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Gelu POPESCU

Let $\mathcal{T}=\{T_1,T_2,\dots\}$ be a sequence of noncommuting operators on a Hilbert space $\mathcal{K}$ such that the matrix $[T_1,T_2,\dots]$ is a contraction. An extension of the Sz.-Nagy-Foias commutant lifting theorem [10,4,6,7] shows that there is a close connection between the commutant of $\mathcal{T}$ and the commutant of a sequence $\mathcal{S}=\{S_1,S_2,\dots\}$ of orthogonal shifts on a Hilbert space $\mathcal{H}\supset\mathcal{K}$ such that the operator matrix $[S_1,S_2,\dots]$ is nonunitary (see Section 2). The operators belonging to the commutant of $\mathcal{S}$ will be called multi-analytic (or $\mathcal{S}$-analytic) operators.

The main aim of this paper is to provide some factorization theorems and to apply them to the study of $\mathcal{S}$-analytic operators and of the lattice of the invariant subspaces of $\mathcal{S}$, i.e.

$\mathrm{Lat}\,\mathcal{S}=\{\mathcal{M}\subset\mathcal{H}:\ S_n\mathcal{M}\subset\mathcal{M}$ for any $n=1,2,\dots\}$.

First, we present a universal model for sequences $\mathcal{T}$ of noncommuting operators in terms of orthogonal shifts $\mathcal{S}$ (see also [9,3,4,5,6,7]) and, in connection with this, we point out the role of the lattice $\mathrm{Lat}\,\mathcal{S}$ in the study of the invariant subspaces of $\mathcal{T}$.

Section 3 is devoted to an extension of the abstract Beurling factorization problem [8, p.8] and to the application of this fact for proving a version of the Beurling-Lax theorem [11,7,8] for $\mathcal{S}$. In other words, we shall show when an operator $T \in B(\mathcal{H})$ admits a factorization of type $T=AA^*$ for some $\mathcal{S}$-analytic operator $A \in B(\mathcal{H})$ and we shall give, as a consequence, a characterization of the elements of Lat $\mathcal{S}$.

The last section deals with the following extension of the abstract Szego factorization problem: when an operator $T \in B(\mathcal{H})$ has a factorization of type $T=A^*A$ for some $\mathcal{S}$-analytic operator $A \in B(\mathcal{H})$. For this, it is easy to see that a necessary condition is that

$$(\mathcal{T}) \qquad \begin{aligned} &S_n^*TS_n=T && \text{for any } n=1,2,\ldots \\ &S_n^*TS_m=0 && \text{for any } n\neq m;\ n,m=1,2,\ldots \end{aligned}$$

Such an operator $T \in B(\mathcal{H})$ satisfying $(\mathcal{T})$ will be called $\mathcal{S}$-Toeplitz operator. We extend the abstract Szego factorization [8, p. 50] for obtaining a similar result for nonnegative $\mathcal{S}$-Toeplitz operators. By applying this theorem, we obtain an $\mathcal{S}$-inner-outer factorization of $\mathcal{S}$-analytic operators and a factorization for the nonnegative invertible $\mathcal{S}$-Toeplitz operator [8, p. 53].

In Section 2 we establish some results concerning the structure of the $\mathcal{S}$-Toeplitz operators and, in particular, of $\mathcal{S}$-analytic operators. As in the classical case, we shall associate to each $\mathcal{S}$-Toeplitz operator a "symbol" which is an operator in our setting.

Let us remark that at this stage of our research the connections between the $\mathcal{S}$-Toeplitz operators and their "symbols" is far from being well understood.

Concerning $\mathcal{S}$-analytic operators, in a subsequent paper we shall give a parametrization of the set of all contractive

$\mathcal{S}$-analytic operators on a Hilbert space, in terms of the choice sequences [2].

I take this opportunity to thank Professor Gr. Arsene for the useful discussions on the subject of this paper.

1. UNIVERSAL MODEL

Throughout this paper Λ stands for the set $\{1,2,\dots,k\}$ ($k\in\mathbb{N}$) or the set $\mathbb{N}=\{1,2,\dots\}$. For every $n\in\mathbb{N}$ let $F(n,\Lambda)$ be the set of all functions from the set $\{1,2,\dots,n\}$ to Λ and

$$\mathcal{F}=\bigcup_{n=0}^{\infty} F(n,\Lambda), \quad \text{where } F(0,\Lambda)=\{0\} .$$

A sequence $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ of unilateral shifts on a Hilbert space $\mathcal{H}$ with orthogonal final spaces is called a Λ-orthogonal shift if the operator matrix $[S_1,S_2,\dots]$ is nonunitary, i.e. $\mathcal{L}:=\mathcal{H}\ominus(\bigoplus_{\lambda\in\Lambda} S_\lambda\mathcal{H})\neq\{0\}$. This definition is essentially the same as that from [6]. The dimension of $\mathcal{L}$ is called the multiplicity of the Λ-orthogonal shift. One can show that a Λ-orthogonal shift is determined up to unitary equivalence by its multiplicity.

For our purpose we need an operator version of the Wold decomposition [11,8] for sequences of isometries with orthogonal final spaces [4,6].

THEOREM 1.1. Let $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ be a sequence of isometries on a Hilbert space $\mathcal{K}$, with orthogonal final spaces. Then:

(i) $P_0:=I_{\mathcal{K}}-\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^*$ is the projection of $\mathcal{K}$ on $\mathcal{L}:=\mathcal{K}\ominus(\bigoplus_{\lambda\in\Lambda} V_\lambda\mathcal{K})$;

(ii) $\sum_{f\in F(n,\Lambda)} V_f V_f^* \to P$ (strongly) as $n\to\infty$, where P is a projection;

(iii) $P\mathcal{K}=\bigcap_{n=0}^{\infty} (\bigoplus_{f\in F(n,\Lambda)} V_f\mathcal{K})$;

(iv) $\sum_{n=0}^{k} \sum_{f\in F(n,\Lambda)} V_f P_o V_f^* \to Q:=I_{\mathcal{K}}-P$ (strongly) as $k\to\infty$;

(v) $Q\mathcal{K}=\left\{k\in\mathcal{K};\ \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \| V_f^* k \|^2=0 \right\}$;

(vi) $P\mathcal{K}$ and $Q\mathcal{K}$ reduce each V_λ $(\lambda\in\Lambda)$;

(vii) $(I_{\mathcal{K}}-\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^*)\,|_{P\mathcal{K}}=0$;

(viii) $\left\{V_\lambda\,|\,Q\mathcal{K}\right\}_{\lambda\in\Lambda}$ is a Λ-orthogonal shift;

(ix) $I_{\mathcal{K}}=P+\sum_{f\in\mathcal{F}} V_f P_o V_f^*$, $\mathcal{K}=P\mathcal{K}\oplus (\bigoplus_{f\in\mathcal{F}} V_f P_o\mathcal{K})$, where for any $f\in F(n,\Lambda)$, V_f stands for the product $V_{f(1)}V_{f(2)}\cdots V_{f(n)}$.

This version can be proved directly or can be deduced from [6, Theorem 1.3]. We omit the proof.

COROLLARY 1.2. A sequence $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ of isometries on $\mathcal{K}$ with orthogonal final spaces is a Λ-orthogonal shift if and only if

$$\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \| V_f^* k \|^2=0 \quad \text{for all } k\in\mathcal{K}.$$

Specializing the Wold decomposition to the case of a Λ-orthogonal shift, we obtain:

COROLLARY 1.3. If $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ is a Λ-orthogonal shift on $\mathcal{K}$

and $\mathcal{L}=\mathcal{K}\ominus(\bigoplus_{\lambda\in\Lambda} S_\lambda\mathcal{K})$, then $\mathcal{K}=\bigoplus_{f\in\mathcal{F}} S_f\mathcal{L}$ and each $k\in\mathcal{K}$ has a unique representation $k=\sum_{f\in\mathcal{F}} S_f l_f$, $l_f\in\mathcal{L}$. Moreover $\|k\|^2=\sum_{f\in\mathcal{F}}\|l_f\|^2$ and $l_f=P_o S_f^* k$ $(f\in\mathcal{F})$, where $P_o:=I_{\mathcal{K}}-\sum_{\lambda\in\Lambda} S_\lambda S_\lambda^*$ is the projection of $\mathcal{K}$ on $\mathcal{L}$. Now one can easily prove the following characterization of the subspaces $\mathcal{M}$ of $\mathcal{K}$ which reduce each S_λ $(\lambda\in\Lambda)$, that is $\mathcal{M}\in(\mathrm{Lat}\,\mathcal{S})\cap(\mathrm{Lat}\,\mathcal{S}^*)$.

COROLLARY 1.4. A subspace $\mathcal{M}$ of $\mathcal{K}$ reduces each S_λ $(\lambda\in\Lambda)$ if and only if

$$\mathcal{M}=\bigoplus_{f\in\mathcal{F}} S_f\mathcal{M}_o ,$$

where $\mathcal{M}_o$ is a subspace of $\mathcal{L}=\mathcal{K}\ominus(\bigoplus_{\lambda\in\Lambda} S_\lambda\mathcal{K})$.

Let us note that the Λ-orthogonal shifts $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ can be thought as universal models for sequences $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ of noncommuting operators.

This fact has been proved in [6,7]. However we next give another proof generalizing the results from [9,3].

THEOREM 1.5. Let $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ be a sequence of operators on $\mathcal{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda^*T_\lambda\leq I_{\mathcal{H}}$ and $\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)}\|T_f h\|^2=0$ for any $h\in\mathcal{H}$. Let $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ be a Λ-orthogonal shift on a Hilbert space $\mathcal{K}$ such that

$$\dim(\mathcal{K}\ominus(\bigoplus_{\lambda\in\Lambda} S_\lambda\mathcal{K}))\geqslant\dim((I_{\mathcal{H}}-\sum_{\lambda\in\Lambda} T_\lambda^*T_\lambda)\mathcal{H})^-$$

Then there exists an invariant subspace $\mathcal{M}$ of each S_λ^* $(\lambda\in\Lambda)$ such $\mathcal{T}$ is unitarily equivalent to $\{S_\lambda^*|_{\mathcal{M}}\}_{\lambda\in\Lambda}$ i.e., there exists a unitary operator U such that

$$UT_\lambda = (S^*_\lambda|\mathcal{M})U \quad \text{for each } \lambda\in\Lambda.$$

Proof. Denote $\mathcal{L}=\mathcal{K}\ominus(\bigoplus_{\lambda\in\Lambda} S_\lambda\mathcal{K})$ and $D=(I_{\mathcal{H}}-\sum_{\lambda\in\Lambda} T^*_\lambda T_\lambda)^{1/2}$. Since $\dim\overline{D\mathcal{H}}\leq\dim\mathcal{L}$ we can define an isometry W from $\overline{D\mathcal{H}}$ to $\mathcal{L}$, and an operator U from $\mathcal{H}$ to $\mathcal{K}$ by

$$Uh=\sum_{f\in\mathcal{F}}\tilde{S}_f WDT_f h, \qquad (h\in\mathcal{H})$$

where $\tilde{S}_f$ stands for the product $S_{f(n)}S_{f(n-1)}\cdots S_{f(1)}$ when $f\in F(n,\Lambda)$. By Corollary 1.3, for any $h\in\mathcal{H}$ we get

$$\|Uh\|^2=\sum_{f\in\mathcal{F}}\|DT_f h\|^2=\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{f\in F(k,\Lambda)}\langle T^*_f(I_{\mathcal{H}}-\sum_{\lambda\in\Lambda}T^*_\lambda T_\lambda)T_f h,h\rangle=$$

$$=\lim_{n\to\infty}(\|h\|^2-\sum_{f\in F(n+1,\Lambda)}\|T_f h\|^2)=\|h\|^2$$

Thus U is an unitary operator from $\mathcal{H}$ onto $\mathcal{M}:=U\mathcal{H}$. For any $h\in\mathcal{H}$ and $\lambda\in\Lambda$ we have

$$S^*_\lambda Uh=\sum_{f\in\mathcal{F}}S^*_\lambda\tilde{S}_f WDT_f h=UT_\lambda h \quad .$$

It follows that $\mathcal{M}$ is invariant for each S^*_λ $(\lambda\in\Lambda)$ and $\mathcal{T}$ is unitarily equivalent to $\{S^*_\lambda|\mathcal{M}\}_{\lambda\in\Lambda}$. The proof is complete.

COROLLARY 1.6. $\mathcal{T}$ has a non-trivial invariant subspace if and only if $\mathcal{S}^*=\{S^*_\lambda\}_{\lambda\in\Lambda}$ has an invariant subspace $\mathcal{N}$ such that $\{0\}\subsetneqq\mathcal{N}\subsetneqq\mathcal{M}$.

REMARK 1.7. If $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ does not satisfy the hypotheses of the theorem, then $\{c_\lambda T_\lambda\}_{\lambda\in\Lambda}$ will satisfy the hypotheses for any scalars c_λ, $0<|c_\lambda|\leq 1$ $(\lambda\in\Lambda)$ such that

$$\sum_{\lambda\in\Lambda} |c_\lambda|^2 T_\lambda^* T_\lambda \leq r I_{\mathcal{H}} \qquad \text{for some } r\in(0,1) .$$

In this case it is necessary to choose a Λ-orthogonal shift $\mathcal{S}$ whose multiplicity is dim $\mathcal{H}$ and as in the above theorem we find

$$U T_\lambda = (c_\lambda^{-1} S_\lambda^*|_{\mathcal{M}})U \text{ for any } \lambda\in\Lambda$$

2. $\mathcal{S}$-TOEPLITZ OPERATORS

The commutant of a sequence $\mathcal{T}=\{T_\lambda\}_{\lambda\in\Lambda}$ of operators on $\mathcal{H}$ is the set

$$C(\mathcal{T})=\{X\in B(\mathcal{H}):\ XT_\lambda=T_\lambda X \text{ for all } \lambda\in\Lambda\}$$

The lifting theorem [10,4,6,7] characterizes $C(\mathcal{T})$ when $\mathcal{T}$ is represented as in the universal model of Section 1. Let us recall it.

THEOREM 2.1. Let $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ be a Λ-orthogonal shift on a Hilbert space $\mathcal{K}$, $\mathcal{H}$ be an invariant subspace for each S_λ^* $(\lambda\in\Lambda)$ and let

$$T_\lambda=S_\lambda^*|_{\mathcal{H}} \qquad (\lambda\in\Lambda) .$$

If $X\in B(\mathcal{H})$ satisfies

$$XT_\lambda=T_\lambda X \quad \text{for all } \lambda\in\Lambda ,$$

then there exists $Y\in B(\mathcal{K})$ with properties:

(i) $Y\mathcal{H}\subset\mathcal{H}$ and $X=Y|_{\mathcal{H}}$;

(ii) $YS_\lambda^*=S_\lambda^*Y$ for all $\lambda\in\Lambda$;

(iii) $\|Y\| = \|X\|$.

As we see, it is important to study the commutant of a Λ-orthogonal shift $\mathcal{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ on a Hilbert space $\mathcal{H}$ and even a larger class of operators, that of the $\mathcal{S}$-Toeplitz operators, which appear in connection with an extension of the abstract Szegö factorization problem (see the introduction). For this, we need the following definitions.

An operator $T \in B(\mathcal{H})$ is called

(i) $\mathcal{S}$-Toeplitz if $S_\lambda^* T S_\lambda = T$ for any $\lambda \in \Lambda$ and $S_\lambda^* T S_\mu = 0$ for $\lambda \neq \mu$; $\lambda, \mu \in \Lambda$;

(ii) $\mathcal{S}$-analytic if $TS_\lambda = S_\lambda T$ for any $\lambda \in \Lambda$,

(iii) $\mathcal{S}$-inner if T is $\mathcal{S}$-analytic and partially isometric,

(iv) $\mathcal{S}$-outer if T is $\mathcal{S}$-analytic and $\overline{T\mathcal{H}}$ reduces each S_λ $(\lambda \in \Lambda)$,

(v) $\mathcal{S}$-constant if T and T^* are $\mathcal{S}$-analytic.

In genral, examples of $\mathcal{S}$-Toeplitz operators are easily constructed from $\mathcal{S}$-analytic operators. If $T_1, T_2 \in B(\mathcal{H})$ are $\mathcal{S}$-analytic, then the operators T_1, T_2^*, $T_2^* T_1$ are $\mathcal{S}$-Toeplitz.

Now, we give a theorem on $\mathcal{S}$-inner operators. We omit the proof which is an easy extension of [8, §1.7, Theorems A,B,C].

THEOREM 2.2. Let $T \in B(\mathcal{H})$ be $\mathcal{S}$-inner and $\mathcal{L} = \mathcal{H} \ominus (\bigoplus_{\lambda\in\Lambda} S_\lambda \mathcal{H})$. Then

(i) The initial space of T reduces each S_λ $(\lambda \in \Lambda)$,

(ii) The final space of T reduces each S_λ $(\lambda \in \Lambda)$ if and only if T is $\mathcal{S}$-constant,

(iii) T is $\mathcal{S}$-constant if and only if has the form

$$Th = \sum_{f\in\mathcal{F}} S_f T_o l_f \qquad (h = \sum_{f\in\mathcal{F}} S_f l_f \ , \ l_f \in \mathcal{L})$$

for some partial isometry $T_o \in B(\mathcal{L})$.

In what follows, to each $T\in B(\mathcal{H})$ we associate a matrix of operators in $B(\mathcal{L})$: $T\sim[A_{f,g}]_{f,g\in\mathcal{F}}$, where

$$A_{f,g}=P_oS_f^*TS_gP_o|_{\mathcal{L}} \quad (f,g\in\mathcal{F})$$

and P_o is the orthogonal projection of $\mathcal{H}$ on $\mathcal{L}=\mathcal{H}\ominus(\bigoplus_{\lambda\in\Lambda} S_\lambda\mathcal{H})$.

For any $h\in\mathcal{H}$, $h=\sum_{q\in\mathcal{F}} S_ql_q$ (see Corollary 1.3) we have $Th=\sum_{f\in\mathcal{F}} S_fl_f'$, where $l_f'=\sum_{q\in\mathcal{F}} A_{f,q}l_q$. Indeed, by Corollary 1.3 it follows that $l_f'=P_oS_f^*Th=P_oS_f^*T\sum_{q\in\mathcal{F}} S_ql_q=\sum_{q\in\mathcal{F}} A_{f,q}l_q$ $(f\in\mathcal{F})$ where the sums are strongly convergent.

It is easy to see that the correspondence $T\sim[A_{f,g}]_{f,g\in\mathcal{F}}$ is linear and that $T^*\sim[B_{f,g}]_{f,g\in\mathcal{F}}$ where $B_{f,g}=A^*_{g,f}$ for all $f,g\in\mathcal{F}$.

PROPOSITION 2.3. If $T_1\sim[A_{f,g}]_{f,g\in\mathcal{F}}$ and $T_2\sim[B_{f,g}]_{f,g\in\mathcal{F}}$ then $T_1T_2\sim[C_{f,g}]_{f,g\in\mathcal{F}}$ where $C_{f,g}=\sum_{q\in\mathcal{F}} A_{f,q}B_{q,g}$ $(f,g\in\mathcal{F})$ with convergence of the sums in the strong operator topology.

<u>Proof</u>. By Theorem 1.1 we have $I_{\mathcal{H}}=\sum_{q\in\mathcal{F}} S_qP_oS_q^*$. For any $f,g\in\mathcal{F}$

$$C_{f,g}=P_oS_f^*T_1T_2S_gP_o|_{\mathcal{L}}=P_oS_f^*T_1(\sum_{q\in\mathcal{F}} S_qP_oS_q^*)T_2S_gP_o|_{\mathcal{L}}=\sum_{q\in\mathcal{F}} A_{f,q}B_{q,g}$$

with convergence in the strong operator topology.

In the sequel we shall need the following notation. When $f\in F(n,\Lambda)$, $g\in F(m,\Lambda)$, $f\supset g$ means that $n\geq m$ and $g=f|_{\{1,2,\dots,m\}}$. In this case $f\setminus g$ stands for the function $h:\{1,2,\dots,n-m\}\to\Lambda$ given by

$h(1)=f(m+1),\ h(2)=f(m+2),\dots,h(n-m)=f(n)$ if $n>m$

and

$h=0$ if $n=m$.

THEOREM 2.4. An operator $T\in B(\mathcal{H})$ is $\mathcal{S}$-Toeplitz if and only if its matrix is of the form $[A_{f,g}]_{f,g\in\mathcal{F}}$ where for $f,g\in\mathcal{F}$

$$(2.1)\qquad \begin{aligned} A_{f,g}&=A_{f\setminus g}; && \text{if } f\supset g\\ &=\tilde{A}_{g\setminus f}; && \text{if } g\supset f\\ &=0; && \text{otherwise}\end{aligned}$$

where $A_q\in B(\mathcal{L})$ $(q\in\mathcal{F})$, $\tilde{A}_q\in B(\mathcal{L})(q\in\mathcal{F}\setminus\{0\})$.

In this case

$$(2.2)\quad A_q=P_oS_q^*TP_o|_{\mathcal{L}}\ (q\in\mathcal{F})\ \text{and}\ \tilde{A}_q=P_oTS_qP_o|_{\mathcal{L}}\quad (q\in\mathcal{F}\setminus\{0\}).$$

<u>Proof</u>. If T is $\mathcal{S}$-Toeplitz with matrix $[A_{f,g}]_{f,g\in\mathcal{F}}$ then

$$\begin{aligned} A_{f,g}=P_oS_f^*TS_gP_o|_{\mathcal{L}}&=P_oS_{f\setminus g}^*TP_o|_{\mathcal{L}} && \text{if } f\supset g\\ &=P_oTS_{g\setminus f}P_o|_{\mathcal{L}} && \text{if } g\supset f\\ &=0 && \text{otherwise}\end{aligned}$$

Thus the matrix of T is given by (2.1) where the entries A_q $(q\in\mathcal{F})$; $\tilde{A}_q$ $(q\in\mathcal{F}\setminus\{0\})$ are defined by (2.2).

Conversely, if the matrix of $T\in B(\mathcal{H})$ has the form (2.1), for some $A_q\in B(\mathcal{L})$ $(q\in\mathcal{F})$, $\tilde{A}_q\in B(\mathcal{L})$ $(q\in\mathcal{F}\setminus\{0\})$ then T, $S_\lambda^*TS_\lambda$ $(\lambda\in\Lambda)$ have the same matrix and for $\lambda,\mu\in\Lambda$, $\lambda\neq\mu$, $S_\lambda^*TS_\mu=0$. Hence T is $\mathcal{S}$-Toeplitz.

COROLLARY 2.5. An operator $T\in B(\mathcal{H})$ is $\mathcal{S}$-analytic if and only if its matrix is of the form $[A_{f,g}]_{f,g\in\mathcal{F}}$ where for $f,g\in\mathcal{F}$

$$(2.3)\qquad \begin{aligned} A_{f,g} &= A_{f\setminus g} \quad \text{if } f\supset g \\ &= 0 \quad\quad\ \ \text{otherwise} \end{aligned} \qquad (A_q\in B(\mathcal{L}),\ q\in\mathcal{F})$$

In this case for any $q\in\mathcal{F}$, $A_q=P_oS_q^*TP_o|_{\mathcal{L}}$. Moreover, T is $\mathcal{S}$-constant if and only if $A_q=0$ for all $q\in\mathcal{F}\setminus\{0\}$ i.e. the matrix of T has the form

$$\operatorname{diag}\{A_o,A_o,\dots\}.$$

In what follows we need the following notations and definitions:

$$(2.4)\qquad l^2(\mathcal{F},\mathcal{L})=(\bigoplus_{f\in\mathcal{F}^*} S_f\mathcal{L})\oplus\mathcal{L}\oplus(\bigoplus_{f\in\mathcal{F}^*} S_f\mathcal{L}),\quad \text{where } \mathcal{F}^*=\mathcal{F}\setminus\{0\};$$

$$l^2_-(\mathcal{F},\mathcal{L})=(\bigoplus_{f\in\mathcal{F}^*} S_f\mathcal{L})\oplus 0\oplus 0;$$

$$l^2_+(\mathcal{F},\mathcal{L})=0\oplus\mathcal{L}\oplus(\bigoplus_{f\in\mathcal{F}^*} S_f\mathcal{L}).$$

We identify $\mathcal{H}$ with $l^2_+(\mathcal{F},\mathcal{L})$ and $l^2(\mathcal{F},\mathcal{L})$ with $l^2_-(\mathcal{F},\mathcal{L})\oplus l^2_+(\mathcal{F},\mathcal{L})$. For each $\lambda\in\Lambda$ let us define $U_\lambda\in B(l^2(\mathcal{F},\mathcal{L}))$ by setting

$$U_\lambda(l_-)=(S_\lambda^-)^*l_-,\qquad \text{for } l_-\in l^2_-(\mathcal{F},\mathcal{L});$$

$$U_\lambda(l_+)=S_\lambda^+l_+,\qquad \text{for } l_+\in l^2_+(\mathcal{F},\mathcal{L}),$$

where $\{S_\lambda^-\}_{\lambda\in\Lambda}$ (resp. $\{S_\lambda^+\}_{\lambda\in\Lambda}$) is the Λ-orthogonal shift acting on $l^2_-(\mathcal{F},\mathcal{L})\oplus\mathcal{L}$ (resp. $l^2_+(\mathcal{F},\mathcal{L})$).

Obviously U_λ is a coupling of S_λ^- and S_λ^+ (see [1]), that is $U_\lambda|l^2_+(\mathcal{F},\mathcal{L})=S_\lambda^+$ and $U_\lambda^*|l^2_-(\mathcal{F},\mathcal{L})\oplus\mathcal{L}=S_\lambda^-$. We also remark that if Λ has a single element then we find again the bilateral shift on $l^2(\mathbb{Z},\mathcal{L})$.

Now, to each $\mathcal{S}$-Toeplitz operator $T\in B(\mathcal{H})$ we associate an operator $\theta:\mathcal{L}\to l^2(\mathcal{F},\mathcal{L})$ defined with respect to decomposition (2.4) by the operator matrix

$$\theta = \begin{bmatrix} [\tilde{A}_q]_{q\in\mathcal{F}^*} \\ \\ [A_q]_{q\in\mathcal{F}} \end{bmatrix} ,$$

where $\tilde{A}_q$, A_q are defined by (2.2). Thus θ is uniquely determined by T and we write $T=T_\theta$. Note that for $l\in\mathcal{L}$

$$\theta l = \sum_{f\in\mathcal{F}^*} U_f^*\tilde{A}_f l + \sum_{f\in\mathcal{F}} U_f A_f l$$

THEOREM 2.6. The following statements are equivalent:

(i) $T_\theta\in B(l_+^2(\mathcal{F},\mathcal{L}))$ is $\mathcal{S}$-Toeplitz,

(ii) There exists $A_\theta\in B(l^2(\mathcal{F},\mathcal{L}))$ uniquely determined with the properties:

1. $A_\theta|_{\mathcal{L}} = \theta$

(2.5) 2. $A_\theta U_\lambda|_{l_+^2(\mathcal{F},\mathcal{L})} = U_\lambda A_\theta|_{l_+^2(\mathcal{F},\mathcal{L})}$;

$A_\theta U_\lambda^*|_{l_-^2(\mathcal{F},\mathcal{L})\oplus\mathcal{L}} = U_\lambda^* A_\theta|_{l_-^2(\mathcal{F},\mathcal{L})\oplus\mathcal{L}}$ for any $\lambda\in\Lambda$.

3. $T_\theta = P_+A_\theta|_{l_+^2(\mathcal{F},\mathcal{L})}$, where P_+ is the orthogonal projection of $l^2(\mathcal{F},\mathcal{L})$ on $l_+^2(\mathcal{F},\mathcal{L})$.

In this case

$$(2.6)\qquad A_\theta\Big(\sum_{f\in\mathcal{F}^*} U_f^* l_f' + \sum_{f\in\mathcal{F}} U_f l_f\Big) = \sum_{f\in\mathcal{F}^*} U_f^*\theta l_f' + \sum_{f\in\mathcal{F}} U_f\theta l_f \qquad (l_f', l_f\in\mathcal{L})$$

and

(2.7) $$T_\theta(\sum_{f\in\mathcal{F}} U_f l_f)=P_+(\sum_{f\in\mathcal{F}} U_f\theta l_f) .$$

Proof. (i)$\Longrightarrow$(ii) An operator $A_\theta\in B(l^2(\mathcal{F},\mathcal{L}))$ which satisfies (2.5) is uniquely determined by $A_\theta|_{\mathcal{L}}=\theta$. This follows because for every $f\in\mathcal{F}^*$, $l\in\mathcal{L}$ we have

(2.8) $$A_\theta U_f l=U_f\theta l \quad , \quad A_\theta U_f^* l=U_f^*\theta l$$

and

$$l^2(\mathcal{F},\mathcal{L})=(\bigoplus_{f\in\mathcal{F}^*} U_f^*\mathcal{L}) \oplus (\bigoplus_{f\in\mathcal{F}} U_f\mathcal{L}) .$$

Taking into account the matrix form of T_θ (see Thm. 2.4) and the definitions of U_λ $(\lambda\in\Lambda)$ and θ, it is easy to see that for any $f,g\in\mathcal{F}$, $l,l'\in\mathcal{L}$ we have

$$\langle P_+A_\theta U_g l, U_f l'\rangle=\langle T_\theta U_g l, U_f l'\rangle .$$

Since $l_+^2(\mathcal{F},\mathcal{L})=\bigoplus_{f\in\mathcal{F}} U_f\mathcal{L}$ it follows that

$$P_+A_\theta|_{l_+^2(\mathcal{F},\mathcal{L})}=T_\theta .$$

Conversely, if $A_\theta\in B(l^2(\mathcal{F},\mathcal{L}))$ is such that (2.5) holds and $\theta=A_\theta|_{\mathcal{L}}$, then the operator $T_\theta=P_+A_\theta|_{l_+^2(\mathcal{F},\mathcal{L})}$ is $\mathcal{S}$-Toeplitz.

Indeed, for any $\lambda,\mu\in\Lambda$, h,k $l_+^2(\mathcal{F},\mathcal{L})$ we have

$$\langle S_\mu^* T_\theta S_\lambda h,k\rangle=\langle A_\theta h,U_\lambda^* U_\mu k\rangle=\langle P_+A_\theta h,k\rangle; \quad \text{if } \lambda=\mu$$
$$=0 \quad ; \quad \text{if } \lambda\neq\mu .$$

The last statement of the theorem is immediately. The proof is complete.

In general if θ is a bounded operator from $\mathcal{L}$ to $l^2(\mathcal{F},\mathcal{L})$ (i.e. $\theta \in B(\mathcal{L}, l^2(\mathcal{F},\mathcal{L}))$), then A_θ defined by (2.8) is an unbounded operator. Note that $T_\theta \in B(l^2_+(\mathcal{F},\mathcal{L}))$ is $\mathcal{S}$-analytic if and only if $\theta \in B(\mathcal{L}, l^2_+(\mathcal{F},\mathcal{L}))$. In this case we have

$$(2.9) \qquad T_\theta\Big(\sum_{f\in\mathcal{F}} S_f l_f\Big) = \sum_{f\in\mathcal{F}} S_f \theta l_f \qquad (l_f \in \mathcal{L})$$

THEOREM 2.7. Let $T_\theta \in B(l^2_+(\mathcal{F},\mathcal{L}))$ be an $\mathcal{S}$-analytic operator. In order that T_θ be:

(i) $\mathcal{S}$-inner,

(ii) $\mathcal{S}$-outer,

(iii) $\mathcal{S}$-constant,

it is necessary and sufficient that the following conditions hold, respectively

(i) θ is a partial isometry with $\theta\mathcal{L}$ an wandering subspace for $\mathcal{S}$, i.e. $S_f\theta\mathcal{L} \perp S_g\theta\mathcal{L}$ for any $f,g\in\mathcal{F}$, $f\neq g$.

(ii) $\bigvee_{f\in\mathcal{F}} S_f\theta\mathcal{L} = l^2_+(\mathcal{F},\mathcal{L}_0)$ for some subspace $\mathcal{L}_0 \subset \mathcal{L}$.

(iii) $\theta \in B(\mathcal{L})$.

We omit the proof of this theorem which is easily to deduce from the results up to now.

The following theorem gives explicit forms for the commutants of all $\mathcal{S}$-analytic operators.

THEOREM 2.8. An operator $X \in B(l^2_+(\mathcal{F},\mathcal{L}))$ commutes with all $\mathcal{S}$-analytic operators if and only if its matrix has the form (2.3), where each entry is a scalar multiple of the identity on $\mathcal{L}$.

Proof. If $X \in B(l^2_+(\mathcal{F},\mathcal{L}))$ commutes with all $\mathcal{S}$-analytic operators on $l^2_+(\mathcal{F},\mathcal{L})$ then X is $\mathcal{S}$-analytic. Its matrix is given by (2.3) and must commute in particular with diag(Y,Y,...) for every $Y \in B(\mathcal{L})$. Therefore the entries of (2.3) commute with all operators in $B(\mathcal{L})$. It follows that each entry of (2.3) is a scalar multiple of the identity on $\mathcal{L}$. Conversely is straightforward.

At the end of this Section we give a concrete realization of a Λ-orthogonal shift $\mathcal{S}$ and we provide explicit forms for the $\mathcal{S}$-analytic operators. For sake of simplicity we only consider the case when $\Lambda=\{1,2\}$ and $\mathcal{S}=\{S_1,S_2\}$.

Let $\mathcal{F}(\mathcal{H})$ be the Fock space defined by

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=1}^{\infty} H^2(\mathbb{D}^n, \mathcal{H}),$$

where $H^2(\mathbb{D}^n,\mathcal{H})$ is the Hardy space of all analytic functions in the unit polydisc $\mathbb{D}^n$ with values in a Hilbert space $\mathcal{H}$. To be more precise $f_n(\lambda_1,\lambda_2,\ldots,\lambda_n)$ belongs to $H^2(\mathbb{D}^n,\mathcal{H})$ if f_n has a power series expansion of the form

$$f_n(\lambda_1,\lambda_2,\ldots,\lambda_n) = \sum_{i_1,\ldots,i_n\geq 0} f_n(i_1,\ldots,i_n)\lambda_1^{i_1}\lambda_2^{i_2}\ldots\lambda_n^{i_n}$$

and

$$\| f_n \|^2_{H^2(\mathbb{D}^n,\mathcal{H})} = \sum_{i_1,\ldots,i_n\geq 0} \| f_n(i_1,\ldots,i_n)\|^2_{\mathcal{H}}$$

For any $f \in \mathcal{F}(\mathcal{H})$, $f=f_1(\lambda_1) \oplus f_2(\lambda_1,\lambda_2) \oplus f_3(\lambda_1,\lambda_2,\lambda_3) \oplus \ldots$ we define

$$S_1 f=\lambda_1 f_1(\lambda_1) \oplus \lambda_1 f_2(\lambda_1,\lambda_2) \oplus \lambda_1 f_3(\lambda_1,\lambda_2,\lambda_3) \oplus \dots$$

$$S_2 f=0 \oplus f_1(\lambda_2) \oplus f_2(\lambda_2,\lambda_3) \oplus f_3(\lambda_2,\lambda_3,\lambda_4) \oplus \dots$$

It is easy to see that S_1, S_2 are orthogonal shifts on $\mathfrak{F}(\mathcal{H})$ and $[S_1,S_2]$ is nonunitary. Moreover $\mathcal{H}=(I-S_1S_1^*-S_2S_2^*)\mathfrak{F}(\mathcal{H})$ and the multiplicity of $\mathcal{S}=\{S_1,S_2\}$ is equal to $\dim\mathcal{H}$.

Let us consider the scalar case when $\mathcal{H}=\mathbb{C}$. If $T\in B(\mathfrak{F}(\mathbb{C}))$ is an $\mathcal{S}$-analytic operator then one can easily show that

(2.10) $\qquad Tf=T_{[\varphi]}f=[\varphi]f \qquad$ for any $f\in\mathfrak{F}(\mathbb{C})$

where $\qquad \varphi=\varphi_1(\lambda_1)\oplus\varphi_2(\lambda_1,\lambda_2)\oplus\varphi_3(\lambda_1,\lambda_2,\lambda_3)\oplus\dots$

stands for $T(1)$ and $[\varphi]$ stands for the matrix of functions

$$\begin{bmatrix} \varphi_1(\lambda_1) & 0 & 0 & 0 & \dots \\ \varphi_2(\lambda_1,\lambda_2) & \varphi_1(\lambda_2) & 0 & 0 & \dots \\ \varphi_3(\lambda_1,\lambda_2,\lambda_3) & \varphi_2(\lambda_2,\lambda_3) & \varphi_1(\lambda_3) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

It is easy to see that if $T_{[\varphi]}$, $T_{[\eta]}$ are $\mathcal{S}$-analytic operators then

$$T_{[\varphi]}+T_{[\eta]}=T_{[\varphi]+[\eta]}$$

$$T_{\lambda[\varphi]}=\lambda T_{[\varphi]} \qquad (\lambda\in\mathbb{C})$$

$$T_{[\varphi]}T_{[\eta]}=T_{[\varphi][\eta]}$$

Now if $\varphi\in\mathfrak{F}(\mathbb{C})$ such that $[\varphi]f\in\mathfrak{F}(\mathbb{C})$ for any $f\in\mathfrak{F}(\mathbb{C})$ and the operator $T_{[\varphi]}$ defined by (2.10) is bounded then $T_{[\varphi]}$ is an $\mathcal{S}$-analytic operator on $\mathcal{F}(\mathbb{C})$.

Example 2.9. Let $\varphi=\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_m \oplus 0 \oplus \dots$ where $m\in\mathbb{N}$, $\varphi_k \in H^\infty(\mathbb{D}^k)$ for any $k=1,2,\dots,m$. Then $T_{[\varphi]}$ is an $\mathcal{S}$-analytic operator on $\mathcal{F}(\mathbb{C})$.

If $\dim\mathcal{H}=n$, we can consider $\mathcal{H}=\mathbb{C}^n$. The form of the $\mathcal{S}$-analytic operators on $\mathcal{F}(\mathbb{C}^n)=\mathcal{F}(\mathbb{C}) \otimes \mathbb{C}^n$ can be easily deduce from the scalar case using the tensor product.

3. BEURLING-TYPE FACTORIZATIONS AND LAT $\mathcal{S}$

Throughout section $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ is a Λ-orthogonal shift on a Hilbert space $\mathcal{H}$ and we keep the definitions from the beginning of Section 2. For $T\in B(\mathcal{H})$ we denote

$$D=T-\sum_{\lambda\in\Lambda} S_\lambda T S_\lambda^* \quad \text{and} \quad \mathcal{L}=\mathcal{H}\ominus\Big(\bigoplus_{\lambda\in\Lambda} S_\lambda\mathcal{H}\Big).$$

The following theorem is a version of [8, Theorem 1.9] in our setting.

THEOREM 3.1. If $T\in B(\mathcal{H})$, then the following are equivalent:

(i) $T=AA^*$ for some $\mathcal{S}$-analytic operator $A\in B(\mathcal{H})$;

(ii) $D=W^*W$ for some operator W from $\mathcal{H}$ to $\mathcal{L}$;

(iii) $D\geq 0$ and $\dim\overline{D\mathcal{H}}\leq\dim\mathcal{L}$.

Proof. (i) $\Longrightarrow$ (ii). If (i) holds, then $D=AP_oA^*=W^*W$, where $P_o=I_{\mathcal{H}}-\sum_{\lambda\in\Lambda} S_\lambda S_\lambda^*$ and $W=P_oA^*$ is an operator from $\mathcal{H}$ to $\mathcal{L}$.

(ii) $\Longrightarrow$ (iii). If (ii) holds, then $D=W^*W\geq 0$. Let $W=VQ$ the polar decomposition of W. Therefore $Q=(W^*W)^{1/2}$ and V is a partial isometry from $\mathcal{H}$ to $\mathcal{L}$, with initial space $\overline{Q\mathcal{H}}$. It is easy to see that $\overline{D\mathcal{H}}\subset\overline{Q\mathcal{H}}$. Since V acts isometrically on $\overline{Q\mathcal{H}}$ it follows that $\dim\overline{D\mathcal{H}}\leq\dim\mathcal{L}$.

(iii) $\Longrightarrow$ (i). If (iii) holds, then $\dim\overline{D^{1/2}\mathcal{H}}\leq\dim\mathcal{L}$. Therefore there exists an isometry V from $\overline{D^{1/2}\mathcal{H}}$ to $\mathcal{L}$. Setting $W=VD^{1/2}$ we find that $W^*W=D$, that is $T-\sum_{\lambda\in\Lambda}S_\lambda TS_\lambda^*=W^*W$. An easy computation shows that

$$T-\sum_{f\in F(n+1,\Lambda)}S_f TS_f^*=\sum_{k=0}^{n}\sum_{f\in F(k,\Lambda)}S_f W^*WS_f^* \quad \text{for any } n=0,1,2,\ldots$$

Hence we obtain that

$$(3.1) \qquad \langle Th_1,h_2\rangle-\sum_{f\in F(n+1,\Lambda)}\langle S_f TS_f^*h_1,h_2\rangle =$$

$$=\left\langle\sum_{k=0}^{n}\sum_{f\in F(k,\Lambda)}S_f WS_f^*h_1\ ,\ \sum_{k=0}^{n}\sum_{f\in F(k,\Lambda)}S_f WS_f^*h_2\right\rangle$$

for any $h_1,h_2\in\mathcal{H}$ and $n=0,1,2,\ldots$.

Define $A\in B(\mathcal{H})$ by setting $A^*=\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{f\in F(k,\Lambda)}S_f WS_f^*$ (strongly). It is easy to see that $AS_\lambda=S_\lambda A$ for any $\lambda\in\Lambda$ and letting $n\to\infty$ in (3.1) we obtain

$$\langle Th_1,\ h_2\rangle = \langle A^*h_1,\ A^*h_2\rangle \qquad \text{for any } h_1,h_2\in\mathcal{H}\ .$$

Therefore $T=AA^*$ where A is $\mathcal{S}$-analytic. The proof is complete.

In the sequel, by the support of an $\mathcal{S}$-analytic operator $T\in B(\mathcal{H})$ we understand the smallest reducing subspace $\mathrm{supp}(T)\subset\mathcal{H}$ for each S_λ $(\lambda\in\Lambda)$ containing $\overline{T^*\mathcal{H}}$. Taking into account Corollary

1.3 one can easily show that

$$\text{supp}\,(T) = \bigoplus_{f\in\mathcal{F}} S_f \overline{P_o T^*\mathcal{H}}\ ,$$

where P_o is the orthogonal projection of $\mathcal{H}$ on $\mathcal{L}$.

THEOREM 3.2. If $T_1, T_2 \in B(\mathcal{H})$ are $\mathcal{S}$-analytic, then

$$T_1T_1^* = T_2T_2^* \tag{3.2}$$

if and only if $T_2 = T_1C$, where C is an $\mathcal{S}$-constant inner operator with initial space supp (T_2) and final space supp (T_1).

Moreover, in this case C is unique and $T_1 = T_2C^*$.

<u>Proof</u>. Setting $P_o = I_{\mathcal{H}} - \sum_{\lambda\in\Lambda} S_\lambda S_\lambda^*$, by (3.2) we get $\|P_oT_1^*h\| = \|P_oT_2^*h\|$ for any $h\in\mathcal{H}$, which implies that there exists a unique partial isometry $C_o \in B(\mathcal{L})$ with initial space $\overline{P_oT_2^*\mathcal{H}}$ and final space $\overline{P_oT_1^*\mathcal{H}}$. We extend C_o to an $\mathcal{S}$-constant inner operator C on $\mathcal{H}$ (see Theorem 2.2) with initial space supp (T_2) and final space supp (T_1). Now it is easy to see that $T_2 = T_1C$. Moreover C is unique since C* is unique determined on $T_1^*\mathcal{H}$. The fact that $T_1 = T_2C^*$ is immediately.

We apply the results up to now for proving a version of the Beurling-Lax theorem [11, 7, 8] for a Λ-orthogonal shift $\mathcal{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ on $\mathcal{H}$.

THEOREM 3.3. A subspace $\mathcal{M}$ of $\mathcal{H}$ is invariant for each S_λ $(\lambda\in\Lambda)$ if and only if

$\mathcal{M} = M\mathcal{H}$

for some $\mathcal{S}$-inner operator $M \in B(\mathcal{H})$. Moreover, this representation is essentially unique.

Proof. An implication is obviously.

Conversely, let $\mathcal{M} \subset \mathcal{H}$ be an invariant subspa ce for each S_λ $(\lambda \in \Lambda)$ and let $P_{\mathcal{M}}$ be the projection of $\mathcal{H}$ on $\mathcal{M}$. Then, $P := P_{\mathcal{M}} - \sum_{\lambda \in \Lambda} S_\lambda P_{\mathcal{M}} S_\lambda^*$ is the projection of $\mathcal{H}$ on $\mathcal{M} \ominus (\bigoplus_{\lambda \in \Lambda} S_\lambda \mathcal{M})$.

Let us show that $\dim P\mathcal{H} \leq \dim \mathcal{L}$, where as usual $\mathcal{L} = \mathcal{H} \ominus (\bigoplus_{\lambda \in \Lambda} S_\lambda \mathcal{H})$. The case when $\mathcal{L}$ is infinite dimensional is clearly. If $\dim \mathcal{L}$ is finite and $\{e_i\}_{i \in I}$ is an orthonormal basis for $\mathcal{L}$, then $\{S_f e_i : i \in I,\ f \in \mathcal{F}\}$ is an orthonormal basis for $\mathcal{H}$ and we have:

$$\dim P\mathcal{H} = \sum_{i \in I} \sum_{f \in \mathcal{F}} \langle PS_f e_i, S_f e_i \rangle =$$

$$= \lim_{n \to \infty} \sum_{i \in I} \sum_{k=0}^{n} \sum_{f \in F(k,\Lambda)} \langle (P_{\mathcal{M}} - \sum_{\lambda \in \Lambda} S_\lambda P_{\mathcal{M}} S_\lambda^*) S_f e_i, S_f e_i \rangle =$$

$$= \lim_{n \to \infty} \sum_{i \in I} \sum_{f \in F(n,\Lambda)} \langle P_{\mathcal{M}} S_f e_i, S_f e_i \rangle \leq$$

$$\leq \lim_{n \to \infty} \sum_{i \in I} \sum_{f \in F(n,\Lambda)} \| S_f e_i \|^2 \leq \sum_{i \in I} \| e_i \|^2 = \dim \mathcal{L}$$

By Theorem 3.1 it follows that $P_{\mathcal{M}} = MM^*$ for some $\mathcal{S}$-analytic operator $M \in B(\mathcal{H})$. Moreover, M is $\mathcal{S}$-inner since $P_{\mathcal{M}}$ is a projection. Thus $\mathcal{M} = P_{\mathcal{M}} \mathcal{H} = M\mathcal{H}$. The uniqueness follows by Theorem 3.2.

4. SZEGO-TYPE FACTORIZATIONS AND $\mathcal{S}$-INNER-OUTER FACTORIZATIONS FOR $\mathcal{S}$-ANALYTIC OPERATORS

Let us consider $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ a Λ-orthogonal shift on $\mathcal{H}$,

$\mathcal{L}=\mathcal{H}\ominus(\bigoplus_{\lambda\in\Lambda} S_\lambda\mathcal{H})$ and let $T\in B(\mathcal{H})$ be a nonnegative $\mathcal{S}$-Toeplitz operator. For each $\lambda\in\Lambda$ we define the Lowdenslager's isometry $S_{T,\lambda}$ on $\mathcal{H}_T:=\overline{T^{1/2}\mathcal{H}}$ by setting $S_{T,\lambda}(T^{1/2}h)=T^{1/2}S_\lambda h$, $(h\in\mathcal{H})$. It is easy to see that $\mathcal{S}_T:=\{S_{T,\lambda}\}_{\lambda\in\Lambda}$ is a sequence of isometries with orthogonal final spaces.

After these preliminaries we can state the following theorem which is a version of [8, Theorem 3.4] in our setting.

THEOREM 4.1. If $T\in B(\mathcal{H})$ is a nonnegative $\mathcal{S}$-Toeplitz operator then the following are equivalent:

(i) $T=A^*A$ for some $\mathcal{S}$-analytic operator $A\in B(\mathcal{H})$;

(ii) $\mathcal{S}_T$ is a Λ-orthogonal shift on $\mathcal{H}_T$;

(iii) There is a dense subset $\mathcal{L}'$ of $\mathcal{L}$ such that for any $l'\in\mathcal{L}'$

$$\lim_{n\to\infty}\left(\sup\left\{\sum_{f\in F(n,\Lambda)}|\langle Tl', S_f h\rangle|^2;\ h\in\mathcal{H},\ \|T^{1/2}h\|=1\right\}\right)=0 .$$

In this case there is an $\mathcal{S}$-outer operator $A\in B(\mathcal{H})$ such that $T=A^*A$ and $A_o:=P_oAP_o|_{\mathcal{L}}\geqslant 0$, where P_o is the projection of $\mathcal{H}$ on $\mathcal{L}$.

Proof. (i)$\Longrightarrow$(iii). If (i) holds and $l\in\mathcal{L}$, $h\in\mathcal{H}$ with $\|T^{1/2}h\|=1$ then $\sum_{f\in F(n,\Lambda)}|\langle Tl,S_f h\rangle|^2=\sum_{f\in F(n,\Lambda)}|\langle S_f^*Al,Ah\rangle|^2\leqslant$ $\leqslant\sum_{f\in F(n,\Lambda)}\|S_f^*Al\|^2$ for any $n=0,1,2,\ldots$. Since $\mathcal{S}$ is a Λ-orthogonal shift, by Corollary 1.2, $\lim_{n\to\infty}\sum_{f\in F(n,\Lambda)}\|S_f^*Al\|^2=0$ and hence (iii) holds.

(iii)$\Longrightarrow$(ii). If (iii) holds then for every $l'\in\mathcal{L}'$ and $n=0,1,2,\ldots$

(4.1) $$\sum_{f\in F(n,\Lambda)} \|S^*_{T,f}T^{1/2}l'\|^2 =$$

$$=\sup\left\{\sum_{f\in F(n,\Lambda)} |\langle Tl',S_f h\rangle|^2 ; h\in\mathcal{H}, \ \|T^{1/2}h\|=1\right\},$$

where $S_{T,f}$ stnds for the product $S_{T,f(1)}S_{T,f(2)}\cdots S_{T,f(n)}$.

Indeed, we have $\sum_{f\in F(n,\Lambda)} |\langle Tl',S_f h\rangle|^2 =$

$=\sum_{f\in F(n,\Lambda)} |\langle S^*_{T,f}T^{1/2}l',T^{1/2}h\rangle|^2$ and since the set $\{T^{1/2}h: \|T^{1/2}h\|=1\}$ is dense in the unit sphere of $\mathcal{H}_T$, (4.1) follows. Therefore (iii) implies

(4.2) $$\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|S^*_{T,f}T^{1/2}l'\|^2=0 \qquad (l'\in\mathcal{L}')$$

On the other hand for $l'\in\mathcal{L}'$ and $g\in F(m,\Lambda)$, $m=1,2,\ldots$, we have

$$\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|S^*_{T,f}T^{1/2}S_g l'\|^2 = \lim_{n\to\infty} \sum_{q\in F(n-m,\Lambda)} \|S^*_{T,q}T^{1/2}l'\|^2 \overset{(4.2)}{=} 0.$$

An approximation argument shows that

$$\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|S^*_{T,f}k\|^2=0 \quad \text{for any } k\in\mathcal{H}_T .$$

Now (ii) follows from Corollary 1.2.

(ii)$\Longrightarrow$(i) Assume that (ii) holds. By the definition of $S_{T,\lambda}$ $(\lambda\in\Lambda)$ we infer that for each $\lambda\in\Lambda$

(4.3) $\qquad T^{1/2}S^*_{T,\lambda}h=S^*_\lambda T^{1/2}h \quad$ for any $h\in\mathcal{H}_T$.

Hence, the operator $X=T^{1/2}|_{\mathcal{L}_T}$ maps $\mathcal{L}_T:=\mathcal{H}_T\ominus(\bigoplus_{\lambda\in\Lambda} S_{T,\lambda}\mathcal{H}_T)$ into $\mathcal{L}$. As in [8, Thm. 3.4] the polar decomposition of X^* gives $X^*=W^*Q$, where $Q=(XX^*)^{1/2}\in B(\mathcal{L})$ and W mapping $\mathcal{L}_T$ into $\mathcal{L}$ is an isometry. We extend W to an isometry V from $\mathcal{H}_T$ to $\mathcal{H}$ such that

(4.4) $\qquad VS_{T,\lambda}=S_\lambda V \qquad (\lambda\in\Lambda)$

as follows. Since $\mathcal{S}_T$ is Λ-orthogonal shift on $\mathcal{H}_T$, by Corollary 1.3, each $k\in\mathcal{H}_T$ has a unique representation

$$k=\sum_{f\in\mathcal{F}} S_{T,f}k_f\ , \qquad (k_f\in\mathcal{L}_T)$$

We now set

$$Vk=\sum_{f\in\mathcal{F}} S_f W k_f\ .$$

Let $A\in B(\mathcal{H})$ be defined by

$$Ah=VT^{1/2}h, \qquad (h\in\mathcal{H})\ .$$

Taking into account (4.3), (4.4) it is easy to verify that $T=A^*A$ and $AS_\lambda=S_\lambda A$ for each $\lambda\in\Lambda$, i.e. (ii) holds.

Finally, let us show that A is $\mathcal{S}$-outer and $A_o:=P_oAP_o|_{\mathcal{L}}\geqslant 0$. Setting $\mathcal{M}=W\mathcal{L}_T$ we find $\overline{A\mathcal{H}}=\bigoplus_{f\in\mathcal{F}} S_f\mathcal{M}$ and by Corollary 1.4 it follows that $\overline{A\mathcal{H}}$ reduces each S_λ $(\lambda\in\Lambda)$ i.e. A is $\mathcal{S}$-outer.

If $l\in\mathcal{L}$, then $A_o l=P_oVT^{1/2}l=WP_{\mathcal{L}_T}T^{1/2}l=WX^*l=Ql$, where $P_{\mathcal{L}_T}$ is the projection of $\mathcal{H}_T$ on $\mathcal{L}_T$.

Therefore $A_o=Q\geqslant 0$ and the proof is complete.

We are now ready to use the above theorem for obtaining $\mathcal{S}$-inner-outer factorizations for $\mathcal{S}$-analytic operators.

THEOREM 4.2. Let $T\in B(\mathcal{H})$ be an $\mathcal{S}$-analytic operator. Then

$$T = BA$$

where $A\in B(\mathcal{H})$ is $\mathcal{S}$-outer and $B\in B(\mathcal{H})$ is $\mathcal{S}$-inner with initial space $\overline{A\mathcal{H}}$.

Moreover, there is a factorization of this type so that the diagonal entry A_o in the matrix of A satisfies $A_o\geq 0$. In this case the factors A and B are unique.

<u>Proof</u>. Applying Thoerem 4.1 to the operator $X=T^*T$ we get an $\mathcal{S}$-outer operator $A\in B(\mathcal{H})$ such that

$$T^*T=A^*A \tag{4.5}$$

and $A_o=P_oAP_o|_{\mathcal{L}}\geq 0$.

By (4.5) there is a unique partial isometry $B\in B(\mathcal{H})$ with initial space $\overline{A\mathcal{H}}$ sucht that $T=BA$. Let us show that for each $\lambda\in\Lambda$, $S_\lambda B=BS_\lambda$.

Obviously $S_\lambda B$ and BS_λ coincide on $A\mathcal{H}$. On the other hand, since A is $\mathcal{S}$-outer, $\overline{A\mathcal{H}}$ is a reducing subspace for each S_λ $(\lambda\in\Lambda)$, which together with $B|_{(C\mathcal{H})^\perp}=0$ implies that $S_\lambda B$ and BS_λ are zero on $(A\mathcal{H})^\perp$. Therefore B is $\mathcal{S}$-inner.

For uniqueness let $T=B_1A=B_2C$ be with the required properties of theorem. Then, $A^*A=C^*C$ where A, C are $\mathcal{S}$-outer and $A_o\geq 0$, $C_o\geq 0$. Similar arguments as above show that there is an $\mathcal{S}$-constant inner operator B with initial space $\overline{C\mathcal{H}}$ and final space $\overline{A\mathcal{H}}$ such that $A=BC$. By Proposition 2.3 we infer that $A_o=B_oC_o$

whence $A_o^2 \leq C_o^2$. Interchanging the roles of A and C we obtain $A_o^2 = C_o^2$ and hence $A_o = C_o$.

Since $A_o = B_o C_o$, B_o coincides with the identity operator on $\overline{C_o \mathcal{L}}$. On the other hand $\overline{C_o \mathcal{L}} = \overline{P_o C \mathcal{L}} = \overline{P_o C \mathcal{H}}$ and since C is $\mathcal{S}$-outer, $\overline{C\mathcal{H}}$ reduces each S_λ $(\lambda \in \Lambda)$.

Now by Corollary 1.4 one easily show that

$$\overline{C\mathcal{H}} = \bigoplus_{f \in \mathcal{F}} S_f(\overline{C_o \mathcal{L}}) \quad .$$

Thus, each $h \in \overline{C\mathcal{H}}$ has the form $h = \sum_{f \in \mathcal{F}} S_f l_f$, where $l_f \in \overline{C_o \mathcal{L}}$. By Theorem 2.2 we have

$$Bh = \sum_{f \in \mathcal{F}} S_f B_o l_f = \sum_{f \in \mathcal{F}} S_f l_f = h \quad , \qquad (h \in \overline{C\mathcal{H}})$$

Since A=BC it follows that A=C.

The fact that $B_1 = B_2$ is immediately. The proof is complete.

Finally, we adapt [8, Theorem 3.7] to our setting.

THEOREM 4.3. If $T \in B(\mathcal{H})$ is a nonnegative $\mathcal{S}$-Toeplitz operator such that $T \geq rI_{\mathcal{H}}$ for some $r > 0$, then $T = A^*A$ for some $\mathcal{S}$-analytic operator $A \in B(\mathcal{H})$.

<u>Proof</u>. Since T is invertible, $\mathcal{H}_T = \mathcal{H}$ and for each $\lambda \in \Lambda$

$$S_{T,\lambda} = T^{1/2} S_\lambda T^{-1/2}$$

Thus

$$\lim_{n\to\infty} \sum_{f \in F(n,\Lambda)} \| S^*_{T,\lambda} h \|^2 = \lim_{n\to\infty} \sum_{f \in F(n,\Lambda)} \| T^{-1/2} S^*_f T^{1/2} h \|^2 = 0$$

for any $h \in \mathcal{H}$.

By Corollary 1.2, $\mathcal{S}_T=\{S_{T,\lambda}\}_{\lambda\in\Lambda}$ is a Λ-orthogonal shift on $\mathcal{H}$. Applying Theorem 4.1 the result follows.

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