow$iii) Obvious (see also proposition 3).

iii)$\Longrightarrow$i) Denote by m the maximal ideal in $A/(A_1)$ and by $\pi : A \to A/(A_1)$ the canonical projection. Then, by proposition 3, $\pi^{-1}(m)$ is the set of noninvertible elements of A. $\pi^{-1}(m)$ is a two sided ideal in A. On the other hand, let $x = x_o + x_1$, $x \in \pi^{-1}(m)$. As x_1 is nilpotent, it follows $x_1 \in \pi^{-1}(m)$. We infer $x_o = x - x_1 \in \pi^{-1}(m)$, and hence $\pi^{-1}(m)$ is a superideal.

Let us consider the K local superalgebras $A = A_o \oplus A_1$ and $B = B_o \oplus B_1$ and let $f : A \to B$ be a K-superalgebra morphism. Then there exists the following commutative diagram:

$$
(*) \qquad
\begin{array}{ccc}
A_o & \xrightarrow{\ f_o\ } & B_o \\
\downarrow{\scriptstyle j_A} & & \downarrow{\scriptstyle j_B} \\
A & \xrightarrow{\ f\ } & B \\
\downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\
A/(A_1) & \xrightarrow{\ \tilde{f}\ } & B/(B_1)
\end{array}
$$

when j_A, j_B are canonical inclusions, $f_o := f|_{A_o}$, π_A, π_B are canonical projections and $\tilde{f}$ is induced by f.

PROPOSITION 6. The following statements are equivalent:

i) f is local;

ii) f_o is local;

iii) $\tilde{f}$ is local.

DEFINITION 16. A K-superalgebra $i_A : K \to A$ is of residue field K-isomorphic to K iff the residue field $i_R : K \to R$ is isomorphic to the K-superalgebra $K \xrightarrow{id} K$.

Then, we have:

PROPOSITION 7. The following statements are equivalent for the local K-superalgebra $i_A : K \to A$:

i) A is of residue field K-isomorphic to K;

ii) A_o is of residue field K-isomorphic to K;

iii) $A/(A_1)$ is of residue field K-isomorphic to K.

PROPOSITION 8. Every morphism between two local K-superalgebras of residue field K-isomorphic to K is local.

Proof. The conclusion follows by the corresponding result for analytic algebras.

REMARK 12. Local K-superalgebras of residue field K-isomorphic to K and their morphisms form in a natural way a full subcategory of the category of commutative K-superalgebras.

§ 2. SUPERANALYTIC ALGEBRAS

Let K be a commutative field, char $K \neq 2$, let $n \in \mathbb{N}$, $n \geq 1$.

We shall denote by $\Lambda_K(\xi_1,\ldots,\xi_n)$, the Grassman algebra over K with canonic generators $\xi_1,\ldots,\xi_n$ (see [3]).

For $I = (i_1,\ldots,i_k)$ with $1 \leq i_1 < \ldots < i_k \leq n$, we set $|I| := k$ and $\xi_I = \xi_{i_1} \cdot \ldots \cdot \xi_{i_k}$. If $I = \emptyset$, we set $|I| = 0$ and $\xi_I = 1$.

Each element of $\Lambda_K(\xi_1,\ldots,\xi_n)$ may be uniquely written as $\sum_I \alpha_I \xi_I$ with $\alpha_I \in K$ (the sum has at most 2^n terms). In the following, for a sum as above $\alpha_\emptyset$ will be denoted by α_o. We set

$$\Lambda_K(\xi_1,\ldots,\xi_n)_o := \{\sum_I \alpha_I \xi_I \mid |I| = 2\ell,\ \ell \in \mathbb{N},\ 2\ell \leq n\}$$

and

$$\Lambda_K(\xi_1,\ldots,\xi_n)_1 := \{\sum_I \alpha_I \xi_I \mid |I| = 2\ell + 1,\ \ell \in \mathbb{N},\ 2\ell + 1 \leq n\}$$

thus obtaining a $\mathbb{Z}_2$-grading on $\Lambda_K(\xi_1,\ldots,\xi_n)$. In this way $\Lambda_K(\xi_1,\ldots,\xi_n)$ becomes a commutative superring. $\Lambda_K(\xi_1)$ is the K-algebra of dual numbers.

DEFINITION 17. For an element $\alpha = \sum_I \alpha_I \xi_I$ of $\Lambda_K(\xi_1,\ldots,\xi_n)$ we define the order of α by

$$\mathrm{ord}(\alpha) := \begin{cases} \infty & \text{if } \alpha_I = 0 \text{ for all } I \\ \min\{|I|\} \alpha_I \neq 0\} \end{cases}$$

REMARK 13. If n is even, then $\Lambda_K(\xi_1,\ldots,\xi_n)_o$ coincides with the center of the ring $\Lambda_K(\xi_1,\ldots,\xi_n)$, and if n is odd, then the center of the ring $\Lambda_K(\xi_1,\ldots,\xi_n)$ is the direct sum between $\Lambda_K(\xi_1,\ldots,\xi_n)_o$ and the subgroup of the elements of order $\geq$ n. (Note that elements of order n are homogeneous).

Invertible elements in $\Lambda_K(\xi_1,\ldots,\xi_n)$ are characterized by $\alpha_o \neq 0$. The ideal generated by $\xi_1,\ldots,\xi_n$ in $\Lambda_K(\xi_1,\ldots,\xi_n)$ is a superideal $\mathfrak{m}$. It is the unique maximal ideal, because its complementary consists of invertible elements. The canonical embedding $K \to \Lambda_K(\xi_1,\ldots,\xi_n)$ and the fact that $\Lambda_K(\xi_1,\ldots,\xi_n)/m \simeq K$ turns (naturally) $\Lambda_K(\xi_1,\ldots,\xi_n)$ into a local K-superalgebra of residue field K-isomorphic to K.

Let $B := \Lambda_K(\xi_1,\ldots,\xi_n)$ and let A be a commutative K-algebra. Denote $\Lambda_A(\xi_1,\ldots,\xi_n) := A \otimes_K B$ (see also definition 11).

The elements of $\Lambda_A(\xi_1,\ldots,\xi_n)$ may be uniquely written as $\alpha = \sum_I \alpha_I \xi_I$ with $\alpha_I \in A$.

$\Lambda_A(\xi_1,\ldots,\xi_n)$ is a commutative K-superalgebra; its grading is given by:

$\Lambda_A(\xi_1,\ldots,\xi_n)_o := \{\sum_I \alpha_I \xi_I \mid |I| = 2\ell, \ell \in \mathbb{N}, 2\ell \leq n\}$ and

$\Lambda_A(\xi_1,\ldots,\xi_n)_1 := \{\sum_I \alpha_I \xi_I \mid |I| = 2\ell + 1, \ell \in \mathbb{N}, 2\ell + 1 \leq n\}$

α is invertible in $\Lambda_A(\xi_1,\ldots,\xi_n)$ iff α_o is invertible in A (see also proposition 3).

PROPOSITION 9. If A is a local K-algebra, then $\Lambda_A(\xi_1,\ldots,\xi_n)$ is a local K-superalgebra.

Let us now consider the embedding $j_A : A \to \Lambda_A(\xi_1,\ldots,\xi_n)$ given by:

$$a \to \sum_I \alpha_I \xi_I,$$

where $\alpha_o = a$, $\alpha_I = 0$ for $I \neq \emptyset$.

With respect to this embedding $\Lambda_A(\xi_1,\ldots,\xi_n)$ becomes an A-supermodule of finite type.

We have

PROPOSITION 10. If A is a noetherian ring, then $\Lambda_A(\xi_1,\ldots,\xi_n)$ is a (left) noetherian ring.

PROPOSITION 11. Let A be a local noetherian K-algebra. Let **a** be a superideal in $\Lambda_A(\xi_1,\ldots,\xi_n)$. Then the K-superalgebra $C := \Lambda_A(\xi_1,\ldots,\xi_n)/a$ is local and is a (left) noetherian ring.

PROPOSITION 12. Let A be a commutative K-algebra, $B = B_o \oplus B_1$ a commutative K-superalgebra, $f : A \to B$ a K-superalgebra morfism and let $\beta_1,\ldots,\beta_n \in B_1$. Then there exists a unique K-superalgebra morphism, $f : \Lambda_A(\xi_1,\ldots,\xi_n) \to B$ such that.

$$F \circ j_A = f \text{ and } F(\xi_i) = \beta_i \quad i = 1,\ldots,n.$$

Proof

Set $F(\sum_I \alpha_I \xi_I) := \sum_I f(\alpha_I)\beta_I$

(where if $I = i \leq i_1 < \ldots < i_k \leq n$ then $\beta_I := \beta_{i_1} \cdot \ldots \cdot \beta_{i_k}$, and $\beta_\emptyset := 1$)

Let us suppose now that $K = \mathbf{R}$ or $K = \mathbb{C}$.

Consider A above to be in turn:

1. $A := K[[X_1,\ldots,X_m]]$. We denote then $\Lambda_A(\xi_1,\ldots,\xi_n)$ by $F_{m,n}$.

2. $A := K\{X_1,\ldots,X_m\}$. We denote then $\Lambda_A(\xi_1,\ldots,\xi_n)$ by $A_{m,n}$.

3. $A := E_{m,o}$:= the K algebra of germs of C^{∞} K- valued functions around the origin in R^m. We denote then $\Lambda_A(\xi_1,\ldots,\xi_n)$ by $E_{m,n}$.

The following properties of $F_{m,n}$, $A_{m,n}$ and $E_{m,n}$ can be infered in a canonical way.

$F_{m,n}$ is a local K-superalgebra of residue field K-isomorphic to K. Its maximal ideal $m_{F_{m,n}}$ is generated by $X_1,\ldots,X_m$, $\xi_1,\ldots,\xi_n$. $F_{m,n}$ is a (left) noetherian ring. The Krull topology is Hausdorff and complete. The canonical embedding $K\{X_1,\ldots,X_m\} \to K[[X_1,\ldots,X_m]]$ furnishes an embedding $i : A_{m,n} \to F_{m,n}$.

$A_{m,n}$ is a local K-superalgebra of residue field K-isomorphic to K and i is a local moprhism $m_{A_{m,n}} = m_{F_{m,n}} \cap A_{m,n}$. $A_{m,n}$ is a left noetherian ring. The Krull topology is Hausdorff.

$E_{m,n}$ is a local K-superalgebra of residue field K-isomorphic to K. Its maximal ideal is generated by the germs of $X_1,\ldots,X_m$ and by $\xi_1,\ldots,\xi_n$.

The canonical embedding $K\{X_1,\ldots,X_m\} \to E_{m,o}$ furnishes a morphism $j : A_{m,n} \to E_{m,n}$, which is local. The Taylor expansion morphism $E_{m,o} \to$ $\to K[[X_1,\ldots,X_m]]$ (which is surjective by th Borel theorem) induces a surjective local K superalgebra morphism $p: E_{m,n} \to F_{m,n}$ with $p \circ j = i$.

THEOREM 1. $(A_{m,n})_o$ is a K-analytic algebra.

Proof. $\Lambda_K(\xi_1,\ldots,\xi_n)_o$ is a local K-algebra of residue field K-isomorphic to K and, as it is a finite dimensional K-vector space, it is an artinian analytic algebra (see e.g. [8]). As $(A_{m,n})_o$ is isomorphic to $K\{X_1,\ldots,X_m\} \otimes_K \Lambda_K(\xi_1,\ldots,\xi_n)_o$, the conclusion follows.

DEFINITION 18. A superanalytic algebra is the K-superalgebra $A := A_{m,n}/a$, where a is a superideal in $A_{m,n}$, $a \neq A_{m,n}$.

REMARK 14. K- superanalytic algebras and their (local K-superalgebra) morphisms form in a natural way a full subcategory of the category of local K--superalgebras of residue field K-isomorphic to K.

THEOREM 2. Let A be a commutative K-superalgebra. Then the following statements are equivalent:

i) A is a superanalytic algebra.

ii) A_o is a K-analytic algebra and A_1 is an A_o-module of finite type.

Proof. i)$\Rightarrow$ii) If we suppose that $A = A_{m,n}/a$ then A_o is the quotient of $(A_{m,n})_o$ by a_o. As $(A_{m,n})_o$ is a K-analytic algebra (by theorem 1), it follows that A_o is a K-analytic algebra (Note that $a \neq A_{m,n}$ implies $a_o \neq (A_{m,n})_o$). The other statement follows from the fact that $(A_{m,n})_1$ is an $(A_{m,n})_o$-module of finite type.

ii)$\Rightarrow$ i) There exists $m \in \mathbb{N}$ and a K-algebra morphism $f : K\{X_1,\ldots,X_m\} \to A_o$ which is onto. Take $\alpha_1,\ldots,\alpha_n$ to be generators for the A_o-module A_1. Then by proposition 12 there exists a K-superalgebra morphism $F : A_{m,n} \to A$ such that

$$F|_{K\{X_1,\ldots,X_m\}} = f \text{ and } F(\xi_i) = \alpha_i \quad i = 1,\ldots,n .$$

F is onto and, as kernels of K-superalgebra morphism are superideals, the statement follows.

DEFINITION 19. Let A and B be two local K-superalgebras of residue field K-isomorphic to K, and let $f : A \to B$ be a morphism. By means of f, B becomes on A--supermodule in a natural way.

f is called quasifinite iff $\dim_K B/m_A B < \infty$

f is called finite iff B is an A-supermodule of finite type.

THEOREM 3. Let A and B be superanalytic algebras. Consider the diagram (*) in § 1. Then the following statements are equivalent:

(i) f is finite.

(ii) f is quasifinite.

(iii) $\tilde{f}$ is finite.

(iv) $\tilde{f}$ is quasifinite.

(v) f_o is finite.

(vi) f_o is quasifinite.

Proof. We sketch the proof of some implications. Let us denote (A_1) by I_A, (B_1) by I_B, A/I_A by A_o, B/I_B by B_o. The diagram (*) becomes:

$$\begin{array}{ccc} A_o & \xrightarrow{f_o} & B_o \\ {\scriptstyle j_A}\downarrow & & \downarrow{\scriptstyle j_B} \\ A & \xrightarrow{f} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ A_o & \xrightarrow{\widetilde{f}} & B_o \end{array}$$

(ii)$\Longrightarrow$(iv) π_B induces a surjective K-vector space morphism

$$\widetilde{\pi}_B : B/m_A B \to B_o/m_{A_o} B_o \quad .$$

(iv)$\Longrightarrow$(iii) By theorem 2, A_o and B_o are K-analytic algebras. Then A_o and B_o are also K-analytic algebras (see also § 1). The result follows then by the classical theorem for analytic algebras (see [4]).

(iii)$\Longrightarrow$(i) There exists $r \in N$ such that $I_B^{r+1} = 0$. Consider then

$B_k := I_B^k / I_B^{k+1}$ for $k = 1, \ldots, r$ and

$$GrB := \bigoplus_{i=0}^{r} B_k$$

By means of f, GrB becomes an A_o module.

B_k are B_o modules of finite type. By the hypotheses it follows that GrB is an A_o module of finite type. It follows that (via π_A), GrB is an A-module of finite type.

A standard decreasing induction reasoning shows then that B is an A-module of finite type.

DEFINITION 20. Let A and B be two local K-superalgebras and $f : A \to B$ a K-superalgebra morphism. f has the Weierstrass property iff for each B-supermodule M of finite type the following statements are equivalent:

(i) M is an A-supermodule (via f) of finite type.

(ii) $\dim_K M/m_A M < \infty$

Remark (i) $\Longrightarrow$ (ii) is always true.

THEOREM 4. Let A and B be two superanalytic algebras and $f : A \to B$ a morphism between them. Then f has the Weierstrass property.

Proof

The proof is similar to the proof of (iii) $\Longrightarrow$ (i) in theorem 3 above by considering $\mathrm{Gr} M = \bigoplus_{k=1}^{r} M_k$ where

$$M_o := M/I_B M, \quad M_k = I_B^k M/I_B^{k+1} M.$$

REMARK 15.

1. An alternative proof of theorem 3 may be given along the following lines:

Due to proposition 12, morphisms from $A = A_{m,n}/a$ to $B := A_{p,q}/b$ "lift" to morphisms from $A_{m,n}$ to $A_{p,q}$. A standard reasoning allows then us to need to prove the equivalence of the finiteness to the quasifiniteness only for the case of $f : A_{m,n} \to A_{p,q}$; here the arguments from (iii) $\Longrightarrow$ (i) may be immediately transposed.

2. One can consider a superdifferentiable algebra to be the K-superalgebra $E = E_{m,n}/a$ where **a** is a superideal of finite type in $E_{m,n}$, $a \neq E_{m,n}$. E_o is then a differentiable algebra in the sense of Malgrange [6]. Morphisms have to be defined as having the "lifting" property mentioned above. Then, an analogous of the proof sketched in 1, above furnishes a proof to the equivalence of finiteness and quasi-finiteness of K-superdifferentiable algebras.

In the following, unless otherwise specified, by a morphism between two K-superanalytic algebras, we shall understand a K-superanalytic algebra morphism.

The following version of the normalization lemma holds:

PROPOSITION 13. Let A be a K-superanalytic algebra, $\mathbf{a}_1 \subset \ldots \subset \mathbf{a}_r$ be a chain of proper superideals in A. Then, there exist an unique $d \in \mathbf{N}$, a finite injective morphism $u_d : K\{X_1,\ldots,X_d\} \to A$, and for each $i \in \{1,\ldots,r\}$, there exists $h(i) \in \mathbf{N}$ such that $u_d^{-1}(\mathbf{a}_i) = (X_1,\ldots,X_{h(i)})$.

Proof. The idea is to show that d is the least among the natural numbers m for each there exists a finite morphism $u_m : K\{X_1,\ldots,X_m\} \to A$. For the second part, one need only consider the case $A = K\{X_1,\ldots,X_d\}$ and then the conclusion follows by [1], Ch. II, preliminaries.

PROPOSITION 14. Let A be a K-superanalytic algebra, **a**, **b**, be two proper prime superideals in **A**. Then, all maximal chains of prime superideals $\mathbf{a} = \mathbf{p}_0 \subset \mathbf{p}_1 \ \ldots \subset \mathbf{p}_r = \mathbf{b}$ have the same length r.

Proof. Prime superideals contain $A_1 + A_1^2$. If **p** is a proper prime superideal in A, then $\mathbf{p} \cap A_0$ is a proper prime ideal in A_0. Moreover, if $\tilde{\mathbf{p}}$ is a proper prime ideal in A_0, **q** a proper prime superideal in A with $\mathbf{q} \cap A_0 \supset \tilde{\mathbf{p}}$, then the proper prime superideal $\mathbf{p} := \tilde{\mathbf{p}} + A_1$ satisfies: $\mathbf{p} \subset \mathbf{q}$. Also if **p** and **q** are different prime superideals in A, then $\mathbf{p} \cap A_0 \neq \mathbf{q} \cap A_0$. The conclusion follows then as in [1] for the K-analytic algebra A_0.

By the Krull dimension of a commutative superring A (denoted by dim A) we mean the supremum length of strictly increasing chains of proper prime superideals of A.

REMARK 16. 1. If A is a commutative ring, considered as a superring by means of the trivial grading, dim A coincides with the usual Krull dimension.

2. $\dim A_{m,n} = \dim E_{m,n} = \dim F_{m,n} = m$.

PROPOSITION 15. Let A be a K-superanalytic algebra. Then $\dim A = \dim A_o = \dim A/I_A = d$ (where d is given by the normalization lemma).

Proof. The first equality holds by (the proof of) proposition 14. There exists a bijective correspondence between ideals in A/I_A and ideals in A_o which contain A_1^2. As prime ideals contain A_1^2, the second equality holds. The last equality follows by the normalization lemma.

REMARK 17. 1. For K-superanalytic algebras, proposition 12 may be restated as follows: If $m,n \in \mathbb{N}$, A is a K-superanalytic algebra, $(\alpha_i)_{i=1,\dots,m}$ are even elements in $\mathfrak{m}_A$ and $(\beta_j)_{j=1,\dots,n}$ are odd elements in A, there exists a unique morphism $u : A_{m,n} \to A$ such that $u(X_i) = \alpha_i, i = 1,\dots,m$ and $u(\xi_j) = \beta_j, j = 1,\dots,n$.

2. Finite fibered sums exist in the category of K-superanalytic algebras.

THEOREM 5. Let A be a K-superanalytic algebra. The following statements are equivalent:

i) There exist $m,n \in \mathbb{N}$ such that $A \simeq A_{m,n}$

ii) For each K-superanalytic algebras B and C, for each surjective morphism $u : C \to B$ and for each morphism $f : A \to B$, there exists a morphism $v : A \to C$ such that $f = u \circ v$ (i.e. A is a projective K-superanalytic algebra).

Proof. Due to proposition 12 and remark 17. 1, we need only to prove ii)$\Rightarrow$i). First, we claim that there exists $m \in \mathbb{N}$, such that $A/I_A \simeq K\{X_1,\dots,X_m\}$. Indeed, let us consider two K-analytic algebras B and C, a surjective morphism $u : C \to B$ and a morphism $\tilde{f} : A/I_A \to B$. Let $f : A \to B$ be given by $f := \tilde{f} \circ \pi_A$. There exists $v : A \to C$

such that $f = u \circ v$. As $I_C = 0$, there exists $\tilde{v} : A/I_A \to C$ such that $\tilde{v} \circ \pi_A = v$. It follows $\tilde{f} = u \circ \tilde{v}$. Hence, A/I_A is a projective K-analytic algebra and (e.g. by [8]) our claim follows. In the following we shall identify A/I_A to $K\{X_1,\dots,X_m\}$.

Next, by remark 17. 1, there exists $s_A : A/I_A \to A$ such that $\pi_A \circ s_A = 1_{A/I_A}$.

Now, let $\beta_1,\dots,\beta_n$ be a minimal system of elements in A_1, which generate A_1 as an A_0-module. By means of s_A, A becomes an A/I_A-module. It is a standard argument then, that A is generated as an A/I_A-module by $(\beta_I)_I$ (where, as usual, $\beta_\emptyset = 1$ and for $I = 1 \leq i_1 < \dots < i_k \leq n$, $\beta_I := \beta_{i_1} \cdot \dots \cdot \beta_{i_k}$). Consider the canonical embedding $j_{m,n} : K\{X_1,\dots,X_m\} \to A_{m,n}$. By proposition 12, there exists a morphism

$u : A_{m,n} = \Lambda_{A/I_A}(\xi_1,\dots,\xi_n) \to A$ such that: $u \circ j_{m,n} = s_A$, $u(\xi_i) = \beta_i$ for $i = 1,\dots,n$. u is obviously onto. We shall show now that u is injective.

By ii) there exists a morphism $v : A \to A_{m,n}$ such that $u \circ v = 1_A$. We show now that $\beta_1 \cdot \dots \cdot \beta_n \neq 0$. If this were not the case, then:

$$v(\beta_i) = \xi_i + \sum_{|I|=2k+1\leq n} \gamma_I^i \xi_I, \text{ where } \gamma_I^i \in K\{X_1,\dots,X_m\} \text{ and}$$

$$u\Big(\sum_{|I|=2k+1\leq n} \gamma_I^i \xi_I\Big) = 0 .$$

For $l \in \{1,2,\dots,n\}$, if we denote $\gamma_{\{l\}}^i$ by γ_l^i, we get:

$$(1) \qquad v(\beta_i) = \xi_i(1 + \gamma_i^i) + \sum_{j=1;j\neq i}^{n} \gamma_j^i \xi_j + \sum_{3\leq|I|=2k+1\leq n} \gamma_I^i \xi_I .$$

We state that for each i and l from $\{1,2,\dots,n\}$, $\gamma_l^i \in m_{K\{X_1,\dots,X_n\}}$. Indeed, by applying u in (1) we get

$$0 = \sum_{j=1}^{n} u(\gamma_j^i)\beta_j + \sum_{3\leq|I|=2k+1\leq n} u(\gamma_I^i)\beta_I$$

If $\gamma_l^i \notin m_{K\{X_1,\dots,X_m\}}$, then it is invertible; hence $u(\gamma_l^i)$ is invertible. One obtains β_l as a combination (with coefficients in A_o) of the other elements β_k, and this contradicts the minimality of the chosen system $\beta_1,\dots,\beta_n$ and hence our statement follows.

If $\beta_1 \cdot \dots \cdot \beta_n$ were equal to zero then, $v(\beta_1) \cdot \dots \cdot v(\beta_n) = 0$ and then we would obtain:

$$\xi_1 \cdot \dots \cdot \xi_n \cdot \delta = 0$$

where δ is a sum between $\prod_{i=1}^{n} (1 + \gamma_i^i)$ and products of factors of type $(1 + \gamma_l^l)$ and γ_k^j where at least one factor of type γ_k^j appears.

It follows that δ is invertible in $K\{X_1,\dots,X_m\}$ and then $\xi_1 \cdot \dots \cdot \xi_n = 0$ which cannot hold. Hence $\beta_1 \cdot \dots \cdot \beta_n \neq 0$.

Next, we show that if $\alpha \in A/I_A$ and $s_A(\alpha) \cdot \beta_1 \cdot \dots \cdot \beta_n = 0$ it follows $\alpha = 0$. We have: $(v \circ s_A)(\alpha) \cdot v(\beta_1) \cdot \dots \cdot v(\beta_n) = 0$, $v(\beta_1) \cdot \dots \cdot v(\beta_n) \neq 0$ and $(v \circ s_A)(\alpha) - j_{m,n}(\alpha) = \gamma := \gamma_o + \sum_{2 \leq |I| = 2k \leq n} \gamma_I \xi_I$ where $u(\gamma) = 0$.

It follows $j_{m,n}(\alpha) + \gamma_o = 0$. As $(\pi_A \circ u)(\gamma - \gamma_o) = 0$, then:

$$\alpha = (\pi_A \circ u \circ v \circ s_A)(\alpha) = (\pi_A \circ u)(j_{m,n}(\alpha) + \gamma_o) + (\pi_A \circ u)(\gamma - \gamma_o) = 0 .$$

Finally, we show that u is injective. Suppose $u(\sum_I \alpha_I \xi_I) = 0$. It follows:

(2) $$\sum_I s_A(\alpha_I)\beta_I = 0$$

By multiplying (2) by $\beta_1 \cdot \dots \cdot \beta_n$, we obtain $s_A(\alpha_o) \cdot \beta_1 \cdot \dots \cdot \beta_n = 0$. Hence $\alpha_o = 0$.

Inductively (with respect to $|I|$), by multiplying (2) by β_L (where $I \cup L = \{1,\dots,n\}$, $I \cap L = \emptyset$) we obtain as in the case $I = \emptyset$ above that $\alpha_I = 0$ for each I and hence u is injective. q.e.d.

An alternative point of view on superrings is as follows:

A commutative superring is a pair (A_o, A_1), where A_o is a commutative unitary ring (whose multiplication is denoted by μ_{A_o}) A_1 is a unitary. A_o-module

(A_o-scalar multiplication is denoted by μ_{A_o,A_1}) and there exists an A_o-module morphism $\mu_{A_1} : \Lambda^2 A_1 \to A_o$, and if we denote by $u : A_1 \times A_1 \to \Lambda^2 A_1$ the canonical map and by $\tilde{\mu}_{A_1} : A_1 \times A_1 \to A_o$ the A_o-module morphism given by $\tilde{\mu}_{A_1} := \mu_{A_1} \circ u$, then the following diagrams are commutative.

$$\begin{array}{ccc} A_1 \times A_1 \times A_1 & \xrightarrow{\mu_{A_1} \times 1_{A_1}} & A_o \times A_1 \\ {\scriptstyle 1_{A_1} \times \mu_{A_1}} \downarrow & & \downarrow {\scriptstyle \mu_{A_o,A_1}} \\ A_1 \times A_o & \xrightarrow[\mu_{A_o,A_1}]{} & A_1 \end{array}$$

$$\begin{array}{ccc} A_o \times A_1 \times A_1 & \xrightarrow{\mu_{A_o,A_1} \times 1_{A_1}} & A_1 \times A_1 \\ {\scriptstyle 1_{A_o} \times \mu_{A_o,A_1}} \downarrow & & \downarrow {\scriptstyle \tilde{\mu}_{A_1}} \\ A_o \times A_o & \xrightarrow[\mu_{A_o}]{} & A_o \end{array}$$

A superring morphism between (A_o,A_1) and (B_o,B_1) is a pair f_o,f_1, where $f_o : A_o \to B_o$ is a ring morphism and $f_1 : A_1 \to B_1$ is a morphism "over f_o" (i.e. f is a group morphism and the following diagrams

$$\begin{array}{ccc} A_o \times A_1 & \xrightarrow{f_o \times f_1} & B_o \times B_1 \\ {\scriptstyle \mu_{A_o,A_1}} \downarrow & & \downarrow {\scriptstyle \mu_{B_o,B_1}} \\ A_1 & \xrightarrow{f_1} & B_1 \end{array}$$

$$\begin{array}{ccc} \Lambda^2 A_1 & \xrightarrow{\Lambda^2 f_1} & \Lambda^2 B_1 \\ {\scriptstyle \mu_{A_1}} \downarrow & & \downarrow {\scriptstyle \mu_{B_1}} \\ A_o & \xrightarrow{f_o} & B_o \end{array}$$

are commutative).

REFERENCES

1. Bănică C. & Stănăşilă O.: Methodes algebriques dans la theorie globale des espaces complexes, Editura Academiei RSR et Gauthier-Villars Editeur, 1977.

2. Berezin, F.A.: Introduction in the algebra and analysis of anticommuting varibles (Russian), Moscow Univ. Press, 1983.

3. Bourbaki, N.: Algebre, ch. 2, Alg. lineaire, Hermann, 1962.

4. Jurchescu, M.: Introduzione agli spazi analitici 1. Quaderno dei Gruppi di Ricerca Matematica del C.N.R., Perugia 1971.

5. Leites, D.A.: Introduction in the theory of supermanifolds (Russian), Uspehi Mat. Nauk 35, 1 (1980), p. 3-37.

6. Malgrange, B.: Ideals of differentiable functions, Oxford Univ. Press, 1966.

7. Manin, Iu.I.: Gauge fields and complex geometry, Nauka, Moscow 1984.

8. Radu, N.: Local rings II (Romanian), Ed. Acad. R.S.R., Bucureşti, 1970.

9. Yankov, C.: Power series with coefficients in Grassmann algebra in "Complex Analysis and applications '85", Publishing House of Bulgarian Academy of Sciences, Sofia, 1986.