INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

A NOTE ON THE CLASS $A_{1,\aleph_0}$

by B. PRUNARU

PREPRINT SERIES IN MATHEMATICS
No. 29/1988

BUCURESTI

A NOTE ON THE CLASS $A_{1,\aleph_0}$

by B. PRUNARU

June 1988

*) *Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 Bucharest, Romania.*

A NOTE ON THE CLASS $A_{1,\aleph_0}$.

B. Prunaru

Abstract. The solvability of certain systems of simultaneous equations in the predual of a dual operator algebra is studied. The main result is a geometric criterion for membership in the class $A_{1,\aleph_0}$ which improves a similar one from [3]. The proof is based on the techniques introduced in [3] and [4].

Let H denote a separable, infinite dimensional complex Hilbert space and let $L(H)$ denote the algebra of all bounded liner operators on H. A dual algebra is by definition a weak* closed unital subalgebra of $L(H)$; the paper [1] contains an excellent account of the theory of dual algebras and a comprehensive bibliography until 1985.

In this note we are concerned with several classes of contractions appearing in the theory of dual algebras. To be more explicity, recall that if A is a dual algebra and m,n are cardinal numbers, $1 \leq m,n \leq \aleph_0$, then A is said to have property $(A_{m,n})$ if each system of simultaneous equations

$$[L_{ij}] = [x_i \otimes y_j] \quad , \; 0 \leq i < m, 0 \leq j < n$$

in the predual Q_A of A has a solution $\{x_i;\ 0 \leq i < m\}$, $\{y_j;\ 0 \leq j < n\}$, where x_i and y_j are vectors from H.
(Here $[x \otimes y]$ denotes the class in Q_A of the rank-one operator defined by $(x \otimes y)(z) = (z, y)x$, $z \in H$).

If $\rho > 0$ then A has property $A_{m,n}(\rho)$ if for each $s > \rho$, vectors x_i and y_j can be chosen to satisfy the above equations and moreover

$$\|x_i\| < (s \sum_{0 \le j < n} \|[L_{ij}]\|)^{1/2}, \quad 0 \le i < m$$

and

$$\|y_j\| < (s \sum_{0 \le i < m} \|[L_{ij}]\|)^{1/2}, \quad 0 \le j < n$$

If $T \in L(H)$ then A_T denotes the dual algebra generated by T in $L(H)$ and Q_T denotes the predual Q_{A_T} of A_T. As usually, $A = A(H)$ denotes the class of all absolutely continuous contractions (i.e. for which the unitary summand is absolutely continuous) for which the Sz.-Nagy-Foiaş functional calculus is an isometry. If m and n are cardinal numbers, $1 \le m, n \le \aleph_0$, then

$A_{m,n} = \{T \in A;\ A_T$ has property $(A_{m,n})\}$ and similarly for

$A_{m,n}(\rho)$.

If m=n, then one usually denotes $A_n = A_{n,n}$ and $A_n(\rho) = A_{n,n}(\rho)$.

In [3], B.Chevreau and C.Pearcy have given a certain sufficient condition for membership in the class $A_1(\rho)$ (cf. [3], Theorem 4.4).

The purpose of this note is to show that this condition is sufficiently strong to ensure the membership in the class $A_{1,\aleph_0}(\rho)$. Before giving the main result we recall some notions introduced in [3].

Let $A \subset L(H)$ be a dual algebra and let $\theta \in [0,1)$. Then $E_\theta^r(A)$ denotes the set of all those $[L]$ in Q_A such that there exist sequences $\{x_n\}$ and $\{y_n\}$ in the unit ball of H satisfying:

a) $\overline{\lim}\|[L] - [x_n \otimes y_n]\| \leq \theta$

b) $\lim \|[x_n \otimes z]\| = 0 \quad \forall z \in H$

and

c) $\{y_n\}$ converges weakly to zero.

If $0 \leq \theta < \gamma$ then a dual algebra A is said to have property $E^r_{\theta,\gamma}$ if the closed absolutely convex hull of the set $E^r_{\theta}(A)$ contains the closed ball in Q_A centered at 0 with radius γ.

Let also recall that if $T \in A(H)$ and if $B \in L(K)$ denotes the minimal coisometric extension of T, then $B \in A(K)$ and $B = S^* \oplus R$, where $S \in L(P)$ is a unilateral shift and $R \in L(R)$ is an absolutely continuous unitary operator. Let also denote by Q and A the orthogonal projections of K onto P and R, respectively. It follows from [3, Proposition 3.10] that if $R \neq 0$, then R contains a reducing subspace R_o for R so that $R_o = R|R_o$ is unitarily equivalent with the multiplication operator $M_{e^{it}}$ on $L^2(\sigma(R))$ and such that the subspace R_o^+ of R_o corresponding to the closure of polynomials in $L^2(\sigma(R))$ is contained in $\overline{AH}$.

If $T \in A(H)$, then there exists a canonical isometry $\varphi : Q_T \to Q_B$ such that

$$(f(T), [L]) = (f(B), \varphi([L])) \quad , [L] \in Q_T$$

It follows that $\varphi([x \otimes y]_T) = [x \otimes y]_B$, for all x and y in H and since $BH \subset H$, we also have $[x \otimes z]_B = [x \otimes P_H z]_B$, $\forall\, x \in H$ and $\forall\, z \in K$.

The main result of the paper is the following

Theorem 1

Suppose $T \in A(H)$ and for some $0 \leq \theta < \gamma$, A_T has property $E^r_{\theta,\gamma}$. Then $T\in A_{1,\aleph_0}(\rho)$, for some $\rho>0$.

After this paper was completed, the author learned that B. Chevreau, G. Exner and C. Pearcy have also proved Theorem 1 seemingly with different methods. Moreover, they showed that all c.n.u., contractions T such that A_T has property $E^r_{\theta,\gamma}$ are reflexive. Their results were announced in [2].

Let A_o denote the orthogonal projection of K onto R_o and let $z \to \{z\}$ denote the isomorphism from R_o onto $L^2(\sigma(R))$. The following lemma is proved in [3].

<u>Lemma 1.</u> ([3 , Theorem 3.11]). Suppose $T\in A(H)$ and for some $0<\theta<\gamma$. A_T has property $E^r_{\theta,\gamma}$. Suppose also that $[L]\in Q_B$, $0<\rho<1$, $\varepsilon>0$, $\delta>0$, $a\in H$, $w\in P$, $b\in R_o$, $\{d_s\}_{s=1}^{t}\subset K$ and $\{z_\ell\}_{\ell=1}^{r}\subset P$ are given so that

$$\|[L]_B - [a\otimes(w+b)]_B\| < \delta$$

Then there exist $a'\in H$, $u\in H$, $w'\in P$, $b'\in R_o$ such that

$$\|[L]_B - [a'\otimes(w'+b')]_B\| < \frac{\theta}{\gamma}\delta$$
$$\|a'-a\| < 3\left(\frac{\delta}{\gamma}\right)^{1/2}$$
$$\|w'-w\| < \left(\frac{\delta}{\gamma}\right)^{1/2}$$
$$\|b'\| < \frac{1}{\rho}\left\{\|b\| + \left(\frac{\delta}{\gamma}\right)^{1/2}\right\}$$
$$|\{A_o a'\}(e^{it})| \geq \rho\,|\{A_o(a+u)(e^{it})\}(e^{it})| \quad , e^{it}\in\mathbb{T}$$
$$\|[u\otimes d_s]\| < \varepsilon \qquad 1\leq s\leq t$$
$$\|[(a'-a)\otimes z_\ell]\| < \varepsilon,\ 1\leq \ell\leq r.$$

We are now prepared to prove the main result. Its proof follows the main ideas from [4 , Lemma 5] and [3, Theorem 4.7].

Proof of Theorem 1. Let $\{[L_j]\}_{j=1}^{\infty} \subset Q_T$, and let also $[L'_j]_B = \varphi([L_j]_T)$ for $j \geq 1$.

Let $\delta_j > 0$ such that $\sum \delta_j^{1/2}$ is finite. Without loss of generality, we may assume that $\|[L'_j]\| < \delta_j$ for each j.

Let us denote $\varepsilon_{jk} = \delta_j \left(\frac{\theta}{\gamma}\right)^k$ for all $j \geq 1$ and $k \geq 0$.

Let $\{s_n\}$ be a sequence of positive numbers strictly decreasing to $\frac{1}{2}$ such that $s_1 = 1$ and let $\rho_n = \frac{s_{n+1}}{s_n}$, $n \geq 1$.

Let $B: N \times N \to N$ be a bijection such that $j \leq j'$ and $k \leq k'$ implies

$$B(j,\ k) \leq B(j',\ k').$$

Let $w_{j,0} = 0$ in P and $b^n_{j,0} = 0$ in R_o, for all $j \geq 1$ and all $n \geq 1$.

We shall construct, by induction (on the range of B) sequences $\{x_n\} \subset H$, $\{w_{j,k}\}_{j,k \geq 1}$ in P and for $n \geq 1$, finite sequences $\{b^n_{j,k}\}_{B(j,k) \leq n}$ in R_o such that:

1) $\|[L'_j]_B - [x_n \otimes (w_{j,k} + b^n_{j,k})]_B\| < \varepsilon_{j,k}$; $B(j,k) \leq n$

2) $\|x_n - x_{n-1}\| < 3\varepsilon^{1/2}_{j,k-1}$, for $n = B(j,k)$

3) $\|w_{j,k} - w_{j,k-1}\| < \varepsilon^{1/2}_{j,k-1}$ $(\forall)\ j,k \geq 1$

4) $\|b^k_{j,k}\| < \frac{1}{\rho_n} \|b^{n-1}_{j,k}\|$ if $n > B(j,k)$

and

5) $\|b^n_{j,k}\| < \frac{1}{\rho}\{\|b^{n-1}_{j,k-1}\| + \varepsilon^{1/2}_{j,k-1}\}$ if $n = B(j,k)$.

For $n=1=B(1,1)$, we apply Lemma 1 , with $L=[L_1']$, $\delta=\delta_1$, $\rho=\rho_1$, $a=0$, $b=0$, $w=0$, $d=0$, $z=0$, to find $x_1\in H$, $w_{11}\in P$, $b_{11}^1\in R_o$ so that

$$\|[L_1'] - [x_1 \otimes (w_{11}+b_{11})]\| < \varepsilon_{11}$$

$$\|x_1\| < 3\delta_1^{1/2}$$

$$\|w_{11}\| < \delta_1^{1/2}$$

and

$$\|b_{11}^1\| < \frac{\lambda}{\rho_1}\delta_1^{1/2}$$

Suppose now that vectors $\{x_1,\dots,x_n\}$ in H , $\{w_{j,k}\}_{B(j,k)\leq n}$ in P and $\{b_{jk}^n\}_{B(j,\ k)\leq n}$ in R_o have been chosen so that 1) - 5) are satisfied.

Let $n+1=B(p,\ q)$.

Apply Lemma 1 with $[L]=[L_p']$,

$a=x_n$, $w=w_{p,q-1}$, $b=b_{p,q-1}^n$, $\rho=\rho_{n+1}$, $\delta=\varepsilon_{p,q-1}$,

$$\{d_s\}=\{b_{jk}^n\}_{B(j,k)\leq n}$$

$$\{z_\ell\}=\{w_{j,k}\}_{B(j,k)\leq n}$$

and $\varepsilon>0$ sufficiently small (to be determined later) to obtain

$x_{n+1}\in H$, $w_{p,q}\in P$, $b_{p,q}^{n+1}\in R_o$ and $u_{n+1}\in H$ such that

$$\|[L_p']_B - [x_{n+1} \otimes (w_{p,q} + b_{p,q}^{n+1})]_B\| < \varepsilon_{p,q}$$

$$\|x_{n+1}-x_n\| < 3\varepsilon_{p,q-1}^{1/2}$$

$$\|w_{p,q} - w_{p,q-1}\| < \varepsilon_{p,q-1}^{1/2}$$

$$\| b^{n+1}_{p,q} \| < \frac{1}{\rho_{n+1}} \{ \| b^{n}_{p,q-1} \| + \varepsilon^{1/2}_{p,q-1} \}$$

$$| \{ A_o x_{n+1} \} (e^{it}) | > \rho_{n+1} | \{ A_o (x_n + u_{n+1}) (e^{it}) \} | \qquad e^{it} \in T.$$

$$\| [(x_{n+1} - x_n) \otimes w_{jk}] \| < \varepsilon \qquad \text{for } B(j,k) \leq n$$

and .

$$\| [u_{n+1} \otimes b^{n}_{j,k}] \| < \varepsilon \qquad \text{for } B(j,k) \leq n$$

Let us define for each (j, k), with $B(j,k) \leq n$

$$\overline{\{ b^{n+1}_{j,k} \}} (e^{it}) = \frac{\{ A_o (x_n + u_{n+1}) \} (e^{it})}{\{ A_o (x_{n+1}) \} (e^{it})} \cdot b^{n}_{j,k} (e^{it})$$

if $\{ A_o (x_{n+1}) (e^{it}) \} \neq 0$

and

$$\overline{\{ b^{n+1}_{j,k} \}} (e^{it}) = 0$$

if $\{ A_o (x_{n+1}) (e^{it}) \} = 0$

It follows that

$$b^{n+1}_{j,k} \in R_o$$

$$\| b^{n+1}_{j,k} \| < \frac{1}{\rho_{n+1}} \| b^{n}_{j,k} \|$$

and

$$[x_{n+1} \otimes b^{n+1}_{j,k}]_B = [(x_n + u_{n+1}) \otimes b^n_{j,k}]_B$$

for all (j,k) such that

$$B(j,\ k) \leq n$$

For $\varepsilon > 0$ sufficiently small, we have

$$\| [L_j]_B - [x_{n+1} \otimes (w_{jk} + b^{n+1}_{jk})]_B \| < \varepsilon_{jk}$$

if $B(j,k) \leq n+1$.

Therefore, relations 1)-5) are fulfilled for $n+1$. It also follows from 2) and 3) that $\{x_n\}$ and $\{w_{jk}\}_{k=1}^{\infty}$ are Cauchy sequences, for all $j \geq 1$ and from 4) and 5) that the sequences $\{b_{jk}\}_{k=1}^{\infty}$ are bounded for all $j \geq 1$, where $b_{jk} = b_{jk}^{B(j,k)}$ for all $j,k \geq 1$. Without loss of generality, we may suppose that $\{b_{jk}\}_{k=1}^{\infty}$ converges weakly to some $b_j \in \mathcal{R}_0$. Let also denote $x = \lim x_n$ and $w_j = \lim_k w_{jk}$, for each $j \geq 1$. Then it follows easily that

$$[L_j] = [x \otimes P_H(w_j + b_j)]$$

for all $j \geq 1$.

It follows from (1) that

$$\| x \| \leq 3 \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \varepsilon_{j,k}^{1/2} = 3 \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \delta_j^{1/2} \left(\frac{\theta}{\gamma}\right)^{k/2} =$$

$$= \frac{3}{1-(\theta/\gamma)^{1/2}} \sum_{j=1} \delta_j^{1/2}$$

Similarly, we obtain:

$$\|w_j\| \leq \frac{\delta_j^{1/2}}{1-(\theta/\gamma)^{1/2}} \quad \text{for all } j \geq 1.$$

From (4) and (5) we infer that

$$s_{n+1}\|b_{j,k}\| \leq s_{B(j,k-1)+1}\|b_{j,k-1}\| + \varepsilon_{j,k-1}^{1/2}$$

$$\leq s_{B(j,1)+1}\|b_{j,1}\| + \sum_{\ell=1}^{k-1} \varepsilon_{j,\ell}^{1/2}$$

$$\leq \sum_{\ell=0}^{k-1} \varepsilon_{j,\ell}^{1/2} = \sum_{\ell=0}^{k-1} \delta_j^{1/2} \left(\frac{\theta}{\gamma}\right)^{\ell/2} \leq \delta_j^{1/2} \frac{1}{1-(\theta/\gamma)^{1/2}}$$

and therefore that

$$\|b_{j,k}\| \leq \frac{2\delta_j^{1/2}}{1-(\theta/\gamma)^{1/2}} \quad \text{for all } j \geq 1 \text{ and } k \geq 0$$

It follows that

$$\|b_j\| \leq \frac{2\delta_j^{1/2}}{1-(\theta/\gamma)^{1/2}} \quad (\forall)\ j \geq 1$$

From the above relations, we obtain that $T \in A_{1,\aleph_0}(\rho)$, where $\rho \leq \frac{3}{1-(\theta/\gamma)^{1/2}}$. The proof is complete.

REFERENCES

1. H.Bercovici, C.Foiaş and C.Pearcy. Dual algebras with applications to invariant subspaces and dilation theory. CBMS Regional Conference Series in Math. No.56, A.M.S. Providence, 1985.

2. B.Chevreau, G.Exner and C.Pearcy. Sur la réflexivité des contractions de l'espace hilbertien. C.R.Acad. Sci.,Paris (to appear).

3. B.Chevreau and C.Pearcy. On the structure of contraction operators I.J.Funct.Anal. (to appear).

4. R.Olin and J.Thomson. Algebras of subnormal operators. J.Funct.Anal.37(1980), p.271-301.

Bebe Prunaru
Department of Mathematics
INCREST
Bd.Păcii 220, 79622
Bucharest, ROMANIA.