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by

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SPATIAL BUCKLED STATES FOR IMMERSED RODS

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Abstract. The equilibrium equations for a heavy nonlinearly elastic rod immersed in a heavy incompressible fluid are derived. The bifurcation of solutions of these equations is analysed and results concerning the equality of the buckling loads for the heavy immersed rod and for a certain heavy rod in void, associated to it, are studied.
In the case of variable cross section with equal moments of inertia the reduced componential form of the governing equations is given.

1. INTRODUCTION

The first model for an elastic beam was given by the Bernoullis and it is the so-called elastica. The analysis of the planar equilibrium configurations of the elastica subjected only to end forces lead Euler (1744) to a problem which is equivalent with determining the functions $[\underline{0},1]\ni S \to (x(S),\ y(S),\ \Theta(S))$, which satisfy the equations:

Θ = theta
0 = zero

$$[B(S)\Theta'(S)]' + \lambda \sin\Theta(S) = \underline{0}, \qquad (1.1)$$

λ = lambda
$\underline{1}$ = one

$$x'(S) = \cos\Theta(S), \quad y'(S) = \sin\Theta(S), \qquad (1.2)$$

supplemented with a suitable set of boundary conditions.

In the preceding formulas S represents a scaled arc length parameter, $(x(S), y(S))$ give the plane configuration of the elastica, $\Theta(S)$ is the angle the tangent to the curve (x, y) at S makes with the x's axis. The other notations λ and $B(S)$ represent respectively the magnitude of the terminal force and the bending stiffness of the rod at S (see [1] for further historical comments).

For the case in which B=const., Euler gave a classification of all solutions of (1.1) and studied the process of buckling. In the case of non-constant B, Krasnoselskiĭ ([2], chapter IV) used his bifurcation theorem to show that non-trivial solutions of (1.1), (1.2) and

$$\Theta'(0) = \Theta'(1) = 0, \tag{1.3}$$

bifurcate from the eigenvalues of the problem liniarized about the trivial solution $\Theta \equiv 0$.

In [3] Antman showed how to apply for problems like (1.1)-(1.3) the global bifurcation theory developed by Rabinowitz in [4].

A theory able to account for the large deflection of rods in space was established by Kirchhoff in 1859 (see [5]).

The problem of buckling of an elastic rod under its own weight was studied by Greenhill (see [6]) and in a much more general framework, by Antman and Kenney in [7]. They considered a nonlinearly elastic rod modeled as an one dimensional Cosserat continuum. The equations given by this model can also be obtained from the three-dimensional theory of elasticity using the projection method. The three-dimensional approach can be used to take into account forces acting on the lateral surface of the rod (see [8], chapter 1).

However, as far as we know, in the framework of Antman's theory of rods there are no published results concerning the buckling problem in which forces acting on the lateral surface are involved.

A problem of this kind was studied by Vâlcovici in [9]. He considered a heavy rod immersed in a fluid. Using a one dimensional approach, essentially the linearized equations of the elastica and several approximations, Vâlcovici obtained the following equivalence result:

Theorem 1. The elastic line and the buckling condition of a heavy rod immersed in a fluid are the same as the elastic line and the buckling condition for a similar rod in void, if we replace its specific weight γ_0 with $\gamma_0-\gamma_1$, where γ_1 is the specific weight of the fluid, and the other conditions are the same.

γ = gama

The aim of this paper is to extend this result for a nonlinearly elastic rod, modeled as in Antman's theory. We shall compare our equations for the immersed rod with those given in [7] for the heavy rod in void in order to test the validity of Vâlcovici's result for various particular cases.

2. KINEMATICAL CONSTRAINTS AND CONSTITUTIVE ASSUMPTIONS

There are three ways of modelling rods in order to derive rod theories (cf. [8] and [10]): (i) one dimensional Cosserat continua, (ii) constrained three dimensional bodies, (iii) very thin three dimensional bodies, for which various asymptotic expansions are used in deriving the governing equations.

There is a straightforward correspondence between approaches (i) and (ii) (see [8], chapter II). We adopt interpretation (ii) as it can be relied with the approach (i), used in [7] for the heavy rod in void and it takes into account forces acting on the lateral surface (cf. [8], chapter I and [10], §§3-4).

Let $\mathcal{B} \subset E$ (three dimensional euclidean space) be the reference configuration of the rod. We denote by (X,Y,S) or by (X,Y,T) the Cartesian coordinates in E of a point from $\mathcal{B}$ and by $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ the corresponding unit base vectors relative to a frame with the S axis in the direction of the gravitational field. The set $\mathcal{B}$ is supposed to be bounded. It follows that the sets

$$A(S)=\left\{(X,Y)\in R^2 \,\middle|\, (X,Y,S)\in\mathcal{B}\right\};\quad V(S)=\left\{(X,Y,T)\in\mathcal{B} \,\middle|\, T\geq S\right\},$$

are also bounded. The set $A(S)$ is called the cross section at S.

We assume that in the reference configuration the density of mass is constant and the sections $A(S)$ obey the conditions:

$$\int_{A(S)} X dX dY = 0,\quad \int_{A(S)} Y dX dY = 0,\quad \int_{A(S)} XY dX dY = 0. \tag{2.1}$$

These hypothesis are satisfied if OS is the axis of centroids of the cross sections and OX, OY are the principal axes of inertia of $A(S)$. The origine 0 of the coordinate system will be taken in the centroid of the lower cross section supposed fixed in the problems we shall tackle.

Without loss of generality we can suppose that the length of the rod is one. Let

$$\begin{aligned}\underset{\sim}{r}(X,Y,S)&=x(X,Y,S)\underset{\sim}{e}_1+y(X,Y,S)\underset{\sim}{e}_2+s(X,Y,S)\underset{\sim}{e}_3,\\ \underset{\sim}{r}(X,Y,T)&=x(X,Y,T)\underset{\sim}{e}_1+y(X,Y,T)\underset{\sim}{e}_2+t(X,Y,T)\underset{\sim}{e}_3,\end{aligned} \tag{2.2}$$

be the position vector in the deformed state of the material point having coordinates (X,Y,S) or (X,Y,T) in the reference configuration.

We impose the following kinematical constraints:

a) There are the smooth vector functions $\underset{\sim}{r}_o$, $\underset{\sim}{a}_1$, $\underset{\sim}{a}_2$: $[0,1]\to R^3$ such that

$$\underset{\sim}{r}(X,Y,S)=\underset{\sim}{r}_o(S)+X\underset{\sim}{a}_1(S)+Y\underset{\sim}{a}_2(S), \tag{2.3}$$

for any $(X,Y,S)\in\mathcal{B}$, where

ξ = esi
η = eta
ζ = dzeta

$$\underset{\sim}{r}_o(S)=\xi(S)\underset{\sim}{e}_1+\eta(S)\underset{\sim}{e}_2+\zeta(S)\underset{\sim}{e}_3, \tag{2.4}$$

and $\xi,\eta,\zeta\in C^2[0,1]$.

b) The vectors $\underset{\sim}{a}_1(S)$ and $\underset{\sim}{a}_2(S)$ are orthonormal and if we denote by ' the derivation with respect to S

$$\underset{\sim}{r}_o'(S)=\underset{\sim}{a}_1(S)\times\underset{\sim}{a}_2(S)\equiv\underset{\sim}{a}_3(S) \tag{2.5}$$

(Kirchhoff's hypothesis).

Assumption a) implies that the configuration of the rod is completely determined by giving the vector functions $\underset{\sim}{r}_o,\underset{\sim}{a}_1,\underset{\sim}{a}_2$ as $\underset{\sim}{r}_o$ gives the position of the line of centroids and $\underset{\sim}{a}_1$, $\underset{\sim}{a}_2$ determine the cross-section at S.

Since $\underset{\sim}{a}_1$, $\underset{\sim}{a}_2$, $\underset{\sim}{a}_3$ is an orthonormal baxis, there exists a vector function $\underset{\sim}{u}$: $[0,1]\to R^3$ such that

$$\underset{\sim}{a}_i'=\underset{\sim}{u}\times\underset{\sim}{a}_i, \qquad i=1,2,3. \tag{2.6}$$

The vector function u is the deformation field of this theory.

Let $-\underset{\sim}{N}(S)$, $-\underset{\sim}{M}(S)$ be the contact force and the contact couple exerted on $v(S)=\underset{\sim}{r}(V(S))$ (the image of $V(S)$ in the deformed configuration) by the rest of the deformed beam.

The constitutive assumption is that there is a function $\hat{\underset{\sim}{M}}$: such that

$$\underset{\sim}{M}(S) = \hat{\underset{\sim}{M}}(\underset{\sim}{u}(S), S). \tag{2.7}$$

We shall suppose that the symmetrical part of the matrix $\frac{\partial\hat{\underset{\sim}{M}}}{\partial\underset{\sim}{u}}(\underset{\sim}{u}_o, S)$ is positive definite for all $S\varepsilon[0,1]$ and for any vector $\underset{\sim}{u}_o\varepsilon R^3$. This hypothesis is related with the Legendre-Hadamard strong elipcity condition of the three dimensional elasticity ([10]). Using a topological degree argument (cf. [11], chapter IV and [7]) it can be proved that the above assumption implies that the function $\hat{\underset{\sim}{M}}$ supports a global implicit function theorem, so that there exists a function $\hat{\underset{\sim}{u}}; R^3 x[0,1] \rightarrow R^3$ such that

$$\underset{\sim}{u}(S) = \hat{\underset{\sim}{u}}(\underset{\sim}{M}(S), S). \tag{2.8}$$

The rod model presented above is the same with the one used by Buzano, Geymonat and Poston in [12] excepting the fact that we do not necessarily consider hyperelastic rods.

By using (2.3), (2.5) and (2.6) we obtain that the Jacobian of the transformation $(X,Y,S) \longrightarrow \underset{\sim}{r}(X,Y,S)$ is

$$J(X,Y,S) = 1-Xu_2(S) + Yu_1(S), \tag{2.9}$$

with

$$u_i(S) = \underset{\sim}{u}.\underset{\sim}{a}_i(S), \quad i=1,2,3 . \tag{2.10}$$

We shall use this expression for J in order to take into account hydrostatic forces.

3. FORMULATION OF THE GOVERNING EQUATIONS

Let $\mathbf{N}(l)$, $\mathbf{M}(l)$ be the total contact force respectively the contact couple exerted from the exterior on $a(l)$, where

$$a(S)=\mathbf{r}(A(S)), \qquad 0\leq S\leq l, \tag{3.1}$$

represents the image set of $A(S)$ in the deformed configuration of the rod.

We shall suppose that the rod is clamped at the lower end $S=0$ and completely immersed, i.e. all the boundary of the deformed configuration, excepting $a(0)$ and a set of zero area, is in contact with the fluid. The last part of the above assumption allows us to study the case where at the upper end concentrated forces and couples are acting.

Let $\mathbf{F}_g(S)$, $\mathbf{F}_h(S)$ be the resultants of gravitational, respectively hydrostatic forces acting on $v(S)$ and $\mathbf{M}_g(S)$, $\mathbf{M}_h(S)$ the resulting moments about 0 of the same systems of forces. We denote by $\mathbf{N}_l$ and $\mathbf{M}_l$ the concentrated contact force, respectively the concentrated contact couple exerted on $a(l)$. The differences $\mathbf{N}(l)-\mathbf{N}_l$, $\mathbf{M}(l)-\mathbf{M}_l$, represent the action of the hydrostatic forces acting on $a(l)$.

= zero

The requirement that the resultant torque about 0 on the material of $v(S)$ vanish, yields the equations:

$$\mathbf{N}(S)=\mathbf{N}_l + \mathbf{F}_g(S) + \mathbf{F}_h(S), \tag{3.2}$$

$$\mathbf{M}(S)+\mathbf{r}_o(S)\times\mathbf{N}(S)=\mathbf{M}_l+\mathbf{r}_o(l)\times\mathbf{N}_l+\mathbf{M}_g(S)+\mathbf{M}_h(S). \tag{3.3}$$

The end concentrated force and couple N_1 and M_1 are supposed to satisfy the relations:

= miu

$$N_1 = -\lambda e_3, \quad M_1 = \mu c, \quad c \in \left\{ e_3,\ a_3(1) \right\}, \tag{3.4}$$

where $\lambda \geq 0$ and $\mu \in R$ are fixed (see [7] for the interpretation of these boundary conditions).

If γ_o and γ_1 are the specific weight of the rod respectively of the fluid, we have:

$$F_g(S) = -\gamma_o F(S) e_3 \tag{3.5}$$

where $F(S)$ is the volume of $V(S)$ and

$$F_h(S) = -\int_{\partial v(S)} (p_o - \gamma_1 t)\, n\, da + \int_{a(S)} (p_o - \gamma_1 t)\, n\, da, \tag{3.6}$$

where p_o is the hydrostatic pressure in 0, t is defined by (2.2), n is the outer unit normal, da is the surface element and $\partial v(S)$ is the boundary of $v(S)$. Henceforward we suppose that γ_1 is constant too.

By appling Stokes' theorem, the change of variables formula and (2.1), (2,3), we obtain:

$$-\int_{\partial v(S)} (p_o - \gamma_1 t)\, n\, da = \gamma_1 F(S) e_3 \tag{3.7}$$

The second term of (3.6) can be written

$$\int_{a(S)} (p_o - \gamma_1 t)\, n\, da = [p_o - \gamma_1 \zeta(S)] F'(S) r'_o(S), \tag{3.8}$$

where $-F'(S)$ is equal with the area of $A(S)$, in the hypothesis that F is smooth.

Using relations (3.6)-(3.8) it follows that

$$\boldsymbol{F}_h(S)=\gamma_1 F(S)\boldsymbol{e}_3+\Big[p_o-\gamma_1\zeta(S)\Big]F'(S)\boldsymbol{r}_o'(S) \qquad (3.9)$$

The equilibrium equation (3.2) becomes

$$\boldsymbol{N}(S)=\boldsymbol{N}_1-(\gamma_o-\gamma_1)F(S)\boldsymbol{e}_3+[p_o-\gamma_1\zeta(S)]F'(S)\boldsymbol{r}_o'(S) \qquad (3.10)$$

If we admit that all the functions appearing in (3.10) are smooth enough, we obtain the following differential form of the equilibrium equation (3.10):

$$\begin{aligned}\boldsymbol{N}'(S)=&-(\gamma_o-\gamma_1)F'(S)\boldsymbol{e}_3+[p_o-\gamma_1\zeta(S)]F'(S)\boldsymbol{r}_o''(S)+\\&+[p_o-\gamma_1\zeta(S)]F''(S)\boldsymbol{r}_o'(S)-\gamma_1\zeta'(S)F'(S)\boldsymbol{r}_o'(S)\end{aligned} \qquad (3.11)$$

We shall now give a suitable form of the relation (3.3).

The moment of the gravitational forces is

$$\boldsymbol{M}_g(S) =-\int_{v(S)}\boldsymbol{r}\times\gamma t\boldsymbol{e}_3\ dxdydt$$

where γ is the specific weight of the deformed rod and depends on x,y,t. Applying (2.3)-(2.6), change of variables formula and Fubini's theorem it follows that

$$\boldsymbol{M}_g(S)=\ \gamma_o\int_S^1 F'(T)[\eta(T)\boldsymbol{e}_1-\xi(T)\boldsymbol{e}_2]dT. \qquad (3.12)$$

The moment of hydrostatic forces is given by the relation

$$\boldsymbol{M}_h(S)=-\int_{\partial v(S)}\boldsymbol{r}\times(p_o-\gamma_1 t)\boldsymbol{n}da+\int_{a(S)}\boldsymbol{r}\times(p_o-\gamma_1 t)\boldsymbol{n}da. \qquad (3.13)$$

By using Kelvin's transformation, change of variables formula, Fubini's theorem and relations (2.3)-(2.6), the first term from the right side of (3.13) becomes

$$
\begin{aligned}
-\int_{\partial v(S)} \mathbf{r}\times(p_0-\gamma_1 t)\mathbf{n}\,da = & -\gamma_1\int_S^1 F'(T)[\eta(T)\mathbf{e}_1-\xi(T)\mathbf{e}_2]dT + \\
& +\gamma_1\int_S^1[I_Y(T)a_{2Y}(T)\mathbf{u}_1(T)-I_X(T)a_{1Y}(T)\mathbf{u}_2(T)]\mathbf{e}_1 dT + \\
& +\gamma_1\int_S^1[I_X(T)a_{1X}(T)\mathbf{u}_2(T)-I_Y(T)a_{2X}(T)\mathbf{u}_1(T)]\mathbf{e}_2 dT,
\end{aligned} \tag{3.14}
$$

where

$$I_X(S)=\int_{A(S)} X^2 dXdY,\quad I_Y(S)=\int_{A(S)} Y^2 dXdY,$$

$$\mathbf{a}_i(T)=a_{iX}(T)\mathbf{e}_1+a_{iY}(T)\mathbf{e}_2+a_{iS}(T)\mathbf{e}_3,\qquad i=1,2,3.$$

The second term from the right side of (3.13) satisfies the relation:

$$
\begin{aligned}
\int_{a(S)} \mathbf{r}\times(p_0-\gamma_1 t)\mathbf{n}\,da = & [p_0-\gamma_1\zeta(S)]F'(S)\mathbf{r}_0(S)\times\mathbf{r}_0'(S) + \\
& +\gamma_1[I_Y(S)a_{2S}(S)\mathbf{a}_1(S)-I_X(S)a_{1S}(S)\mathbf{a}_2(S)].
\end{aligned} \tag{3.15}
$$

Using relations (3.13)-(3.15), the equilibrium condition (3.3) becomes

$$
\begin{aligned}
\mathbf{M}(S)+\mathbf{r}_0(S)\times\mathbf{N}(S) = & \mathbf{M}_1+\mathbf{r}_0(1)\times\mathbf{N}_1 + \\
& +(\gamma_0-\gamma_1)\int_S^1 F'(T)[\eta(T)\mathbf{e}_1-\xi(T)\mathbf{e}_2]dT + \\
& +\gamma_1\int_S^1[I_Y(T)a_{2Y}(T)\mathbf{u}_1(T)-I_X(T)a_{1Y}(T)\mathbf{u}_2(T)]\mathbf{e}_1 dT + \\
& +\gamma_1\int_S^1[I_X(T)a_{1X}(T)\mathbf{u}_2(T)-I_Y(T)a_{2X}(T)\mathbf{u}_1(T)]\mathbf{e}_2 dT + \\
& +[p_0-\gamma_1\zeta(S)]F'(S)\mathbf{r}_0(S)\times\mathbf{r}_0'(S)+\gamma_1[I_Y(S)a_{2S}(S)\mathbf{a}_1(S)-I_X(S)a_{1S}(S)\mathbf{a}_2(S)]
\end{aligned} \tag{3.16}
$$

The smoothness assumption for the function from the above relation and equation (3.11) allows us to derive the local form of (3.16):

$$\mathbf{M}'(S)+\mathbf{r}_o'(S)\times\mathbf{N}(S)=\gamma_1[\mathbf{T}_1(S)+\mathbf{T}_2(S)] \tag{3.17}$$

where

$$\begin{aligned}\mathbf{T}_1(s)=&[I_x(S)a_{1y}(S)\mathbf{u}_2(S)-I_y(S)a_{2y}(S)\mathbf{u}_1(S)]\mathbf{e}_1+\\&+[I_y(S)a_{2x}(S)\mathbf{u}_1(S)-I_x(S)a_{1x}(S)\mathbf{u}_2(S)]\mathbf{e}_2+\\&+I_y(S)a'_{2S}(S)\mathbf{a}_1(S)+I_y(S)a_{2S}(S)\mathbf{a}'_1(S)-I_x(S)a'_{1S}(S)\mathbf{a}_2(S)-I_x(S)a_{1S}(S)\mathbf{a}'_2(S)\\\mathbf{T}_2(S)=&I'_y(S)a_{2S}(S)\mathbf{a}_1(S)-I'_x(S)a_{1S}(S)\mathbf{a}_2(S).\end{aligned}$$

The projection of $\mathbf{T}_1$ on the basis $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ yields

$$\begin{aligned}\mathbf{T}_1(S)=&(I_x(S)-I_y(S))[u_3(S)a_{1S}(S)\mathbf{a}_1(S)-u_3(S)a_{2S}(S)\mathbf{a}_2(S)+\\&+[u_2(S)a_{2S}(S)-u_1(S)a_{1S}(S)]\mathbf{a}_3(S)\end{aligned}$$

and equation (3.17) becomes:

$$\begin{aligned}\mathbf{M}'(S)+\mathbf{r}_o'(S)\times\mathbf{N}(S)=&\gamma_1(I_x(S)-I_y(S))[u_3(S)a_{1S}(S)\mathbf{a}_1(S)-u_3(S)a_{2S}(S)\mathbf{a}_2(S)+\\&+(u_2(S)a_{2S}(S)-u_1(S)a_{1S}(S))\mathbf{a}_3(S)\\&+\gamma_1(I'_y(S)a_{2S}(S)\mathbf{a}_1(S)-I'_x(S)a_{1S}(S)\mathbf{a}_2(S))\end{aligned} \tag{3.18}$$

For rods with constant cross-section which satisfy the symmetry condition $I_x=I_y$, the last equation can be written

$$\mathbf{M}'(S)+\mathbf{r}_o'(S)\times\mathbf{N}(S)=\mathbf{0} \tag{3.19}$$

The relations (3.11), (3.19) are identical with the differential form of the equilibrium equation for a heavy rod of specific weight $\gamma_0 - \gamma_1$ in void (cf. [7]). It follows that Vâlcovici's theorem holds in this particular case for nonlinearly elastic rods, too.

4. BIFURCATION ANALYSIS OF THE EQUILIBRIUM EQUATIONS

In this section we state the necessary general results from bifurcation theory and show how to apply them for our problem.

Let W be a real Banach space. We shall denote by K(W) the space of linear compact operators from W into W endowed with the usual norm of bounded linear operators.

Let D be an open subset of R^n and $L: D \longrightarrow K(W)$ a family of compact linear operators depending in a continuously differentiable manner on $\nu \in D$ i.e. $L \in C^1(D, K(W))$.

Let $G: D \times W \rightarrow W$ be completely continuous and satisfying the requirement

$\nu = niu$

$$\lim_{\|v\| \to \underline{\underline{0}}} G(\nu, v)/\|v\| = \underline{\underline{0}}$$

uniformely for ν in bounded subsets of D.

We shall cast our problem in the operatorial form:

$$v = L(\nu)v + G(\nu, v) \tag{4.1}$$

with L and G satisfying the above requirements.

The existence of buckled states for our problem will follow as a consequence of the existence of bifurcation points for the equation (4.1), i.e. $(\nu, \underline{\underline{0}}) \in D \times W$ such that in any neibourghood of this point there is a nontrivial solution of the preceding equation.

Bifurcation problems of this kind were studied by Rabinowitz ([4]), Magnus ([13]), Esquinas and López-Gómez ([14]) for $\nu \varepsilon R$ and by Alexander and Antman ([15]) for $\nu \varepsilon R^n$.

It is a well-known fact (e.g. [2], chapter IV, §2) that if $(\nu, \underline{\underline{0}})$ is a bifurcation point of (4.1), then the linear equation

$$v = L(\nu)v \, , \tag{4.2}$$

admits nontrivial solutions. This observation restricts the set of bifurcation points. We shall use the following result (cf [7]):

Theorem 2. If ν is an eigenvalue of the problem (4.2) with odd multiplicity then from $(\nu \, , \underline{\underline{0}})$ there bifurcates a branch of nontrivial solutions of (4.1) satisfying one of the following conditions:

a) It is not confined to any closed and bounded subset of D x W.

b) Its closure contains a point $(\rho, \underline{\underline{0}})$ where ρ is an eigenvalue of (4.2) distinct from ν.

We shall now show how to cast our problem in the form (4.1).

From (2.5), (2.6), (2.8), (3.10), (3.4) and (3.18) it follows that:

$$\underset{\sim}{r}_{\underline{\underline{o}}}(S) = \int_{\underline{\underline{o}}}^{S} \underset{\sim}{a}_3(T)\,dT, \tag{4.3}$$

$$\underset{\sim}{a}_i(S)=\underset{\sim}{e}_i + \int_0^S \hat{\underset{\sim}{u}}(\underset{\sim}{M}(T),T)\times\underset{\sim}{a}_i(T)\,dT, \tag{4.4}$$

$$\begin{aligned}\underset{\sim}{M}(S)=\underset{\sim}{M}(1)&-\int_S^1[\lambda+(\gamma_o-\gamma_1)F(T)]\,\underset{\sim}{a}_3(T)\times\underset{\sim}{e}_3(T)\,dT-\\ &-\gamma_1\int_S^1[I'_y(T)a_{2S}(T)\underset{\sim}{a}_1(T)-I'_x(T)a_{1S}(T)\underset{\sim}{a}_2(T)]\,dT-\\ &-\gamma_1\int_S^1[I_x(T)-I_y(T)]\Big\{u_3(T)a_{1S}(T)\underset{\sim}{a}_1(T)-\\ &-u_3(T)a_{2S}(T)\underset{\sim}{a}_2(T)+[u_2(T)a_{2S}(T)-u_1(T)a_{1S}(T)]\underset{\sim}{a}_3\Big\}dT\end{aligned} \tag{4.5}$$

where $\underset{\sim}{M}(1)$ is given by

$$\begin{aligned}\underset{\sim}{M}(1)&=\underset{\sim}{M}_1-\int_{a(1)}[X\underset{\sim}{a}_1(1)+Y\underset{\sim}{a}_2(1)]\times[p_o-\gamma_1(a_{1S}(1)X+a_{2S}(1)Y+\zeta(1)]\,\underset{\sim}{r}'_o(1)\,dXdY=\\ &=\underset{\sim}{\mu c}+\gamma_1[I_X(1)a_{1S}(1)\underset{\sim}{a}_2(1)-I_Y(1)a_{2S}(1)\underset{\sim}{a}_1(1)]\end{aligned} \tag{4.6}$$

These relations, excepting the last term from (4.5) are similar with the one obtained by Antman and Kenney in [7] and their methods hold in this case too.

The above equations represent a system of fifteen scalar integral equations for the fifteen dimensional vector

$$X=\left\{\underset{\sim}{r}_o,\ \underset{\sim}{a}_i,\ \underset{\sim}{M}\right\}$$

It is easy to check that this system admits the trivial solution

$$X^*=X^*(\lambda,\mu,\gamma_o,\gamma_1)=\left\{\underset{\sim}{r}^*_o,\underset{\sim}{a}^*_i,\underset{\sim}{M}^*\right\}$$

where

$$\underset{\sim}{r}^*_o(S)=S\underset{\sim}{e}_3\ ,\quad \underset{\sim}{a}^*_3(S)=\underset{\sim}{e}_3\ , \tag{4.7}$$

δ = delta

$$a_1^*(S)=\cos\delta(S)e_1+\sin\delta(S)e_2 \quad , \tag{4.8}$$

$$a_2^*(S)=-\sin\delta(S)e_1+\cos\delta(S)e_2 \, , \; M^*=\mu e_3 , \tag{4.9}$$

with:

$$\delta(S)=\int_0^S \hat{u}_3(M^*,T)\,dT \quad ,$$

for all the values of the parameters $\lambda,\mu,\gamma_0,\gamma_1$.

Let now $W = C^0([0,1])^{15}$ be the Banach space obtained by taking the direct sum of fifteen copies of the Banach space of continuous real functions defined on $[0,1]$.

If we put $X = \{r_0, \; a_i, \; M\} \in W$, the above system of integral equations can be written

$$X = H(X,\lambda,\mu, \; \gamma_0,\gamma_1).$$

It is easy to see that H is a Fréchet differentiable operator and using Arzela-Ascoli theorem it follows that it is a completely continuous operator acting on W x R^4.

Let Y be the difference between X and the trivial solution X^*. It follows that Y satisfies the equation:

$$\begin{aligned} Y &= H_X(X^*(\lambda,\mu,\gamma_0,\gamma_1),\lambda,\mu,\gamma_0,\gamma_1)Y+ \\ &+\Big\{H(X^*(\lambda,\mu,\gamma_0,\gamma_1)+Y,\lambda,\mu,\gamma_0,\gamma_1)-H(X^*(\lambda,\mu,\gamma_0,\gamma_1),\lambda,\mu,\gamma_0,\gamma_1)- \\ &\quad -H_X(X^*(\lambda,\mu,\gamma_0,\gamma_1),\lambda,\mu,\gamma_0,\gamma_1)Y\Big\}, \end{aligned} \tag{4.10}$$

where $H_X(X^*(\lambda,\mu,\gamma_0,\gamma_1),\lambda,\mu,\gamma_0,\gamma_1)Y$ is the Fréchet differential of $H(\cdot,\lambda,\mu,\gamma_0,\gamma_1)$ at X^* in the direction Y.

Equation (4.9) has the form (4.1) with $W=(C^{0}[0,1))^{15}$ and:

$$L(\lambda,\mu,\gamma_0,\gamma_1)=H_{\chi}(\chi^{*}(\lambda,\mu,\gamma_0,\gamma_1),\lambda,\mu,\gamma_0,\lambda_1), \tag{4.11}$$

$$G(\lambda,\mu,\gamma_0,\gamma_1,y)=H(\chi^{*}(\lambda,\mu,\gamma_0,\gamma_1)+y,\lambda,\mu,\gamma_0,\gamma_1)- \tag{4.12}$$

$$-H(\chi^{*}(\lambda,\mu,\gamma_0,\gamma_1),\lambda,\mu,\gamma_0,\gamma_1)-H_{\chi}(\chi^{*}(\lambda,\mu,\gamma_0,\gamma_1),\lambda,\mu,\gamma_0,\gamma_1)y.$$

In order to apply theorem 2 it is enough to prove that L and G defined by (4.11) and (4.12) are compact. The compactness of L follows from the fact that the Fréchet differential of a compact operator is still compact (cf.[2], chapter II, §4). The compactness of G is a consequence of the complete continuity of H, L and of the relation (4.12).

Thus the determination of the global behaviour of continua of solutions of the equations (4.3)-(4.5) is based on the study of the eigensurfaces of the linear compact operator - valued function L(.).

In the rest of this section we shall study the case of a rod with constant cross section but not necessarily with equal moments of inertia.

First, let us notice that the system (4.3)-(4.5) differs from the equilibrium equations for a heavy rod of specific weight $\gamma_0-\gamma_1$ (cf.[7]) in void only by the term

$$-\gamma_1\int_S^1 \mathbf{T}_1(T)\,dT=-\gamma_1(I_x-I_y)\int_S^1[u_3(T)a_{1S}(T)\mathbf{a}_1(T)-$$
$$-u_3(T)a_{2S}(T)\mathbf{a}_2(T)+(u_2(T)a_{2S}(T)-u_1(T)a_{1S}(T))\mathbf{a}_3(T)]dT$$

This term has no influence on the bifurcation at odd multiplicity eigenvalue (defined in [13] or [14]); more precisely the following nonlinear counterpart of Vâlcovici's result holds:

Theorem 3. If ν is an odd multiplicity eigenvalue for the liniarized problem corresponding to a rod of specific weight $\gamma_0-\gamma_1$ in void, then $(\nu, 0)$ is a bifurcation point for the equilibrium equations of a rod of specific weight γ_0 immersed in a fluid of specific weight γ_1 if $M(1)$ and $N(1)$ satisfy the same conditions and both rods are clamped at the lower end.

Proof. We have only to prove that the linearizations near the common trivial solution of both problems are the same. Then by applying theorem 2 the conclusion follows in a straightforward way.

In order to do this it is enough to prove that the linearization of $T_1(S)$ is identically zero. For its first term, we have

ε = epsilon

$$\lim_{\varepsilon\to 0} \hat{u}(\varepsilon M(S), S)[e_3+\varepsilon a_3(S)]\varepsilon a_{1S}[e_1+\varepsilon a_1(S)]\ \varepsilon^{-1}=$$
$$=\lim_{\varepsilon\to 0} \hat{u}(\varepsilon M(S), S)\, a_{1S}(S)\, e_1\cdot e_3 = 0,$$

if we suppose that the rod is unstressed in its reference configuration. In a similar way it follows that the liniarization of the other terms from $T_1(S)$ is zero.

5. A COMPONENTIAL FORM OF THE EQUILIBRIUM EQUATIONS

In this section we study the case of a rod which is transversely isotropic i.e., the constitutive function $\hat{u}$ satisfies the relation

$$\hat{u}(QM, S) = Q\hat{u}(M, S), \quad 0\leq S\leq 1, \tag{5.1}$$

for each constant orthogonal matrix with components with respect to the basis $\left\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3\right\}$ are:

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} & 0 \\ \Omega_{21} & \Omega_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The transverse isotropy condition (5.1) ensures that the moments of inertia about all axes passing through the centroid of the cross section are equal (if we accept the assumption that the cross section has the same properties as the constitutive function , stated in [12]), in particular

$$I_x(S) = I_y(S) = I(S), \qquad 0 \leq S \leq 1 \tag{5.2}$$

Let m_i and $\hat{u}_i$ be the components of $\underset{\sim}{M}$ respectively $\underset{\sim}{\hat{u}}$ with respect to the basis $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3$, $i=1,2,3$. Another consequence of the condition (5.1) is that $\hat{u}_3$ depends only on:

$$m_1^2 + m_2^2 \ ; \ S, \ m_3 \ , \tag{5.3}$$

and that

$$\hat{u}_\beta = \hat{\sigma} m_\beta \quad , \quad \beta = 1,2 \tag{5.4}$$

where $\hat{\sigma}$ depends only on the arguments from (5.3) (cf.[7]). The constitutive assumption (5.1) and its consequences allow us to use the method developed by Antman and Kenney in the preceding quoted paper, in order to obtain a more tractable form of the equilibrum equation (3.10).

Taking into account (3.4), (3.10) and (5.2), equation (3.19) becomes

$$M'(S)-[\lambda+(\gamma_o-\gamma_1)F(S)]\mathbf{a}_3(S)\times\mathbf{e}_3= \\ =\gamma_1 I'(S)[a_{2S}(S)\mathbf{a}_1(S)-a_{1S}(S)\mathbf{a}_2(S)]\,. \tag{5.5}$$

φ = fi
ψ = psi

The basis $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3$ is related to the fixed basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ by the Eulerian angles Θ, φ, ψ through the relations:

$$\mathbf{a}_1=(-\sin\varphi\sin\psi+\cos\varphi\cos\psi\cos\Theta)\mathbf{e}_1+(\cos\varphi\cos\psi-\sin\varphi\sin\psi\cos\Theta)\mathbf{e}_2- \\ -\cos\psi\sin\Theta\mathbf{e}_3,$$

$$\mathbf{a}_2=(-\sin\varphi\cos\psi-\cos\varphi\sin\psi\cos\Theta)\mathbf{e}_1+(\cos\varphi\cos\psi-\sin\varphi\sin\psi\cos\Theta)\mathbf{e}_2+ \\ +\sin\psi\sin\Theta\mathbf{e}_3\,,$$

$$\mathbf{a}_3=\cos\varphi\sin\Theta\mathbf{e}_1+\sin\varphi\sin\Theta\mathbf{e}_2+\cos\Theta\mathbf{e}_3\,. \tag{5.6}$$

The above equations and (2.6) imply that the components of u with respect to $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3$ are

$$\begin{aligned} u_1 &= -\varphi'\sin\Theta\cos\psi+\Theta'\sin\psi\,, \\ u_2 &= \varphi'\sin\Theta\sin\psi+\Theta'\cos\psi\,, \\ u_3 &= \psi'+\varphi'\cos\Theta\,. \end{aligned} \tag{5.7}$$

In the following, we shall use the basis $\mathbf{D}_1,\mathbf{D}_2,\mathbf{D}_3$ defined by

$$\mathbf{D}_1=\cos\psi\mathbf{a}_1-\sin\psi\mathbf{a}_2,\quad \mathbf{D}_2=\sin\psi\mathbf{a}_1+\cos\psi\mathbf{a}_2,\quad \mathbf{D}_3=\mathbf{a}_3\,. \tag{5.8}$$

If we dot (5.5) with $\mathbf{e}_3$ it follows that

α = alpha

$$\mathbf{M}(s)\cdot\mathbf{e}_3=\mathbf{M}(1)\cdot\mathbf{e}_3=\alpha=\text{const.} \tag{5.9}$$

By taking the scalar product of (5.5) with $\underset{\sim}{a}_3$ we obtain

β = beta

$$m_3(s) = m_3(1) = \beta = \text{const.}, \tag{5.10}$$

since (5.4) implies that

$$\hat{u}_1 m_2 - \hat{u}_2 m_1 = 0.$$

Let us denote by U_i, M_i the components of $\underset{\sim}{U}$ and $\underset{\sim}{M}$ with respect to the basis $\{\underset{\sim}{D}_i\}$.

From (5.7) and (5.8) it follows that

$$U_1 = -\varphi' \sin\Theta, \quad U_2 = \Theta', \quad U_3 = u_3 = \psi' + \varphi' \cos\Theta. \tag{5.11}$$

Relations (5.6), (5.8), (5.9) and (5.10) imply that

$$\begin{aligned} \alpha &= \underset{\sim}{M}(S) . \underset{\sim}{e}_3 = \underset{\sim}{M}(S) \left[-\sin\Theta(S) \underset{\sim}{D}_1(S) + \cos\Theta(S) \underset{\sim}{D}_3(S) \right] \\ &= -M_1(S) \sin\Theta(S) + \beta \cos\Theta(S), \qquad 0 \leq S \leq 1 . \end{aligned} \tag{5.12}$$

Taking in the above relations $S=0$ it follows that

$$\alpha = \beta, \tag{5.13}$$

since the rod is clamped at the lower end. Relations (5.12) and (5.13) imply that

$$M_1 = -\beta(1-\cos\Theta)/\sin\Theta, \tag{5.14}$$

or equivalently

$$M_1 = -\beta\sin\Theta/(1 + \cos\Theta) \qquad (5.15)$$

for all $S \varepsilon [0,1]$ where $\sin\Theta(S)$ does not vanish.

By using (5.6) and (5.8) the equilibrium equation (5.5) becomes:

$$\underset{\sim}{M}'(S)+[\lambda+(\gamma_0-\gamma_1)F(S)-\gamma_1 I'(S)]\sin\Theta(S)\,\underset{\sim}{D}_2(S)=\underline{0} \qquad (5.16)$$

As a consequence of (2.6), (5.4), (5.6), (5.11) we obtain that the relation

$$\underset{\sim}{M}'.\underset{\sim}{D}_2=M_2' + \hat{\sigma}\beta^2\sin\Theta\,/(1+\cos\Theta)^2, \qquad (5.17)$$

is valid where $\sin\Theta$ does not vanish.

By dotting (5.16) with $\underset{\sim}{D}_2$ and using (5.17) we obtain

$$M_2'+[\hat{\sigma}\beta^2/(1+\cos\Theta)^2+\lambda+(\gamma_0-\gamma_1)F-\gamma_1 I']\sin\Theta=\underline{0}\;. \qquad (5.18)$$

From (5.4), (5.11) and (5.13) it follows that:

$$\Theta' = \hat{\sigma}M_2\,, \qquad (5.19)$$

$$\varphi' = \hat{\sigma}\beta/(1+\cos\Theta)\,, \qquad (5.20)$$

$$\psi' = \hat{u}_3-\hat{\sigma}\beta\cos\Theta/(1+\cos\Theta), \qquad (5.21)$$

where $\hat{\sigma}$ depends on $M_1=-\beta\sin\Theta/(1+\cos\Theta)$, M_2, $M_3=\beta$ and the last relations (5.19)-(5.21) are valid only where $\sin\Theta$ does not vanish.

In [7] it is shown that either the solution of the system (4.3)-(4.5) is trivial or relations (5.15), (5.16), (5.18), (5.19), (5.20) and (5.21) hold for each $S\varepsilon[0,1]$. By using the fact that the rod is clamped at the lower end and relations (4.6) we obtain the boundary conditions:

$$\Theta(0) = 0, \quad M_2(1) = + \gamma_1 I \sin\Theta(1) \tag{5.22}$$

when the alternative $c=a_3(1)$ holds in (3.4).

By comparing the above relation with the reduced componential form of the governing equations for a heavy rod of specific weight $\gamma_0-\gamma_1$ (cf[7]) we notice that the only difference occurs in (5.18) which contains the term $-\gamma_1 I' \sin\Theta$, not appearing in the corresponding equation for the heavy rod. By linearization at the trivial solution $\Theta\equiv 0$ this term becomes $-\gamma_1 I'\Theta$ which is not identically zero, unless the cross section is constant. We conclude that Vâlcovici's result doesn't hold for nonlinearly elastic rods with variable cross-section. However, equation (5.18) is of the same kind with the corresponding equation for the rod in void so that for $I'(s)$ given, one can state equivalence theorems of Vâlcovici type for immersed rods.

From (5.20) it follows that if $\beta=0$, i.e. the rod is not torsioned at the upper end, we have φ=const., which means that the solutions are planar. The planar problem for rods subjected to forces and couples acting only at its extremities was studied by Antman and Rosenfeld([16]) in the framework of a model which doesn't use Kirchhoff's hypothesis. In [17], Tucsnak used the above hypothesis in modelling the planar version of the situation considered in the present paper and showed that the linearization of the system (5.18) and (5.19) with the boundary conditions (5.22) represents a Sturm-Liouville problem, having only simple eigenvalues, so that in this case the hypothesis of theorem 2 are fulfilled.

The results of this section show that the methods developed by Antman and Kenney for the heavy rod in void are entirely applicable for immersed rods.

6. CONCLUDING REMARKS

The equivalence result of Vâlcovici stated in theorem 1 rests valid for nonlinearly elastic rods if the cross section of the rod is constant and has equal principal moments of inertia in the reference configuration. If the principal moments of inertia are different but the cross section is still constant the result holds in the sense of theorem 3.

For rods with variable cross section but with equal principal moments of inertia, the reduced componential form of the governing equations, obtained in section 5, has the same form with the one obtained by Antman and Kenney for the heavy rod in void. This shows that for particular functions I(s) the problem of determining the critical loads for an immersed rod can be reduced to a buckling problem for a heavy rod in void.

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