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Wanda Morariu*) and Dorin Popescu**)

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Wanda Morariu*) and Dorin Popescu**)

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*)*Institute of Mathematics, Str. Academiei 14, 70109 Bucharest*

**) *Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.*

IMMEDIATE EXTENSIONS OF FILTERED RINGS

Wanda Morariu and Dorin Popescu

A separated filtered ring has a maximally complete immediate extension. This is an analogue of Krull Theorem from valuation ring theory in the frame of filtered rings.

1. INTRODUCTION

In Noetherian rings an important role is played by adic filtrations (all the rings are here commutative with identity). Unfortunately such filtrations are not necessarily separated in non-Noetherian case. Then we are obliged to consider other filtrations which are separated. This is already the case when we deal with non-Noetherian valuation rings, indeed then we consider filtrations indexed by the positive parts of some totally ordered groups.

It is the purpose of our paper to try to extend some concepts such as: immediate (or dense) extension, pseudo-convergent, pseudo limit, maximally completeness as well as results from valuation theory to general (non-Noetherian) filtered rings. For instance Theorem (3.14) below is an analogue of Kaplansky's Theorem (see [2] Theorem 1) and Theorem (4.8) below says that every separated filtered ring has a maximally complete immediate extension (i.e. an analogue of Krull Theorem, see [3] Propositions 24, 25 or [10] ch. 1, §3 Lemma 5.) Let $R \subset R'$ be an immediate extension of valuation rings and T a variable. Then

$R[T] \subset R'[T]$ is a simple example of immediate extension (of non-valuation rings) with respect to the filtration induced on $R[T]$ by the natural filtration of R (see (5.5) ii)). But our interest for this study was given by the following more sophisticated example.

Let K be a field and $K[X]$, $X = (X_1, X_2, \ldots X_n)$ the polynomial K-algebra in X. Let $\mathbb{N}^*$, K^*, $K[X]^*$ be the ultrapowers of $\mathbb{N}$, K and $K[X]$ with respect to a certain nonprincipal ultrafilter on $\mathbb{N}$ (see (5.1)). Since the X-adic filtration of $K[X]$ induces a non-separated filtration in $K[X]^*$ it would be nice to find a sub-$K[X]$-algebra A of $K[X]^*$ such that

i) A has a canonical separated filtration inducing a separated one in $K[X]^*$ and $A \subset K[X]^*$ is immediate,

ii) A is constructed in a nice way from $K[X]$, for example it is a filtered inductive limit of polynomial (or smooth) $K[X]$-algebras (see [5] § 1 for an analogue question).

Such an algebra seems to be the monoid K^*-algebra $A := K^*[X, \mathbb{N}^*]$ which is a filtered inductive limit of polynomial $K^*[X]$-algebras and a filtered inductive limit of smooth finite type $K[X]$-algebras (see (5.6) and (5.7)). The filtration $\{(X^r)\}_{r \in \mathbb{N}}$ is separated on A and induces a separated one on $K[X]^*$. Moreover, $K[X]^*$ is an immediate extension of A and A is in fact the graded ring associated to $K[X]^*$ with respect to the above filtration (see (5.5) iii)).

When R is a regular local ring containing its residue field k and $x = (x_1, \ldots x_n)$ is a regular system of parameters in R then the map $k[X] \to R$, $X = (X_1, \ldots, X_n) \to x$ shows that R is an immediate extension of $k[X]$ (the graded ring of R). Thus, A plays with respect to $K^*[X] \subset K[X]^*$ more or less the same role as $k[X]$ with respect to $k \subset R$.

2. PRELIMINARIES ON VALUATION RINGS

(2.1) Let F be a field, Γ a totally ordered group and $v : F \setminus \{0\} \to \Gamma$ a surjective valuation. $\mathcal{F} = (F, v, \Gamma)$ is called a valued field and $R = \{x \in F \mid v(x) \geq 0\} \cup \{0\}$ is its valuation ring. Clearly R is a filtered ring, the filtration being given by the ideals $\{E_\gamma\}_{\gamma \in \Gamma_+}$, where $E_\gamma := \{x \in F \mid v(x) \geq \gamma\}$ and $\Gamma_+ :=$ $:= \{\gamma \geq 0 \mid \gamma \in \Gamma\}$. The valuation ring R' of $\mathcal{F}' = (F', v', \Gamma')$ is an extension of R (shortly we write $R \subseteq R'$) if $F \subseteq F'$, $\Gamma \subseteq \Gamma'$ and v is given by restriction from v'.

Let k, k' be the residue fields of the valuation rings R resp. R' of $\mathcal{F}$ resp. $\mathcal{F}'$. The extension $R \subseteq R'$ is called immediate if $\Gamma = \Gamma'$ and $k = k'$. The immediate extension $R \subseteq R'$ is dense if for every $x \in R'$ and every $\gamma \in \Gamma_+$ there exists an element $y \in R$ such that $v(x - y) \geq \gamma$.

(2.2) A well ordered sequence $a = (a_i)_{i<\theta}$ of elements from F is called pseudo-convergent (shortly we write a is a p.c.s.) if it satisfies:

i) a has not a last element, i.e. θ is a limit ordinal

ii) $v(a_j - a_i) < v(a_t - a_j)$ for all $i < j < t < \theta$

(2.2.1) A p.c.s. $a = (a_i)_{i<\theta}$ of elements of R is fundamental (shortly we write a is a f.s.) if the set $\{v(a_i - a_j) \mid i < j < \theta\}$ is cofinal in Γ_+, i.e. for every $\gamma \in \Gamma_+$ there exists a $t < \theta$ such that $v(a_j - a_i) > \gamma$ for all $i < j$ with $i > t$.

(2.2.2) An element $y \in R'$ is called a pseudo-limit (shortly we write a p.l.) of a p.c.s. $a = (a_i)_{i<\theta}$ from R if $v(y - a_i) =$ $= v(a_{i+1} - a_i)$ for all $i < \theta$. A p.l. is not unique in general. Indeed, if y is a p.l. of a in R' and there exists an element $b \in R'$ such that $v(b) \geq v(a_j - a_i)$ for all $i < j < \theta$ then $y + b$ is another p.l. of a.

(2.2.3) A p.l. of a f.s. is necessary unique and so we called it limit. If the extension $R \subseteq R'$ is dense then every element from R' is a limit of a f.s. from R.

(2.2.4) If the extension $R \subseteq R'$ is immediate, then every element from $R' \setminus R$ is a p.l. of a p.c.s. from R having no p.l. in R (see [2]).

(2.2.5) R is called maximally complete if every immediate extension of it is trivial or equivalent if every p.c.s. from R has a p.l. in R. Every valuation ring has a maximally complete immediate extension (see [3], [10]) which is not necessary unique if char $R > 0$.

(2.3) If $a = (a_i)_{i<\theta}$ is a p.c.s. from R then the following statements hold (see [6], [2], [9], or [10] for proofs).

(2.3.1) $v(a_j - a_i) = v(a_{i+1} - a_i)$ for all $i < j < \theta$

(2.3.2) either

i) $v(a_i) < v(a_j)$ for all $i < j < \theta$, or

ii) $v(a_i) = v(a_j)$ for $j > i >> 0$

(2.3.3) if y is a p.l. of a then either

i) $v(y) > v(a_i)$ for all i in the case (2.3.2) i), or

ii) $v(y) = v(a_i)$ for $i >> 0$ in the case (2.3.2) ii)

3. IMMEDIATE EXTENSIONS OF FILTERED RINGS

(3.1) Let A be a ring, Γ a totally ordered group and $E = (E_\gamma)_{\gamma \in \Gamma_+}$ a strictly decreasing filtration of ideals on A such that

i) $E_0 = A$

ii) $E_\gamma E_{\gamma'} \subseteq E_{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma_+$,

By a filtered couple (A, E) we mean a ring A, a totally ordered group Γ and a filtration E on A as above. A subset $L \subset \Gamma_+$ is called convex if holds $(*)$ $\alpha \in L$, $\beta \leq \alpha$, $\beta \in \Gamma_+ \Rightarrow \beta \in L$. These subset form obviously with the inclusion a complete totally ordered lattice $(L(\Gamma), \leq)$. To every subset $S \subset \Gamma_+$ we can associate an element from $L(\Gamma)$ namely the convex closure $\overline{S} := \{\gamma \in \Gamma_+ \mid \exists\, \alpha \in S \text{ with } \gamma \leq \alpha\}$ of S, and so the additive operation of Γ induces an additive operation $\oplus$ on $L(\Gamma)$ by

$$L \oplus L' := \overline{L + L'}$$

An element $L \in L(\Gamma)$ is principal iff there exists $\gamma \in \Gamma_+$ such that $L = \{\tau \in \Gamma_+ \mid 0 \leq \tau \leq \gamma\}$. Given an element $x \in A$ then $\eta_x :=$ $:= \{\gamma \in \Gamma_+ \mid x \in E_\gamma\}$ belongs to $L(\Gamma)$ and $\eta : A \to L(\Gamma), x \to \eta_x$ defines a canonical map which is analog to the order map for adic filtrations in Noetherian case, as shows the following elementary:

(3.2) LEMMA

The following statements are equivalent

(i) there exists an order map with respect to E, i.e. a function $\nu : A \setminus \{0\} \to \Gamma_+$ such that every nonzero element $x \in A$ belongs to $E_{\nu(x)} \setminus I_{\nu(x)}$, where $I_\gamma := \bigcup_{\lambda > \gamma} E_\lambda$

ii) for every nonzero element $x \in A$ the set η_x is principal.

(3.3) LEMMA.

Let $x, y \in A$. Then the following statements hold:

i) $\eta(x) = \eta(-x)$

ii) $\eta(x + y) \geq \eta(x)$ if $\eta(x) \leq \eta(y)$

iii) $\eta(x + y) = \eta(x)$ if $\eta(x) < \eta(y)$

iv) $\eta(xy) \geq \eta(x) \oplus \eta(y)$

v) $\eta(0) = \Gamma_+$, $\eta(1) = 0$

PROOF. Clearly $x \in E_\gamma$ iff $(-x) \in E_\gamma$ and so i) holds.

ii) Let $\gamma \in \eta(x)$. Then $\gamma \in \eta(y)$ and so $x, y \in E_\gamma$. Thus $x + y \in E_\gamma$, i.e. $\gamma \in \eta(x + y)$.

iii) Using ii) we get $\eta(x + y) \geq \eta(x)$. Suppose that there exists $\tau \in \eta(x + y) \setminus \eta(x)$. Changing τ by a smaller one if necessary we may suppose also $\tau \in \eta(y)$. Then $y \in E_\tau$ but $x \notin E_\tau$ and so we get $x + y \notin E_\tau$, i.e. $\tau \notin \eta(x + y)$. Contradiction! Thus iii) holds.

iv) It is enough to see that

$$\eta(x) \oplus \eta(y) \subset \eta(xy)$$

Let $\alpha \in \eta(x)$, $\beta \in \eta(y)$. Then $x \in E_\alpha$, $y \in E_\beta$ and so $xy \in E_\alpha E_\beta \subset$ $\subset E_{\alpha+\beta}$, i.e. $\alpha + \beta \in \eta(xy)$

v) is obvious.

(3.3.1) REMARK.

Suppose that an order map $\nu : A \setminus \{0\} \longrightarrow \Gamma_+$ is given with respect to E. Then the above Lemma says that for every nonzero elements $x, y \in A$ it holds

(i) $\nu(x) = \nu(-x)$

(ii) $\nu(x + y) \geq \nu(x)$ if $\nu(x) \leq \nu(y)$

(iii) $\nu(x + y) = \nu(x)$ if $\nu(x) < \nu(y)$

(iv) $\nu(xy) \geq \nu(x) + \nu(y)$

(v) $\nu(1) = 0$

which are certainly the usual properties of the order maps.

(3.4) Let θ be a limit ordinal, $a = (a_i)_{i<\theta}$ a well ordered sequence of elements from A and $\mu_i := \eta(a_{i+1} - a_i)$. We call a <u>pseudo-convergent</u> (shortly p.c.s.) if

i) $\mu_i = \eta(a_j - a_i)$ for all j, $i < j < \theta$

ii) $(\mu_i)_{i<\theta}$ form a strictly increasing sequence of elements from $L(\Gamma)$.

A p.c.s. $a = (a_i)_{i<\theta}$ is <u>fundamental</u> (shortly a f.s.) if $\Gamma_+ = \bigcup_{i<\theta} \mu_i$.

An element $y \in A$ is a <u>pseudo-limit</u> (shortly a p.l.) of a p.c.s. $a = (a_i)_{i<\theta}$ if $\eta(y - a_i) = \mu_i$ for all $i < \theta$. Clearly a p.l. is not necessary unique, as shows the following

(3.5) LEMMA. If y is a p.l. of a p.c.s. $a = (a_i)_{i<\theta}$ from A and $z \varepsilon A$ satisfies $\eta(z) \geq \bigcup_{i<\theta} \mu_i$ then $y + z$ is another p.l. of a. Conversely every two p.l. differ by such an element z.

PROOF. We have

$\eta(y + z - a_j) = \eta(y - a_j + z) = \eta(y - a_j) = \mu_j$ for each $j < \theta$, because $\eta(z) > \mu_j$ (see (3.3) (iii) and so $y + z$ is a p.l. of a.

Conversely, if y and y' are two p.l. of a then for $b := y - y'$ we have $\eta(b) = \eta(y - y') = \eta((y - a_i) - (y' - a_i)) \geq \mu_i$ for each $i < \theta$ (see (3.3) (ii)) and so $\eta(b) \geq \bigcup_{i<\theta} \mu_i$.

(3.6) PROPOSITION. Let $a = (a_i)_{i<\theta}$ be a p.c.s. from A and $y \varepsilon A$ a p.l. of a. Then one and only one of the following statements holds:

i) $\eta(a_i) = \eta(a_j) = \eta(y)$ for an $i >> 0$ and every j, $i < j < \theta$

ii) $\eta(a_i) < \eta(a_j) < \eta(y)$ for all i,j with $i < j < \theta$.

PROOF. We have the following two cases

(1) there exists $i < \theta$ such that $\mu_i > \eta(a_j)$ for all j, $i < j < \theta$,

(2) for all $i < \theta$ there exists j, $i < j < \theta$ such that $\mu_i < \eta(a_j)$.

In the first case we get

$$\eta(a_i) = \eta(a_i - a_j + a_j) = \eta(a_j)$$

since $\eta(a_i - a_j) = \mu_i > \eta(a_j)$ (see (3.3) iii), i) and (3.4) i)). Also it follows

$$\eta(y) = \eta(y - a_i + a_i) = \eta(a_i) \text{ since}$$

$$\eta(y - a_i) = \mu_i > \eta(a_j) = \eta(a_i) \text{ (see above).}$$

In case (2) fix an $i < \theta$ and take j, $i < j < \theta$ such that $\mu_i < \eta(a_j)$. Then

$$\eta(a_i) = \eta(a_i - a_j + a_j) = \mu_i \text{ since } \eta(a_i - a_j) = \mu_i < \eta(a_j)$$

Thus $(\eta(a_i))_{i<\theta}$ is a strictly increasing sequence because $(\mu_i)_i$ is so. Also

$$\eta(y) = \eta(y - a_i + a_i) \geq \eta(a_i) \text{ since } \eta(y - a_i) = \mu_i = \eta(a_i).$$

As $(\eta(a_i))_{i<\theta}$ is strictly increasing we get $\eta(y) > \eta(a_i)$ for all i.

(3.7) A ring morphism $u : A \to B$ is a morphism of filtered couples $(A, E) \longrightarrow (B, F)$, $E = (E_\gamma)_{\gamma\in\Gamma_+}$, $F = (F_\lambda)_{\lambda\in\Lambda_+}$, Γ, Λ being some totally ordered groups, if $\Lambda = \Gamma$ and $u^{-1}(F_\gamma) = E_\gamma$ for every $\gamma \in \Gamma_+$. If u is an injective morphism (of filtered couples) we say that (B, F) is an extension of (A, E). If u is injective and F is the filtration induced by E on B then we say B is an extension of (A, E). The morphism u is trivial if A = B and $u = 1_A$ (then follows also $E = F$). If E are separated i.e. $\bigcap_{\gamma\in\Gamma_+} E_\gamma = 0$ and u is a morphism (of filtered couples) then u is injective. Indeed, then we have

$$u^{-1}(0) \subset u^{-1}\Big(\bigcap_{\gamma\in\Gamma_+} F_\gamma\Big) = \bigcap_{\gamma\in\Gamma_+} u^{-1}(F_\gamma) = \bigcap_{\gamma\in\Gamma_+} E_\gamma = 0$$

(3.8) LEMMA. Let $u : (A, E) \to (B, F)$ be a morphism of filtered couples and $a = (a_i)_{i<\theta}$ a p.c.s. in A. Then the following statements hold:

(i) $\eta_B u = \eta_A$, where $\eta_A : A \to L(\Gamma)$, $\eta_B : B \to L(\Gamma)$ are given as above with respect to E, F.

(ii) $\mu_i = \eta_A(a_i - a_j) = \eta_B(u(a_i) - u(a_j))$ for all $i < j < \theta$

(iii) u(a) is a p.c.s. (with respect to F)

iv) if $x \in A$ is a p.l. of a (with respect to E) then u(x) is a p.l. of u(a) (with respect to F).

For the proof it is enough to note that if $x \in A$ then $x \in E_\gamma$ iff $u(x) \in F_\gamma$.

(3.9) LEMMA. Let $u : (A,E) \to (B,F)$ be a morphism of filtered couples, y an element from B and $x = (x_i)_{i<\theta}$ an well ordered sequence from A such that the sequence

$$\{\eta_B(y - u(x_i))\}_{i<\theta}$$

is stricly increasing in $L(\Gamma)$. Then x is a p.c.s. in A and y is a p.l. of $u(x)$. Moreover if x has a p.l. z in A then

P t

$$\eta_B(y - u(z)) > \eta_B(y - u(x_i)) \text{ for all } i < \theta.$$

PROOF. We have

$$\eta_A(x_i - x_j) = \eta_B(u(x_i) - u(x_j)) = \eta_B(u(x_i) - y + y - u(x_j)) = \eta_B(y - u(x_i))$$

for all $i < j < \theta$ since $\eta_B(y - u(x_i)) < \eta_B(y - u(x_j))$ by hypothesis. In particular $\eta_A(x_i - x_j) = \eta_A(x_i - x_{i+1}) =: \mu_i$ and $(\mu_i)_i$ is strictly increasing. Thus x is a p.c.s. and y is a p.l. of $u(x)$.

Now suppose that x has a p.l. z in A. Since y, $u(z)$ are both p.l. of $u(x)$ we must have

$$\eta_B(y - u(z)) > \mu_i \text{ for all } i < \theta \text{ (see Lemma 3.5).}$$

But $\mu_i = \eta_B(y - u(x_i))$ for all $i < \theta$ (see above).

(3.10) LEMMA. Let $u : (A,E) \to (B,F)$ be a morphism of filtered couples and y an element from $B \setminus u(A)$. Then one (and only one) of the following statements holds:

i) there exists a p.c.s. $a = (a_i)_{i<\theta}$ from A having no p.l. in A such that y is a p.l. of $u(a)$,

ii) the subset

$$\Lambda = \bigcup_{x \in A} \eta_B(y - u(x)) \subset \Gamma$$

has a maximum (in Λ).

PROOF. Let $L := \{\eta_B(y - u(x)) \mid x \in A\} \subset L(\Gamma)$.

First suppose that L has no maximum (in L). Then we choose in L a well ordered cofinal subset

$$L' = \{\eta_B(y - u(x_i))\}_{i<\theta}$$

where $x = (x_i)_{i<\theta}$ is a well ordered sequence of elements from A. Certainly we can suppose that L' is strictly increasing and by Lemma (3.9) we conclude that x is a p.c.s. and y is a p.l. of u(x). Since L' is cofinal in L we note that x has no p.l. in A.

Now suppose that L has a maximum $\eta_B(y - u(z))$ for a certain $z \in A$ but $\Lambda = \eta_B(y - u(z))$ has no maximum. Since Λ has no maximum let $(\gamma_i)_{i<\theta}$ be a well ordered strictly increasing sequence of elements from Λ which forms a cofinal set in Λ. Choose some elements $t = (t_i)_{i<\theta}$ in A such that $t_i \in E_{\gamma_i} \setminus E_{\gamma_{i+1}}$ and put $x_i = z + t_i$. Since $y - u(z) \in F_{\gamma_i}$ for all i, we get $\gamma_i \in \eta_B(y - u(x_i))$ but $\gamma_{i+1} \notin \eta_B(y - u(x_i))$. Thus $\eta_B(y - u(x_i))\}_{i<\theta}$ forms a strictly increasing sequence in $L(\Gamma)$ and by Lemma (3.9) it follows that $x = (x_i)_{i<\theta}$ is a p.c.s. in A and y is a p.l. of u(x). Since $\{\eta_B(y - u(x_i))\}_{i<\theta}$ forms a cofinal set in Λ we note that x has no p.l. in A.

(3.10.1) REMARK. In the notations and hypothesis of lemma (3.10) suppose that i) holds. Then $y \notin u(A) + \bigcap_{\gamma\in\Lambda} F_\gamma$.

(3.11) PROPOSITION. Let $u : (A,\mathcal{E}) \to (B,\mathcal{F})$ be a morphism of filtered couples,

$$\mathcal{E} = (E_\gamma)_{\gamma\in\Gamma_+},\ \mathcal{F} = (F_\gamma)_{\gamma\in\Gamma_+},\ I_\gamma := \bigcup_{\lambda>\gamma} E_\lambda,\ J_\gamma := \bigcup_{\lambda>\gamma} F_\lambda.$$

Suppose that u induces an isomorphism $u_\gamma : E_\gamma/I_\gamma \to F_\gamma/J_\gamma$ for every $\gamma \in \Gamma_+$. Then for every element y from $B\setminus u(A)$ there exists a p.c.s. a in A such that

(∗) y is a p.l. of u(a),

(∗∗) a has no p.l. in A.

PROOF. Let y be an element from $B\setminus u(A)$. By Lemma (3.10) it is enough to show that

$$\Lambda = \bigcup_{x \in A} \eta_B(y - u(x)) \subset \Gamma_+,$$

has no maximum (in Λ). Suppose that Λ has a maximum $\gamma \in \Lambda$. Then there exists $z \in A$ such that $\gamma \in \eta_B(y - u(z))$. Thus $y - u(z) \in F_\gamma$ and so there exists $x \in E_\gamma$ such that $y - u(z) - u(x) \in J_\gamma$, u_γ being surjective. Then $\eta_B(y - u(z'))$, $z' = z + x$ contains an element bigger than γ. Contradiction! Consequently Λ has no maximum.

(3.12) Let $u : (A,E) \to (B,F)$ be a morphism of filtered couples and $L \in \mathcal{L}(\Gamma)$. Denote $E_L = \bigcap_{\gamma \in L} E_\gamma$, $I_L = \bigcup_{\gamma > L} E_\gamma$, $F_L = \bigcap_{\gamma \in L} F_\gamma$, $J_L = \bigcup_{\gamma > L} F_\gamma$. If $L \notin \eta_A(A)$ we claim that $E_L = I_L$. Indeed, if $x \in E_L$ then $\eta_A(x) \supset L$. Since $L \notin \eta_A(A)$ we have $\eta_A(x) \neq L$ and so there exists $\gamma \in \eta_A(x) \setminus L$. It follows $\gamma > L$ and $x \in E_\gamma \subset I_L$.

If L is principal, let us say $L = \{\gamma \mid 0 \leq \gamma \leq \tau\}$, then $E_L = E_\tau$ and we denote I_L by I_τ. Clearly if Γ contains $\mathbb{Z}$ as an isolated subgroup then $I_\tau = E_{\tau+1}$.

Since u is a morphism of filtered couples we have easily $u(E_L) \subset F_L$, $u(I_L) \subset J_L$, $u^{-1}(F_L) = E_L$ and $u^{-1}(J_L) = I_L$. Then the morphism

$$u_L : E_L/I_L \longrightarrow F_L/J_L$$

induced by u is injective but u_L can be not surjective even u_γ is so for every $\gamma \in L$.

We define the <u>graded</u> <u>ring</u> of A with respect to E by

$$Gr_E(A) := \bigoplus_{L \in \mathcal{L}(\Gamma)} E_L/I_L = \bigoplus_{L \in \eta_A(A)} E_L/I_L.$$

$Gr_E(A)$ is a ring with the following multiplicative structure obtained by linearity from

$(a \bmod I_L)(b \bmod I_{L'}) = ab \bmod I_{L \oplus L'}$, where $a \in E_L$, $b \in E_{L'}$ and

$L, L' \in \mathcal{L}(\Gamma)$. Similarly $Gr_F(B) = \bigoplus_{L \in \mathcal{L}(\Gamma)} F_L/J_L$

and u induces a ring morphism (given by u_L)

$$Gr(u) : Gr_E(A) \to Gr_F(B).$$

A morphism of filtered couples $u : (A,E) \to (B,F)$ is pseudo immediate if

(i) $Gr(u)$ is an isomorphism,

(ii) E, F are separated.

Using (ii) we note that pseudo immediate morphisms are injective (see (3.7)) and so they are in fact extensions. Usually we say that (B,F) is a pseudo immediate extension of (A,E). If F is the filtration induced by E on B, i.e. $F = (u(E_\gamma)B)_{\gamma \in \Gamma_+}$, then we say that B is an immediate extension of (A,E).

When A,B are Noetherian rings, E is an adic filtration and B is an immediate extension of (A,E) then B is flat over A by [4] Theorem (22.3). This is not true in general as shows the following.

(3.13) EXAMPLE. Let A be a ring which is not coerent and X a variable. Then the formal power series A-algebra A [[X]] is not flat. Let E be the (X)-adic filtration of the polynomial A-algebra A[X] and $u : A[X] \to A[[X]]$ the canonical inclusion. Then A[[X]] is an immediate extension of $(A[X],E)$ which is not flat.

(3.14) THEOREM. Let $u : (A,E) \to (B,F)$ be a pseudo immediate morphism of filtered couples. Then for every element y from $B \setminus u(A)$ there exists a p.c.s. a in A such that

(i) y is a p.l. of u(a)

(ii) a has no p.l. in A.

The Theorem is a consequence of Proposition (3.11).

(3.15) A morphism of filtered couples $u : (A,E) \to (B,F)$ is <u>dense</u> if

(i) u induces an isomorphism

$$A/E_\gamma \to B/F_\gamma$$

for every $\gamma \in \Gamma_+$.

(ii) E and F are separated.

If F is the filtration E_B induced by E on B we say that B is a dense immediate extension of (A,E).

Clearly a dense extension is pseudo immediate. Now suppose that E is separated. Then take $\hat{A} := \varprojlim_{\gamma\in\Gamma_+} A/E_\gamma$ and let $v : A \to \hat{A}$ be the canonical injection and $\hat{E}_\gamma$ the closure of $v(E_\gamma)\hat{A}$ in the natural topology of $\hat{A}$. Clearly v defines a dense extension $(A,E) \to (\hat{A},\hat{E})$, where $\hat{E} = (\hat{E}_\gamma)_{\gamma\in\Gamma_+}$. But $\hat{A}$ is not in general a dense immediate extension of (A,E) because usually $\hat{E}_\gamma \neq v(E_\gamma)\hat{A}$.

(3.16) LEMMA. <u>Let</u> $u : (A,E) \to (B,F)$ <u>be a dense extension. Then every element from</u> B <u>is a limit of a f.s. from</u> A.

PROOF. Let y be an element from $B\setminus u(A)$. By Theorem (3.14) y is a p.l. of a p.c.s. a from A having no p.l. in A. But then a must be f.s. (see (3.10.1)).

(3.16.1) REMARK. The above Lemma is certainly well known but we included here just to illustrate our Theorem (3.14).

4. MAXIMALLY COMPLETE IMMEDIATE EXTENSIONS OF FILTERED RINGS

(4.1) Let (A,E) be a filtered couple and suppose that E is separated. Then (A,E) is called <u>maximally complete</u> if every immediate extension of it is trivial. The couple (A,E) is <u>complete</u> if every dense extension of it is trivial.

(4.2) LEMMA. <u>Suppose that E is separated and every p.c.s from A has a p.l. in A. Then (A,E) is maximally complete.</u>

PROOF. Let B be an immediate extension of (A,E) and y an element from B\A. By Theorem (3.14) there exist a p.c.s. a in A having no p.l. in A but for which y is a p.l. in B. Since by hypothesis a must have a p.l. in A we get a contradiction. Thus A = B.

(4.3) LEMMA. <u>Suppose that E is separated. Then (A,E) is complete iff every f.s. from A has a limit in A.</u>

PROOF. The sufficiency goes like in (4.2) using Lemma (3.16). Now, if there exists a f.s. from A which has no limit in A then the dense extension $(A,E) \longrightarrow (\hat{A},\hat{E})$ (see (3.15) for notations) is not trivial and so (A,E) is not complete.

(4.3.1) REMARK. The above Lemma is certainly well known. The necessity goes here because given a f.s. a from A having no limit in A we know to construct a dense extension $(A \longrightarrow \hat{A})$ where a has a limit. Unfortunately given a p.c.s. b in A having no p.l. in A we do not know to construct an immediate extension where b has a p.l. (this is not very easy for valuation rings, see [2] Theorems 2,3). Fir this reason the converse of Lemma (4.2) is still open.

(4.4) Let (A,E) be a separated filtered couple, $(L_i)_{i<\theta}$ a well ordered strictly increasing sequence from $\eta(A)$ and $\bar{x}_{L_i} \in E_{L_i}/I_{L_i}$,

$i < \theta$ some elements. The well ordered formal sum $f = \sum_{i<\theta} \bar{x}_{L_i}$ can be identified with an element $(\bar{y}_L)_{L\in\eta(A)}$ from $P_E(A) = \prod_{L\in\eta(A)} E_L/I_L$, where

$$\bar{y}_L = \begin{cases} \bar{x}_{L_i} & \text{if } L = L_i \text{ for certain } i < \theta \\ 0 & \text{if } L \neq L_i \text{ for all } i < \theta. \end{cases}$$

Thus the abelian group S of these well ordered formal sums (the operation is defined canonically) is isomorph with a subgroup of $P_E(A)$ (the operation in $P_E(A)$ is given componentwisely).

Let $j < \theta$. The formal sum $g := \sum_{i<j} \bar{x}_{L_i}$ is called a <u>truncation</u> of f. A well ordered sequence $(g_r)_{r<\omega}$ of elements from S is called a <u>sequence of truncations</u> if for every $r < r' < \omega$, g_r is a truncation of $g_{r'}$. If $(g_r)_{r<\omega}$ is a sequence of truncations, $g_r = \sum_{i<j_r} \bar{z}_{L_i}$ where $(j_r)_{r<\omega}$ is a strictly increasing sequence of ordinals and $\theta = \inf\{s \text{ ordinal} \mid s > j_r \text{ for all } r < \theta\}$ then $g = \sum_{i<\theta} \bar{z}_{L_i}$ is called the <u>limit</u> of $(g_r)_{r<\omega}$ (shortly we write $g = \lim_{r<\omega} g_r$). Clearly g_r is a truncation of g for every $r < \omega$. Let $\mathrm{ord} : S \to L(\Gamma)$ be the map given by $\mathrm{ord}\, f = \inf\{L_i \mid \bar{x}_{L_i} \neq 0\}$. Then g_r is a truncation of f iff $\mathrm{ord}\,(f - g_r) > \bigcup_{i<j_r} L_i$.

(4.5) LEMMA. $\mathrm{card}\, A \leq \mathrm{card}\, P_E(A)$

PROOF. Let $\pi_L : E_L \to E_L/I_L$ be the canonical surjection, v_L a section of π_L and $v : Gr_E(A) \to A$ the sum -map given by $(v_L)_{L\in\eta(A)}$. Let $D_1 = \bigcup_{L\in\eta(A)} v_L(E_L/I_L)$ and $M_1 = v^{-1}(D_1) \subset Gr_E(A) \subset S$. Since D_1 is in fact a disjoint union and $(v_L)_{L\in\eta(A)}$

are injective by construction it follows that v induces a bijective map $w_1 : M_1 \to D_1$. Let $t_1 = w_1^{-1}$. Fix a structure of well ordered set on $A\setminus D_1$, let us say $A\setminus D_1 = \{a_i \mid i < \theta,\ i$ is not a limit ordinal$\}$.

By transfinite induction we will construct in $S\setminus M_1$ a well ordered sequence $\{f_i \mid i < \theta,\ i$ is not a limit ordinal$\}$ such that for every $j < \theta$ the sets

$$D_j = D_1 \cup \{a_i \mid i < j\},\ M_j = M_1 \cup \{f_i \mid i < j\}$$

satisfy

i) if $f \in M_j$ then all its truncations belong to M_j,

ii) the map $t_j : D_j \longrightarrow M_j$ extending t_1 by $a_i \longrightarrow f_i$ is a bijection,

iii) for every $a, a' \in D_j$. $a \neq a'$ such that

$t_j(a) - t_j(a') = \overline{x}_L + \sum_{L'>L} \overline{x}_{L'}$ it holds

$$\eta(a - a' - v_L(\overline{x}_L)) > L = \eta(a - a').$$

(4.5.1) REMARK. In particular iii) says that

$$\eta(a - a') = \mathrm{ord}\,(t_j(a) - t_j(a'))\ \text{for every}\ a,a' \in D_j.$$

First we note that D_1 and M_1 satisfy iii) (i) and ii) are trivially fulfilled in this case). Let $a = v_L(\overline{x}_L)$, $a' = v_{L'}(\overline{x}'_{L'})$. If $L = L'$ then $t_1(a) - t_1(a') = \overline{x}_L - \overline{x}'_L$ and we have

$$\eta(a - a' - v_L(\overline{x}_L - \overline{x}'_L)) > L = \eta(a - a')$$

If $L < L'$ then $t_1(a) - t_1(a') = \overline{x}_L - \overline{x}'_{L'}$ and

$$\eta(a - a' - v_L(\overline{x}_L)) = \eta(a - a' - a) = \eta(a') = L' > L = \eta(a - a')$$

Now suppose given $(D_i, M_i)_{i<j}$ for a certain ordinal $j < \theta$ which is not a limit one. We will construct a sequence of truncations $(g_r)_{r<\omega}$, $g_r := \sum_{s<r} \overline{y}_{L_s}$ in M_{j-1} such that

$$(*)\ \eta(a_j - t_{j-1}^{-1}(g_r)) \geq \bigcup_{s<r} L_s\ \text{for}\ r < \omega.$$

Apply induction on r. Take $L_o := \eta(a_j)$, $\bar{y}_{L_o} := \pi_{L_o}(a_j)$ and $g_1 := \bar{y}_{L_o} \in M_1$. We have

$$\eta(a_j - v_{L_o}(\bar{y}_{L_o})) > L_o = \eta(a_j)$$

Suppose given g_s for $s < r$. If r is a limit ordinal then put $g_r = \lim_{s<r} g_s$. If r is not a limit ordinal then we have

$$\eta(a_j - t^{-1}_{j-1}(g_{r-1})) > \bigcup_{s<r-1} L_s$$

Like above for $a'_j := a_j - t^{-1}_{j-1}(g_{r-1})$ there exist $L_{r-1} \in \eta(A)$, $L_{r-1} \supset \bigcup_{s<r-1} L_s$ and $\bar{y}_{L_{r-1}} \in M_1$ such that

$$(**)\ \eta(a'_j - v_{L_{r-1}}(\bar{y}_{L_{r-1}})) > L_{r-1} = \eta(a'_j)$$

Take $g_r := g_{r-1} + \bar{y}_{L_{r-1}}$. We have two cases

1) $g_r \in M_{j-1}$
2) $g_r \notin M_{j-1}$

In case 1) denote $b_s := t^{-1}_{j-1}(g_s)$, $s \leq r$ and using iii) for b_r and b_s we get

$$\eta(b_r - b_s - v_{L_s}(\bar{y}_{L_s})) > L_s = \eta(b_r - b'_s) \text{ because}$$

$g_r - g_s = \bar{y}_{L_s} + \ldots$. By $(*)$ we get $\eta(a_j - b_r) =$

$= \eta(a_j - b_{s+1} + b_{s+1} - b_r) \geq L_s$ for all s with $s+1 < r$ because $\eta(a_j - b_{s+1}) \geq L_s$ and $\eta(b_{s+1} - b_r) = L_{s+1}$. If r is a limit ordinal it follows that g_r satisfies $(*)$. If r is not a limit ordinal this follows from

$$\eta(a_j - b_r) = \eta((a'_j - v_{L_{r-1}}(\bar{y}'_{L_{r-1}})) - (b_r - b_{r-1} - v_{L_{r-1}}(\bar{y}_{L_{r-1}}))) \geq L_{r-1}$$

(see $(**)$).

In case 2) take $\omega = r$, $f_j := g_r$, $M_j = M_{j-1} \cup \{g_r\}$. It is clear that D_j, M_j satisfy i), ii). For iii) it is enough to take $a \in D_j$ with $t_{j-1}(a) - g_r = \bar{x}_L + \dots$ and to show that

$$(***)\ \eta(a - a_j - v_L(\bar{x}_L)) > L = L' := \eta(a - a_j)$$

We claim that $L' < \bigcup_{s<r} L_s$. Indeed if $L' \geq \bigcup_{s<r} L_s$ then we shall prove that g_r is a truncation of $t_{j-1}(a)$ and so $g_r \in M_{j-1}$ (see i)). Contradiction!

We have

$\eta(a - b_{s+1}) = \eta(a - a_j + a_j - b_{s+1}) > L_s$ for $s + 1 < r$ because $\eta(a_j - b_{s+1}) > L_s$ by $(*)$. Thus ord $(t_j(a) - g_{s+1}) > L_s$ (see (4.5.1)) and so g_{s+1} is a truncation of $t_j(a)$ for $s + 1 < r$. If r is a limit ordinal then g_r is also a truncation of $t_j(a)$. If r is not a limit ordinal then by construction (see $(**)$) we have

$$\eta(a_j - b_{r-1}) = L_{r-1}$$

Then

$$\eta(a - b_{r-1}) = \eta(a - a_j + a_j - b_{r-1}) = L_{r-1}$$

and so ord $(t_j(a) - g_{r-1}) = L_{r-1}$ by Remark (4.5.1). Then $t_j(a) = g_{r-1} + \bar{z}_{L_{r-1}} + \dots$ for a certain $\bar{z}_{L_{r-1}} \in E_{L_{r-1}}/I_{L_{r-1}}$ and by iii) we get

$$\eta(a - b_{r-1} - v_{L_{r-1}}(\bar{z}_{L_{r-1}})) > L_{r-1}$$

But

$$\eta(v_{L_{r-1}}(\bar{z}_{L_{r-1}}) - v_{L_{r-1}}(\bar{y}_{L_{r-1}})) = \eta[(v_{L_{r-1}}(\bar{z}_{L_{r-1}}) - a + b_{r-1}) + (a - a_j) +$$

$$+ (a_j - b_{r-1} - v_{L_{r-1}}(\bar{y}_{L_{r-1}}))] > L_{r-1} \text{ (see } (**)).$$

Thus $\bar{z}_{L_{r-1}} = \bar{y}_{L_{r-1}}$ and so g_r is a truncation of $t_j(a)$.

Consequently $L' < \bigcup_{s<r} L_s$, let us say $L' < L_s$ for a certain $s < r$. We have

$$\eta(a - b_s) = \eta(a - a_j + a_j - b_s) = L'$$

because $\eta(a_j - b_s) = L_s$ by $(**)$ ($s + 1$ is not a limit ordinal). Thus $\mathrm{ord}(t_{j-1}(a) - g_s) = L'$ by Remark (4.5.1) and so $L = \mathrm{ord}(t_{j-1}(a) - g_r) = L'$ since g_s is a truncation of g_r and $L' < L_s$.

Applying iii) to a, b_s we get

$$\eta(a - b_s - v_L(\bar{x}_L)) > L$$

(clearly $\bar{x}_L$ is still the leading term in $t_{j-1}(a) - g_s$). Then $\eta(a - a_j - v_L(\bar{x}_L)) = \eta(a - b_s - v_L(\bar{x}_L) + b_s - a_j) > L$ because $\eta(b_s - a_j) = L_s$ by $(**)$ and so $(***)$ holds. Finally note that using the above construction we should arive in the case 2) because in the worse situation we will get $\eta(a_j - b_r) = \Gamma_+$ for $r \gg 0$ (see $(*)$) and so $a_j = b_r$ because E is separated. Thus if $g_r \in M_{j-1}$ then $b_r \in D_{j-1}$. Contradiction!

If j is a limit ordinal it is trivial to see that D_j, M_j satisfy i) - iii). Whole construction gives an injective function

$$A = \bigcup_{j<\theta} D_j \to \bigcup_{j<\theta} M_j \subset S \subset P_E(A)$$

Thus $\mathrm{card}\, A \leq \mathrm{card}\, P_E(A)$.

(4.5.2) REMARK. When A is Noetherian then $\eta(A)$ contains just principal elements (see (3.2)) and $(E_\tau/I_\tau)_{\tau\in\Gamma_+}$ are finitely generated over A/I_0 and so we can say that card A is bounded by a cardinal depending just of $\mathrm{card}(A/I_0)$ and $\mathrm{card}\,\Gamma_+$ (the last one being usually $\aleph_0$). If A is a valuation ring then again $\eta(A)$ contains just principal elements and $E_\tau/I_\tau \cong A/I_0$ as A-modules for every $\tau \in \Gamma_+$. Thus card A is again bounded by a cardinal depending just of card(A/I_o) and card$\,\Gamma_+$.

(4.6) COROLLARY. There exists an injective function $t : A \to S$ such that $\eta = \mathrm{ord}\cdot t$. The proof is a consequence of Remark (4.5.1) and of the above construction.

(4.7) COROLLARY. The cardinal of every pseudo immediate extension (B,F) of (A,E) is bounded by card $P_E(A)$.
For the proof note that if (B,F) is an immediate extension of (A,E) then $P_F(B) \cong P_E(A)$ and apply Lemma (4.5).

(4.8) THEOREM. (A,E) has a maximally complete immediate extension.

PROOF. By the above Corollary the isomorphism classes of immediate extensions B of (A,E) form a set $I(A,E)$ which is nonempty because at least $(1_A, A) \in I(A,E)$. Let U be a totally ordered subset of $I(A,E)$ and $D := \bigcup_{B \in U} B$.
We have

$$A \cap E_\gamma D = A \cap \Big(\bigcup_{B\in U} E_\gamma B\Big) = \bigcup_{B\in U} (A \cap E_\gamma B) = \bigcup_{B\in U} E_\gamma = E_\gamma .$$

Since the map $Gr_E(A) \to Gr_{E_D}(D)$ is an isomorphism being a filtered inductive limit of the isomorphisms $Gr_E(A) \to Gr_{E_B}(B)$, $B \in U$, we get $D \in U$.
Thus totally ordered subsets of $I(A,E)$ have superiors and by Zorn's Lemma $I(A, E)$ have maximal elements. But these maximal elements must be maximally complete.

5. SOME EXAMPLES

(5.1) A set D of subsets of $\mathbb{N}$ is a filter on $\mathbb{N}$ if

i) $\emptyset \notin D$,

ii) if $S, T \in D$ then $S \cap T \in D$,

iii) if $S \in D$ and $S \leq T \leq N$, then $T \in D$.

The set of cofinite subsets of N form on N the Fréchet filter. D is an ultrafilter on N if it satisfies one of the following two equivalent conditions

1) D is maximal in the ordered set of all filters on N

2) a subset R of N belongs to D iff $N \setminus R$ does not.

D is a principal filter if there exists a set $R \in D$ such that $D = \{T \mid R \leq T \leq N\}$. D is a nonprincipal ultrafilter (on N) iff it includes the Fréchet filter. In particular, the elements of a nonprincipal ultrafilter are infinite subsets of N. Let $(A_i)_{i \in N}$ be a family of rings, $R = \prod_{i \in N} A_i$ its ring product, D an ultrafilter on N and P_d the ideal of R given by $P_D = \{(a_i)_{i \in N} \mid \{i \mid a_i = 0\} \in D\}$. The quotient ring $A^* := R/P_D$ is the ultraproduct of $(A_i)_{i \in N}$ with respect to D (sometimes we denote A^* by $\prod_{i \in N} A_i/D$). If $A_i = A$ for every $i \in N$ then A^* is called the ultrapower of A with respect to D. We denote by $[(a_i)_{i \in N}]$ the element induced in A^* by $(a_i)_{i \in N} \in R$. Let $u_i : A_i \longrightarrow B_i$, $i \in I$ be some ring morphisms. Then the map $u^* : A^* \longrightarrow B^*$ given by $[(a_i)_{i \in N}] \longrightarrow [(u(a_i))_{i \in N}]$ is called the ultraproduct of $(u_i)_{i \in N}$ with respect to D.

(5.2) Let $(A_n, E_n)_{n \in N}$ be a family of filtered couples, $E_n = (E_{n,\gamma_n})_{\gamma_n \in (\Gamma_n)_+}$, where $\Gamma = (\Gamma_n)_{n \in N}$ is a family of totally ordered groups. Let D be a nonprincipal ultrafilter on N, A^* (resp. Γ^*) the ultraproducts of $(A_n)_{n \in N}$ (resp. Γ) with respect to D and $E^* = (E^*_\tau)_{\tau \in \Gamma^*_+}$ the filtration given on A^* by $E^*_\tau = \prod_{n \in N} E_{n,\gamma_n}/D$, $\tau = [(\gamma_n)_{n \in N}] \in \Gamma^*_+$. We call E^* the ultraproduct filtration of $(E_n)_{n \in N}$ with respect to D.

Let $(B_n, F_n)_{n \in N}$ be a family of filtered couples such that

$B_n \supset A_n$ for every $n \in N$, B^* the ultraproduct of $(B_n)_{n \in N}$ with respect to D and F^* the ultraproduct filtration of $(F_n)_{n \in N}$.

(5.3) LEMMA. The following statements hold

(i) for every $x = [(x_n)_n] \in A^*$, $\eta_{A^*}(x) = \prod_{n \in N} \eta_{A_n}(x_n)/D \in L(\Gamma^*)$, in particular $\eta_{A^*}(A^*) = \prod_{n \in N} \eta_{A_n}(A_n)/D \subset \prod_{n \in N} L(\Gamma_n)/D \subset L(\Gamma^*)$,

(ii) E^* is separated iff $\delta := \{n \in N \mid E_n \text{ separated}\} \in D$,

(iii) for every $\tau = [(\tau_n)_n] \in \Gamma^*$, $A^* \cap F^*_\tau \subset E^*_\tau$ iff

$$\{n \in N \mid A_n \cap F_{n,\tau_n} \subset E_{n,\tau_n}\} \in D$$

(iv) for every $L^* = \prod_{n \in N} L_n/D$, $L_n \in L(\Gamma_n)$ it holds $E^*_{L^*} = \prod_{n \in N} E_{n,L_n}/D$, $I^*_{L^*} = \prod_{n \in N} I_{n,L_n}/D$, where

$$E^*_{L^*} := \bigcap_{\tau \in L^*} E^*_\tau, \; I^*_{L^*} := \bigcup_{\tau > L^*} E^*_\tau$$

(v) for every L^* like in (iv) $F^*_{L^*} \subset E^*_{L^*} + J^*_{L^*}$ iff

$$\omega := \{n \in N \mid F_{n,L_n} \subset E_{n,L_n} + J_{n,L_n}\} \in D$$

(vi) for every L^* like in (iv) the canonical map

$$u_{L^*} : E^*_{L^*}/I^*_{L^*} \to F^*_{L^*}/J^*_{L^*}$$

is an isomorphism iff

$$\{n \in N \mid u_{L_n} : E_{n,L_n}/I_{n,L_n} \to F_{n,L_n}/J_{n,L_n}$$

is an isomorphism$\} \in D$.

PROOF. (i) We have $\tau = [(\tau_n)_n] \in \eta^*(x)$ iff $x \in E^*_\tau$. This holds iff

$$\{n \in N \mid \tau_n \in \eta_{A_n}(x_n)\} = \{n \in N \mid x_n \in E_{n,\tau_n}\} \in D \quad \text{i.e.} \quad \text{iff}$$

$\tau \in \prod_{n \in N} \eta_{A_n}(x_n)/D$.

(ii) E^* is separated iff $\eta_{A^*}(x) = \Gamma_+^*$ just for $x = 0$. This holds iff

$\delta = \{n \in N \mid \eta_{A_n}(x_n) = (\Gamma_n)_+$ just for $x_n = 0\} \in D$

(iii) It is enough to note that

$$A^* \cap F_\tau^* = \prod_{n\in N} (A \cap F_{n,\tau_n})/D$$

(iv) Let $x = [(x_n)_n] \in E^*_{L^*}$. We have $\eta_{B^*}(x) \supset L^*$. But $\eta_{B^*}(x) = \prod_{n\in N} \eta_{B_n}(x_n)/D$ and so the above inclusion says that

$$\{n \in N \mid x_n \in F_{n,L_n}\} = \{n \in N \mid \eta_{B_n}(x_n) \supset L_n\} \in D$$

i.e. $x \in \prod_{n\in N} F_{n,L_n}/D$. Thus $E^*_{L^*} \in \prod_{n\in N} E_{n,L_n}/D$. Since $\prod_{n\in N} E_{n,L_n}/D \subset E_\tau^*$ for every $\tau \in L^*$ we get also the other inclusion.

Clearly $I_\gamma^* \subset \prod_{n\in N} I_{n,L_n}/D$ for every $\gamma > L^*$ and so $I^*_{L^*} \subset \prod_{n\in N} I_{n,L_n}/D$. If $x \in \prod_{n\in N} I_{n,L_n}/D$ then

$$\theta := \{n \in N \mid x_n \in I_{n,L_n}\} \in D.$$

Thus for every $n \in \theta$ there exists $\gamma_n > L_n$ such that $x_n \in E_{n,\gamma_n}$. Take $\gamma = [(\gamma_n)_n]$, $\gamma_n = 0$ for $n \notin \theta$. Then $x \in E_\gamma^*$ and $\gamma > L^*$.

(v) Using (iv) we note that $F^*_{L^*} \subset E^*_{L^*} + J^*_{L^*}$ iff

$$\prod_{n\in N} F_{n,L_n}/D \subset \prod_{n\in N} (E_{n,L_n} + J_{n,L_n})/D$$

which is enough.

(vi) Apply (iii) and (v).

(5.4) COROLLARY. The following statements holds:

(i) (B^*, F^*) is an extension of (A^*, E^*) iff $\{n \in N \mid (B_n, F_n)$ is an extension of $(A_n, E_n)\} \in D$

(ii) (B^*, F^*) is a pseudo immediate extension of (A^*, E^*) if

$\{n \in N \mid (B_n, F_n)$ is a pseudo immediate extension of $(A_n, E_n)\} \in D$

(iii) (B^*, F^*) is a dense extension of (A^*, E^*) if

$$\{n \in N \mid (B_n, F_n) \text{ is a dense extension of } (A_n, E_n)\} \in D$$

PROOF (i) follows from Lemma (5.3) (iii) and for (ii) apply (5.3) (ii), (vi). For (iii) it is enough to note that for all $\tau \in \Gamma_+^*$ the map $A^*/E_\tau^* \to B^*/F_\tau^*$ is the ultraproduct of the maps $A_n/E_{n,\tau_n} \to B_n/F_{n,\tau_n}$, $n \in N$ with respect to D.

(5. 5) EXAMPLE i) Let A be a Notherian ring, $\underline{a} \subset A$ a proper ideal and $\hat{A}$ the completion of A in the $\underline{a}$-adic topology. Then $A \subset \hat{A}$ is dense with respect to the $\underline{a}$-adic topology and so the ultrapower $A^* \subset \hat{A}^*$ is dense too with respect to the ultrapower filtration (see 5.4 iii))

ii) Let $R \subset R'$ be an immediate extension of valuation rings which is not dense and T a variable. Then $R[T] \subset R'[T]$ is immediate but not dense with respect to the filtration induced on R[T] by the natural filtration of R, i.e. given by $\underline{a}_\gamma := \{x \in R \mid v(x) \geq \gamma\}$, $\gamma \in \Gamma_+$, where $v : R \setminus \{0\} \to \Gamma_+$ is the valuation of R. Certainly $R[T]^* \to R'[T]^*$ is still an immediate extension but not dense with respect to the ultrapower filtration (see (5.4) (ii), (iii)).

iii) Let K be a field and K[X], $X = (X_1, \ldots, X_n)$ the polynomial K-algebra in X. Let $N^*, \mathbb{Z}^*$, K^*, $K[X]^*$ be the ultrapowers of N, $\mathbb{Z}$, K and K[X] with respect to D. Clearly $\mathbb{Z}$ induces componentwisely a structure of totally ordered group on $\mathbb{Z}^*$ and $N^* = \mathbb{Z}_+^*$. Given $r_t = [(r_{ti})_{i \in N}] \in N^*$ we consider the nonstandard power $X_t^{r_t} = [(X_t^{r_{ti}})_{i \in N}]$. The sub – K^*-algebra of $K[X]^*$ generated by

$$\{X_t^{r_t} \mid t = 1,2,\ldots,n;\ r_t \in N^*\}$$

is in fact the monoid K^*-algebra $A := K^*[X, N^*]$. Consider on

$K^*[X,N^*]$ the filtration $E' = (E_r)_{r\in N^*}$ where E_r is generated by $\{X_1^{r_1}\dots X_n^{r_n}\}_{r_1+r_2+\dots+r_n = r}$. The filtration E_B induced by E on $B := K[X]^*$ is exactly the ultrapower of the (X)-adic filtration. Thus E_B is separated by (5.3) ii) and in particular E is too. By Lemma (3.2) $\eta_A(A)$ contains just principal elements and it is enough to see that the canonical map $u_\tau : E_\tau/E_{\tau+1} \to E_\tau B/E_{\tau+1}B$ (in this case $I_\tau = E_{\tau+1}$) is an isomorphism for all $\tau \in N^*$. This is true because

$$\{X_1^{r_1}\dots X_n^{r_n} \mid r = (r_1,\dots,r_n) \in N^{*n},\ \sum_{t=1}^{n} r_t = \tau\}$$

induces bases in the linear K^*-space $E_\tau/E_{\tau+1}$, $E_\tau B/E_{\tau+1}B$ on which u_τ acts identically. Note that $Gr_E(A) = Gr_{E_B}(B) = A$ and so B is an immediate extension of (A,E) (unfortunately $A \subset B$ is not flat).

iv) Let $K, K[X]$ be like in iii), T a new variable,

$$A := \bigcup_{s\in N} K[X/T^s, T] \subset K[X,T,T^{-1}] \text{ and}$$

$$B := \bigcup_{s\in N} K[[X/T^s]][T].$$

Consider on $\Gamma = \mathbb{Z}^2$ the lexicographically order and for $(r,s) \in \Gamma_+ = (\{0\}\times N) \cup \bigcup_{r\geq 1} (\{r\}\times\mathbb{Z})$ let $E_{r,s}$ be the ideal generated by

$$\{X_1^{r_1}\dots X_n^{r_n} T^s\}_{\substack{r_1+\dots+r_n = r\\ r_i\geq 0}}$$

and $E = (E_{r,s})_{(r,s)\in\Gamma_+}$. In fact E is given by the following strictly decreasing sequence of ideals

$$A \supset (T) \supset \dots \supset (T^s) \supset \dots \supset (X/T^s) \supset \dots \supset (X) \supset \dots \supset (XT^s) \supset \dots$$

$$\supset (X^r/T^s) \supset \dots \supset (X^r) \supset \dots \supset (X^rT^s) \supset \dots$$

Clearly E and E_B are separated. As in Lemma (3.2) $\eta(A)$ contains just principal elements and we must show only that the map $u_{(r,s)} : E_{(r,s)}/E_{(r,s+1)} \longrightarrow E_{(r,s)}B/E_{(r,s+1)}B$ is an isomorphism. Like in iii) it is enough to note that $\{X_1^{r_1} \dots X_n^{r_n}T^s | r_1, \dots, r_n \geq 0, \sum_{i=1}^{n} r_i = r\}$ induces bases in the K-linear spaces between which $u_{(r,s)}$ acts bijectively. Note that $Gr_E(A) = Gr_{E_B}(B) = A$ and the extension $A \subset B$ is flat because it is a filtered inductive limit of flat inclusions $K[X/T^s,T] \subset K[[X/T^s]][T]$, $s \in N$. Clearly $A \subset B$ is immediate.

We end this Section with a result of different type (announced in introduction).

(5.6) PROPOSITION. $K^*[X,N^*]$, $X = (X_1, \dots, X_n)$ is a filtered inductive limit of smooth finite type $K[X]$-algebras.

PROOF. Since $K \subset K^*$ is a separable extension, K^* is a filtered inductive limit of smooth finite type K-algebras. By base change it is enough to show the following .

(5. 7) LEMMA. Let R be a ring. The monoid R - algebra $R[X,N^*]$, $X = (X_1, \dots, X_n)$ is a filtered inductive limit of polynomial $R[X]$ - algebras in a finite number of variables.

PROOF.Let $\mathbb{Z}^*$ be the ultrapower of $\mathbb{Z}$ with respect to D. Clearly the injective map $\mathbb{Z} \to \mathbb{Z}^*$, $p \to (p, \dots p, \dots)$ identifies $\mathbb{Z}$ with an isolated subgroup of $\mathbb{Z}^*$. Express $\mathbb{Z}^*$ as a filtered inductive union of its finitely generated subgroups containing $\mathbb{Z}$, let us say $\mathbb{Z}^* = \bigcup_{i \in I} \Gamma_i$. Then $R[X,N^*] = \varinjlim_{i \in I} R[X,(\Gamma_i)_+]$ (since $\mathbb{Z}^*_+ = N^*$) and it is enough to apply the following:

(5. 8) LEMMA. Let Γ be a finitely generated totally ordered

group containing $\mathbb{Z}$ as an isolated subgroup. Then $R[X,\Gamma_+]$, $X = (X_1, \ldots, X_n)$ is a filtered inductive limit of polynomial $R[X]$ - algebras in a finite number of variables.

PROOF. We need the following

(5. 9) LEMMA. Let H be a finitely generated totally ordered group. Then $R[X,H_+]$, $X = (X_1, \ldots, X_n)$ is a filtered inductive union of its sub - R - algebras of type $B_h = R[X_1^{h_1}, \ldots, X_n^{h_1}, \ldots, X_1^{h_t}, \ldots, X_n^{h_t}]$ for a positive basis $h = (h_1, \ldots, h_t)$, $t = \mathrm{rk}\, H$ of H (in particular B is a "polynomial" R - algebra).

Take $H := \Gamma/\mathbb{Z}$. Then Γ is in fact the product $H \times \mathbb{Z}$ lexicographically ordered (see for example [7] (7.5)). Applying Lemma (5.9) we express $R[X,H_+]$ as a filtered inductive union of its sub - R - algebras of type B_h, where $h = (h_1, \ldots, h_t)$, $t = \mathrm{rk} H$ runs in a set $\mathcal{H}$ of positive basis of H. Let $B_{h,s}, s = (s_1, \ldots, s_n) \in N^n$ be the sub - R - algebra of $B_h[X,X^{-1}]$ generated by

$$\{X_i^{h_j}/X_1^{s_1} \ldots X_n^{s_n} \mid 1 \leq i \leq n, 1 \leq j \leq t\}$$

Since $\Gamma_+ = (\{0\} \times N) \cup (\bigcup_{h \in H \setminus \{0\}} \{h\} \times \mathbb{Z})$ we get easily that

$$C := \bigcup_{\substack{h \in \mathcal{H} \\ s \in N^n}} B_{h,s}[X]$$

is isomorphic with $R[X,\Gamma_+]$.

Now it is enough to note that $B_{h,s}[X]$ is in fact a polynomial $R[X]$-algebra in n t-variables and the union expressing C is filtered inductive.

(5. 10) PROOF OF LEMMA (5. 9). Using [7] Lemma (4.6.1) we express H_+ as a filtered inductive union of its submonoids generated (as monoids) by some positive basis of H, let us say

$H_+ = \bigcup M_h, M_h = \langle h \rangle$, where h runs in a set H of positive basis of H. then we have

$$R[X,H_+] = \bigcup_{h\in H} R[X,M_h] = \bigcup_{h\in H} B_h,$$

the union being filtered inductive.

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Wanda Morariu
Institute of Mathematics
Str. Academiei 14
70109 Bucharest
ROMANIA

Dorin Popescu
INCREST
Bd. Pacii 220
79622 Bucharest
ROMANIA