INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

INFINITESIMAL AUTOMORPHISMS OF AFFINE ALGEBRAIC GROUPS

by A. BUIUM

PREPRINT SERIES IN MATHEMATICS
No. 27/1988

BUCURESTI

INFINITESIMAL AUTOMORPHISMS OF AFFINE ALGEBRAIC GROUPS

by A. BUIUM

June 1988

*) Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.

INFINITESIMAL AUTOMORPHISMS OF AFFINE ALGEBRAIC GROUPS

by

A.Buium

0. INTRODUCTION. PRELIMINARIES

It is a fact that the automorphism functor $\mathcal{A}ut\,G$ of an affine algebraic F-group G (char F=0) need not be representable on the category of all F-schemes, but (as shown by Borel and Serre [BS]) it is representable when restricted to the category of reduced F-schemes (by some locally algebraic group, call it Aut G). In particular there exist examples of G's (e.g. $G=G_a \times G_m$) for which the map between the corresponding Lie algebras $\lambda : \mathcal{L}(\mathrm{Aut}\ G) \to \mathcal{L}(\mathcal{A}ut\,G)$ is not surjective (cf. [BS]) (in other words for which there exist "infinitesimal automorphisms" which do not come from an algebraic group action). The primary goal of this paper is to provide a better understanding of the above Lie algebra map λ. For instance we will prove that there exists a complex which is exact in the first two terms:

$$0 \to \mathcal{L}(\mathrm{Aut}\ G) \xrightarrow{\lambda} \mathcal{L}(\mathcal{A}ut\,G) \xrightarrow{\alpha} H^1(r/[r,r],r) \xrightarrow{\beta} H^2(u,u)$$

where r (respectively u) is the Lie algebra of the radical (respectively of the unipotent radical) of G and r has a structure of $r/[r,r]$- module which is canonical up to interior automorphisms of r; our map α should be viewed as "the cohomological obstruction to algebraicity of infinitesimal automorphisms of G", while our β should be viewed as an "integrability condition" for the α - obstruction.

As we shall see, β will be related to the geometry of linear subspaces of the variety M of all Lie algebra multiplications on the underlying vector space of u.

The non-representability phenomenon described above has an interesting effect: for each G with non-surjective λ we are provided with a "non-trivial" example of linear Δ-algebraic group in the sense of Cassidy and Kolchin [C_1][K_2] (e.g. $G=G_o \times G_m$ leads to the Δ- algebraic group $\Gamma = \{yy''-(y')^2=0\} \subset GL_1(\mathcal{U})$ in the notations of [C_1]). This suggests to slightly generalize our setting and instead of looking at the space $\mathcal{L}(\mathcal{A}ut\, G)$ of infinitesimal automorphisms to look at the space $\Delta(G)$ of all k-derivations (k a subfield of F) on the coordinate algebra of G which send F into F and "commute" with comultiplication, antipode and counit. The present paper is devoted to the study of $\Delta(G)$ (clearly if k=F the space $\Delta(G)$ can be identified with $\mathcal{L}(\mathcal{A}ut\, G)$). To each derivation in $\Delta(G)$ we associate in section 1 its "α- class" lying in $H^1(r/[r,r],r)$ and study the analogue of the above complex in this setting (cf. Theorem (1.6) whose proof will be given in section 2).

In section 3 we apply the preceeding theory to Δ-algebraic groups. As proved in [B_3] any irreducible linear Δ-$\mathcal{F}$- group Γ (in the sense of [C_1]) of finite transcendence degree (i.e. with tr.deg. $\mathcal{F}\langle\Gamma\rangle/\mathcal{F}<\infty$) has a finitely generated (in the non-differential sense) coordinate algebra hence derives from some affine algebraic group G equipped with m commuting elements of $\Delta(G)$. Consequently by our theory we will attach to Γ cohomology classes $\alpha_1,\dots,\alpha_m$; the images of these classes via β (which we call $\beta_1,\dots,\beta_m$) do not vanish apriori anymore. It will turn out that the vanishing of the α's is equivalent to Γ being "splitable" (cf. [B_3] i.e. Δ- isomorphic to a Δ- algebraic group of the form $\Gamma^* \cap GL_n(\mathcal{K})$ where Γ^* is a $\mathcal{C}$-closed subgroup of $GL_n(\mathcal{U})$ in notations of [C_1], cf. also (0.4) below) while the vanishing of the β's is equivalent to Γ being "semisplitable" (i.e. Δ- isomorphic to a Δ-algebraic group of the form $\Gamma^* \cap \{\delta_i y_{jp} - P_{ijp} = 0\}$ where Γ^* is a $\mathcal{C}$-closed subgroup of $GL_n(\mathcal{U})$ and P_{ijp} are non-differential polynomials in the y_{jp}'s); see section 3 for precise statements.

In what follows we fix some notations and recall some basic facts of differential algebra and algebraic groups.

(0.1) Terminology of affine algebraic F-groups is borrowed from [H]; however we shall also look at affine algebraic F-groups as "group schemes of finite type over F". We will often denote by the same letter an affine algebraic F-group and its "underlying" abstract group (= its group of F-points). Recall that $\mathcal{L}(A)$, $\mathcal{L}(G)$ denote the Lie algebras associated to an associative algebra A and to an affine algebraic F-group G respectively. $\mathcal{O}(G)$ will denote the coordinate algebra of G. $\mathcal{G}(H)$ will denote the affine algebraic F-group associated to an affine Hopf algebra H. $\mathcal{U}(h)$ will denote the universal enveloping algebra of the Lie algebra h. Lie algebras of algebraic groups G,R,U,T,S,... will be denoted by g,r,u,t,s,...

(0.2) In sections 1-2 terminology of differential algebra is that from [B_1]. So if Δ will be an arbitrary (non necessary finite) set of (non necessary commuting) derivation operators we will speak about Δ-fields, Δ-F-vector spaces,... Recall that a Δ-F-vector space over a Δ-field F is an F-vector space together with a map $\Delta \to \mathrm{End}_{\Omega} V$, $\delta \mapsto \delta_V$ with the property that $\delta(\lambda x) = (\delta\lambda)x + \lambda\delta x$ for all $\delta \in \Delta$, $\lambda \in F$, $x \in V$, where we have writen δv instead of $\delta_V v$ for all $v \in V$. Recall that if V,W are Δ-F- vector spaces then $V \otimes W$ and Hom(V,W) have natural structures of Δ-F-vector spaces ($\delta(x \otimes y) = \delta x \otimes y + x \otimes \delta y$ for $\delta \in \Delta$, $x \in V$, $y \in W$ and $(\delta f)(x) = \delta(f(x)) - f(\delta x)$ for $\delta \in \Delta$, $f \in \mathrm{Hom}(V,W)$, $x \in V$). By a Δ- Lie F-algebra we understand a Δ-F-vector space h which is a Lie F-algebra such that the multiplication map $h \otimes h \to h$ is a Δ- map (this is the concept from [B_1] and is different from that of Δ- F-Lie algebra in [C_1][K_2]).

A Δ-F-vector space is called locally finite if it is a union of finite dimensional Δ-F-vector subspaces. If V and W are locally finite so is $V \otimes W$ but Hom(V,W) (and even the dual $V^o = \mathrm{Hom}(V,F)$) won't be in general.

A Δ-F-vector space V will be called split if it has an F-basis contained in V^{Δ}, the Ω-linear space of Δ- constants of V. If V is split then it is locally finite. Conversely, recall the following fact from [B_1] p. 85 :

(0.3) LEMMA. Let V be a finite dimensional Δ-F-vector space. Then there exists a Δ-field extension F_1/F with F_1^{Δ}/F^{Δ} algebraic such that $V \otimes_F F_1$ is a split Δ-F_1-vector space. If F is algebraically closed and V is a "partial" Δ-F-vector space over the "partial" Δ-field F (i.e. Δ is finite and acts by commuting derivations) then F_1/F can be taken to be a Picard-Vessiot extension.

(0.4) In section 3 we will use terminology and notations of differential algebra from [C_1][K_1]. So all Δ- fields will be "partial" (card Δ=m). $\mathcal{U}$ will denote a universal Δ-field of characteristic zero with field of constants $\mathcal{K}$ and $\mathcal{F}$ will be a Δ-subfield of $\mathcal{U}$ (over which $\mathcal{U}$ is universal) with field of constants

$\mathcal{C}$. $\mathcal{F}$ will always be assumed algebraically closed. We close our introduction by discussing a (rather curious) Lie algebra theoretic construction which will play a role in our paper. It consists in producing a family of Lie algebra structures starting from a given Lie algebra structure on a vector space.

(0.5) Let h be a Lie F-algebra (F a field), h_o an abelian Lie subalgebra of h and h_1 an ideal of h. Denote by $A^1(h_1)$ the space of all bilinear alternating maps $h_1 \times h_1 \longrightarrow h_1$ (notation cf. [NR]). Then let's define a linear map $\mathcal{D}$: $\mathrm{Hom}(h/[h,h],h_o) \longrightarrow$ $\longrightarrow A^1(h_1)$, $\varphi \mapsto \langle\ ,\ \rangle_\varphi$ by the formula $\langle x,y\rangle_\varphi = [\varphi px,y]-[\varphi py,x]$, for all $x,y \in h_1$, where $p: h \rightarrow h/[h,h]$ is the canonical projection. Moreover let $\mathcal{D}(h,h_o,h_1) \subset A^1(h_1)$ denote the image of the map $\mathcal{D}$ and call it the "derived space" associated to $(h\ ,h_o,h_1)$. This derived space has same remarkable properties:

(0.6) LEMMA. For every bilinear map $\langle\ ,\ \rangle_\varphi$ in the derived space the following hold:

1) $\langle\ ,\ \rangle_\varphi \in Z^2(h_1,h_1)$

2) $\langle\ ,\ \rangle_\varphi$ is a Lie algebra multiplication on h_1

3) $(h_1, \langle\ ,\ \rangle_\varphi)$ is a metaabelian Lie algebra (i.e. solvable in 2 steps)

4) for every $z \in h_o$, ad z induces a derivation of the Lie algebra $(h_1, \langle\ ,\ \rangle_\varphi)$.

Proof. A straightforward computation. So the derived space $\mathcal{D}(h,h_o,h_1)$ is a linear subspace of the variety M of all Lie algebra multiplications on the underlying vector space of h_1. It would be interesting for our purpose to understand this space better (for instance to dispose of apriori bounds for the dimension of its projection in $H^2(h_1,h_1)$, cf. (1.7) below).

1. The space $\Delta(G)$

(1.1) Let k be a field of characteristic zero, F an algebraically closed field containing k and let G be an affine algebraic F-group. Denote by $\Delta(G)$ the F-vector space of all k-derivations $\delta : \mathcal{P}(G) \to \mathcal{P}(G)$ enjoying the following properties:

1) $\delta(F) \subset F$
2) $\mu \circ \delta = (\delta \otimes 1 + 1 \otimes \delta) \circ \mu : \mathcal{P}(G) \longrightarrow \mathcal{P}(G) \otimes \mathcal{P}(G)$
3) $\tau \circ \delta = \delta \circ \tau : \mathcal{P}(G) \longrightarrow \mathcal{P}(G)$
4) $\varepsilon \circ \delta = \delta \circ \varepsilon : \mathcal{P}(G) \longrightarrow F$

where μ, τ, ε are the comultiplication, antipode and counit respectively on $\mathcal{P}(G)$ [Sw]. Then $\Delta(G)$ is also a Lie k-algebra equiped with a natural map $\partial : \Delta(G) \to \mathrm{Der}(F/k)$ and with respect to this structure it is a Lie space over F (i.e. $[\lambda\delta_1, \delta_2] = \lambda[\delta_1, \delta_2] - (\delta_2\lambda)\delta_1$ for $\lambda \in F$, $\delta_1, \delta_2 \in \Delta(G)$ where $\delta_2\lambda = \partial(\delta_2)(\lambda)$, see [NW], [$C_2$]). For any intermediate field E between k and F we may consider the Lie subspace $\Delta(G/E)$ of $\Delta(G)$ consisting of all $\delta \in \Delta(G)$ which vanish on E; then $\Delta(G/E)$ is a Lie E-algebra. The F-Lie algebra $\Delta(G/F)$ has a remarkable interpretation in terms of the automorphism functor of G.

Indeed, let $\mathcal{A}ut\,G : \{\text{F-schemes}\} \to \{\text{groups}\}$ be the functor defined by $S \mapsto \mathrm{Aut}(G \times S/S)$. This functor is not generally representable cf. [BS]; its restriction to $\{\text{reduced F-schemes}\}$ is however representable cf. [BS] by a locally algebraic group scheme, call it Aut G, with affine connected component of the identity $\mathrm{Aut}^o G$. We may view Aut G as a functor $\{\text{F-schemes}\} \to \{\text{groups}\}$ by identifying it with its functor of points. Then there is an obvious homomorphism Aut $G \to \mathcal{A}ut\,G$ inducing a homomorphism $\mathcal{L}(\mathrm{Aut}\,G) \to \mathcal{L}(\mathcal{A}ut\,G)$ (here if $\mathcal{A}$ is any functor $\{\text{F-schemes}\} \to \{\text{groups}\}$, $\mathcal{L}(\mathcal{A})$ is defined to be the kernel of the map $\mathcal{A}(\mathrm{Spec}\,F_\varepsilon) \to \mathcal{A}(\mathrm{Spec}\,F)$ induced by projection of the ring $F_\varepsilon =$

$=F+\varepsilon F$ of dual numbers onto F given by $\varepsilon \mapsto 0$; $\mathcal{L}(\mathcal{A})$ is apriori only a group, not a Lie algebra, cf [DG]). Now the map $\theta \mapsto \mathrm{id}+\varepsilon\theta$ clearly identifies $\Delta(G/F)$ with $\mathcal{L}(\mathcal{A}ut\ G)$ making the latter a Lie F-algebra and making the map $\mathcal{L}(\mathrm{Aut}\ G) \to \mathcal{L}(\mathcal{A}ut\,G)$ a Lie algebra map.

An important role in our paper will be played by the set $\Delta(G,\mathrm{fin})$ of all locally finite derivations in $\Delta(G)$ (here $\delta \in \Delta(G)$ is called locally finite if $\mathcal{P}(G)$ is a locally finite $\{\delta\}$-F-vector space). Apriori this set is not even a vector subspace of $\Delta(G)$; but as we shall see below (cf. Theorem (1.6)) it is in fact a Lie subspace of $\Delta(G)$.

(1.2) LEMMA. The map $\lambda: \mathcal{L}(\mathrm{Aut}\ G) \to \mathcal{L}(\mathcal{A}ut\ G) = \Delta(G/F)$ is injective and its image equals $\Delta(G/F) \cap \Delta(G,\mathrm{fin})$ (viewed as a subset of $\Delta(G)$).

Proof. Start with a preparation. Assume W is a finite dimensional vector subspace of $\mathcal{P}(G)$ generating $\mathcal{P}(G)$ as an F-algebra and let $\mathcal{A}ut(G,W)$: {affine F-schemes} $\to$ {groups} be the subfunctor of $\mathcal{A}ut\ G$ defined by $S=\mathrm{Spec}\ B \mapsto \{f \in \mathrm{Aut}(G \times S/S);\ f^*: \mathcal{P}(G) \otimes B \to \mathcal{P}(G) \otimes B$ preserves $W \otimes B\}$. We claim that $\mathcal{A}ut(G,W)$ is representable; note that the affine group scheme Aut(G,W) representing it is reduced by [Sw] p.280. To prove our claim let W_1 be the intersection of all sub-F-coalgebras of $\mathcal{P}(G)$ containing $W_o = W + \tau W$; by [Sw] W_1 is a finite dimensional coalgebra. Now define inductively the increasing sequence of subspaces W_i of $\mathcal{P}(G)$ by the formula $W_{i+1} = \mu(W_i \otimes W_i)$ for $i \geq 1$ and define functors $\mathcal{A}_o, \mathcal{A}_1, \mathcal{A}_2, \ldots$ from {affine F-schemes} to {groups} as follows. We let $\mathcal{A}_o(\mathrm{Spec}\ B)$ be the group of those B-linear automorphisms σ_o of $W_o \otimes B$ such that $\sigma_o \circ \tau_B = \tau_B \circ \sigma_o$, where $\tau_B = \tau \otimes 1_B$. For $i \geq 1$, let $\mathcal{A}_i(\mathrm{Spec}\ B)$ be the group of those B-linear automorphisms σ_i of $W_i \otimes B$ such that $\sigma_i(W_{i-1} \otimes B) = W_{i-1} \otimes B$,

$\sigma_i|_{W_{i-1}\otimes B} \in \mathcal{A}_{i-1}(\mathrm{Spec}\ B)$ and $\sigma_i \circ \mu_B = \mu_B \circ (\sigma_{i-1} \otimes \sigma_{i-1})$ where $\mu_B = \mu \otimes 1_B$. We have canonical restriction maps $\mathcal{A}_i \to \mathcal{A}_{i-1}$ for all $i \geq 1$. Now clearly all $\mathcal{A}_i$'s are representable by affine algebraic F-groups A_i, hence we have a projective system $\cdots \to A_i \to A_{i-1} \to \cdots \to A_1 \to A_o$. One checks that $\mathcal{A}ut(G,W) = \varprojlim \mathcal{A}_i$. Consequently $\mathcal{A}ut(G,W)$ is represented by $\mathrm{Spec}(\varinjlim \mathcal{O}(A_i))$ and our claim is proved.

Let's prove that $\mathrm{Im}\,\lambda = \Delta(G/F) \cap \Delta(G,\mathrm{fin})$. The inclusion "$\subset$" is clear. Conversely if $\delta \in \Delta(G/F) \cap \Delta(G,\ \mathrm{fin})$ we may choose W above such that $\delta W \subset W$. Then $\mathrm{id}+\varepsilon\delta \in \mathcal{A}ut(G,W)(\mathrm{Spec}\ F_\varepsilon)$, hence we get a morphism $f:\mathrm{Spec}\ F_\varepsilon \to \mathrm{Aut}(G,W)$ such that $\mathrm{id}+\varepsilon\delta = f^*\varphi_{G,W}$ where $\mathrm{Aut}(G,W)$ is the affine group scheme representing $\mathcal{A}ut(G,W)$ and $\varphi_{G,W}$ is the universal $\mathrm{Aut}(G,W)$-automorphism of $G \times \mathrm{Aut}(G,W)$. Now $\mathrm{Aut}(G,W)$ being reduced there exists a morphism $h:\mathrm{Aut}(G,W) \to \mathrm{Aut}\ G$ such that $\varphi_{G,W} = h^*\varphi_G$ where φ_G is the universal Aut G-automorphism of $G \times \mathrm{Aut}\ G$. Consequently $\mathrm{id}+\varepsilon\delta = (h\circ f)^*\varphi_G$ hence $\delta \in \mathrm{Im}\,\lambda$.

Finally, let's prove that λ is injective. Let $\mathrm{Aut}^oG = \mathrm{Spec}\ R$; we may choose a finitely dimensional subspace W of $\mathcal{O}(G)$ generating $\mathcal{O}(G)$ as an F-algebra such that $\varphi_G^*(W \otimes R) = W \otimes R$ (where $\varphi_G^*:\mathcal{O}(G) \otimes R \to \mathcal{O}(G) \otimes R$ is induced by φ_G). Exactly as above, there exists a morphism $h:\mathrm{Aut}(G,W) \to \mathrm{Aut}\ G$ such that $h^*\varphi_G = \varphi_{G,W}$. There is also a natural morphism $c:\mathrm{Aut}^oG \to \mathrm{Aut}(G,W)$ defined at the level of S-points by $(\mathrm{Aut}^oG)(S) \to \mathrm{Aut}(G,W)(S)$, $s \mapsto \tilde{s}$ where $s^*\varphi_G = \tilde{s}^*\varphi_{G,W}$. Consider the affine group scheme $A = \mathrm{Aut}(G,W) \times_{\mathrm{Aut}\ G} \mathrm{Aut}^oG$; note that the projection $p_1:A \to \mathrm{Aut}(G,W)$ is a closed embedding. Now the map $A \xrightarrow{p_2} \mathrm{Aut}^oG \xrightarrow{c} \mathrm{Aut}(G,W)$ equals the map $p_1:A \to \mathrm{Aut}(G,W)$ for if $(f,f') \in A(S)$ is an S-point of A we have $h\circ f = i\circ f'$ (where $i:\mathrm{Aut}^oG \to \mathrm{Aut}\ G$ is the inclusion) so the image of (f,f') via $(c\circ p_2)(S)$ is a map $s \in \mathrm{Aut}(G,W)(S)$ such that $s^*\varphi_{G,W} = f'^*i^*\varphi_G = f^*h^*\varphi_G = f^*\varphi_{G,W}$; consequently $s=f$ by the universality of $\mathrm{Aut}(G,W)$. We get that p_2 is a closed embedding, so A is an affine algebraic group. Since the map $p_2 : A \longrightarrow \mathrm{Aut}^{\circ}G$ induces a bijection at the level of F-points this map is an isomorphism. Consequently c is a closed embedding hence $\mathrm{Aut}^{\circ}G$ is a subfunctor of Aut(G,W) hence of $\mathcal{A}ut\, G$ and injectivity of λ follows.

(1.3) Let R,U,T be the radical of G, the unipotent radical of G and a fixed maximal torus of R respectively and denote by g, r,u,t the Lie algbras of G,R,U,T.

As in [B_3] g has a natural structure of $\Delta(G)$-Lie F-algebra; it is defined as follows. $\mathcal{P}(G)$ is a $\Delta(G)$-F-vector space hence so is its dual $\mathcal{P}(G)^{\circ}$. One can check that $\mathcal{P}(G)^{\circ}$ with convolution product is a $\Delta(G)$-F-algebra, hence $\mathcal{L}(\mathcal{P}(G)^{\circ})$ will be a $\Delta(G)$-Lie F-algebra. Finally one can check that $g = \mathcal{L}(G)$ (which is defined as a subspace of $\mathcal{L}(\mathcal{P}(G)^{\circ})$ cf. [H] p.36) is $\Delta(G)$-stable hence is a $\Delta(G)$-Lie F-algebra.

Now the universal enveloping algebra $\mathcal{U}(g)$ has a natural structure of $\Delta(G)$-F-algebra induced from the tensor algebra on g. The dual $\mathcal{U}(g)^{\circ}$ becomes then a $\Delta(G)$-F-algebra. Inside $\mathcal{U}(g)^{\circ}$ lies the continuous dual $\mathcal{U}(g)'$ (cf. [H] p.228); it is defined to be the space of all functionals on $\mathcal{U}(g)$ whose kernel contains some two-sided ideal of finite codimension; one checks that $\mathcal{U}(g)'$ is a sub $\Delta(G)$-F-algebra of $\mathcal{U}(g)^{\circ}$ (for if $f \in \mathcal{U}(g)^{\circ}$ vanishes on the ideal J then δf will vanish on J^2 for all $\delta \in \Delta(G)$). Note that $\mathcal{U}(g)'$ need not be locally finite! Now inside $\mathcal{U}(g)'$ lies the algebra $\mathcal{B}(g)$ of g-nilpotent representative functions (which consists of all functionals on $\mathcal{U}(g)$ which vanish on some power of $g.\mathcal{U}(g)$). Clearly $\mathcal{B}(g)$ is a $\Delta(G)$-F-subalgebra of $\mathcal{U}(g)'$ and is locally finite (for $\mathcal{B}(g) = \cup V_n$ where V_n consists of all $f \in \mathcal{U}(g)'$ which vanish on $(g.\mathcal{U}(g))^n$), see also [B_3]).

The construction above shows that the natural embedding

$e_G : \mathcal{P}(G) \to \mathcal{U}(g)'$ (cf. [H] p.230) is a map of $\Delta(G)$-F-algebras. In particular $\Delta(G/F)$ embeds into $gl(g)$ hence has finite dimension over F. Consequently, if tr.deg.$F/k < \infty$ then $\Delta(G)$ itself has finite dimension over F.

(1.4) LEMMA. 1) r is a $\Delta(G)$-ideal of g; hence so is $[r,r]$.

2) Assume we have a map $\Delta \to \Delta(G)$ from some set Δ and H is an irreducible affine algebraic subgroup of G whose Lie algebra h is a Δ-ideal of g; then the ideal defining H in G is a Δ-ideal of $\mathcal{P}(G)$. In particular, the ideal defining R in G is a $\Delta(G)$-ideal in $\mathcal{P}(G)$.

3) There is a naturally induced restriction map $\Delta(G) \to \Delta(R)$.

Proof. Arguments are similar to those in $[B_3]$. We reproduce them for the sake of completeness.

1) By (0.3) there is an extension F_1/F of $\Delta(G)$-fields such that $g \otimes F_1$ splits so $g \otimes_F F_1 = g_o \otimes_C F_1$ where $C = F_1^{\Delta(G)}$ and $g_o = (g \otimes F_1)^{\Delta(G)}$. Let r_o be the radical of g_o. Then both $r_o \otimes_C F_1$ and $r \otimes_F F_1$ coincide with the radical of $g \otimes F_1$. Now $r = (r \otimes_F F_1) \cap g = (r_o \otimes_C F_1) \cap g$ and the latter is clearly preserved by $\Delta(G)$.

2) There is a commutative diagram

$$\begin{array}{ccc} \mathcal{P}(G) & \xrightarrow{e_G} & \mathcal{U}(g)' \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \mathcal{P}(H) & \xrightarrow[e_H]{} & \mathcal{U}(h)' \end{array}$$

We have $\ker f_1 = e_G^{-1}(\ker f_2)$ and we are done since e_G and f_2 are Δ-algebra maps.

3) follows immediately from 2).

We should emphasize that, unlike r, the Lie algberas u and t are not generally $\Delta(G)$-subalgebras of g!

(1.5) We are in a position to define our basic maps α and β. So in notations from (1.3) we have a (vector space direct sum) decomposition $r=u+t$ and let's denote by $e_1 : r \longrightarrow r$ the F-linear endomorphism of r obtained by projecting on u and by $e_2 : r \longrightarrow r$ the Lie F-algebra endomorphism obtained by projecting on t. Clearly, there is a factorisation $e_2 : r \xrightarrow{p} r/[r,r] \xrightarrow{\pi} r$ (p being the canonical projection). Note that by (1.4) $r/[r,r]$ is a $\Delta(G)$-F-vector space and p is a $\Delta(G)$-map but π need not be a $\Delta(G)$-map! Now define linear maps

$$\Delta(G) \xrightarrow{a} Z^1(r/[r,r],r) \xrightarrow{b} Z^2(u,u)$$

where r is viewed as an $r/[r,r]$ - module via the representation $ad \circ \pi : r/[r,r] \to gl(r)$. We put $a(\delta)(x) = \pi\delta x - \delta\pi x$ for all $\delta \in \Delta(G)$, $x \in r/[r,r]$ in other words $a(\delta) = \delta\pi$, where we view π as an element of the $\Delta(G)$-F-vector space $Hom(r/[r,r],r)$. Moreover, put $b(f)(x,y) = [e_2 fpx, y] - [e_2 fpy, x]$ for all $x,y \in u$ and all $f \in Z^1(r/[r,r],r)$, in other words $b(f) = \langle , \rangle_{e_2 \circ f}$ in notations of (0.5) where $(h,h_0,h_1)=(r,t,u)$. So in particular the image of b lies the derived space $\mathcal{D}(r,t,u)$. Moreover b sends $B^1(r/[r,r],r)$ into zero (because any element $f \in B^1$ can be written as $f = ad(x_0) \circ \pi$ for some fixed $x_0 \in r$, hence f vanishes on $u/[r,r]$). In particular we have induced maps to the cohomology:

$$\Delta(G) \xrightarrow{\alpha} H^1(r/[r,r],r) \xrightarrow{\beta} H^2(u,u)$$

It worths noting that our maps above are not intrinsically associated to G but depend on the choice of the maximal torus T of R. One can check (but this will not be used in what follows) that a and b behave nicely when we change the torus; more precisely if $T'=v^{-1}Tv$ (for some $v\in U$) then upon letting a' and b' be the maps corresponding to T' and upon letting $\sigma=\mathrm{Ad}(v)$ (viewed either as an element of $\mathfrak{gl}(r)$ or as an element of $\mathfrak{gl}(u)$) we have for any $\delta\in\Delta(G)$:

$$\sigma^{-1}\circ a'(\delta)-a(\delta)\in B^1(r/[r,r],r) \text{ and}$$
$$\sigma\circ b'(a'(\delta))=b(a(\delta))\circ(\sigma\times\sigma)$$

From now on (unless otherwise specified) we fix our torus T. Here is our main result on the maps α and β.

(1.6) THEOREM. 1) $\Delta(G,\mathrm{fin})$ is a Lie subspace of $\Delta(G)$ and we have an exact sequence

$$0\to\Delta(G,\mathrm{fin})\to\Delta(G)\xrightarrow{\alpha}H^1(r/[r,r],r)$$

Moreover, $\mathcal{O}(G)$ is locally finite as a $\Delta(G,\mathrm{fin})$-F-vector space.

2) If F_G is the smallest algebraically closed field of definition for G contained in F and containing k (which exists by [B_2]), then there are exact sequences:

$$0\to\Delta(G/F_G)\to\Delta(G)\xrightarrow{\beta\circ\alpha}H^2(u,u)$$
$$0\to\Delta(G/F)\to\Delta(G/F_G)\to\mathrm{Der}(F/F_G)\to 0$$

the second being split in the category of Lie spaces over F.

Before proving this theorem let's give some consequences of it

(1.7) COROLLARY. The following hold:

1) There is a complex, exact in the first two terms:

$$0 \to \mathcal{L}(\mathrm{Aut}\ G) \xrightarrow{\lambda} \mathcal{L}(\mathcal{A}ut\, G) \xrightarrow{\alpha} H^1(r/[r,r],r) \xrightarrow{\beta} H^2(u,u)$$

in particular $\dim(\mathcal{L}(\mathcal{A}ut\, G)/\mathcal{L}(\mathrm{Aut}\ G)) \leq \dim(\ker \beta)$.

2) The natural map $\Delta(G)/\Delta(G,\mathrm{fin}) \to \Delta(R)/\Delta(R,\mathrm{fin})$ is injective in particular the map $\mathcal{L}(\mathcal{A}ut\, G)/\mathcal{L}(\mathrm{Aut}\ G) \to \mathcal{L}(\mathcal{A}ut\, R)/\mathcal{L}(\mathrm{Aut}\ R)$ is injective.

3) If the radical of G is unipotent then $\Delta(G)=\Delta(G,\mathrm{fin})$, in particular λ is an isomorphism.

4) If $i:G\to G'$ is an isogeny then there is a natural injective lifting map $i^*:\Delta(G') \to \Delta(G)$ such that $(i^*)^{-1}(\Delta(G,\mathrm{fin}))=\Delta(G',\mathrm{fin})$.

5) If G has finite center then $\Delta(G)=\Delta(G,\mathrm{fin})$, in particular λ is an isomorphism.

6) If the radical of G is nilpotent then $\Delta(G)=\Delta(G/F_G)$, hence $F_G=F^{\Delta(G)}$.

7) $\dim(\Delta(G)/\Delta(G/F)) \leq \mathrm{tr.deg}(F/F_G)+\dim(\mathcal{D}(r,t,u)/\mathcal{D}(r,t,u)\cap B^2(u,u))$.

Morally, the last assertion says that $\dim(\Delta(G)/\Delta(G/F))$ is restricted by the geometry of linear subspaces of M; unfortunately this geometry is quite misterious (cf. [NR]).

(1.8) Proof of (1.7). Assertion 1) is clear from (1.2)(1.6).

Assertion 2) follows from (1.6) by looking at the commutative diagram

$$\begin{tikzcd} \Delta(G) \arrow[r, "\alpha_G"] \arrow[d] & H^1(r/[r,r],r) \\ \Delta(R) \arrow[ur, "\alpha_R"'] & \end{tikzcd}$$

Assertion 3) follows because if R is unipotent then $\pi=0$ hence a=0. To check assertion 4) note that $G \to G'$ being étale, any k-derivation δ' on $\mathcal{P}(G')$ lifts uniquely to a k-derivation δ on $\mathcal{P}(G)$ (see $[B_1]$ p.13). If $\delta' \in \Delta(G')$ then δ must belong to $\Delta(G)$ because the map $\mu\circ\delta-(\delta\otimes 1+1\otimes\delta)\circ\mu$ (respectively $\tau\circ\delta-\delta\circ\tau$, $\varepsilon\circ\delta-\delta\circ\varepsilon$) is an μ-F-derivation $\mathcal{P}(G)\to\mathcal{P}(G)\otimes\mathcal{P}(G)$ (respectively a τ- F- derivation, an ε- derivation) vanishing on $\mathcal{P}(G')$; such a derivation must vanish on the whole of $\mathcal{P}(G)$. So we get the lifting map $i^*: \Delta(G')\to\Delta(G)$ and clearly $(i^*)^{-1}(\Delta(G,\text{fin}))\subset\Delta(G',\text{fin})$. To check that $\Delta(G',\text{fin})$ is mapped into $\Delta(G,\text{fin})$ let R',U',r',u' be the corresponding objects for G'. Then we get an isogeny $R\to R'$ and an isomorphism $U\to U'$: moreover there exists a maximal torus T of R and a maximal torus T' of R' such that we have an induced isogeny $T\to T'$. Passing to Lie algebras we get induced isomorphisms $r\to r'$, $u\to u'$, $t\to t'$ and a commutative diagram

$$\begin{tikzcd} \Delta(G) \arrow[r, "\alpha"] & H^1(r\ r,r\ ,r) \\ \Delta(G') \arrow[u] \arrow[r, "\alpha'"] & H^1(r'/\ r';r'\ ,r') \arrow[u, "\wr"'] \end{tikzcd}$$

which together with (1.6) yields our claim.

To prove assertion 5) start with a preparation. Assume V is an N-dimensional Δ-F-vector space (Δ an arbitrary set). Then the coordinate algebra $\mathcal{P}(GL(V))$ of GL(V) has a natural structure of Δ-F-algebra defined by identifying $\mathcal{P}(GL(V))$ with $S(gl(V)^{o})[1/d]$ where S="symmetric algebra" and $d \in S^N(gl(V)^o)$ is "the determinant". We claim that $\mathcal{P}(GL(V))$ is locally finite; indeed $S(gl(V)^o)$ clearly is so and we are done by noting that d is a Δ-constant (to check this replace F by some Δ-field extension of it such that V splits; then V has a Δ-constant F-basis. Associated to this basis there is a Δ- constant basis X_{ij} of $gl(V)^o$; now d is a polynomial in the X_{ij}'s with Q-coefficients hence is Δ-constant). Coming back to our group G, let $Ad: G \rightarrow GL(g)$ be its adjoint representation. Using the description of Ad in [H] p.51 one cheks that $Ad^*: \mathcal{P}(GL(g)) \rightarrow \mathcal{P}(G)$ is a $\Delta(G)$-algebra map. Consequently if Z(G) is the center of G, $\mathcal{P}(G/Z(G))$ is a locally finite $\Delta(G)$-F-vector space (being identified with $\mathcal{P}(GL(g))/\ker Ad^*$). By assertion 4) $\mathcal{P}(G)$ must be locally finite as a $\{\delta\}$-F-vector space for all $\delta \in \Delta(G)$ and we are done.

To prove assertion 6) note that if R is nilpotent there is a unique maximal torus T in it and R is a direct product of U by T, so b vanishes identically which implies 6).

Assertion 7) follows immediately from (1.6) and the definition of b.

From the discussion of assertion 5) above we get the following useful:

(1.9) Remark. Let G be an irreducible affine algebraic F-group. Then the ideal defining its center is a $\Delta(G)$-ideal of $\mathcal{P}(G)$.

Proof. Indeed one checks that the ideal m defining 1_g in GL(g) is a $\Delta(G)$-ideal of $\mathcal{P}(GL(g))$ (use once again a splitting of g). Then the ideal defining the center of G is $(Ad^* m)\mathcal{P}(G)$ hence it is a $\Delta(G)$-ideal of $\mathcal{P}(G)$.

2. Proof of the Theorem

(2.1) To prove assertion 1) in Theorem (1.6) it is sufficient (and convenient) to prove that for any map $\gamma: \Delta \to \Delta(G)$ the following are equivalent

i) $\mathrm{Im}\,\gamma \subset \ker \alpha$

ii) $\mathcal{P}(G)$ is locally finite as a Δ-F-vector space

and also to prove that

iii) $\ker \alpha$ is a Lie subring of $\Delta(G)$.

We need a preparation

(2.2) Remark. Let V be a Δ-F-vector space of finite dimension and let $\delta \in \Delta$; we also denote by δ the Ω-endomorphism δ_V of V defined by δ . Then the map $GL(V) \longrightarrow gl(V)$ defined by the formula $\sigma \mapsto \sigma^{-1} \circ \delta \circ \sigma - \delta$ coincides with Kolchin's logarithmic derivative $\ell\delta$ (cf [K_1] p. 394). Indeed, we may enlarge F and assume V splits over F. Then take a Δ- constant basis of V; if the matrix of σ with respect to this basis is $S \in GL_n(F)$ then the matrix of $\sigma^{-1} \circ \delta \circ \sigma - \delta$ will be $\delta S \cdot S^{-1} \in gl_n(F)$ and our assertion is proved.

(2.3) Now let G be as in Theorem (1.6). For any $v \in U$ let $\sigma = Ad(v)$ be viewed as an automorphism of r; clearly it induces the identity on $r/[r,r]$.

Since σ belongs to the algebraic group $\mathrm{Im}(Ad: U \to GL(r))$, by Kolchin's theory [$K_1$]p.394, $\ell\delta\sigma$ will belong to the Lie algebra of this group which is $\mathrm{Im}(ad: u \longrightarrow gl(r))$, so we get that $\ell\delta\sigma = ad(x(\delta,\sigma))$ for some $x(\delta,\sigma) \in u$. Now for $\delta \in \Delta$ let's still denote by δ the k-endomorphism of r (respectively $r/[r,r]$) defined by δ . Using (2.2) we get the following equality of maps from $r/[r,r]$ to r:

$$(*) \qquad \sigma^{-1}\circ(\delta\circ\sigma\circ\pi - \sigma\circ\pi\circ\delta) = a(\delta) + (\ell\delta\sigma)\circ\pi$$

We are in a position to prove ii) $\Rightarrow$ i) in (2.1). Since $\mathcal{P}(G)$ is finitely generated as an F-algebra and locally finite as a Δ-F-linear space, one can (after enlarging F) assume that $\mathcal{P}(G)$ splits (cf. (0.3)) so $\mathcal{P}(G)=\mathcal{P}(G)^{\Delta}\otimes_{F_o}F$, $F_o=F^{\Delta}$. $\mathcal{P}(G)^{\Delta}$ is easily seen to be a finitely generated Hopf F_o-algebra; let $G_o=\mathcal{G}(\mathcal{P}(G)^{\Delta})$ and let R_o, U_o, T_o be the radical of G_o, the unipotent radical of G_o and a maximal torus of R_o. We have $R=R_o\otimes F$, $U=U_o\otimes F$, $v^{-1}Tv=$ $=T_o\otimes F$ for some $v\in U$. Consequently if $\sigma=Ad(v)\in End(r)$, then $\sigma\circ\pi : r/[r,r]\to r$ is a Δ-map so the first member of the formula (*) above vanishes which shows that $a(\delta)\in B^1(r/[r,r],r)$ and i) follows.

(2.4) To prove i) $\Rightarrow$ ii) in (2.1) note that if i) holds then for any $\delta\in\Delta$, there exists $x(\delta)\in r$ such that $a(\delta)=ad(x(\delta))\circ\pi$. Since $ad(z)\circ\pi=0$ for all $z\in t$ we may assume $x(\delta)\in u$ for all $\delta\in\Delta$. By Kolchin's surjectivity theorem of the logarithmic derivative (cf. its version in $[B_1]$ p.51) we may find (after enlarging the Δ- field F once again) a point $\sigma=Ad(v)\in GL(r)$ with $v\in U$ such that $\ell\delta\sigma=-ad(x(\delta))$ for all $\delta\in\Delta$. Then the second member of formula (*) vanishes which implies that $\sigma\circ\pi$ is a Δ-map. Consequently the kernel $u/[r,r]$ and the image σt of this map are Δ-subalgebras of r. Since the projection $p:r\to r/[r,r]$ is a Δ-map it follows that u itself is a Δ-ideal of r. We also may assume that g splits over F and take a decomposition $g^{\Delta}=r_o+s_o$ where r_o is the radical of g^{Δ} and s_o is a complementary semisimple Lie F_o-subalgebra of g^{Δ} ($F_o=F^{\Delta}$). Clearly, $r=$ $=r_o\otimes F$; letting $s=s_o\otimes F$ and $t'=\sigma t$ we have a decomposition

Med 24828

$g=u+t'+s$ where u,t',s are all Δ-subalgebras of g and all three are algebraic Lie subalgebras: $u=\mathcal{L}(U)$, $t'=\mathcal{L}(T')$ where $T'=v^{-1}Tv$ and $s=\mathcal{L}(S)$ (for algebraicity of s, see [H]p.112). By (1.4) the ideals defining U,T',S in G are Δ-ideals in $\mathcal{P}(G)$. Consequently, the (surjective) map of algebraic varieties $U \times T' \times S \to G$ defined by multiplication induces an (injective) Δ-F-algebra map $\mathcal{P}(G) \to \mathcal{P}(U) \otimes \mathcal{P}(T') \otimes \mathcal{P}(S)$. So to prove that $\mathcal{P}(G)$ is locally finite, it is sufficient to check that each of $\mathcal{P}(U)$, $\mathcal{P}(T')$, $\mathcal{P}(S)$ is locally finite. Let's consider these cases separately below.

To check that $\mathcal{P}(U)$ is locally finite, use the fact [H], p.232 that the image of the $\Delta(U)$-map $e_U: \mathcal{P}(U) \to \mathcal{U}(u)'$ is contained in the algebra of u-nilpotent representative functions $\mathcal{B}(u)$ which by (1.3) is locally finite and we are done.

To check that $\mathcal{P}(T')$ is locally finite note that $\mathcal{P}(T')$ is Hopf algebra isomorphic to $F[\Sigma]$, the group F-algebra of some group Σ. For each $\delta \in \Delta$ let δ^* be the unique k-derivation on $F[\Sigma]$ which kills Σ and restricted to F acts like δ. Then $id+\varepsilon(\delta-\delta^*): F_\varepsilon[\Sigma] \to F_\varepsilon[\Sigma]$ is a Hopf F_ε -algebra isomorphism. In particular it takes Σ (which is the group of group-like elements of $F_\varepsilon[\Sigma]$, cf [Sw]) onto itself. Since after reduction modulo ε we get the identity it follows that $id+(\delta-\delta^*)=id$ hence $\delta=\delta^*$, which shows that $F[\Sigma]$ is locally finite.

To check that $\mathcal{P}(S)$ is locally finite, one can proceed by a similar argument as above invoking the rather difficult (yet now standard) theory of reductive group schemes over non-reduced base schemes such as F_ε (cf. [DG],[D]), more precisely the representability of $\mathcal{A}ut\, S$ plus the fact that S is defined over $\mathbb{Q}$. But one can also give a quite elementary argument which runs as follows: it was shown in [B_3] (using only the vanishing of the H^1 for finitely dimensional s-modules) that $\mathcal{U}(s)'$ is locally finite for any semisimple Δ-Lie F-algebra s; by (1.3) $\mathcal{P}(S)$ embeds into $\mathcal{U}(s)'$ as a Δ-subspace so must be locally finite.

This closes the proof of i)$\Leftrightarrow$ii) in (2.1).

(2.5) To check assertion iii) in (2.1) let $\delta_1,\delta_2 \in \Delta(G)$ such that $\alpha(\delta_1)=\alpha(\delta_2)=0$. Then $a(\delta_i)=ad(x_i)\circ\pi$ for some $x_i \in r$, $i=1,2$. One can easily check that

$$a([\delta_1,\delta_2])=ad([x_1,x_2]+\delta_2 x_1-\delta_1 x_2)$$

and assertion 1 in Theorem (1.6) is proved.

(2.6) Let's prove assertion 2) in Theorem (1.6). The second exact sequence is clear; indeed if $G=G_o\otimes_{F_G}F$ (G_o an affine algebraic F_G-group) and if for any $d\in Der(F/F_G)$ we define $\delta=1\otimes d \in \Delta(G_o\otimes F)\simeq\Delta(G)$, then $d\mapsto\delta$ is a section for the map $\Delta(G/F_G)\to Der(F/F_G)$. For the first exact sequence we must prove that the following assertions are equivalent (for any $\delta\in\Delta(G)$):

i) $\beta(\alpha(\delta))=0$

ii) δ vanishes on F_G.

Now ii) is equivalent to :

iii) F^{δ} is a field of definition for G.

By [B_2] this is equivalent to :

iv) F^{δ} is a field of definition for u.

Let's prove i)$\Leftrightarrow$iv). If i) holds then there exists an F-linear map $\theta: u\to u$ such that

(*) $b(a(\delta))(x,y)=\theta[x,y]-[\theta x,y]-[x,\theta y]$ for all $x,y\in u$

Now letting δ denote also the k-endomorphisms of r and $r/[r,r]$ induced by δ we get that

$$e_2\circ a(\delta)\circ p=e_2\circ(\delta\circ\pi-\pi\circ\delta)\circ p=e_2\circ\delta\circ e_2-e_2\circ\pi\circ\delta\circ p=$$
$$=e_2\circ\delta\circ e_2-e_2\circ\pi\circ p\circ\delta=e_2\circ\delta\circ e_2-e_2\circ\delta$$

In particular if $x\in u$ then $e_2\, a(\delta)px=-e_2\delta x$ so we get that

$$(**)\qquad b(a(\delta))(x,y)=[e_2\delta y,x]-[e_2\delta x,y]$$

Projecting the equality $\delta[x,y]=[\delta x,y]+[x,\delta y]$ on u and using (**) we get

$$(***)\qquad e_1\delta[x,y]=[e_1\delta x,y]+[x,e_1\delta y]-\, b(a(\delta))(x,y)$$

for all $x,y\in u$. From (*) and (***) we get that $\tilde{\delta}:=e_1\circ\delta+\theta$ is a k-derivation on u and one checks immediately that $\tilde{\delta}(\lambda x)=(\delta\lambda)x+\lambda\delta x$ for all $\lambda\in F$, $x\in u$. By $[B_1]$ p.86, K^{δ} is a field of definition for u, hence i) $\Rightarrow$ ii) is proved.

Conversely, if ii) holds, then writing $G=G_1\otimes_{F_1}F$ ($F_1=F^{\delta}$, G_1 an affine algebraic F_1-group) we may consider $\partial(\delta)\in Der(F/F_1)$ and lift it to an F_1-derivation $\delta^*:=1\otimes\partial(\delta)$ on $\mathcal{O}(G)$. Clearly δ^* preserves U, hence preserves u. Then view $\theta=\delta^*-e_1\circ\delta$ as a map from u to u; clearly, it is F-linear. Substracting the equality (***) from the equality $\delta^*[x,y]=[\delta^*x.y]+[x,\delta^*y]$ we get that formula (*) holds for our θ just defined, hence $b(a(\delta))$ is a coboundary and we are done.

3. Application to Δ-algebraic groups

We use standard terminology of differential algebra from $[C_1]$ $[K_1]$; so $\Delta,\mathcal{U},\mathcal{K},\mathcal{F},\mathcal{C}$ have the meaning explained in (0.4). Recall from $[C_1]$ that a linear Δ-$\mathcal{F}$- group means simply a Δ-$\mathcal{F}$- closed subgroup of some $GL_n(\mathcal{U})$.

(3.1) Definitions

1) By an f-group we will mean an irreducible linear Δ-$\mathcal{F}$-group Γ such that tr.deg. $\mathcal{F}\langle\Gamma\rangle/\mathcal{F} < \infty$.

2) An f-group $\Gamma \subset GL_n(\mathcal{U})$ is called split (cf [B_3]) if $\Gamma = \Gamma^* \cap GL_n(\mathcal{K})$ where Γ^* is some $\mathcal{C}$-closed subgroup of $GL_n(\mathcal{U})$.

3) An f-group $\Gamma \subset GL_n(\mathcal{U})$ is called semisplit if $\Gamma = \Gamma^* \cap X$ where Γ^* is a $\mathcal{C}$-closed subgroup of $GL_n(\mathcal{U})$ and X is a Δ-$\mathcal{F}$-closed subset of $GL_n(\mathcal{U})$ defined by equations of the form $0 = \delta_i y_{jk} - P_{ijk}$ with $P_{ijk} \in \mathcal{F}[(y_{jk})]$, $1 \le i \le m$, $1 \le j,k \le n$. Clearly "split" implies "semisplit" (take $P_{ijk}=0$).

4) An f-group is called splitable (respectively semisplitable over a Δ- extension $\mathcal{F}_1$ of $\mathcal{F}$ if it is Δ-$\mathcal{F}_1$-isomorphic to a split (respectively semisplit) f-group. An f-group will be called splitable (respectively semisplitable) if it is so over $\mathcal{F}_1 = \mathcal{U}$.

(3.2) In spite of Cassidy's deep results in [C_1][C_2][C_3] a satisfactory picture of f-groups is still missing. What we intend to do here is initiate a study of f-groups based on the concepts introduced in (3.1)(cf. also [B_3])and using our theory developed in the preceeding sections.

By [B_3] $\Gamma = \{yy'' - (y')^2 = 0\} \subset GL_1(\mathcal{U}) = \mathcal{U}^*$ is an example of a semisplitable f-group which is not splitable.

The relation between f-groups and the theory from section 1 is given by the following.

(3.3) THEOREM. [B_3] . Let Γ be an f-group. Then the Δ-coordinate algebra $\mathcal{F}\{\Gamma\}$ is finitely generated as a non-differential $\mathcal{F}$ - algebra.

Thus to any f-group Γ one can associate an affine algebraic $\mathcal{F}$-group $G=\mathcal{G}(\mathcal{F}\{\Gamma\})$, together with m commuting derivations $\delta_1,\dots,\delta_m \in \Delta(G)$ acting on $\mathcal{P}(G)=\mathcal{F}\{\Gamma\}$. By (1.5) above (with $F=\mathcal{F}, k=\mathcal{C}$) we may associate to Γ cohomology classes $\alpha_1,\dots,\alpha_m \in H^1(r/[r,r],r)$, $\alpha_i=\alpha(\delta_i)$, $1\le i\le m$ and cohomology classes $\beta_1,\dots,\beta_m \in H^2(u,u)$, $\beta_i=\beta(\alpha_i)$, $1\le i\le m$ (where r and u have the same meaning as in (1.5)).

(3.4) Since rational maps between algebraic groups commuting with multiplications must be everywhere defined one gets that any surjective homomorphism $\Gamma\to\Gamma'$ of f-groups induces a natural surjective homomorphism between the corresponding affine algebraic groups $G\to G'$: clearly if $\Gamma\to\Gamma'$ is a Δ - $\mathcal{F}$ - isomorphism (respectively a Δ - $\mathcal{F}$ - isogeny) then $G\to G'$ is an $\mathcal{F}$- isomorphism (respectively an $\mathcal{F}$- isogeny). Here $\Gamma\to\Gamma'$ is called isogeny if (it is surjective and) $[\mathcal{F}\langle\Gamma\rangle : \mathcal{F}\langle\Gamma'\rangle] < \infty$.

Recall that the radical $R(\Gamma)$ of a linear Δ - $\mathcal{F}$ - group Γ is the unique maximal element in the set of all Δ-closed normal irreducible solvable subgroups of G; it is Δ - $\mathcal{F}$ - closed.

(3.5) LEMMA. Let Γ be an f-group, let $R(\Gamma)$ be its radical and $Z^o(\Gamma)$ be the connected component of the center of Γ. Moreover, let $G=\mathcal{G}(\mathcal{F}\{\Gamma\})$, let $R(G)$ be the radical of G and $Z^o(G)$ the connected component of the center of G. Then

1) $R(G)=\mathcal{G}(\mathcal{F}\{R(\Gamma)\})$ as subgroups of G.
2) $Z^o(G)=\mathcal{G}(\mathcal{F}\{Z^o(\Gamma)\})$ as subgroups of G.

Proof. 1) was proved in $[B_3]$. 2) can be proved similarily by using (1.9).

(3.6) In what follows a linear Δ-$\mathcal{F}$-group is called unipotent [C2] if it consists of unipotent matrices and will be called nilpotent if it so as an abstract group. One can easily check that an f-group Γ is unipotent (respectively nilpotent) if and only if $G=\mathcal{G}(\mathcal{F}\{\Gamma\})$ is so (see also [B3]).

Here is a consequence of assertion 1) in Theorem (1.6).

(3.7) COROLLARY. For an f-group Γ the following are equivalent:

1) Γ is spitable
2) Γ is splitable over some Picard-Vessiot extension of $\mathcal{F}$
3) $\alpha_1=\ldots=\alpha_m=0$
4) $\mathcal{F}\{\Gamma\}$ is locally finite as a Δ-$\mathcal{F}$-vector space.

Proof. Note that 3)$\Leftrightarrow$4) is given by (1.6). The rest of the implications were proved in [B3].

Here is what assertion 2) in Theorem (1.6) gives:

(3.8) COROLLARY. For an f-group Γ the following hold:
1) Γ is semisplitable
2) Γ is semisplitable over $\mathcal{F}$
3) $\beta_1=\ldots=\beta_m=0$
4) $\mathcal{C}$ is a field of definition for $G=\mathcal{G}(\mathcal{F}\{\Gamma\})$.

Proof. 3)$\Leftrightarrow$4) is given by (1.6). The rest of the implications are easy.

Putting together (1.7) and (3.5)-(3.8) we get

(3.9) COROLLARY. For an f-group Γ the following hold:

1) Γ is splitable (respectively semisplitable) iff its radical is so.

2) If Γ has a unipotent (respectively nilpotent) radical then Γ is splitable (respectively semisplitable), cf. also [B_3].

3) If there is an isogeny $\Gamma \to \Gamma'$ then Γ is splitable (respectively semisplitable) iff Γ' is so.

4) If Γ has a finite center, then it is splitable.

(3.10) We close our paper by discussing the classification problem for f-groups.

The set S of Δ-isomorphism classes of splitable f-groups clearly identifies with the set $H_{\mathcal{C}}$ of $\mathcal{C}$-isomorphism classes of affine algebraic $\mathcal{C}$-groups.

Let's examine the set SS of Δ-isomorphism classes of semisplitable f-groups. There is a well defined surjective map $SS \to H_{\mathcal{C}}$, $[\Gamma] \to [G_o]$ defined as follows: if Γ is a semisplitable f-group, take $G = \mathcal{G}(\mathcal{F}\{\Gamma\})$ and let G_o be any affine algebraic $\mathcal{C}$-group such that G is $\mathcal{F}$-isomorphic to $G_o \otimes_{\mathcal{C}} \mathcal{F}$. So it is natural to try to describe the fibres of the above map; i.e. to describe, for a given G_o, the set $SS(G_o)$ of all Δ-isomorphism classes of semisplitable f-groups "associated" to G_o. $SS(G_o)$ has the following description.

Let $\Delta(G_o \otimes \mathcal{F}/\mathcal{F}) = \Delta(G_o/\mathcal{C}) \otimes \mathcal{F}$ (respectively $\Delta(G_o \otimes \mathcal{U}/\mathcal{U}) = \Delta(G_o/\mathcal{C}) \otimes \mathcal{U}$) be viewed as a Δ-Lie $\mathcal{F}$-algebra (respectively as a Δ-Lie $\mathcal{U}$-algebra) by letting Δ act trivially on $\Delta(G_o/\mathcal{C})$. Let $V_{\mathcal{F}}$ (respectively $V_{\mathcal{U}}$) be the set of m-uples $\theta_1, \dots, \theta_m$ with $\theta_i \in \Delta(G_o \otimes \mathcal{F}/\mathcal{F})$ (respectively with $\theta_i \in \Delta(G_o \otimes \mathcal{U}/\mathcal{U})$) satisfying the integrability conditions:

$$\delta_i\theta_j - \delta_j\theta_i + [\theta_i, \theta_j] = 0 \quad \text{for all } i,j=1,\dots,m$$

On $V_{\mathcal{U}}$ acts the group $A(\mathcal{U})$ of $\mathcal{U}$-points of the group $\mathcal{C}$-scheme $A=\mathrm{Aut}\, G_o$ by the "Loewy-type" formula [C1]:

$$(\sigma,(\theta_1,\dots,\theta_m)) \mapsto (\sigma^{-1}\circ\theta_1\circ\sigma+\ell\delta_1\sigma,\dots,\sigma^{-1}\circ\theta_m\circ\sigma+\ell\delta_m\sigma)$$

where $\ell\delta_i\sigma=\sigma^{-1}\circ\delta_i^*\circ\sigma-\delta_i^*\in\Delta(G_o\otimes\mathcal{U}/\mathcal{U})$, δ_i^* being the lifting of δ_i from $\mathcal{U}$ to $\mathcal{P}(G_o)\otimes\mathcal{U}$. We say that $\theta,\theta'\in V_{\mathcal{F}}$ are equivalent if, as elements of $V_{\mathcal{U}}$ are in the same $A(\mathcal{U})$-orbit. Then it is easy to see that $SS(G_o)$ identifies with $V_{\mathcal{F}}/\sim$ where $\sim$ is the above equivalence relation: for each $\theta=(\theta_1,\dots,\theta_m)$ in $V_{\mathcal{F}}$ we may associate a semisplitable f-group Γ by letting $\mathcal{P}(G_o)\otimes\mathcal{F}$ be viewed as a Δ-$\mathcal{F}$-algebra with derivations $\delta_1^*+\theta_1,\dots,\delta_m^*+\theta_m$ and putting $\Gamma=\mathrm{Hom}_{\Delta\text{-}\mathcal{F}\text{-alg}}(\mathcal{P}(G_o)\otimes\mathcal{F},\mathcal{U})$. Note that if $\alpha:\Delta(G_o\otimes\mathcal{F})\to H^1(r/[r,r],r)$, then by (1.6) $\alpha(\delta_i^*)=0$ so $\alpha(\delta_i^*+\theta_i)=\alpha(\theta_i)$. Consequently $(\theta_1,\dots,\theta_m)\in V_{\mathcal{F}}$ is equivalent to $(0,\dots,0)$ if and only if $\alpha(\theta_1)=\dots=\alpha(\theta_m)=0$. It would be interesting to have a cohomological test for equivalence of elements in $V_{\mathcal{F}}$ (not only for equivalence to $(0,\dots,0)$).

4. Final comments and questions

(4.1) We do not know any example of an f-group which is not semisplitable (equivalently, by our theory, we do not know examples of affine algebraic F-groups G for which the map $\beta\circ\alpha:\Delta(G)\to H^2(u,u)$ is non-zero, in other words for which $\Delta(G/F_G)\neq\Delta(G)$). Nevertheless we are tempted to believe that such examples exist.

(4.2) It would be interesting to generalize the present theory to the case of non-linear algebraic groups G. Here $\mathcal{A}ut\, G$ restricted to reduced F-schemes is still representable by [B_4] so we still have a map $\lambda : \mathcal{L}(\mathrm{Aut}\ G) \to \mathcal{L}(\mathcal{A}ut\ G)$ and may ask for its kernel and image. One of the first questions which arrise is the following: given such a group G and given an infinitesimal automorphism of it $\theta \in \mathcal{L}(\mathcal{A}ut\ G)$, does θ preserve the maixmal affine irreducible subgroup of G?

(4.3) It would be interesting to characterize those algebraic F-groups G for which $\mathcal{A}ut\, G$ is representable on the category of all F-schemes (cf. [DG] [D] [MO]).

References

[BS] A.Borel, J.P.Serre, Théorèmes de finitude en cohomologie galoisienne, Comment.Math.Helvetici 39(1964), 111-164.

[B_1] A.Buium, Differential Function Fields and Moduli of Algebraic Varieties, Lecture Notes in Math.1226, Springer 1986.

[B_2] A.Buium, Birational moduli and nonabelian cohomology, Compositio Math., to appear.

[B_3] A.Buium, Splitting differential algebraic groups, Preprint INCREST 19/1988

[B_4] A.Buium, The automorphism group of a non-linear algebraic group, Preprint INCREST 19/1988.

[C_1] P.Cassidy, Differential algebraic groups, Amer.J.Math. 94(1972), 891-954.

[C_2] P.Cassidy, The classification of the semisimple differential algebraic groups and the linear semisimple differential algebraic Lie algebras, Preprint 1987.

[C_3] P.Cassidy, Unipotent differential algebraic groups, in: Contributions to Algebra, Academic Pres, New York 1977.

[D] M.Demazure, Schémas en groupes réductifs, Bull. Soc. Math. France, 93(1965), 369-413.

[DG] M.Demazure, A.Grothendieck, Schémas en groupes I, Lecture Notes in Math.151, Springer 1970.

[H] G.Hochschild , Basic theory of Algebraic Groups and Lie Algebras, Springer 1981.

[K_1] E.Kolchin, Differential Algebra and Algebraic Groups, Academic Press, New York 1973.

[K_2] E.Kolchin, Differential Algebraic Groups, Academic Press New York, 1985.

[MO] H.Matsumura, F.Oort, Representability of group functors and automorphisms of algebraic schemes, Invent.Math.4(1967),1-25.

[NW] W.Nichols, B.Weisfeiler, Differential formal groups of J.F.Ritt, Amer.J.Math.104(5)(1982), 943-1005.

[NR] A.Nijenhuis, R.W.Richardson, Cohomology and deformations in graded Lie algebras, Bull.A.M.S.72(1966),1,1-29.

[Sw] M.E.Sweedler, Hopf Algebras, Benjamin, New York 1969.

A. Buium

Department of Mathematics
Bd.Păcii 220
79622 Bucharest, Romania.