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I. SPLITTING DIFFERENTIAL ALGEBRAIC GROUPS
II. THE AUTOMORPHISM GROUP OF A NON-LINEAR ALGEBRAIC GROUP

by

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I.SPLITTING DIFFERENTIAL ALGEBRAIC GROUPS

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0. INTRODUCTION

Throughout this paper we use standard terminology of differential algebra from Kolchin's books [K_1][K_2]. So we denote by $\mathcal{U}$ a universal Δ-field of characteristic zero with field of constants $\mathcal{K}$ and consider a Δ-subfield $\mathcal{F}$ of $\mathcal{U}$ (over which $\mathcal{U}$ is universal) with field of constants $\mathcal{C}$. By a linear Δ-$\mathcal{F}$-group [C_1] we mean a Δ-$\mathcal{F}$-closed subgroup G of some $GL_n(\mathcal{U})$. Let's make the following:

DEFINITION. A linear Δ-$\mathcal{F}$-group $G \subset GL_n(\mathcal{U})$ is called split if it is of the form $G=G^* \cap GL_n(\mathcal{K})$ where G^* is a $\mathcal{C}$-closed subgroup of $GL_n(\mathcal{U})$ (G^* coincides then with the $\mathcal{C}$-closure of G in $GL_n(\mathcal{U})$). G is called splitable over an extension $\mathcal{F}_1$ of $\mathcal{F}$ if it is Δ-$\mathcal{F}_1$-isomorphic to a split linear Δ-$\mathcal{F}$-group; it will be called splitable if it is splitable over some extension of $\mathcal{F}$.

Splitable groups naturally appeared in Cassidy's work [C_1][C_2][C_3] on semisimple and unipotent Δ-algebraic group. To simplify our exposition assume throught the paper that $\mathcal{F}$ is algebraically closed. Moreover we will concentrate ourselves on irreducible linear Δ-$\mathcal{F}$-groups. Clearly, if such a group G is splitable then tr.deg. $\mathcal{F}\langle G\rangle/\mathcal{F} < \infty$. The converse fails as shown by the example of the Δ-subgroup of $GL_1(\mathcal{U})$ defined by the equation $y''y-(y')^2=0$ (cf. (2.2) below). Our aim in this paper is to exhibit a large class of G's for which the converse holds.

Recall that for a linear Δ-$\mathcal{F}$-group $G \subset GL_n(\mathcal{U})$ the set of all Δ-closed normal irreducible solvable sugroups of G has a unique maximal element which obviously is Δ-$\mathcal{F}$-closed and will be called the radical of G. Moreover a linear Δ-$\mathcal{F}$-group $G \subset GL_n(\mathcal{U})$ is called unipotent (cf. [C_1]) if it consists of unipotent matrices. Now we can state our

MAIN THEOREM. Let G be an irreducible linear Δ-$\mathcal{F}$-group with tr.deg. $\mathcal{F}\langle G\rangle/\mathcal{F}<\infty$. If the radical of G is unipotent then G is splitable over some Picard-Vessiot extension of $\mathcal{F}$.

The "extreme" case when G is semisimple is due to P. Cassidy [C_2] ; however we won't use here her results and develop instead a quite different method based on the interplay between differential algebra and the Hopf-algebra machinery in [H]. Our method has an interest in itself since it relates splitability with representation theory of Lie algebras and with representative functions. Consequently we will borrow our terminology of affine algebraic groups from [H] (rather than from [K_1]).

So $\mathcal{L}(A)$, $\mathcal{L}(\mathcal{G})$ denote the Lie algebra associated to an associative algebra A and to an affine algebraic group $\mathcal{G}$ respectively. $\mathcal{P}(\mathcal{G})$ will denote the affine Hopf algebra associated to $\mathcal{G}$. Moreover $\mathcal{G}(H)$ will denote the affine algebraic group associated to an affine Hopf algebra H; the letter $\mathcal{G}$ will never be used to denote a Δ-field (like in [C_1], [K_1]). To avoid confusion with our universal Δ-field $\mathcal{U}$ we denote by U(L) (rather than $\mathcal{U}$(L) as in [H]) the universal enveloping algebra of the Lie algebra L. Finally note that by a Δ-Lie $\mathcal{F}$-algebra we understand a Δ-algebra over $\mathcal{F}$ which is a Lie algebra; this is the concept in [B_1] and is different from that of Δ-$\mathcal{F}$-Lie algebra in [C_2][K_2].

1. FINITE GENERATION

The first step in our approach is the following

(1.1) THEOREM. Let G be an irreducible linear Δ-$\mathcal{F}$-group with tr.deg. $\mathcal{F}\langle G\rangle/\mathcal{F}<\infty$. Then the Δ-coordinate algebra $\mathcal{F}\{G\}$ is finitely generated as a (non-differential) Δ-algebra.

The above theorem allows us to consider the affine algebraic $\mathcal{F}$-group $\mathcal{G}(\mathcal{F}\{G\})$, where $\mathcal{F}\{G\}$ is viewed with the natural Hopf Δ-algebra structure induced from that of $\mathcal{F}\{GL_n(\mathcal{U})\}$ via the given embedding $G\subset GL_n(\mathcal{U})$; similarily we get an affine algebraic $\mathcal{U}$-group $\mathcal{G}(\mathcal{U}\{G\})$. Note that G can be naturally seen as a subgroup of $\mathcal{G}(\mathcal{U}\{G\})$ via the identifications:

$$G=\mathrm{Hom}_{\Delta\text{-alg}}(\mathcal{U}\{G\},\mathcal{U})$$

$$\mathcal{G}(\mathcal{U}\{G\})=\mathrm{Hom}_{\mathrm{alg}}(\mathcal{U}\{G\},\mathcal{U})$$

For the proof of (1.1) we need the following lemma (in which Q(R) means the quotient field of the integral domain R):

(1.2) LEMMA. Let $\mathcal{A}\subset\mathcal{B}$ be a Δ-finitely generated extension of Δ-$\mathbb{Q}$-algebras. Assume $\mathcal{B}$ is an integral domain and tr.deg. $Q(\mathcal{B})/Q(\mathcal{A})<\infty$. Then there exists a non-zero element $s\in\mathcal{B}$ such that $\mathcal{B}[1/s]$ is finitely generated over $\mathcal{A}$ as a (non-differential) algebra.

Proof. Proceeding by induction on the number of Δ-generators of $\mathcal{B}$ over $\mathcal{A}$ we may assume that $\mathcal{B}=\mathcal{A}\{b\}$. Let Θ denote as usual the free commutative monoid built on Δ. If $\theta=\delta_1^{\alpha_1}\dots\delta_m^{\alpha_m}$ and $\eta=\delta_1^{\beta_1}\dots\delta_m^{\beta_m}$ we write $\theta<\eta$ if and only if $(\sum\alpha_i,\alpha_1,\dots,\alpha_m)<$ $<(\sum\beta_i,\beta_1,\dots,\beta_m)$ in the lexicographic order. We write $\theta\leq\eta$ if either $\theta<\eta$ or $\theta=\eta$. Finally we write $\theta\leqq\eta$ if $\alpha_i\leq\beta_i$ for all i. For any $\theta\in\Theta$ put $\mathcal{B}^\theta=\mathcal{A}[\eta b;\eta<\theta]$ and construct inductively (with respect to $\leq$) subsets Σ^θ of Θ in the following way: $\Sigma^0=\emptyset$ and if

η is the succesor of θ put

$$\Sigma^{\eta} = \begin{cases} \Sigma^{\theta} & \text{if } \eta b \text{ is algebraic over } \mathcal{B}^{\theta} \\ \Sigma^{\theta} \cup \{\eta\} & \text{if } \eta b \text{ is transcendental over } \mathcal{B}^{\theta} \end{cases}$$

Put $\Sigma = \bigcup_{\theta} \Sigma^{\theta}$, $\Lambda = \Theta \setminus \Sigma$ and let Λ_{min} be the set of minimal elements of Λ with respect to the order "$\leq$". Clearly Λ_{min} is a finite set $\{\theta_1, \ldots, \theta_M\}$. Define $R = \mathcal{A}[\theta b, \theta \in \Sigma]$; it is a polynomial algebra over $\mathcal{A}$ (in finitely many variables), which is not a Δ-subalgebra of $\mathcal{B}$. Now for any $i \in \{1, \ldots, M\}$ let F_i be a non-zero polynomial of minimum degree in $\mathcal{B}^{\theta_i}[T]$ such that $F_i(\theta_i b) = 0$. Since $\mathbb{Q} \subset \mathcal{A}$, $dF_i/dT \neq 0$ and so $s_i = (dF_i/dT)(\theta_i b) \in \mathcal{B}^{\theta_i}[\theta_i b]$, $s_i \neq 0$ hence $s = s_1 \ldots s_M \in \mathcal{B}$ is a non-zero element. Now it is easy to check that $\theta b \in \mathcal{B}^{\theta}[\theta_i b, 1/s_i]$ for all $1 \leq i \leq M$ and $\theta_i \leq \theta$.

This immediately implies that $\mathcal{B}[1/s] = R[\theta_1 b, \ldots, \theta_M b, 1/s_1, \ldots, 1/s_M]$ and we are done.

(1.3) Proof of Theorem (1.1). We may assume $\mathcal{F}$ is uncountable. By the above Lemma, the scheme $X = \operatorname{Spec} \mathcal{R}$ ($\mathcal{R} = \mathcal{F}\{G\}$) contains an open set X_o of finite type over $\mathcal{F}$. Now X is a group scheme over $\mathcal{F}$. Let $M_1 \in X \setminus X_o$ and look for a neighbourhood X_1 of M_1 of finite type over $\mathcal{F}$. We may assume M_1 is a maximal ideal. Since $\mathcal{F}$ is algebraically closed, uncountable and $\mathcal{R}/M_1$ is a countably generated $\mathcal{F}$-vector space, a well known argument shows that $\mathcal{R}/M_1 \simeq \mathcal{F}$, hence $M_1 = \ker g_1$ for some $\mathcal{F}$-point of X, $g_1 \in X(\mathcal{F})$. Now take any $g_o \in X(\mathcal{F})$ such that $M_o = \ker g_o \in X_o$ and conclude by letting X_1 be the image of X_o via translation from the right with $g_1 g_o^{-1} \in X(\mathcal{F})$.

(1.4) Note that if we are given a Δ-$\mathcal{F}$-isomorphism $G \to G'$ between linear Δ-$\mathcal{F}$-groups with tr.deg. $\mathcal{F}\langle G\rangle/\mathcal{F} < \infty$ we get an induced birational map from $\mathcal{G}(\mathcal{F}\{G\})$ to $\mathcal{G}(\mathcal{F}\{G'\})$ which agrees with multiplication maps whenever operations make sense. Such a map must be an isomorphism (cf. [L] p. 5). This is a remarkable property which does not hold apriori for Δ-$\mathcal{F}$-groups of "infinite transcendence degree".

2. SPLITTING

In this section we prove our Main Theorem.

A Δ-$\mathcal{F}$-vector space V is said to split over an extension $\mathcal{F}_1$ of $\mathcal{F}$ if $V \otimes \mathcal{F}_1$ possesses an $\mathcal{F}_1$-basis (e_α) with $\delta e_\alpha = 0$ for all $\delta \in \Delta$. Start recalling from [B] p. 79 a basic fact on splitting Δ-$\mathcal{F}$-vector spaces (cf. [T] for a generalisation to Hopf algebra actions more general than derivations).

(2.1) LEMMA. Any finite dimensional Δ-$\mathcal{F}$-vector space splits over some Picard-Vessiot extension of $\mathcal{F}$.

For the sake of completness we give the argument. If V is a Δ-$\mathcal{F}$-vector space with basis $e_1, \dots, e_N$, write $\delta_k e_i = \sum a^k_{ij} e_j$.

Let $a^k = (a^k_{ij})$ be viewed as an element of $\mathfrak{gl}_N(\mathcal{F})$. Commutativity of the δ_k's implies that $\delta_p a^k - \delta_k a^p + [a^k, a^p] = 0$ for all p and k. By Kolchin's surjectivity of the logarithmic derivative (cf. its form in [B_1] p. 51, Corollary (2.9)) there is a Picard-Vessiot extension $\mathcal{F}_1/\mathcal{F}$ and a matrix $g = (g_{ij}) \in GL_N(\mathcal{F}_1)$ such that $\delta_k g = a^k g$ for all k. Now the elements $f_1, \dots, f_N$ of $V \otimes \mathcal{F}_1$ defined by $e_i = \sum g_{ij} f_j$ clearly form an $\mathcal{F}_1$-basis of the latter space and we easily check $\delta_k f_j = 0$ for all k and j.

The next lemma translates splitability in terms of locally finitness; recall that a Δ - $\mathcal{F}$-vector space V is called locally finite if it a union of Δ - $\mathcal{F}$-vector spaces of finite dimension.

(2.2) LEMMA. Let G be a connected linear Δ - $\mathcal{F}$-group with tr.deg. $\mathcal{F}\langle G\rangle/\mathcal{F}<\infty$. Then the following are equivalent:

1) G is splitable.

2) G is splitable over some Picard-Vessiot extension of $\mathcal{F}$.

3) The Δ-coordinate algebra $\mathcal{F}\{G\}$ is locally finite as a Δ - $\mathcal{F}$-vector space.

Proof. 2) $\Longrightarrow$ 1) is trivial.

1) $\Longrightarrow$ 3) Assume G is Δ - $\mathcal{F}_1$-isomorphic ($\mathcal{F}_1$ algebraically closed) with a split Δ - $\mathcal{F}$-group $H \subset GL_m(\mathcal{U})$. In order to prove that $\mathcal{F}\{G\}$ is locally finite as a Δ - $\mathcal{F}$-vector space it is sufficient to check that $\mathcal{F}_1\{G\}$ is locally finite as a Δ - $\mathcal{F}_1$-vector space. By (1.4) we have $\mathcal{F}_1\{G\} \simeq \mathcal{F}_1\{H\}$ so it is sufficient to check that $\mathcal{F}\{H\}$ is locally finite as a Δ - $\mathcal{F}$-vector space. Write $H=H^* \cap GL_m(\mathcal{K})$, hence $\mathcal{F}\{H\} =$ $= \mathcal{F}\{y\}/[\delta y, g_\alpha] = \mathcal{F}[y]/(g_\alpha)$ where $y=(y_{ij})$ and $g_\alpha \in \mathcal{C}[y]$; now conclude by noting that the $\mathcal{F}$-linear subspaces of $\mathcal{F}[y]/(g_\alpha)$ generated by monomials in y of bounded degree are Δ - $\mathcal{F}$-vector subspaces.

3) $\Longrightarrow$ 2) There exists a finite dimensional Δ - $\mathcal{F}$-vector subspace V of $\mathcal{F}\{G\}$ generating $\mathcal{F}\{G\}$ as an $\mathcal{F}$-algebra. By Lemma (2.1) V splits over some Picard-Vessiot extension $\mathcal{F}_1$ of $\mathcal{F}$. It follows that the whole of $\mathcal{F}\{G\}$ splits over $\mathcal{F}_1$. So upon letting $\mathcal{R} = \mathcal{F}_1\{G\}$ we have $\mathcal{R} = \mathcal{R}^\Delta \otimes_{\mathcal{C}} \mathcal{F}_1$ where the upper Δ means "taking constants". Clearly $\mathcal{R}^\Delta$ is a $\mathcal{C}$-subalgebra of $\mathcal{R}$. Since $(\mathcal{R} \otimes_{\mathcal{F}_1} \mathcal{R})^\Delta = \mathcal{R}^\Delta \otimes_{\mathcal{C}} \mathcal{R}^\Delta$, the comultiplication map $\mathcal{R} \to \mathcal{R} \otimes_{\mathcal{F}_1} \mathcal{R}$ takes $\mathcal{R}^\Delta$ into $\mathcal{R}^\Delta \otimes_{\mathcal{C}} \mathcal{R}^\Delta$ so $\mathcal{R}^\Delta$ becomes a finitely generated Hopf $\mathcal{C}$-algebra. Take any embedding

$\mathcal{G}(\mathcal{R}^{\Delta}) \subset GL_n(\mathcal{C})$ and let H* be the $\mathcal{C}$-closure of $\mathcal{G}(\mathcal{R}^{\Delta})$ in $GL_n(\mathcal{U})$ and $H=H^* \cap GL_n(\mathcal{K})$. Then it is trivial to check that G is Δ-$\mathcal{F}_1$-isomorphic with H. This closed the proof of the lemma.

Let's apply the implication 1) $\Longrightarrow$ 3) above to show that the Δ-subgroup G of $GL_1(\mathcal{U})$ defined by $y''y-(y')^2=0$ is not splitable. Indeed $\mathcal{F}\{G\}=\mathcal{F}[y,1/y,y']$. Put $\gamma=y'/y$; then $\gamma'=0$ so for all $n \geq 0$ $y^{(n+1)}=\gamma^n y'$ which shows that $\mathcal{F}\{G\}$ is not locally finite as a Δ-$\mathcal{F}$-vector space.

(2.3) Let V, W be Δ-$\mathcal{F}$-vector spaces. Recall that $V \otimes_{\mathcal{F}} W$ and $\mathrm{Hom}_{\mathcal{F}}(V,W)$ have natural structures of Δ-$\mathcal{F}$-vector spaces given by $\delta(x \otimes y)=(\delta x) \otimes y+x \otimes (\delta y)$ and $(\delta f)(x)=\delta(f(x))-f(\delta x)$ for $x \in V$, $y \in W$, $f \in \mathrm{Hom}_{\mathcal{F}}(V,W)$, $\delta \in \Delta$; in particular the linear dual $V^o=\mathrm{Hom}_{\mathcal{F}}(V,\mathcal{F})$ is a Δ-$\mathcal{F}$-vector space. Note that if V and W are locally finite, so is $V \otimes_{\mathcal{F}} W$; but $\mathrm{Hom}_{\mathcal{F}}(V,W)$ and V^o need not be locally finite.

Now start with a finite dimensional Δ-Lie $\mathcal{F}$-algebra L. Then the universal enveloping algebra U(L) inherits from the tensor algebra $\otimes(L)$ a structure of Δ-$\mathcal{F}$-algebra. So the dual $U(L)^o$ becomes a Δ-$\mathcal{F}$-vector space which is easily seen to be a Δ-$\mathcal{F}$--algebra with respect to convolution. Inside $U(L)^o$ lies the continuous dual $U(L)'$ (cf. [H] p.228); recall that $U(L)'$ is defined as the space of functionals whose kernel contains some two-sided ideal of finite codimension and that $U(L)'$ is a subalgebra of $U(L)^o$. One checks that $U(L)'$ is preserved by Δ: if $f \in U(L)'$ vanishes on an ideal J then δf must vanish on J^2. But even $U(L)'$ need not be locally finite (e.g. take L to be abelian of dimension ≥ 2).

Next assume the radical L_r of L is nilpotent and denote it by R. Then in $U(L)'$ lies the algebra $\mathcal{B}(L)$ of R-nilpotent representative functions, which by definition is the space of all functionals in $U(L)'$ vanishing on some power of $R\cdot U(R)$ (cf. [H] p.258). We claim that $\mathcal{B}(L)$ is preserved by Δ. Indeed this follows from:

(2.4) LEMMA. If L is a Δ-Lie $\mathcal{F}$-algebra (of finite dimension), its radical R is a Δ-ideal.

Proof. By (2.1) L splits over some Picard-Vessiot extension $\mathcal{F}_1$ so $L\otimes \mathcal{F}_1=L_o\otimes_{\mathcal{C}}\mathcal{F}_1$ where $L_o=(L\otimes_{\mathcal{F}}\mathcal{F}_1)^{\Delta}$. Let R_o be the radical of L_o. Then both $R_o\otimes_{\mathcal{C}}\mathcal{F}_1$ and $R\otimes_{\mathcal{F}}\mathcal{F}_1$ coincide with the radical of $L\otimes_{\mathcal{F}}\mathcal{F}_1$. Now $R=(R\otimes_{\mathcal{F}}\mathcal{F}_1)\cap L=(R_o\otimes_{\mathcal{C}}\mathcal{F}_1)\cap L$ and the latter space clearly is preserved by Δ.

(2.5) PROPOSITION. If L is a Δ-Lie $\mathcal{F}$-algebra (of finite dimension) whose radical is nilpotent, $\mathcal{B}(L)$ is locally finite as a Δ-$\mathcal{F}$-vector space.

Proof. First we claim that one can assume L splits over $\mathcal{F}$. Indeed by (2.1) L splits over some $\mathcal{F}_1$; suppose we know that $\mathcal{B}(L\otimes\mathcal{F}_1)=\cup V_\alpha$ where the V_α's are finite dimensional Δ-$\mathcal{F}_1$-vector subspaces of $U(L\otimes\mathcal{F}_1)^o$. Then $\mathcal{B}(L)=\cup(V_\alpha\cap\mathcal{B}(L))$; but one checks that $\dim_{\mathcal{F}}(V_\alpha\cap\mathcal{B}(L))\leq\dim_{\mathcal{F}_1}V_\alpha$ and our claim is proved.

So assume $L=L_o\otimes_{\mathcal{C}}\mathcal{F}$, $L_o=L^{\Delta}$. Let $L_o=R_o+S_o$ where R_o is the radical of L_o and S_o is a complementary semisimple Lie $\mathcal{C}$-algebra;

then $R=R_o \otimes_{\mathscr{C}} \mathscr{F}$ is the radical of L and $S=S_o \otimes_{\mathscr{C}} \mathscr{F}$ is a complementary semisimple Lie $\mathscr{F}$-algebra, both R and S being Δ - $\mathscr{F}$-vector subspaces of L. Recall by [H] pp. 256-259 that the multiplication map $\mu: (U(L)')^R \otimes {}^S(U(L)') \longrightarrow U(L)'$ is an isomorphism of $\mathscr{F}$-algebras where $(U(L)')^R$ is the R-annihilated subalgebra of $U(L)'$ with respect to the left L-module structure of $U(L)'$ defined by $(x\cdot f)(u)=f(ux)$ $(x \in L,\ f \in U(L)',\ u \in U(L))$ and ${}^S(U(L)')$ is the S-annihilated subalgebra of $U(L)'$ with respect to the right L-module structure of $U(L)'$ defined by $(f\cdot x)(u)=f(xu)$ $(x \in L,\ f \in U(L)',\ u \in U(L))$. Moreover the following properties hold:

1) The isomorphism μ induces an $\mathscr{F}$-algebra isomorphism $\tilde{\mu}: (U(L)')^R \otimes {}^S(\mathscr{B}(L)) \rightarrow \mathscr{B}(L)$,

2) $(U(L)')^R$ coincides with the image of the natural injection $\alpha : U(S)' \longrightarrow U(L)'$ and

3) The restriction map $U(L)' \longrightarrow U(R)'$ induces an isomorphism $\beta : {}^S(\mathscr{B}(L)) \longrightarrow \mathscr{B}(R)$, where $\mathscr{B}(R)$ is the algebra of R-nilpotent representative functions on $U(R)$.

Since S is a Δ-subalgebra of L it follows that ${}^S(U(L)')$ is a Δ - $\mathscr{F}$-vector subspace of $U(L)'$ so one sees that the map μ is a Δ-map, hence so is $\tilde{\mu}$. Since α and β above are obviously Δ-maps it follows that the induced isomorphism of $\mathscr{F}$-algebras $\mathscr{B}(L) \simeq$ $\simeq U(S)' \otimes \mathscr{B}(R)$ is a Δ-map. So it is sufficient to check that each of $U(S)'$ and $\mathscr{B}(R)$ are locally finite as Δ - $\mathscr{F}$-vector spaces. Now $\mathscr{B}(R)=\cup V_n$ where V_n is the subspace of all functionals on $U(R)$ vanishing on $(R\cdot U(R))^n$; clearly V_n are finite dimensional Δ - $\mathscr{F}$-vector subspaces of $\mathscr{B}(R)$. To check the assertion for $U(S)'$ we prove the following (apriori) more general:

(2.6) LEMMA. Let S be Δ-Lie $\mathcal{F}$-algebra (of finite dimension). Assume that for any S-module V of finite dimension we have $\mathrm{Ext}^1_S(V,V)=0$. Then $U(S)'$ is locally finite as a Δ-$\mathcal{F}$-vector space.

Proof. We have $U(S)'=\cup V_J$ where J runs through the set Σ of all two-sided ideals of finite codimension and $V_J=\{f\in U(S)';\ f(J)=0\}$. We shall be done if we show that the V_J's are preserved by Δ. For this it is sufficient to check that any ideal $J\in\Sigma$ is a Δ-ideal.

Let $J\in\Sigma$, put $N=\dim_{\mathcal{F}} U(S)/J$, let $V=\mathcal{F}^N$ viewed with its natural structure of Δ-$\mathcal{F}$-vector space and fix an $\mathcal{F}$-linear isomorphism $V\cong U(S)/J$. Moreover consider the algebra map $\varphi: U(S)\longrightarrow \mathrm{End}(V)$ which takes any $u\in U(S)$ into the endomorphism of V corresponding to the multiplication from the left by u in $U(S)/J$; clearly $\ker\varphi=J$. Now φ restricted to S yields a representation $\rho: S\longrightarrow gl(V)$. Since $\mathrm{Hom}_{\mathcal{F}}(S, gl(V))$ is a Δ-$\mathcal{F}$-vector space we may consider for any $\delta\in\Delta$ the linear map $\delta\rho\in \mathrm{Hom}_{\mathcal{F}}(S, gl(V))$. It is easy to check that $\delta\rho$ are in fact cocycles for S in $gl(V)$ where $gl(V)$ is viewed as an S-module via the representation $S\xrightarrow{\rho} gl(V)\xrightarrow{\mathrm{ad}} gl(gl(V))$. Since $\mathrm{Ext}^1_S(V,V)=H^1(S, gl(V))$ is assumed to vanish, $\delta\rho$ must be coboundaries so there exist $h_1,\dots,h_m\in gl(V)$ such that for any $x\in S$:

$$\delta_i(\rho(x))-\rho(\delta_i x)=[\rho(x),h_i]$$

For ecah index i consider the $\mathcal{F}$-linear maps $D_1,D_2: U(S)\longrightarrow \mathrm{End}(V)$ defined by $D_1(u)=\delta_i(\varphi(u))-\varphi(\delta_i u)$ and $D_2(u)=\varphi(u)h_i-h_i\varphi(u)$. One checks that both D_1 and D_2 are φ-derivations i.e. satisfy the formula:

$$D(uv)=D(u)\varphi(v)+\varphi(u)D(v), \qquad u,v\in U(S)$$

where $D=D_1,D_2$. Since D_1 and D_2 agree on S they agree on all of U(S); but this shows that if $\varphi(u)=0$ for some $u\in U(S)$ then $\varphi(\delta_i u)=0$. Since this holds for all indices i, $\ker\varphi$ is a Δ-ideal and we are done.

(2.7) Next we relate groups and Lie algebras. Start with an irreducible linear Δ-$\mathcal{F}$-group G and let $\mathcal{G}=\mathcal{G}(\mathcal{F}\{G\})$, so $\mathcal{P}(\mathcal{G})=\mathcal{F}\{G\}$. Let's put a structure of Δ-Lie $\mathcal{F}$-algebra on $\mathcal{L}(\mathcal{G})$ as follows. First consider the Δ-$\mathcal{F}$-vector space structure on $\mathcal{P}(\mathcal{G})^{\circ}$; next check that with respect to convolution $\mathcal{P}(\mathcal{G})^{\circ}$ becomes a Δ-$\mathcal{F}$-algebra, hence $\mathcal{L}(\mathcal{P}(\mathcal{G})^{\circ})$ becomes a Δ-Lie $\mathcal{F}$-algebra. Finally check that $\mathcal{L}(\mathcal{G})$ (which is defined as a Lie subalgebra of $\mathcal{L}(\mathcal{P}(\mathcal{G})^{\circ})$ cf. [H] p.36) is preserved by Δ. From this construction we see that the naturally induced embedding $e_{\mathcal{G}}:\mathcal{P}(\mathcal{G})\longrightarrow$ $\longrightarrow U(\mathcal{L}(\mathcal{G}))'$ [H] p. 230 is a Δ-algebra map.

(2.8) LEMMA. Let $\mathcal{G}_r$ be the radical of $\mathcal{G}$ and G_r be the radical of G. Then:

1) The defining ideal of $\mathcal{G}_r$ in $\mathcal{P}(\mathcal{G})$ is a Δ-ideal.

2) $\mathcal{G}(\mathcal{U}\{G_r\})=\mathcal{G}(\mathcal{U}\{G\})_r$.

3) G_r is unipotent if and only if $\mathcal{G}_r$ is unipotent.

Proof. 1) Consider the embeddings $e_{\mathcal{G}}$ and $e_{\mathcal{G}_r}:\mathcal{P}(\mathcal{G}_r)\longrightarrow$ $\longrightarrow U(\mathcal{L}(\mathcal{G}_r))'$ as inclusions. Then the defining ideal of $\mathcal{G}_r$ in $\mathcal{P}(\mathcal{G})$ is precisely the intersection (taken in $U(\mathcal{L}(\mathcal{G}))'$) of $\mathcal{P}(\mathcal{G})$ with the kernel of the map

$$\pi: U(\mathcal{L}(\mathcal{G}))'\longrightarrow U(\mathcal{L}(\mathcal{G}_r))'=U(\mathcal{L}(\mathcal{G})_r)'$$

But since by (2.4) $\mathcal{L}(\mathcal{G})_r$ is a Δ-ideal in $\mathcal{L}(\mathcal{G})$, π is a Δ-map and we are done.

2) We have group inclusions

$$\begin{array}{ccc} G_r & \subset & \mathcal{G}(\mathcal{U}\{G_r\}) \\ \cap & & \cap \\ G & \subset & \mathcal{G}(\mathcal{U}\{G\}) \end{array}$$

From the fact that G_r (respectively G) is Zariski-dense in $\mathcal{G}(\mathcal{U}\{G_r\})$ (respectively in $\mathcal{G}(\mathcal{U}\{G\})$), it follows immediately that $\mathcal{G}(\mathcal{U}\{G_r\})$ is an irreducible normal solvable subgroup of $\mathcal{G}(\mathcal{U}\{G\})$ hence it is contained in $\mathcal{G}(\mathcal{U}\{G\})_r$. On the other hand, by assertion 1) the Δ-$\mathcal{F}$-group $G' = \mathcal{G}(\mathcal{U}\{G\})_r \cap G$ is irreducible and dense in $\mathcal{G}(\mathcal{U}\{G\})_r$. Clearly G' is normal in G and solvable so $G' \subset G_r$. Taking Zariski closure we get $\mathcal{G}(\mathcal{U}\{G\})_r \subset \mathcal{G}(\mathcal{U}\{G_r\})$ and we are done.

3) If G_r is unipotent, $\mathcal{U}\{G\}$ is locally unipotent as a G_r-module [H] p. 65 so it will also be so as a $\mathcal{G}(\mathcal{U}\{G\})_r$-module by assertion 2). So $\mathcal{G}(\mathcal{U}\{G\})_r$ (and hence also $\mathcal{G}_r$) is unipotent. The converse is obvious.

(2.9) We are in a position to conclude the proof of the Main Theorem. Indeed if G_r is unipotent, by Lemma (2.8) above $\mathcal{G}$ has a unipotent radical. By [H] p.260 the image of $\mathcal{P}(\mathcal{G})$ via the map $e: \mathcal{P}(\mathcal{G}) \longrightarrow U(\mathcal{L}(\mathcal{G}))'$ is contained in $\mathcal{B}(\mathcal{L}(\mathcal{G}))$. Since by Proposition (2.5), $\mathcal{B}(\mathcal{L}(\mathcal{G}))$ is locally finite as a Δ-$\mathcal{F}$-vector space so will be $\mathcal{P}(\mathcal{G})$ and we may conclude by Lemma (2.2).

3. FINAL REMARK

In proving our Lemma (1.2) we in fact proved the following useful "dévisage" property: let $\mathcal{A} \subset \mathcal{B}$ be an extension of integral Δ-$\mathbb{Q}$-algebras such that $\mathcal{B}$ is Δ-generated over $\mathcal{A}$ by one element; then there exists a non-zero element $s \in \mathcal{B}$ and a (non-differential) sub $\mathcal{A}$-algebra R of $\mathcal{B}[1/s]$ such that R is a polynomial

$\mathcal{A}$-algebra (in possibly infinitely many variables) and $\mathcal{B}[1/s]$ is finitely generated as a (non-differential) R-algebra. Here is an application. Let $t \in \mathcal{B}$, $t \neq 0$; since $\mathcal{B}[1/st]$ is finitely generated as an R-algebra there is a non-zero element $F \in R$ such that any prime in R not containing F is the trace on R of some prime in $\mathcal{B}[1/t]$. Viewing F as a polynomial with coefficients in $\mathcal{A}$ and picking any non-zero coefficient f of it we get that any prime P in $\mathcal{A}$ not containing f is the trace on $\mathcal{A}$ of some prime Q in $\mathcal{B}$ not containing t, i.e. the ring $B[1/t] \otimes_{\mathcal{A}} Q(\mathcal{A}/P)$ is non-zero. But if P is a Δ-ideal the latter ring is a Δ-$\mathbb{Q}$-algebra hence possesses at least one prime Δ-ideal. Consequently Q above can be chosen to be a Δ-ideal. Using an obvious induction we get a quite elementary and short proof of Seidenberg's theorem on "extending differential specialisations" (cf. [K_1] p. 140 for an arbitrary characteristic generalisation) saying if $\mathcal{A} \subset \mathcal{B}$ is a Δ-finitely generated extension of integral Δ-$\mathbb{Q}$-algebras then for any non-zero $t \in \mathcal{B}$ there exists a non-zero $f \in \mathcal{A}$ such that any prime Δ-ideal in $\mathcal{A}$ not containing f is the trace in $\mathcal{A}$ of some prime Δ-ideal in $\mathcal{B}$ not containing t. Now exactly as in [B_2] this implies a "differential Chevalley constructibility theorem".

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THE AUTOMORPHISM GROUP OF A NON-LINEAR ALGEBRAIC GROUP

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The aim of this paper is to prove the following:

Theorem. Let G be an algebraic group (non necessary linear) over a field k of characteristic zero. Then:

1) Aut G is a locally algebraic group,

2) Aut° G is linear,

3) If L is the largest connected linear subgroup of G and $A = G/L$ then the kernel of the homomorphism Aut G $\to$ Aut L $\times$ Aut A is an algebraic group.

Note that assertion 1) above answers a question of Borel and Serre in [BS] p. 152. Moreover 3) shows in particular that if Aut L and Aut A are algebraic groups, so is Aut G.

In [BS] the above theorem is proved under the assumption that G is linear. Our terminology and background are those of [BS]; in particular all locally algebraic schemes are assumed to be geometrically reduced and the assertion "Aut G is a (locally) algebraic group" means that the corresponding functor defined on the category of locally algebraic schemes is representable by a (locally) algebraic group. As explained in [BS] it is not reasonable to look for representability in the category of non-reduced schemes. Nor should one expect that for non-linear G, Aut G is an extension of an arithmetic group by an algebraic group.

The main ingredient in the proof of the above theorem will be our construction in [Bu] p.96 of an equivariant completion $\overline{G}$ of G; the idea is to show that, upon choosing $\overline{G}$ carefully, any automorphism of G which can be "connected" with the identity lifts to an automorphism of $\overline{G}$.

The proof of the theorem will be done in several steps.

§ 1. Assume first that k is algebraically closed and let Y be any locally algebraic scheme acting on G in the sense of [BS]. Then Y will also act on L and since by [BS]

Aut L is a locally algebraic group there is an induced morphism $\gamma : Y \longrightarrow \mathrm{Aut}\, L$. Assume that $\gamma(Y) \subset \mathrm{Aut}^\circ L$. Under this hypothesis one can put a Y-action on Chevalley's construction of orbit spaces as follows. Start with a finite dimension al k-subspace E of $k[L]$ such that $(E \cap M)k[L] = M$ (where M is the ideal in $k[L]$ of the unit of L) and E is both L-invariant (with respect to the action of L on $k\,[L]$ via left translations) and $\mathrm{Aut}^\circ L$ - invariant (with respect to the natural $\mathrm{Aut}^\circ L$ - action on $k[L]$). By the way all group actions we are going to consider in this paper are left actions. That such an E exists can be viewed by considering the semidirect product $L \times_\rho \mathrm{Aut}^\circ L$ (where ρ is the natural action of $\mathrm{Aut}^\circ L$ on L) acting naturally on L; then the semidirect product above acts rationally on $k[L]$. Now if $d = \dim(E \cap M)$, $P = P(\overset{d}{\wedge} E)$, $p_0 = P(\overset{d}{\wedge}(E \cap M)) \in P$ and $\psi : L \times P \longrightarrow P$ is the induced action map then Y naturally acts on P and L (via $\mathrm{Aut}^\circ L$) fixing p_0. One checks that ψ is Y-equivariant. Next we put a Y-action on our construction in [Bu] p. 96. Recall that we defined actions $\tau : L \times (G \times P) \longrightarrow G \times P$, $\tau(x,(g,p)) = (gx^{-1}, \psi(x,p))$ and $\theta = G \times (G \times P) \longrightarrow G \times P$, $\theta(h,(g,p)) = (hg,p)$ and using [Mu] p. 127 we constructed a projective morphism $w : Z \longrightarrow A$ such that the first projection $G \times P \longrightarrow G$ is the pull back of w via the natural projection $v : G \longrightarrow A$ and such that the resulting projection $u : G \times P \longrightarrow Z$ is a principal bundle for (L,τ). Moreover θ is seen to descend to an action $\overline{\theta} : G \times Z \longrightarrow Z$ and upon letting $z_0 = u(1,p_0)$ we have that the map $\phi : G \longrightarrow Z$, $\phi(g) = \overline{\theta}(g,z_0)$ is an immersion and that $w \circ \phi = v$ (see [Bu] p.96 for details). Now τ and θ are clearly Y-equivariant this providing a Y-action on Z making u and $\overline{\theta}$ Y-equivariant maps. In particular z_0 is fixed by Y so ϕ is Y-invariant so we have an induced Y-action on the closure $\overline{G}$ of $\phi(G)$ in Z as well as on the "boundary" $\overline{D} = \overline{G} \setminus \phi(G)$.

§ 2. Let us prove assertion 1) in the theorem (with k algebraically closed). Since we want to apply the criterion in [BS] p. 140 we first construct a certain connected algebraic group H° as follows. Let $\Gamma \subset G \times G \times G$ be the graph of the multiplication map of G and $\overline{\Gamma}$ the closure of Γ in $\overline{G} \times \overline{G} \times \overline{G}$. By [FGA] the functor

$$S \mapsto \{\alpha \in \mathrm{Aut}_S(\overline{G} \times S) \mid \alpha(\overline{D} \times S) = \overline{D} \times S,\ (\alpha \times \alpha \times \alpha)(\overline{\Gamma} \times S) = \overline{\Gamma} \times S\}$$

is representable on the category of locally algebraic schemes by a locally algebraic group H; we let H° be its connected component. There is a natural action $\eta : H^\circ \times G \rightarrow G$ which is faithfull and hence effective in the sense of [BS] p. 139. Let now Y be any connected algebraic scheme acting on G and $y_0 \in Y$ be such that the corresponding automorphism of G is given by some $\alpha_0 \in H^\circ$; in order for Aut G to be locally algebraic (with $\mathrm{Aut}^\circ G = H^\circ$) it is sufficient by [BS] p.140 to prove that for any $y \in Y$ the corresponding automorphism of G is given by some point of H°. Now both H and Y act on G hence on L so by representability of Aut L we get morphisms $\beta : H \to \mathrm{Aut}\, L$, $\gamma : Y \to \mathrm{Aut}\, L$. Since $\gamma(y_o) = \beta(\alpha_o) \in \mathrm{Aut}^\circ L$ we get that $\gamma(Y) \subset \mathrm{Aut}^\circ L$ so our discussion in §1 applies. In particular Y acts on $\bar{G}$ letting $\bar{D}$ and $\bar{F}$ globally fixed so there is a morphism $\delta : Y \to H$. Since $\delta(y_o) = \alpha_o \in H^\circ$ we get that $\delta(Y) \subset H^\circ$ and we are done.

§3. To prove assertion 1) in the theorem for general k it is sufficient by [BS] p. 140 to prove the following: assume in §2 that G descends to a subfield k_o of k such that k is the algebraic closure of k_o; then both H° and its action on G descend to k_o. To prove this we have to be more careful about our choosing E in §1. To find a good E note first that L descends to k_o: $L \simeq L_o \otimes_{k_o} k$. Then choose a finite dimensional subspace E_o of $k_o[L_o]$ such that $E_o \otimes k$ contains a system of generators of M and put $E = \sum s(E_o \otimes k) \subset k[L]$ where the sum is taken for all $s \in L x_\rho \mathrm{Aut}^\circ L$. It is easy to see that $E^\sigma = E$ for all elements σ of the Galois group $g(k/k_o)$. Consequently $g(k/k_o)$ acts on all our schemes $L, P, p_o, Z, \bar{G}, \bar{D}, \bar{F}$ such that the maps ψ, τ, θ, $\bar{\theta}$, ϕ, η are $g(k/k_o)$-equivariant and we are done by Weil descent.

§4. To prove assertion 2) in the theorem we may assume k is algebraically closed. We must show that H° is linear. Let $\tilde{G} \to \bar{G}$ be a H°-equivariant resolution of $\bar{G}$; then the map $\tilde{v} : \tilde{G} \to A$ is nothing but the Albanese map of $\tilde{G}$ and is H°-equivariant (with respect to the trivial action of H° an A). So $H^\circ \subset \ker(\mathrm{Aut}^\circ\tilde{G} \to \mathrm{Aut}^\circ(\mathrm{Alb}(\tilde{G})))$, which is linear by [Li] and we are done.

A different argument for 2) using neither [Li] nor equivariant resolution is implicitely contained in §5 below.

§5. Now we prove assertion 3) in the theorem. Let K' be the kernel of $\mathrm{Aut}\, G \to \mathrm{Aut}\, L \times \mathrm{Aut}\, A$. We will construct a linear algebraic group K acting on G such that the image of the corresponding homomorphism $K \to \mathrm{Aut}\, G$ is K'.

Start by considering the normalization $\hat{G}$ of $\bar{G}$ and denote by $\hat{w} : \hat{G} \to A$ the morphism induced from $w : Z \to A$; the morphism $\hat{\phi} : G \to \hat{G}$ induced by $\phi : G \to Z$ will be an open immersion and $\hat{w} \circ \hat{\phi} = v$.

Moreover let $\hat{D}$ be the effective reduced Weil divisor on $\hat{G}$ whose support is $\hat{G} \setminus \hat{\phi}(G)$ and let $\mathcal{O}(n\hat{D})$ be the coherent reflective sheaf on $\hat{G}$ corresponding to $n\hat{D}$, $n \geq 0$. Note that $F_n := \hat{w}_* \mathcal{O}(n\hat{D})$ is a subsheaf of $v_* \mathcal{O}_G$ and since v is affine F_n will generate $v_* \mathcal{O}_G$ as an $\mathcal{O}_A$-algebra for $n \geq N$ (N a suitable integer). Then the symmetric algebra S of F_N is equiped with a natural surjection $S \to v_* \mathcal{O}_G$ inducing a closed embedding $G \to X := \mathrm{Spec}\, S$ of A-schemes. Moreover consider the natural open embedding

$X \to X^* := \operatorname{Proj} S[T]$, let G^* be the closure of G in X^*, L^* the closure of L in X^* and Γ^* the closure of Γ (= graph of the multiplication map) in $X^* \times X^* \times X^*$. Note that the group Aut (F_N) of automorphisms of the coherent $\mathcal{O}_\Lambda$-module F_N has a natural structure of linear algebraic group and it acts (algebraically) on X and X^*. Define the linear algebraic group

$$K = \{\alpha \in \operatorname{Aut}(F_N) \mid \alpha G^* = G^*,\ \alpha|_{L^*} = \mathrm{id}_{L^*},\ (\alpha \times \alpha \times \alpha)\Gamma^* = \Gamma^*\}$$

Clearly K acts on G and the image of $K \to \operatorname{Aut} G$ is contained in K'. To prove that it actually coincides with K' it is sufficient to note that any automorphism of G contained in K' induces by §1 an automorphism of $\bar{G}$ and hence of $\hat{G}$ which fixes $\hat{D}$. Hence any such automorphism induces an automorphism of F_N and of X^* thus coming from some element of K. Our theorem is proved.

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