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Pseudoconvex domains on complex spaces with singularities

§1. Introduction

This short note deals with pseudoconvex domains (and more generally locally hyperconvex domains) on complex spaces with singularities, especially on Stein spaces or ramified coverings of $\mathbb{C}^n$. For strongly pseudoconvex domains the results are well known [13]. Also for Stein manifolds stronger results (for locally Stein open subsets) have been proved by Docquier and Grauert [3]. Their proof is in fact a reduction of the problem to Oka's theorem [15] by imbedding the given manifold X in $\mathbb{C}^N$ and then showing that there is a neighbourhood U of X and a holomorphic retraction $r: U \to X$. When X is singular such a retraction does not exist [6]. Another difficulty in the case of singular Stein spaces is the lack of a "good distance" (with certain convexity properties) to the boundary of a Stein open subset $D \subset X$ [17].

We state now our results :

Theorem 1. Let X be a complex space, $D \subset\subset X$ a relatively compact open subset which is locally hyperconvex and assume that there exists a continuous strongly plurisubharmonic function in a neighbourhood of $\bar{D}$. Then D is Stein.

As a direct consequence we obtain :

Corollary 1. Let X be a K-complete space and $D \subset\subset X$ a relatively compact open subset which is locally hyperconvex. Then D is Stein. In particular any pseudoconvex domain $D \subset\subset X$ is Stein.

Corollary 2. Let X be a Stein space and $D \subset X$ an open subset which is locally hyperconvex. Then D is Stein. In particular any pseudoconvex domain $D \subset X$ is Stein.

When X is a Stein space the above corollary can be strengthened as follows :

Theorem 2. Let X be a Stein space and $D \subset X$ a locally Stein open subset. Assume that D is locally hyperconvex at $\partial D \cap \mathrm{Sing}(X)$. Then D is a Stein space.

Remark 1.

a) The problem whether pseudoconvex domains on Stein spaces are themselves Stein was raised by Narasimhan [12].

b) Corollary 1 for pseudoconvex domains was proved in ([1], Theorem 2) under the additional assumption that D has a globally defined boundary.

c) A stronger version of theorem 1 is proved in [4] for complex manifolds. In this case the condition "D is locally hyperconvex" may be replaced by the weaker assumption "D is locally Stein".

d) A weaker result than theorem 2 is proved in ([1], Corollary 2). Namely it is assumed that D is strongly pseudoconvex at $\partial D \cap \mathrm{Sing}(X)$.

§2. Preliminaries

All complex spaces considered are supposed reduced and countable at infinity.

A Stein space X is called hyperconvex [18] if there exists a continuous plurisubharmonic exhaustion function $\varphi : X \to (-\infty, 0)$ (the empty set is considered hyperconvex).

Examples of hyperconvex spaces.

Let $D\subset\mathbb{C}^n$ be a Stein open set. Each of the following conditions are sufficient for the hyperconvexity of D :

a) D is bounded and convex [18]

b) D is bounded and has C^2 boundary [2] or C^1 boundary [11]

c) D is a bounded Reinhardt domain containing the origin [5]

d) D is a tube whose base $Re(D)\subset\mathbb{R}^n$ is bounded and convex [5]

Other examples can be found in ([5], [18]). In fact, for bounded domains of $\mathbb{C}^n$, the hyperconvexity is a local property [11]. To get examples of hyperconvex spaces in the singular case one may take subspaces or finite morphisms into the nonsingular ones given above. In particular any relatively compact analytic polyhedron in a Stein space is hyperconvex and any Stein space can be exhausted with hyperconvex open sets.

Definition 1. Let X be a complex space, $D\subset X$ an open subset and $A\subset\partial D$ any subset. We say that D is locally hyperconvex at A if for any $x_o\in A$ there exists an open neighbourhood U of x_o such that $U\cap D$ is hyperconvex. When $A=\partial D$ D is called locally hyperconvex.

Definition 2 ([1], [12], [13]) Let X be a complex space and $D\subset X$ an open subset. D is called pseudoconvex if for any $x_o\in\partial D$ there exists an open neighbourhood U of x_o and a continuous plurisubharmonic function $\varphi:U\to\mathbb{R}$ such that $U\cap D=\{x\in U\mid \varphi(x)<o\}$.

It is clear from the above definitions that any pseudoconvex domain is locally hyperconvex.

The proof of theorem 1 relies on a patching technique developed by M.Peternell in [16] (see also [11]) which allows us to produce a continuous strongly plurisubharmonic exhaustion function $\varphi:D\to\mathbb{R}$. To obtain the Steiness of D we invoke the following result of

Narasimhan [13] :

Theorem 3. Let D be a complex space and assume that there exists a continuous strongly plurisubharmonic exhaustion function $\varphi: D \to \mathbb{R}$. Then D is a Stein space.

For the proof of theorem 2 we shall need the following two results :

Theorem 4 ([1], Theorem 4) Let X be a Stein space and $D \subset X$ a locally Stein open subset. Assume that there is an open neighbourhood U of $\partial D \cap Sing(X)$ such that $D \cap U$ is a Stein space. Then D itself is a Stein space.

Theorem 5 ([14], Theorem 2) Let X be a Stein space, $A \subset X$ a closed analytic subset and V an open neighbourhood of A. Then there exists a continuous plurisubharmonic function $p: X \to \mathbb{R}$ such that $A \subset \{p < 0\} \subset V$.

Let us recall also the following :

Definition 3. A complex space is called K-complete if for any $x_0 \in X$ there is a holomorphic map $f: X \to \mathbb{C}^p$, $p = p(x_0)$ such that x_0 is an isolated point of $f^{-1}(f(x_0))$.

It is known [9] that a complex space of pure dimension n is K-complete iff X can be realised as a ramified domain over $\mathbb{C}^n$, but we shall not need this result.

In ([1], Lemma 5) it was proved :

Theorem 6. Every relatively compact open subset of a K-complete space X carries a C^∞ strongly plurisubharmonic function.

§3. Proof of the main results

In the proof of theorem 1 the existence of some special convex increasing functions on $(-\infty,0)$ will play an important role. So we state :

Lemma 1. Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of negative real numbers such that $a_n \to 0$. Then there exists a function $\tau:(-\infty,0)\to\mathbb{R}$ with the following properties :

1) τ is continuous, increasing and convex
2) $\tau \geq 0$
3) $\lim_{x\to 0} \tau(x)=\infty$
4) $\tau(a_{n+1})-\tau(a_n)<1$ for every $n\in\mathbb{N}$

Proof

We define τ to be linear on each interval $[a_n,a_{n+1}]$ and to vanish identically near $-\infty$. The precise definition is as follows :

$$\tau(x)=\begin{cases} n-\left(\frac{a_2}{a_1}+\dots+\frac{a_n}{a_{n-1}}\right)-\frac{x}{a_n} & \text{if } a_n\leq x\leq a_{n+1} \\ 0 & \text{if } x\leq a_1 \end{cases}$$

Properties 1),2) and 4) follow easily from the definition of τ so it remains to verify 3). Since τ is increasing it suffices to show that $\tau(a_n)\to\infty$. Now $\tau(a_{n+p})-\tau(a_n)=$ $\frac{a_n-a_{n+1}}{a_n}+\dots+\frac{a_{n+p-1}-a_{n+p}}{a_{n+p-1}}\geq\frac{a_n-a_{n+p}}{a_n}$, hence for a given n $\tau(a_{n+p})-\tau(a_n)\geq\frac{1}{2}$ if p is sufficiently large (depending on n). It follows that $\tau(a_n)\to\infty$ which proves the lemma.

Lemma 2. Let $f_1,\dots,f_n:(-\infty,0)\to(-\infty,0)$ be increasing functions such that for any $i\in\{1,\dots,n\}$ $\lim_{x\to 0} f_i(x)=0$. Then there exists a continuous convex increasing function $\tau:(-\infty,0)\to\mathbb{R}$ such that :

a) $\lim_{x\to 0}\tau(x)=\infty$

b) $\tau\circ f_i-\tau\circ f_j$ is bounded for any $i,j\in\{1,\dots,n\}$

Proof

From the assumption " $\lim_{x\to 0} f_i(x)=0$ for any $i\in\{1,\dots,n\}$ " it follows that there exists an increasing sequence $\{\alpha_\nu\}_{\nu\in\mathbb{N}}$ of negative real numbers, $\alpha_\nu\to 0$ such that :

(*) $\max\{f_1(\alpha_\nu),\dots,f_n(\alpha_\nu)\}<\min\{f_1(\alpha_{\nu+1}),\dots,f_n(\alpha_{\nu+1})\}$ for any $\nu\in\mathbb{N}$.

If we set $a_\nu=\min\{f_1(\alpha_\nu),\dots,f_n(\alpha_\nu)\}$ for odd ν and $a_\nu=\max\{f_1(\alpha_\nu),\dots,f_n(\alpha_\nu)\}$ for even ν then $a_1<\dots a_\nu<a_{\nu+1}<\dots<0$ and $a_\nu\to 0$.

By lemma 1 there is a continuous convex increasing function $\tau:(-\infty,0)\to\mathbb{R}$, $\tau\geq 0$, $\lim_{x\to 0}\tau(x)=\infty$ and $\tau(a_{\nu+1})-\tau(a_\nu)<1$ for any $\nu\in\mathbb{N}$. To prove lemma 2 it remains to verify that $\tau\circ f_i-\tau\circ f_j$ is bounded. Since τ is bounded below ($\tau\geq 0$) it suffices to check that $\tau(f_i(x))-\tau(f_j(x))$ is bounded for $x<0$ sufficiently close to 0. If $\alpha_{2\nu}\leq x\leq\alpha_{2\nu+2}$ then $a_{2\nu-1}\leq\min\{f_i(x),f_j(x)\}\leq\max\{f_i(x),f_j(x)\}\leq a_{2\nu+2}$, hence $\tau(f_i(x))-\tau(f_j(x))<3$, which proves lemma 2.

Lemma 3. Let Y be a complex space which carries a continuous strongly plurisubharmonic function and let $D\subset\subset Y$ be a relatively compact open subset. Assume that there exist open subsets of Y $A_i\subset\subset B_i\subset\subset C_i$ $i\in\{1,\dots,k\}$, $D\subset\bigcup_{i=1}^{k}A_i$ and continuous plurisubharmonic exhaustion functions $\varphi_i:C_i\cap D\to\mathbb{R}$

such that $\varphi_i|_{B_i\cap B_j\cap D} - \varphi_j|_{B_i\cap B_j\cap D}$ is bounded for any $i,j\in\{1,\dots,k\}$. Then D is a Stein space.

Proof

The proof is obtained by a slight modification of the arguments given by M.Peternell in ([16], Lemma 1o). For the sake of completeness we shall indicate the modifications to be done.

Take $p_i'\in C_o^\infty(Y)$ with $p_i'\geq o$, supp $p_i'\subset B_i$ and $p_i'|_{A_i}=1$. We define the functions $p_i\in C_o^\infty(Y)$ in the following way : for each i the functions $\varphi_j-\varphi_i$ $j\in\{1,\dots,k\}$ are bounded on $\partial B_j\cap A_i\cap D$ so we can choose a sufficiently large constant $\lambda_i>o$ with $\lambda_i p_i'>\varphi_j-\varphi_i$ on $\partial B_j\cap A_i\cap D$. We set $p_i=\lambda_i p_i'$. Since $p_j=o$ on ∂B_j we have :

(*) $\quad p_i+\varphi_i>p_j+\varphi_j \quad$ on $\partial B_j\cap A_i\cap D$

Let now φ be a continuous strongly plurisubharmonic function on Y and let $A>o$ be a sufficiently large constant such that $A\varphi+p_i$ is strongly plurisubharmonic for any $i\in\{1,\dots,k\}$.

We set $I=\{1,\dots,k\}$ and for $x\in D$ we define $I(x)\subset I$ by $I(x)=\{i\in I\mid x\in B_i\}$. If $x\in D$ we set $u(x)=\max_{i\in I(x)}\{p_i(x)+\varphi_i(x)\}$. We show that $\psi=A\varphi+u$ is a continuous strongly plurisubharmonic exhaustion function on D. It is clear that ψ is an exhaustion function because φ_i are exhaustion functions on $C_i\cap D$, hence it remains to verify that ψ is a continuous strongly plurisubharmonic function on D. Let $x_o\in D$ and set $I'(x_o)=\{i\in I\mid x_o\in\partial B_i\}$. Choose a neighbourhood $D_{x_o}\subset D$ of x_o such that $D_{x_o}\cap B_i=\emptyset$ if $i\notin I(x_o)\cup I'(x_o)$ and let $i_o\in I(x_o)$ with $x_o\in A_{i_o}$. For each $j\in I'(x_o)$ it follows from (*) that $p_{i_o}+\varphi_{i_o}>p_j+\varphi_j$ on D_{x_o} if $D_{x_o}\subset A_{i_o}$ is chosen small enough. We get $u|_{D_{x_o}}=\max_{i\in I(x_o)}\{p_i+\varphi_i\}$ hence

$\psi\big|_{D_{x_0}} = \max_{i\in I(x_0)}\{A\varphi + p_i + \varphi_i\}$ which shows that ψ is a continuous strongly plurisubharmonic function. By theorem 3 D is Stein and the proof of lemma 3 is complete.

Theorem 1. Let X be a complex space and $D \subset\subset X$ a relatively compact open subset which is locally hyperconvex and assume that there exists a continuous strongly plurisubharmonic function in a neighbourhood of $\overline{D}$. Then D is Stein.

Proof

Let Y be a neighbourhood of $\overline{D}$ and φ a continuous strongly plurisubharmonic function on Y. Choose open subsets $A_i \subset\subset B_i \subset\subset C_i \subset Y$ $i \in \{1,\dots,k\}$ such that

1) $D \subset \bigcup_{i=1}^{k} A_i$

2) for any $i \in \{1,\dots,k\}$ there exists a continuous plurisubharmonic exhaustion function $v_i : C_i \cap D \to (-\infty, o)$.

For every $i, j \in \{1,\dots,k\}$ such that $B_i \cap B_j \cap D \neq \emptyset$ we define the function $E_{ij} : (-\infty, o) \to (-\infty, o)$ by $E_{ij}(x) = \inf\{v_j(z) \mid z \in B_i \cap B_j \cap D \quad v_i(z) \geq x\}$

E_{ij} are increasing functions and $\lim_{x\to 0} E_{ij}(x) = o$ because v_i are exhaustion functions. Let $h : (-\infty, o) \to (-\infty, o)$ be the identity map. Now we use lemma 2 for the finite set of functions $\{E_{ij}, h\}$ and we get a continuous increasing function $\tau : (-\infty, o) \to \mathbb{R}$ such that :

1) $\lim_{x\to 0} \tau(x) = \infty$

2) $\tau - \tau\circ E_{ij}$ is bounded for any $i, j \in \{1,\dots,k\}$ with $B_i \cap B_j \cap D \neq \emptyset$.

Setting $\varphi_i = \tau \circ v_i$ we get continuous plurisubharmonic exhaustion functions on $C_i \cap D$. Moreover, if $z \in B_i \cap B_j \cap D$ $\quad E_{ij}(v_i(z)) \leq v_j(z)$, therefore $\varphi_i(z) - \varphi_j(z) \leq (\tau - \tau\circ E_{ij})(v_i(z))$. From lemma 3 D is

Stein and the proof of theorem 1 is complete.

We give now some immediate consequences of theorem 1. By theorem 6 we know that any relatively compact open subset of a K-complete space carries a C^∞ strongly plurisubharmonic function. Therefore we obtain :

Corollary 1. Let X be a K-complete space and $D \subset\subset X$ a relatively compact open subset which is locally hyperconvex. Then D is Stein. In particular any pseudoconvex domain $D \subset\subset X$ is Stein.

When X is a Stein space by an exhaustion argument we get :

Corollary 2. Let X be a Stein space and $D \subset X$ an open subset which is locally hyperconvex. Then D is Stein. In particular any pseudoconvex domain $D \subset X$ is Stein.

Corollary 2 is a particular case of the following open problem (see [1] , [8], [17]) :

Levi Problem Let X be a Stein space and $D \subset X$ a locally Stein open subset. Is D itself a Stein space ?

We show that this is the case at least when D is locally hyperconvex at $\partial D \cap \mathrm{Sing}(X)$, namely we prove :

Theorem 2. Let X be a Stein space and $D \subset X$ a locally Stein open subset. Assume that D is locally hyperconvex at $\partial D \cap \mathrm{Sing}(X)$. Then D is a Stein space.

Proof

For each $x \in \mathrm{Sing}(X)$ we choose a hyperconvex neighbourhood $V_x \subset\subset X$ of x such that $V_x \cap D$ is hyperconvex. Then $V = \bigcup_{x \in \mathrm{Sing}(X)} V_x$ is an open neighbourhood of Sing(X) and by theorem 5 there is a continuous plurisubharmonic function p on X such that $B = \{p < o\}$

contains Sing(X) and $\overline{B}\subset V$. We show that $B\cap D$ is locally hyperconvex. Indeed, for any $x_0\in\overline{B\cap D}\subset\overline{B}\subset V$ there exists $x\in Sing(X)$ with $x_0\in V_x$. On the other hand $V_x\cap B\cap D=(V_x\cap B)\cap(V_x\cap D)$ which is hyperconvex as an intersection of two hyperconvex open subsets. Therefore $B\cap D$ is locally hyperconvex and by corollary 2 $B\cap D$ is Stein. In view of theorem 4 D itself is a Stein space and the proof is complete.

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