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by

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Mihai BAKONYI

In this paper we point out two properties of outer and *-outer spectral factors of an operator valued analytic contractive function. Spectral factorization has been investigated extensively for its applications in various areas including electrical engineering. We are based on interpolation ideas originated to Schur [11] and developed further in a very large number of works, from which we use here [2], [5] and [6].

In §2 we give an approximation of spectral factors extending a well known result on the rational approximation from the scalar case, see for instance [4] and [9].

In §3 we solve an analytic completion problem in the case of functions which admit meromorphic pseudo-continuation of bounded type. The existence of inner dilations for that class of operator valued analytic contractive functions was proved in [8] and [1] , situation related to Darlington synthesis. For such a function we solve an analytic contractive completion problem for a 2x2 operator matrix valued function which extends the constant case, problem solved in [3], [7], [13].

Thanks are due to Professor T.Constantinescu for suggesting these problems and for helpful discussions.

§1. INTRODUCTION

We shall begin by reminding some facts and notations from [12], [2] and [5].

Thus, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ we denote by $B_1(\mathcal{H},\mathcal{H}')$ the set of all contractions from $\mathcal{H}$ into $\mathcal{H}'$ and by $B_1(\mathcal{H})$ the set of the contractions on $\mathcal{H}$. As in [12], for $T\in B_1(\mathcal{H},\mathcal{H}')$ let $D_T=(I-T^*T)^{1/2}$ and $\mathcal{D}_T=\overline{D_T\mathcal{H}}$ the defect operator, respectively the defect space of T.

By the definition given in [2], for two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, a $(\mathcal{H},\mathcal{H}')$ choice sequence is a sequence of contractions $\{\Gamma_n\}_{n=1}^{\infty}$, $\Gamma_1\in B_1(\mathcal{H},\mathcal{H}')$ and $\Gamma_n\in B_1(\mathcal{D}_{\Gamma_{n-1}},\mathcal{D}_{\Gamma^*_{n-1}})$ for all $n\geq 2$.

We present now some facts from [5] which are used in the construction of the Naimark dilation of a semispectral measure.

For a fixed $(\mathcal{H},\mathcal{H}')$ choice sequence, we define:

$$(1.1)_n \qquad \mathcal{H}_n=\mathcal{H}\oplus\bigoplus_{k=1}^{n-1}\mathcal{D}_{\Gamma_k}$$

and the contractions:

$$(1.2)_n \begin{cases} X_n:\mathcal{H}_n\longrightarrow\mathcal{H} \\ X_n=(\Gamma_1, D_{\Gamma_1^*}\Gamma_2, D_{\Gamma_1^*}D_{\Gamma_2^*}\Gamma_3,\ldots,D_{\Gamma_1^*}\ldots D_{\Gamma_{n-1}^*}\Gamma_n) \end{cases}$$

There exist the unitary operators (see [5]):

$$(1.3)_n \qquad \begin{cases} \alpha_n : \mathcal{D}_{X_n} \longrightarrow \overset{n}{\underset{k=1}{\oplus}} \mathcal{D}_{\Gamma_k} \\ \alpha_n D_{X_n} = D_n \end{cases}$$

where:

$$(1.4)_n \qquad D_n : \mathcal{X}_n \longrightarrow \overset{n}{\underset{k=1}{\oplus}} \mathcal{D}_{\Gamma_n}$$

$$\begin{bmatrix} D_{\Gamma_1}, & -\Gamma_1^* \Gamma_2, & -\Gamma_1^* D_{\Gamma_2^*} \Gamma_3 & \cdots\cdots & -\Gamma_1^* D_{\Gamma_2^*} \cdots D_{\Gamma_{n-1}^*} \Gamma_n \\ 0, & D_{\Gamma_2}, & -\Gamma_2^* \Gamma_3 & \cdots\cdots, & -\Gamma_2^* D_{\Gamma_3^*} \cdots D_{\Gamma_{n-1}^*} \Gamma_n \\ 0, & 0, & D_{\Gamma_3}, & \cdots\cdots & \\ \vdots & \vdots & \vdots & & \vdots \\ 0, & 0, & 0, & \cdots\cdots, & D_{\Gamma_n} \end{bmatrix}$$

and:

$$(1.5)_n \begin{cases} \tilde{\alpha}_n : \mathcal{D}_{X_n^*} \longrightarrow \overline{\mathcal{R}(H_n^{1/2})} \\ \tilde{\alpha}_n D_{X_n^*} = H_n^{1/2} \end{cases}$$

where:

$$(1.6)_n \begin{cases} H_n : \mathcal{H} \longrightarrow \mathcal{H} \\ H_n = D_{\Gamma_1^*} D_{\Gamma_2^*} \cdots D^2_{\Gamma_n^*} \cdots D_{\Gamma_2^*} D_{\Gamma_1^*} \end{cases}$$

We also need the unitary operators:

$$(1.7_n)\begin{cases}\beta_n : \overline{\mathcal{R}(H_n^{1/2})} \longrightarrow \mathcal{D}_{\Gamma_n^*} \\ \beta_n H_n^{1/2} = D_{\Gamma_n^*} D_{\Gamma_{n-1}^*} \cdots D_{\Gamma_1^*}\end{cases}$$

and:

$$(1.8)_n\begin{cases}\tilde{\beta}_n : \overline{\mathcal{R}((D_{\Gamma_1} D_{\Gamma_2} \cdots D^2_{\Gamma_n} \cdots D_{\Gamma_2} D_{\Gamma_1})^{1/2})} \longrightarrow \mathcal{D}_{\Gamma_n} \\ \tilde{\beta}_n (D_{\Gamma_1} D_{\Gamma_2} \cdots D^2_{\Gamma_n} \cdots D_{\Gamma_2} D_{\Gamma_1})^{1/2} = D_{\Gamma_n} D_{\Gamma_{n-1}} \cdots D_{\Gamma_1}\end{cases}$$

Let us define:

$$(1.9) \qquad \mathcal{K}_+ = \mathcal{H} \oplus \bigoplus_{n=1}^{\infty} \mathcal{D}_{\Gamma_n}$$

and we denote by $\tilde{P}_n = P^{\mathcal{K}_+}_{\mathcal{K}_n}$ the ortogonal projection of $\mathcal{K}_+$ onto $\mathcal{K}_n$ ($\mathcal{K}_n$ regarded as being embedded in $\mathcal{K}_+$).

In [5] it is proved that there exist the strong operatorial limits:

$$(1.10) \qquad X_\infty = S\text{-}\lim_n X_n \tilde{P}_n : \mathcal{K}_+ \longrightarrow \mathcal{H}$$

$$(1.11) \qquad D_\infty = S\text{-}\lim_n D_n \tilde{P}_n : \mathcal{K}_+ \longrightarrow \bigoplus_{k=1} \mathcal{D}_{\Gamma_k}$$

$$(1.12) \qquad H = S\text{-}\lim_n H_n : \mathcal{H} \longrightarrow \mathcal{H}$$

and the unitary operators:

$$(1.13)\begin{cases}\alpha : \mathcal{D}_{X_\infty} \longrightarrow \bigoplus_{k=1}^{\infty} \mathcal{D}_{\Gamma_k} \\ \alpha D_{X_\infty} = D_\infty\end{cases}$$

and:

$$(1.14)\begin{cases}\widetilde{\alpha}:\mathcal{D}_{X^*_\infty}\longrightarrow\mathcal{D}_*\\ \widetilde{\alpha}\quad D_{X^*_\infty}=D_*\end{cases}$$

where $D_* = H^{1/2}$ and $\mathcal{D}_* = \overline{\mathcal{R}(D_*)}$.

Let us define the unitary operator:

$$(1.15)\begin{cases}W_{red}=W_{red}(\Gamma_1,\Gamma_2,\dots,\Gamma_n,\dots):\mathcal{D}_*\oplus\mathcal{K}_+\to\mathcal{H}\oplus\bigoplus_{k=1}^{\infty}\mathcal{D}_{\Gamma_k}\\ W_{red}=\begin{pmatrix}I & 0\\ 0 & \alpha\end{pmatrix}\begin{pmatrix}D_{X^*_\infty} & X_\infty\\ -X^*_\infty & D_{X_\infty}\end{pmatrix}\begin{pmatrix}\widetilde{\alpha}^* & 0\\ 0 & I\end{pmatrix}\end{cases}$$

Let us consider:

$$(1.16)\ \mathcal{K}=\dots\oplus\mathcal{D}_*\oplus\mathcal{D}_*\oplus\boxed{\mathcal{H}}\oplus\bigoplus_{k=1}^{\infty}\mathcal{D}_{\Gamma_k}$$

and the unitary:

$$(1.17)\begin{cases}W=W(\Gamma_1,\Gamma_2,\dots\Gamma_n,\dots):\mathcal{K}\longrightarrow\mathcal{K}\\ W=I\oplus W_{red}\end{cases}$$

where W is written with respect to the decompositions:

$$\mathcal{K}=(\dots\oplus\mathcal{D}_*)\oplus\mathcal{D}_*\oplus\mathcal{K}_+)\ \text{and}\ \mathcal{K}=(\dots\oplus\mathcal{D}_*)\oplus(\mathcal{H}\oplus\bigoplus_{n=1}^{\infty}\mathcal{D}_{\Gamma_n})$$

The matricial form of W is:

$$(1.18)\quad W = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & I & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & I & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & D_* & \Gamma_1 & D_{\Gamma_1^*}\Gamma_2 & D_{\Gamma_1^*}D_{\Gamma_2^*}\Gamma_3 & \cdots \\ \cdots & 0 & 0 & -Z_1 & D_{\Gamma_1} & -\Gamma_1^*\Gamma_2 & -\Gamma_1^* D_{\Gamma_2^*}\Gamma_3 & \cdots \\ \cdots & 0 & 0 & -Z_2 & 0 & D_{\Gamma_2} & -\Gamma_2^*\Gamma_3 & \cdots \\ \cdots & 0 & 0 & -Z_3 & 0 & 0 & D_{\Gamma_3} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where in fact the column $\begin{bmatrix} -Z_1 \\ -Z_2 \\ \vdots \\ -Z_n \\ \vdots \end{bmatrix}$

denoted by R is:

$$(1.19)\quad \begin{cases} R : \mathcal{D}_* \to \bigoplus_{k=1}^{\infty} \mathcal{D}_{\Gamma_k} \\ R = -\alpha X_\infty^{*}\mathfrak{X}^{*} \end{cases}$$

We will denote by Q_{-1} the ortogonal projection of $\mathfrak{K}_+$ onto the subspace: $\ldots 0 \oplus \mathcal{D}_* \oplus \boxed{0} \oplus 0 \ldots$

Taking into consideration the matricial form (1.18) of W, we define: $Q = \Gamma_1$, $N=(\ldots 0, D_{\Gamma_1}, 0, 0, \ldots)^t$ ("t" means the matrix transpose),

$$P = (\ldots, 0, D_*, D_{\Gamma_1^*}\Gamma_2, D_{\Gamma_1^*}D_{\Gamma_2^*}\Gamma_3, \ldots)$$

M=what remains in W after deleting Q, N and P. Let be E the operator obtained from D by deleting the first column.

We denote by $W_n = W(\{\Gamma_1, \Gamma_2, \ldots \Gamma_n, 0, 0 \ldots\})$ (W defined for the truncated choice sequence) and by R_n, M_n, P_n and E_n the operators obtained from W_n in the same way as R, M, P and E from W.

*

* *

For two Hilbert spaces $\mathcal{E}$ and $\mathcal{F}$ we denote by $\{\mathcal{E}, \mathcal{F}, \Theta(z)\}$ an analytic bounded operator valued function on the unit disk $\mathbb{D}$ with values in $B(\mathcal{E}, \mathcal{F})$ and by $\{\mathcal{E}, \mathcal{F}, \Theta(z)\}_1$ such a contractive function.

It is known from [12], Ch. V Prop. 4.2 that for a given $\{\mathcal{E}, \mathcal{F}, \Theta(z)\}_1$ exists a function denoted by $\{\mathcal{E}, \mathcal{F}_1, R_\Theta(z)\}$ maximal for the relation: $I - \Theta(e^{it})^* \Theta(e^{it}) \geq R_\Theta(e^{it})^* R_\Theta(e^{it})$, a.e. on the unit circle $\mathbb{T}$, uniquely determined by this condition modulo an unitary constant left factor. As in [10] we shall call R_Θ the right spectral factor of Θ.

Also, there exists a function $\{\mathcal{E}_1, \mathcal{F}, L_\Theta(z)\}$ maximal for the relation: $I - \Theta(e^{it}) \Theta(e^{it})^* \geq L_\Theta(e^{it}) L_\Theta(e^{it})^*$, a.e. on $\mathbb{T}$, uniquely determined by this condition modulo an unitary constant right factor. We shall call L_Θ the left spectral factor of Θ.

If for $\{\mathcal{E}, \mathcal{F}, \Theta(z)\}$ with $\Theta(z) = \sum_{n=0}^{\infty} z^n \Theta_n$ we denote as in [12] by $\{\mathcal{F}, \mathcal{E}, \tilde{\Theta}(z)\}$ the function defined by $\tilde{\Theta}(z) = \sum_{n=0}^{\infty} z^n \Theta_n^*$, an easy computation shows that for every $\{\mathcal{E}, \mathcal{F}, \Theta(z)\}_1$ we have that:

(1.20) $$R_{(\mathcal{H})}^{\sim} = \widetilde{L_{(\mathcal{H})}}$$

§2. SPECTRAL FACTORIZATION

For an analytic scalar function f , $f \in H^{\infty}$ with $\|f\|_{\infty} \leq 1$, there are well known results on rational approximation of its spectral factor, also related to the Schur algorithm, see for instance [4] and [9] . In this section we extend the result concerning the approximation of the spectral factors for the operator valued analytic contractive case. We prove that the spectral factors can be approximated (strong operatorial) on coefficients by some operator valued functions given by the Schur algorithm and which admit inner dilation.

Let $\{\mathcal{H} \oplus \mathcal{K}, \mathcal{K} \oplus \mathcal{H}', R(z)\}_1$, $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$

and $\{\mathcal{K}', \mathcal{K}, f(z)\}_1$ be given. Define the R-cascade transformation of f as being:

(2.1) $$C_R(f)(z) = R_{11}(z) + R_{12}(z) f(z) (I - R_{22}(z) f(z))^{-1} R_{21}(z)$$

where the inverse is assumed to exist. This is an analytic contractive function with values in $B(\mathcal{H}, \mathcal{H}')$.

Let $\{\mathcal{H}, \mathcal{H}, f_1(z)\}_1$ be given and $\Gamma_1 = f_1(0) \in B_1(\mathcal{H})$. We consider $\{\mathcal{H} \oplus \mathcal{D}_{\Gamma_1^*}, \mathcal{H} \oplus \mathcal{D}_{\Gamma_1}, J_1(z)\}$ be given by:

$(2.2)_1$ $$J_1(z) = \begin{bmatrix} \Gamma_1 & z D_{\Gamma_1^*} \\ D_{\Gamma_1} & -z \Gamma_1^* \end{bmatrix}$$

which is inner from both sides (for definition see [12] Ch. V).

It is known from [6] that the equation: $f_1 = C_{J_1}(f_2)$ has a unique solution $\{\mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_1^*}, f_2(z)\}_1$.

We define by induction for $n \geq 2$, $\Gamma_n = f_n(0) \in B_1(\mathcal{D}_{\Gamma_{n-1}}, \mathcal{D}_{\Gamma_{n-1}^*})$

$$(2.2)_n \qquad J_n(z) = \begin{bmatrix} \Gamma_n & zD_{\Gamma_n^*} \\ D_{\Gamma_n} & -z\Gamma_n^* \end{bmatrix}$$

and $\{\mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma_n^*}, f_{n+1}(z)\}_1$ the unique solution of the equation: $f_n = C_{J_n}(f_{n+1})$.

We shall call $\{\Gamma_n\}_{n\geq 1}$ the choice sequence of the function f_1, because it coincides with a similar object in [2].

Using the so-called Redheffer product $*$, defined for two block-matrices:

$$R' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} : \begin{matrix} \mathcal{H}_1 \\ \oplus \\ \mathcal{H}_2 \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}_1 \\ \oplus \\ \mathcal{K}_2 \end{matrix} \qquad \text{and}$$

$$R = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{matrix} \mathcal{K}_2 \\ \oplus \\ \mathcal{H}_3 \end{matrix} \longrightarrow \begin{matrix} \mathcal{H}_2 \\ \oplus \\ \mathcal{K}_3 \end{matrix} \qquad \text{by:}$$

$$R' * R = \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \text{ where:}$$

$$x = a' + b'a(I - d'a)^{-1}c'$$
$$y = b'(I - ad')^{-1}b$$
$$z = c(I - d'a)^{-1}c'$$
$$w = c(I - d'a)^{-1}d'b + d$$

we have that: $C_{R'}(C_R(f))=C_{R'*R}(f)$. We note that $*$ is associative.

We obtain in our case:

$$(2.3)\quad (J_1*J_2*\dots*J_n)(z)=\begin{bmatrix} a_n(z) & b_n(z) \\ c_n(z) & d_n(z) \end{bmatrix} : \begin{matrix}\mathcal{H}\\ \oplus\\ \mathcal{D}_{\Gamma_n^*}\end{matrix} \to \begin{matrix}\mathcal{H}\\ \oplus\\ \mathcal{D}_{\Gamma_n}\end{matrix}$$

and $f_1=C_{\begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}}(f_{n+1})$

Because if R and R' are inner ($*$-inner) it results that $R'*R$ is inner ($*$-inner), we deduce that $\begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$ is inner from both sides.

We have that:

$$f_1(z)-a_n(z)=C_{\begin{bmatrix} a_{n-1} & b_{n-1} \\ c_{n-1} & d_{n-1} \end{bmatrix}}(f_n)(z)-C_{\begin{bmatrix} a_{n-1} & b_{n-1} \\ c_{n-1} & d_{n-1} \end{bmatrix}}(\Gamma_n)(z) =$$

$$=b_{n-1}(z)(f_n(z)(I-d_{n-1}(z)f_n(z))^{-1}-\Gamma_n(I-d_{n-1}(z)\Gamma_n)^{-1})c_{n-1}(z).$$

From the definition of b_n it follows that $b_n(z)=z^n v_n(z)$ where $v_n(z)$ is analytic and $*$-outer. From this, and from $f_n(0)=\Gamma_n$ we obtain that:

$$f_1(z)-a_n(z)=z^n\hat{a}_n(z)$$

with $\hat{a}_n(z)$ analytic. So, f_1 and a_n have the first n Fourier coefficients identic, and both being contractive, a_n converges coefficientwise to f_1.

Because a_n depends only on Γ_k, $k=1,2,\ldots,n$ we conclude that f_1 is uniquely determined by its choice sequence.

In [6], Corollary 4.4 it is proved that the right spectral factor of $\tilde{f}_1$ with the notations from §1 is:

$$R_{\tilde{f}_1}(z)=Q_{-1}(I-zM^*)^{-1}P^*=\sum_{k=0}^{\infty} z^k Q_{-1}M^{*k}P^*$$

Taking into account the matricial form (1.18) of W and the definition of M, we deduce that:

$$(2.4)\qquad R_{\tilde{f}_1}(z)=D_* + \sum_{k=1}^{\infty} z^k R^* E^{*k-1}P^*$$

With the notations of the previous sections we have the following result:

PROPOSITION 2.1. For a given analytic contractive operator valued function $\{\mathcal{H}, \mathcal{H}, f_1(z)\}_1$, with right and left spectral factors denoted by R_{f_1} and L_{f_1}, $c_n\beta_n$ and $\tilde{v}_n\beta_n$ converge (strong operatorial) on coefficients to R_{f_1} and $\tilde{L}_{f_1}$ respectively.

Proof. Because $I-a_n(e^{it})\, a_n(e^{it})^* = b_n(e^{it})b_n(e^{it})^*$ a.e. on $\mathbb{T}$ and $b_n(z)=z^n v_n(z)$ with $v_n(z)$ *-outer, we have that $L_{f_1}=v_n$ and with (1.20) $R_{\tilde{f}_1} = \tilde{v}_n$.

From this and from $(1.7)_n$ we obtain applying (2.4) for f_n that:

$$(2.5)\qquad (\tilde{V}_n|\beta_n)(z)=H_n^{1/2}+\sum_{k=1}^{\infty} z^k R_n^* \hat{E}_n^{*k-1} P_n^*$$

where we used the notations from §1 and $\hat{E}_n$ is the compression of E_n to the space $\bigoplus_{k=1}^{n-1}\mathcal{D}_{\Gamma_k^*}$.

As in [5], Lemma 1.3 and Proposition 1.4 there exist the strong operatorial limits:

$$(2.6)\qquad \operatorname*{s-lim}_n P_n^* = P^*,\ \operatorname*{s-lim}_n \tilde{E}_n^* P_n' = E^*\ ,$$

where P_n' is the ortogonal projection of $\mathcal{K}'=\bigoplus_{k=1}^{\infty}\mathcal{D}_{\Gamma_k}$ onto $\mathcal{K}_n'=\bigoplus_{k=1}^{n-1}\mathcal{D}_{\Gamma_k}$

From [5], Prop.1.4: $\mathcal{K}_+'=\overline{\bigcup_{n=1}^{\infty} D_n\mathcal{K}_n}$.

Let be $k_+\in\bigcup_{n=1}^{\infty} D_n\mathcal{K}_n$. So $k_+=D_{n_o}k_{n_o}$ with $k_{n_o}\in\mathcal{K}_{n_o}$. For $n_o\leq p\leq\infty$ we have that:

$$(2.7)_p\qquad k_+ = D_p k_{n_o} \quad \text{and}$$

$$(2.8)_p\qquad X_p k_{n_o} = X_{n_o} k_{n_o}\ .$$

Then:

$$R^* k_+ = -\tilde{\alpha} X_\infty \alpha^* k_+ = -\tilde{\alpha} X_\infty \alpha^* D_\infty k_{n_o} =$$

$$=-\tilde{\alpha} X_\infty D_{X_\infty} k_{n_o} = -\tilde{\alpha} D_{X_\infty^*} X_\infty k_{n_o} = -D_{*\alpha} X_\infty k_{n_o}\ .$$

We used here $(2.7)_\infty$, (1.13) and (1.14).

For $p\geq n_o$, we have that:

$$R_p^* k_+ = -\widetilde{\alpha}_p X_p \alpha_p^* k_+ = -\widetilde{\alpha}_p X_p D_{X_p} k_{n_o} = -\widetilde{\alpha}_p D_{X_p^*} X_p k_{n_o} =$$
$$= -H_p^{1/2} X_p k_{n_o} = -H_p^{1/2} X_\infty k_{n_o}$$

We used here $(2.7)_p$, $(1.3)_p$, $(1.5)_p$ and $(2.8)_\infty$

Because $S\text{-}\lim H_p^{1/2} = D_*$, the last two relations show that:

$$(2.9) \qquad s - \lim_n R_n^* = R^*$$

Using now (2.6), (2.9), (2.5) and (2.4) we obtain that $\widetilde{v}_n \beta_n$ converges (strong operatorial) coefficientwise to $\widetilde{L}_{f_1}$.

Because $I - a_n(e^{it})^* a_n(e^{it}) = c_n(e^{it})^* c_n(e^{it})$, a.e. and c_n is outer, using (1.20) and the unitary operator $\hat{W}$ defined as W for the sequence $\{\Gamma_n^*\}_{n\geq 1}$ we obtain that $c_n \widetilde{\beta}_n$ converges strong operatorial coefficientwise to R_{f_1} the right spectral factor of f_1, which completes the proof of the proposition. ∎

Remark 2.2. About d_n given by the algorithm we cannot say that it converges or not. It has always the following choice sequence: $\{ -\Gamma_n^*, -\Gamma_{n-1}^*, \ldots, -\Gamma_1^*, 0, 0, \ldots \}$.

Every limit point D of d_n makes the operator valued matrix function $\begin{bmatrix} f & 0 \\ R_f & D \end{bmatrix}$ to be contractive. If $I - f(e^{it})^* f(e^{it}) = R_f(e^{it})^* R_f(e^{it})$ a.e. on $\mathbb{T}$, ($I - f^* f$ is called in this case factorable), D must be 0, so d_n converges weak to 0.

§3. ANALYTIC COMPLETION

In [3], [7] and [13] it is solved the problem of contractive completion of a 2x2 matrix. So, it is proved that for a contraction $A \in B_1(\mathcal{H},\mathcal{K})$ and $\Gamma_1 \in B_1(\mathcal{K}',\mathcal{D}_{A^*})$ and $\Gamma_2 \in B_1(\mathcal{D}_A,\mathcal{K}'')$, the matrix $\begin{bmatrix} A & D_{A^*}\Gamma_1 \\ \Gamma_2 D_A & X \end{bmatrix}$ is a contraction iff $X = -\Gamma_2 A^* \Gamma_1 + D_{\Gamma_2^*}\Gamma D_{\Gamma_1}$ with $\Gamma \in B_1(\mathcal{D}_{\Gamma_1},\mathcal{D}_{\Gamma_2^*})$.

We will consider here in place of A an operator valued analytic contractive function which admit meromorphic pseudo-continuation of bounded type and we want in this case to solve the contractive analytic completion of a 2x2 matrix. The notion of inner dilation was considered in [8] and [1] and in these works it is also demonstrated the existence of the inner dilation for the above mentioned class of contractive analytic functions. If it exists, the inner dilation replace the elementary rotation generated by a contraction A, namely $\begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix}$ which is unitary operator.

For a given function $\{\mathcal{K},\mathcal{H}, f_1(z)\}_1$ using Prop.4.1 Ch.V from [12], we obtain as in [2] that an analytic function $[f,H]$ with values in $B(\mathcal{H}\oplus\mathcal{K}',\mathcal{H})$ is contractive iff there exists $\{\mathcal{K}',\mathcal{D}_*,G_1(z)\}_1$ such that $H = L_f G_1$, L_f being the left spectral factor of f.

Also an analytic function $\begin{bmatrix} f \\ J \end{bmatrix}$ with values in $B(\mathcal{K},\mathcal{H}\oplus\mathcal{K}'')$ is contractive iff there exists $\{\mathcal{D},\mathcal{K}'', G_2(z)\}_1$ such that $J=G_2R_f$, R_f being the right spectral factor of f.

We put the following problem: describe all $\{\mathcal{K}, \mathcal{K}', X(z)\}_1$ such that the function:

$$(3.1) \qquad \mathcal{Z} = \begin{bmatrix} f & L_fG_1 \\ G_2R_f & X \end{bmatrix}$$

for given f, G_1, G_2 is contractive with values in $B(\mathcal{H} \oplus \mathcal{K}, \mathcal{K} \oplus \mathcal{K}')$.

In general does not exist such a function. First we give a necessary ans sufficient condition for the existence.

Let be $\Theta = \begin{bmatrix} f \\ G_2R_f \end{bmatrix}$ and $\{\mathcal{C}, \mathcal{K} \oplus \mathcal{K}', L_\Theta\}$

$L_\Theta = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ its left spectral factor.

Because $I - \Theta(e^{it})\Theta(e^{it})^* \geq L_\Theta(e^{it})L_\Theta(e^{it})^*$ a.e. on $\mathbb{T}$, we have by considering the (1.1) component of the above inequality written in matricial form, that:

$$L_f(e^{it})L_f(e^{it})^* \geq L_1(e^{it})L_1(e^{it})^* \text{ , a.e.}$$

Because L_f is $*$-outer, there exists $\{\mathcal{C}, \mathcal{D}_*, V(z)\}_1$ such that:

$$(3.2) \qquad L_f V = L_1$$

To be contractive, $\mathcal{Z}$ must have the form:

$$\mathcal{G} = [\Theta , L_{\Theta} G]$$

with some $\{\mathcal{K}, \mathcal{C}, G(z)\}_1$, from where:

$$\mathcal{G} = \begin{bmatrix} f & L_1 G \\ G_2 R_f & L_2 G \end{bmatrix}$$

So:

(3.3) $$L_f G_1 = L_1 G.$$

From (3.2) and (3.3) we have that $L_f VG = L_f G_1$. Because L_f is *-outer it results:

(3.5) $$VG = G_1 .$$

Because V and G_1 are determined by Θ, the existence of a $\{\mathcal{K}, \mathcal{C}, G(z)\}_1$ with (3.5) is a necessary condition for the existence of X.

The condition is also sufficient because if exists such a G we can take $X = L_2 G$.

We shall give a solution for the completion problem in a special case in which f has inner dilation, which as in [8] means that there exists an analytic function

$$U = \begin{bmatrix} f & B \\ C & D \end{bmatrix}$$

which is inner from both sides.

In [8] and [1] it is proved the existence of inner dilation for functions which have meromorphic pseudo-continuation of bounded type to the exterior of unit disk.

From §2, the functions having finite suported choice sequence $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n, 0, 0 \ldots\}$ corespond to the functions a_n constructed there, and because $\begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$ is inner from both sides, we conclude that this functions admit inner dilation also.

If f admits a meromorphic pseudo-continuation of bounded type we consider (as in [10]) the following inner dilation:

$$(3.6) \qquad \Delta_f = \begin{bmatrix} f & L_f \\ C & D \end{bmatrix}$$

with values in $B(\mathcal{H} \oplus \mathcal{D}_{*}, \mathcal{H} \oplus \mathcal{E})$, $[C, D]$ being the right spectral factor of $[f \;\; L_f]$.

As in [10], §2, all other inner dilations of f have the form:

$$\begin{pmatrix} I & 0 \\ 0 & \alpha \end{pmatrix} \Delta_f \begin{pmatrix} I & 0 \\ 0 & \beta \end{pmatrix}$$

where β is inner and α is pseudomeromorphic function satisfying $\alpha(e^{it})\alpha(e^{it})^{*} = I$ a.e. on $\mathbb{T}$.

Consider for a given f with meromorphic pseudo-continuation of bounded type the following problem: find all analytic

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contractive completions

$$\mathcal{T} = \begin{bmatrix} f & L_f G_1 \\ G_2 C & X \end{bmatrix}$$

of f, where $\{\mathcal{H}, \mathcal{D}_{*}, G_1(z)\}_1$ and $\{\mathcal{E}, \mathcal{K}', G_2(z)\}_1$ are also given.

The problem has always solutions because we can consider $X = G_2 D G_1$.

The main point is the following result:

Proposition 3.1. The left spectral factor of $\Theta = \begin{bmatrix} f \\ G_2 C \end{bmatrix}$ is $\psi = \begin{bmatrix} L_f & 0 \\ G_2 D & L_{G_2} \end{bmatrix}$ where $\{\mathcal{L}_{G_2}, \mathcal{K}', L_{G_2}\}$ is the left spectral factor of G_2.

Proof. It is clear that ψ is $*$-outer. Using the fact that $\Delta_f(e^{it})$ is unitary a.e. we have that:

$$\psi(e^{it})\psi(e^{it})^* = \begin{bmatrix} L_f(e^{it}) & 0 \\ G_2(e^{it})D(e^{it}), & L_{G_2}(e^{it}) \end{bmatrix} \begin{bmatrix} L_f(e^{it})^*, & D(e^{it})^* G_2(e^{it})^* \\ 0 & L_{G_2}(e^{it})^* \end{bmatrix} =$$

$$\begin{bmatrix} L_f(e^{it})L_f(e^{it})^*, & -f(e^{it})C(e^{it})^* G_2(e^{it})^* \\ -G_2(e^{it})C(e^{it})f(e^{it})^*, & L_{G_2}(e^{it})L_{G_2}(e^{it})^* + G_2(e^{it})D(e^{it})D(e^{it})^* G_2(e^{it})^* \end{bmatrix} \leq$$

$$\leq \begin{bmatrix} L_f(e^{it})L_f(e^{it})^* & -f(e^{it})C(e^{it})^* G_2(e^{it})^* \\ -G_2(e^{it})C(e^{it})f(e^{it})^* & I - G_2(e^{it})C(e^{it})C(e^{it})^* G_2(e^{it})^* \end{bmatrix} =$$

$$= I - \Theta(e^{it})\Theta(e^{it})^* \quad \text{a.e. on } \mathbb{T}.$$

Let consider now $\{\mathcal{E}, \mathcal{K} \oplus \mathcal{K}', \Theta'(z)\}$, $\Theta' = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}$ such that:

$$(3.7)\quad \psi(e^{it})\psi(e^{it})^* \leq \Theta'(e^{it})\Theta'(e^{it})^* \leq I - \Theta(e^{it})\Theta(e^{it})^*$$

a.e. on $\mathbb{T}$.

Taking into account the (1.1) component of the inequality (3.7) written in matricial form, it results that

$\Theta_1(e^{it})\Theta_1(e^{it})^* = L_f(e^{it})L_f(e^{it})^*$, a.e. so exists $\{\mathcal{C}, \mathcal{D}_*, \Omega(z)\}_1$ such that:

(3.8) $$\Theta_1 = L_f\Omega$$

From (3.7) it follows that there exists $\{\mathcal{D}_*\oplus\mathcal{L}_{G_2}, \mathcal{C}, [w_1(z), w_2(z)]\}_1$ such that:

(3.9) $$\Psi = \begin{bmatrix}\Theta_1\\ \Theta_2\end{bmatrix}[w_1, w_2]$$

so

(3.10) $$L_f = \Theta_1 w_1$$

From (3.8) and (3.10) it follows that $L_f = L_f\Omega w_1$ and because L_f is $*$ - outer it results that $\Omega w_1 = I$, so $\Omega(0)w_1(0) = I$, and so $w_1(0)$ is an isometry. Taking into account the decomposition theorem from [12] Ch.V, we obtain that: $w_1(z) = w_1(0)$ for all $z \in D$.

Denoting by $\mathcal{R}(w_1)$ the range of $w_1 = w_1(0)$ and considering $\mathcal{C} = \mathcal{R}(w_1) \oplus \mathcal{R}(w_1)^{\perp}$ and $\Theta_1 = [\Theta_{11}, \Theta_{12}]$, $\Theta_2 = [\Theta_{21}, \Theta_{22}]$ the form of Θ_1 and Θ_2 with respect to this decomposition, from (3.9) we have that: $\Theta_{11}w_1 = L_f$ and $\Theta_{21}w_1 = G_2D_1$, so: $\Theta_{21} = G_2Dw_1^*$.

Thus:

$$(3.11)\quad \Theta_2(e^{it})\Theta_2(e^{it})^* = G_2(e^{it})D(e^{it})D(e^{it})^*G_2(e^{it})^* + \Theta_{22}(e^{it})\Theta_{22}(e^{it})^*$$

From the (2,2) component of (3.7) it results:

$$(3.12)\quad \Theta_2(e^{it})\Theta_2(e^{it})^* \leq G_2(e^{it})D(e^{it})D(e^{it})^*G_2(e^{it})^* + I - G_2(e^{it})G_2(e^{it})^*, \text{ a.e. on } \mathbb{T}.$$

From (3.11) and (3.12), it follows:

$$L_{G_2}(e^{it})L_{G_2}(e^{it})^* \leq \Theta_{22}(e^{it})\Theta_{22}(e^{it})^* \leq$$

$$\leq I - G_2(e^{it})G_2(e^{it})^* \text{ a.e. on } \mathbb{T}$$

so: $\Theta_{22}(e^{it})\Theta_{22}(e^{it})^* = L_{G_2}(e^{it})L_{G_2}(e^{it})^*$;

It results: $\Theta'(e^{it})\Theta'(e^{it})^* = \psi(e^{it})\psi(e^{it})^*$ a.e., so with (3.7) ψ is the left spectral factor of Θ. ∎

To be contractive $\mathcal{G}$ must have the form:

$$\mathcal{G} = \left[\Theta \,,\, L_{\Theta}\begin{bmatrix} G_1 \\ \Gamma R_{G_1} \end{bmatrix}\right],$$

where $\{\mathcal{R}, \mathcal{R}_{G_1}, R_{G_1}\}$ is the right spectral factor of G_1, and Γ is analytic contractive with values in $B(\mathcal{R}_{G_2}, \mathcal{L}_{G_2})$.

With Prop.3.1, it results:

$$\mathcal{T} = \begin{bmatrix} f & L_f G_1 \\ G_2 C & G_2 D G_1 + L_{G_2} \Gamma R_{G_2} \end{bmatrix}$$

We established that:

<u>Theorem 3.2.</u> If f has a meromorphic pseudo-continuation of bounded type, with the above notations, the formula

$$X = G_2 D G_1 + L_{G_2} \Gamma R_{G_2}$$

establishes an one-to-one correspondence between all $\{\mathcal{K}, \mathcal{K}', X(z)\}_1$ such that:

$$\mathcal{T} = \begin{bmatrix} f & L_f G_1 \\ G_2 C & X \end{bmatrix}$$

is contractive and all $\{\mathcal{R}_{G_2}, \mathcal{L}_{G_2}, \Gamma(z)\}_1$. ∎

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