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by

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Spectral Theory and Sheaf Theory. IV

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Introduction

In recent years, the Hilbert space operators which extend to operators with a rich spectral decomposition behaviour have begun to be studied and understood. One reason for this is that at present such operators are serious canditates for testing the yet unsolved invariant subspace problem, cf. S. Brown [4], Eschmeier [5], Thomson [22].

In the previous papers of this series [16], [17], [6] a sheaf model of an operator or a commutative system of operators was introduced. This established a dictionary relating spectral theory and sheaf theory, which was particularly relevant for operators with a property called Bishop's condition (β). Recently it has been proved that this condition or a variation of it characterizes sübdecomposability, respectively subscalarity, cf. Albrecht and Eschmeier [1] and Eschmeier and Putinar [7].

It is the aim of this paper to analyze the homological and sheaf theoretical aspects of these characterizations. As a matter of fact we obtain without any extra effort the Fréchet multidimensional analogues of these results. From the point of view of sheaf theory, the topics discussed in the present paper concern the existence of some distinguished soft resolutions of ceratin analytic non-coherent sheaves.

At a certain point we shall focus on the topological flatness of the sheaf of smooth functions on a complex manifold over the sheaf of analytic functions. This question was settled by Malgrange [14] in the category of analytic coherent modules.

The paper is written in terms of analytic modules, so that in spite of the above introduction the reader may well ignore spectral theory. On the other hand, the operatorial counterparts of the main results are listed at the end of the paper, or else they can be easily derived as indicated in [17].

The main objects to deal with in the sequel and which correspond to systems of commuting operators with property (β) are the Fréchet quasi-coherent analytic modules on a Stein space X. They were introduced by Ramis and Ruget [19] in order to avoid some technical difficulties related to analytic duality theory. To give a possibly new definition, an analytic sheaf of Fréchet modules $\mathcal{F}$ is quasi-coherent if there exists a global resolution of $\mathcal{F}$ with topologically free Fréchet $\mathcal{O}_X$-modules:

$$\cdots \longrightarrow \mathcal{O}_X \hat{\otimes} E_1 \longrightarrow \mathcal{O}_X \hat{\otimes} E_0 \longrightarrow \mathcal{F} \longrightarrow 0.$$

This condition is equivalent to the acyclicity of $\mathcal{F}$ on X and the existence, locally on X, of such topologically free resolutions. The equivalence between these definitions and the original one is discussed in Section 1 below.

Of course any coherent $\mathcal{O}_X$-module is quasi-coherent, and similarly for any topologically free module of the form $\mathcal{O}_X \hat{\otimes} E$, where E is a Fréchet space. Quite unexpected is a result in [17] which asserts that any Fréchet soft $\mathcal{O}_X$-module is quasi-coherent (see also [18] for some consequences of that result). It is well known that any sheaf of abelian groups possesses a right resolution with soft sheaves. The main result of Section 2 is a strengthening of this fact in the category of $\mathcal{O}_X$-modules, as follows.

Theorem 1. <u>Let $\mathcal{F}$ be a Fréchet $\mathcal{O}_X$-module on a Stein space of finite Zariski dimension. The sheaf $\mathcal{F}$ is quasi-coherent if and only if $\mathcal{F}$ is acyclic on X and admits a finite resolution to the right with Fréchet soft $\mathcal{O}_X$-modules.</u>

The proof of Theorem 1 is quite simple, the only deep result required being the separation (and flatness) theorem of Malgrange [14] Thm.VI.1.1. The latter result asserts in the complex case that, on a Stein manifold X, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_X$ is a sheaf of Fréchet spaces whenever $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. As usually, $\mathcal{E}_X$ denotes the sheaf of smooth functions on X. The flatness conditions $\mathrm{Tor}_k^{\mathcal{O}}(\mathcal{F}, \mathcal{E}) = 0, k \geq 1$, are then easily derived from the separation of $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}$ and a classical theorem of Whitney, cf [14].

A simple example given in [7] shows that the Malgrange theorem fails to be true beyond the category of coherent sheaves. Section 3 contains some results which give necessary and sufficient conditions for certain quasi-coherent modules on a Stein manifold to remain topologically flat over the algebra of smooth functions. As a matter of fact we obtain simple criteria to decide whether a quasi-coherent module admits resolutions to the right with $\mathcal{E}$ -modules.

The last section restates some previous results in terms of systems of operators acting on a Fréchet space.

1. Preliminaries

1.1 Throughout this paper all Stein spaces $(X, \mathcal{O})$ are supposed to be separable and of finite Zariski dimension.

Let $\mathcal{F}$ be an analytic coherent sheaf on the Stein space $(X, \mathcal{O})$. Cartan's theorem A asserts that the Fréchet $\mathcal{O}(X)$-module $\mathcal{F}(X)$ generates every stalk $\mathcal{F}_x$, $x \in X$. A constructive and more refined way of localizing the module $\mathcal{F}(X)$ at an arbitrary Stein open set $U \subset X$ is described in [2]. Quite specifically, the natural multiplication and restriction map

$$(1) \qquad \mathcal{F}(X) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U) \longrightarrow \mathcal{F}(U)$$

turns out to be an isomorphism of Fréchet $\mathcal{O}(X)$-modules. Moreover, the derived functors of the left hand side relative topological tensor product (which are usually denoted by $\mathrm{Tor}_{\cdot}^{\mathcal{O}(X)}(\mathcal{F}(X), \mathcal{O}(U))$) vanish in positive dimensions. For definitions and comments the reader may consult [2] and [19].

The same localizing procedure works for more general analytic modules. For instance, a Fréchet $\mathcal{O}$-module $\mathcal{F}$ is called <u>quasi-coherent</u> if the natural map (1) is a topological isomorphism and $\mathrm{Tor}_q^{\mathcal{O}(X)}(\mathcal{F}(X), \mathcal{O}(U))=0$, for $q \geq 1$ and every Stein open set $U \subset X$, cf. [19]. We shall write in short after Taylor [21], $\mathcal{F}(X) \perp_{\mathcal{O}(X)} \mathcal{O}(U)$ whenever the latter vanishing conditions hold and the locally convex space $\mathcal{F}(X) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U)$ is separated. This is a topological transversality condition, as explained in [2].

A few of the intrinsic properties of analytic quasi-coherent sheaves are discussed in [19], [17], [18]. Next we present for later use some equivalent definitions of quasi-coherence.

Recall that a topologically free $\mathcal{O}$-module is the sheaf $\mathcal{O}\hat{\otimes} E$ associated to the presheaf $U \longmapsto \mathcal{O}(U)\hat{\otimes} E$, where E is a locally convex space and "$\hat{\otimes}$" denotes a complete topological tensor product. Throughout this paper we assume that E is a Fréchet space; also quasi-coherent means in the sequel Fréchet quasi-coherent.

Proposition 1.1. *Let $\mathcal{F}$ be a Fréchet $\mathcal{O}_X$-module on a Stein space X. The following assertions are equivalent:*

a) *$\mathcal{F}$ is quasi-coherent,*

b) *$\mathcal{F}$ admits global topologically free resolutions to the left,*

c) *$\mathcal{F}$ is acyclic on X and admits, locally on X, topologically free resolutions to the left.*

Proof. a)$\Rightarrow$b). The $\mathcal{O}(X)$-module $\mathcal{F}(X)$ possesses a canonical left resolution (the Bar resolution):

$$\ldots \longrightarrow \mathcal{O}(X)\hat{\otimes}\mathcal{O}(X)\hat{\otimes}\mathcal{F}(X) \longrightarrow \mathcal{O}(X)\hat{\otimes}\mathcal{F}(X) \longrightarrow \mathcal{F}(X) \longrightarrow 0.$$

Let U be an open Stein subset of X. Since every $\mathcal{O}(X)$-module of the form $\mathcal{O}(X)\hat{\otimes} E$ is $\mathcal{O}(U)\hat{\otimes}_{\mathcal{O}(X)}*$-acyclic and $\mathcal{F}$ is supposed to be quasi-coherent, it follows that the complex of $\mathcal{O}$-modules

$$\ldots \longrightarrow \mathcal{O}\hat{\otimes}\mathcal{O}(X)\hat{\otimes}\mathcal{F}(X) \longrightarrow \mathcal{O}\hat{\otimes}\mathcal{F}(X) \longrightarrow \mathcal{F} \longrightarrow 0$$

is still exact.

b)$\Rightarrow$c). This implication is standard.

c)$\Rightarrow$a). Let V be a Stein open set on which the sheaf $\mathcal{F}$ possesses a topologically free left resolution $\mathcal{L}. \longrightarrow \mathcal{F}|_V$. Since dim(X) is finite, a repeated application of the long exact sequence of sheaf cohomology yields the exactness of the complex

(2) $\quad \mathcal{L}.(V) \longrightarrow (V) \longrightarrow 0.$

Take a Stein open subset U of X. The main transversality theorem of [2] (or a direct argument in this simple case) shows that $\mathcal{L}_q(V) \perp_{\mathcal{O}(X)} \mathcal{O}(U)$ for every $q \geq 0$. Moreover, $\mathcal{L}_q(V) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U) \cong \mathcal{L}_q(V), q \geq 0$, as expected. Thus (2) is an acyclic resolution of $\mathcal{F}(V)$ with respect to the functor $* \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U)$. Accordingly, the complex

$$(3) \qquad \mathcal{L}.(U \cap V) \longrightarrow \mathcal{F}(V) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U) \longrightarrow 0$$

is exact.

By comparing the exact complex (3) with (2), written for $U \cap V$ instead of V, the five lemma gives $\mathcal{F}(U \cap V) \cong \mathcal{F}(V) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U)$, and finally $\mathcal{F}(V) \perp_{\mathcal{O}(X)} \mathcal{O}(U)$.

Next consider an open covering with Stein sets $\mathcal{V} = (V_i)_{i \in I}$, so that the sheaf $\mathcal{F}$ admits topologically free resolutions on V_i for every $i \in I$. Due to the assumption on the dimension on X we may assume that the covering $\mathcal{V}$ has finite dimensional nerve. The exactness of the complexes like (2) shows that the sheaf $\mathcal{F}$ is acyclic on any finite intersection $V_i \cap V_j \cap \ldots \cap V_k$. Consequently the Čech complex of alternating cochains $\mathcal{C}^{\cdot}(\mathcal{V}, \mathcal{F})$ computes the cohomology groups $H^q(X, \mathcal{F})$. As this cohomology vanishes by assumption for $q \geq 1$, we get an exact sequence of the form

$$(4) \qquad 0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \ldots \longrightarrow \mathcal{C}^N \longrightarrow 0.$$

Since every component $\mathcal{C}^q$ of the Čech complex is a direct sum of the form $\bigoplus_{i,j,\ldots,k} \mathcal{F}(V_i \cap V_j \cap \ldots \cap V_k)$, the above considerations imply

$$\mathcal{C}^q \perp_{\mathcal{O}(X)} \mathcal{O}(U) \text{ and } \mathcal{C}^q(\mathcal{V};\mathcal{F}) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U) \cong \mathcal{C}^q(\mathcal{V} \cap U, \mathcal{F})$$

for every Stein open subset $U \subset X$. Therefore the exact sequence (4) localizes to an exact complex, too:

$$0 \longrightarrow \mathcal{F}(X) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U) \longrightarrow \mathcal{C}^{\cdot}(\mathcal{V} \cap U, \mathcal{F}).$$

Whence the map (1) is a topological isomorphism and $\mathcal{F}(X) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U)$.

This concludes the proof of Proposition 1.1.

Though in this paper we do not use the following result, we have included it to round off the above characterizations by a purely local criterium for a sheaf to be quasi-coherent. A sheaf of the form $\mathcal{O}\hat{\otimes} E$ with E a Banach space will be called a Banach free $\mathcal{O}$-module.

Proposition 1.2. (Leiterer [13]) Let $\mathcal{F}$ be a Fréchet analytic module on a Stein space X. If $\mathcal{F}$ possesses, locally on X, Banach free resolutions to the left, then $\mathcal{F}$ is quasi-coherent.

Moreover, in the same paper it is proved that a sheaf which fulfills the condition in the above statement has a global resolution with Banach free modules.

We mention that, when X is the affine space $\mathbb{C}^n$, there exists a canonical finite topological resolution of any quasi-coherent sheaf $\mathcal{F}$, given by the Koszul complex of the diagonal $\Delta = \{(z,w)\in \mathbb{C}^n\times \mathbb{C}^n; z=w\}$:

$$K.(z-w,\ \mathcal{O}\hat{\otimes}\mathcal{F}(\mathbb{C}^n)) \longrightarrow \mathcal{F} \longrightarrow 0.$$

This fact has been exploited in [6] and [17].

1.2 To finish the preliminaries on quasi-coherent sheaves, let us remark that the functor which assigns the global sections $\mathcal{F}\longmapsto\mathcal{F}(X)$ establishes an equivalence of categories between the category of quasi-coherent modules on a Stein space X and a full subcategory of the category of Fréchet $\mathcal{O}(X)$-modules. Accordingly, in the sequel we shall shift freely our point of view, depending on the context.

1.3 Originating in the problem of division of distributions by analytic functions and in questions of analytic duality theory, the following result turned out to be a basic tool in modern analytic geometry. We state only its complex analytic version, the real analytic case being not needed in the present paper.

Theorem 1.3. (Malgrange [14]) Let D be a polydisc in $\mathbb{C}^n$ and let $\mathcal{F}$ be an analytic coherent sheaf on D. Then $\mathcal{F}\otimes_{\mathcal{O}}\mathcal{E}$ is a sheaf of Fréchet spaces and $\mathrm{Tor}_q^{\mathcal{O}_x}(\mathcal{F}_x;\mathcal{E}_x) = 0$ for $q\geq 1$ and $x\in X$.

Recall that $\mathcal{E}$ denotes the sheaf of smooth functions on D.

A standard application of Malgrange's theorem provides the $\bar{\partial}$-resolution of a coherent sheaf. Namely, if

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{E}^{(0,0)} \xrightarrow{\bar{\partial}} \mathcal{E}^{(0,1)} \longrightarrow \dots \longrightarrow \mathcal{E}^{(0,n)} \longrightarrow 0$$

denotes the Dolbeault complex on the plydisc D and $\mathcal{F}$ is coherent, then according to Theorem 1.3 the sequence of Fréchet $\mathcal{O}$-modules

$$(5) \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}^{(0,0)} \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \dots \longrightarrow \mathcal{E}^{(0,n)} \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow 0$$

is also exact. This gives a very useful resolution with $\mathcal{E}$-modules, of the sheaf $\mathcal{F}$.

It is straightforward to extend Theorem 1.3 to Stein manifolds. Moreover, the quality of the sheaf $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}$ of being Fréchet is local, so that on a Stein manifold X the sheaf $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is Fréchet whenever $\mathcal{F}$ is a coherent $\mathcal{O}$-module and $\mathcal{G}$ is a locally free $\mathcal{E}$-module. Therefore we may state the following:

Corollary 1.4. Let X be a Stein manifold of complex dimension n and let $\mathcal{F}$ be an analytic coherent sheaf on X. Then $\mathcal{F}$ possesses a resolution of length n to the right with Fréchet $\mathcal{E}$-modules.

In Sections 2 and 3 below several generalizations of this classical fact are given.

1.4 The last topic to be discussed in this paragraph concerns inverse systems of Banach modules. The inverse limit functor is not in general exact in the category of locally convex spaces. Nevertheless some classical criteria for its exactity are known, cf. Palamodov [15]. One of them is reproduced below.

Proposition 1.5. Let $0 \longrightarrow E_n \longrightarrow F_n \longrightarrow G_n \longrightarrow 0$ be an inverse system, indexed over $n \in \mathbb{N}$, of short exact sequences of Fréchet spaces. If for every $n \leq m$ the structure map $E_m \longrightarrow E_n$ has dense range, then the inverse limit sequence $0 \longrightarrow \varprojlim E_n \longrightarrow \varprojlim F_n \longrightarrow \varprojlim G_n \longrightarrow 0$ is also exact.

For a proof of this fact see [15] Theorem 5.2. Such a phenomenon appeared for the first time in the proof of Mittag-Leffler's theorem on meromorphic functions with prescribed singularities. Afterwards it was used in many other complex analysis theorems, especially related to Runge exhaustions. For this reason the phenomenon described in Proposition 1.5 is often attributed to Mittag-Leffler.

Definition 1.6. Let X be a Stein space. An inverse system $(E_n)_{n\in\mathbb{N}}$ of Fréchet $\mathcal{O}(X)$-modules is said to have the <u>Mittag-Leffler property</u> (in short the (ML) property) if the structure maps $E_m \longrightarrow E_n$ have dense range for any $n \leq m$.

A Fréchet $\mathcal{O}(X)$-module is said to have the (ML) <u>property</u> if it is an inverse limit of a sequence of Banach $\mathcal{O}(X)$-modules and this system has the (ML) property.

A first simple example of module with the (ML) property is given in the affine case $X=\mathbb{C}^n$ by the space of analyitc functions $\mathcal{O}(U)$ defined on a Runge open subset U of $\mathbb{C}^n$. Indeed, in this case there exists a compact exhaustion of U with $\mathbb{C}^n$-holomorphically convex sets $K_n \uparrow U$, and denoting by E_n the separate completion of the space $\mathcal{O}(U)$ in the uniform norm on K_n, one gets $\mathcal{O}(U)=\varprojlim E_n$. Obviously, E_n are Banach algebras and the maps $E_m \longrightarrow E_n$ have dense range.

Before listing some other examples of modules with the (ML) property we isolate the next technical result.

Lemma 1.7. <u>Let</u> X <u>be a Stein space and let</u>

$$\cdots \longrightarrow F_n^1 \longrightarrow F_n^0 \longrightarrow G_n \longrightarrow 0$$

<u>be an inverse system of exact sequences of Fréchet</u> $\mathcal{O}(X)$-<u>modules</u>.

<u>If the inverse systems</u> $(F_n^j)_n$ <u>have the</u> (ML) <u>property</u>, $j \geq 0$, <u>then so does the system</u> (G_n), <u>and the sequence</u>

$$\cdots \longrightarrow \varprojlim F_n^1 \longrightarrow \varprojlim F_n^0 \longrightarrow \varprojlim G_n \longrightarrow 0$$

<u>is exact</u>.

Proof. Let K_n^0 denote the kernel of the map $F_n^0 \longrightarrow G_n$. The exactness of the sequence at level n shows that K_n^0 is a quotient $\mathcal{O}(X)$-module of F_n^1. In particular the map $K_m^0 \longrightarrow K_n^0$ has dense range for any $n \leqslant m$. Therefore $K^0 = \varprojlim K_n^0$ is an $\mathcal{O}(X)$-module with the (ML) property. Similarly the modules G and $\varprojlim \operatorname{Ker}(F_n^i \longrightarrow F_n^{i-1})$ have the (ML) property for any $i \geqslant 0$.

The last assertion follows by a repeated use of Proposition 1.5.

Corollary 1.8. <u>Let $\mathcal{F}$ be an analytic coherent sheaf on a Stein space</u> X. <u>Then the Fréchet</u> $\mathcal{O}(X)$-<u>module</u> $\mathcal{F}(X)$ <u>has property</u> (ML).

Proof. Embed X into an affine space $\mathbb{C}^n$ and consider an open bounded polydisc D in $\mathbb{C}^n$. Then there are free resolutions on D:

$$(6) \qquad \dots \longrightarrow \mathcal{O}_X^p|_{D\cap X} \longrightarrow \mathcal{O}_X^q|_{D\cap X} \longrightarrow \mathcal{F}|_{D\cap X} \longrightarrow 0,$$

and

$$(7) \qquad \dots \longrightarrow \mathcal{O}_D^r \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_X|_{D\cap X} \longrightarrow 0.$$

In virtue of Lemma 1.7 and the example following Definition 1.6, the exact sequence (7) shows that $\mathcal{O}_X(D\cap X)$ is an inverse limit of a system of Banach modules with property (ML). By applying again Lemma 1.7 to the sequence (6) one finds that the $\mathcal{O}(X)$-module $\mathcal{F}(D\cap X)$ has the (ML) property.

But $\mathcal{F}(X) \cong \varprojlim \mathcal{F}(D_m \cap X)$, where (D_m) is a relatively compact exhaustion of $\mathbb{C}^n$ with polydiscs. As every restriction map $\mathcal{F}(D_k \cap X) \longrightarrow \mathcal{F}(D_m \cap X)$, $k \geqslant m$, has dense range in a complete system of seminorms of $\mathcal{F}(D_m \cap X)$ which does not depend on k, one can choose by a diagonal selection an inverse system of Banach $\mathcal{O}(X)$-modules with the (ML) property which converges to $\mathcal{F}(X)$. This finishes the proof of Corollary 1.8.

Remark 1.9. Another proof of Corollary 1.8 can be derived from the observation which follows after Proposition 1.2. Namely, the coherent sheaf $\mathcal{F}$ defined on X admits a presentation

$$\mathcal{O}_X \hat{\otimes} E \longrightarrow \mathcal{O}_X \hat{\otimes} F \longrightarrow \mathcal{F} \longrightarrow 0$$

with Banach free modules. Whence the $\mathcal{O}(X)$-module $\mathcal{F}(X)$ has the (ML) property.

2. Soft Resolutions of Quasi-Coherent Modules

This section is devoted to proving Theorem 1 and to deriving a few of its corollaries.

Proof of Theorem 1. Let X be a Stein space and let $\mathcal{F}$ be a Fréchet $\mathcal{O}_X$-module. We embed X into an affine space $\mathbb{C}^n$, so that $\mathcal{F}$ becomes an $\mathcal{O}_{\mathbb{C}^n}$-module by the restriction of scalars. Next $\mathcal{O}$ denotes for simplicity $\mathcal{O}_X$.

We shall prove the equivalence of the following assertions:

(i) $\mathcal{F}$ is quasicoherent;

(ii) $\mathcal{F}(X)$ admits finite resolutions to the right with $\mathcal{O}(X)$-modules of the form $\mathcal{S}(X)$, where $\mathcal{S}$ is a Fréchet soft $\mathcal{O}$-module;

(iii) $\mathcal{F}$ is acyclic on X and admits finite resolutions to the right with Fréchet soft $\mathcal{O}$-modules.

(ii)$\Rightarrow$(i). Let

$$(8) \qquad 0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{S}^0(X) \longrightarrow \dots \longrightarrow \mathcal{S}^N(X) \longrightarrow 0$$

be an exact sequence of topological $\mathcal{O}(X)$-modules, where $\mathcal{S}^i$ are Fréchet soft $\mathcal{O}$-modules, $i=0,1,\dots,N$.

If U denotes an open Stein subset of X, then $\mathcal{S}^i(X) \perp_{\mathcal{O}(X)} \mathcal{O}(U)$ and $\mathcal{S}^i(U) \cong \mathcal{S}^i(X) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U)$ for $i=0,\dots,N$, because the $\mathcal{S}^i$ are quasi-coherent $\mathcal{O}$-modules, see [19] Theorem 2.1. By applying the functor $* \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U)$ to the exact sequence (8), one obtains by a repeated use of the long exact sequence of its derived functors the next exact complex:

$$0 \longrightarrow \mathcal{F}(X) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U) \longrightarrow \mathcal{S}^0(U) \longrightarrow \dots \longrightarrow \mathcal{S}^N(U) \longrightarrow 0.$$

Thus $\mathcal{F}(X) \perp_{\mathcal{O}(X)} \mathcal{O}(U)$. Moreover, denoting by $\mathcal{F}'$ the sheaf associated to the presheaf $U \longmapsto \mathcal{F}(X) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(U)$, one finds that $\mathcal{F}'$ is a quasi-coherent $\mathcal{O}$-module and $\mathcal{F}(X) = \mathcal{F}'(X)$. Therefore $\mathcal{F}$ is quasi-coherent.

(iii)$\Rightarrow$(ii) This implication is evident.

(i) ⇒ (iii). Suppose that the sheaf $\mathcal{F}$ is quasi-coherent. According to Proposition 1.1, $\mathcal{F}$ has a topologically free resolution $\mathcal{L}_\cdot \longrightarrow \mathcal{F} \longrightarrow 0$. This shows in particular that $\mathcal{F}$ is acyclic on Stein open subsets of X.

The $\bar{\partial}$-resolution (5) of the $\mathcal{O}_{\mathbb{C}^n}$-module $\mathcal{O}$ contains only nuclear Fréchet sheaves. Consequently, if E denotes a Fréchet space, then this resolution tensorized with $*\hat{\otimes} E$ becomes a fine resolution of the $\mathcal{O}$-module $\mathcal{O}\hat{\otimes} E$. In particular every $\mathcal{O}$-module $\mathcal{L}_q$ has such a resolution, say

$$0 \longrightarrow \mathcal{L}_q \longrightarrow \mathcal{M}_q^{\cdot} \quad , \; q \geq 0.$$

The naturality of the $\bar{\partial}$-resolution shows that $\mathcal{M}_{\cdot}^{\cdot}$ is a double complex. Moreover, the construction of this double complex yields

$$\mathcal{H}^p(\mathcal{H}_q(\mathcal{M}_{\cdot}^{\cdot})) = \begin{cases} \mathcal{F} & \text{if } p=q=0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $\mathcal{C}^{\cdot}$ denotes the simple complex associated to the double complex $\mathcal{M}_{\cdot}^{\cdot}$, with the canonical convention on the indices, one obtains:

$$\mathcal{H}^p(\mathcal{C}^{\cdot}) = \begin{cases} \mathcal{F} & \text{if } p=0, \\ 0 & \text{if } p\neq 0. \end{cases} \tag{9}$$

Moreover, the complex $\mathcal{C}^{\cdot}$ is bounded to the right and its terms are only Fréchet fine $\mathcal{O}$-modules. Denoting $\mathcal{B}^q = \mathrm{Im}(\mathcal{C}^{q-1} \longrightarrow \mathcal{C}^q)$, one derives from (9) two exact sequences of $\mathcal{O}$-modules:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0/\mathcal{B}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \dots \longrightarrow \mathcal{C}^N \longrightarrow 0, \tag{10}$$

and

$$\dots \longrightarrow \mathcal{C}^{-2} \longrightarrow \mathcal{C}^{-1} \longrightarrow \mathcal{B}^0 \longrightarrow 0. \tag{11}$$

Let V denote an open subset of X. The acyclicity of the sheaves $\mathcal{C}^{-q}$, $q \geq 1$, on V and the long exact sequence of sheaf cohomology imply:

$$H^j(V,\mathcal{B}^0) \cong H^{j+1}(V,\mathcal{B}^1) \cong \ldots \cong H^{j+k}(V,\mathcal{B}^k) = \ldots,$$

for every $j \geq 1$. Since X is a finite dimensional Stein space, one finds $H^j(V,\mathcal{B}^0) = 0$ for $j \geq 1$. Then the same argument applied to the sheaf $\mathcal{C}^0/\mathcal{B}^0$ via the exact sequence $0 \longrightarrow \mathcal{B}^0 \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^0/\mathcal{B}^0 \longrightarrow 0$ finally gives $H^j(V,\mathcal{C}^0/\mathcal{B}^0) = 0, j \geq 1$. Whence the sheaf $\mathcal{C}^0/\mathcal{B}^0$ is soft, cf. for instance [3] Proposition II.1.5.1.

In order to prove that (10) is the desired soft resolution, it remains to check that $\mathcal{C}^0/\mathcal{B}^0$ is a sheaf of Fréchet $\mathcal{O}$-modules. Of course, it suffices to prove that $\Gamma(U,\mathcal{C}^0/\mathcal{B}^0)$ is a separated locally convex space for every Stein open subset $U \subset X$. This assertion follows from the exact sequence derived from (10)

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \Gamma(U,\mathcal{C}^0/\mathcal{B}^0) \longrightarrow \mathcal{C}^1(U)$$

by noticing that the extreme terms are Fréchet spaces.

Assertion (iii) is verified and thus the proof of Theorem 1 is complete.

Let us remark that the length of the soft resolution (10) is equal to the dimension of the affine space in which the underlying Stein space embeds.

By taking into account the observation made in paragraph 1.2, one can unambigously call a module E over a Stein algebra $\mathcal{O}(X)$ a *Fréchet soft $\mathcal{O}(X)$-module* whenever $E = \mathcal{S}(X)$, where $\mathcal{S}$ is a soft sheaf of Fréchet $\mathcal{O}(X)$-modules. Similarly, the Fréchet $\mathcal{O}(X)$-module E is simply called *quasi-coherent* if $E = \mathcal{F}(X)$ and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}$-module.

Corollary 2.1. *A Fréchet module over a Stein algebra is quasi-coherent if and only if it admits a finite resolution to the right with Fréchet soft modules.*

Next we consider a few particular cases of Theorem 1, in which the soft resolutions may be chosen with additional properties.

Proposition 2.2. *Let E be a Fréchet $\mathcal{O}(\mathbb{C}^n)$-module. Then E is quasi-coherent if and only if there exists an exact sequence of Fréchet $\mathcal{O}(\mathbb{C}^n)$-modules:*

(12) $\quad 0 \longrightarrow E \longrightarrow S^0 \longrightarrow S^1 \longrightarrow \dots \longrightarrow S^{n-1} \longrightarrow C \longrightarrow 0,$

where the S^j are Fréchet soft $\mathcal{O}(\mathbb{C}^n)$-modules for $j=0,1,\dots,n-1$.

Proof. The necessity follows from Theorem 1.

In order to prove the sufficiency of the condition, assume that the exact complex (12) exists and let us consider a Stein open subset $U \subset \mathbb{C}^n$. We have to prove that $E \perp_A \mathcal{O}(U)$, where we denote for simplicity $A = \mathcal{O}(\mathbb{C}^n)$.

By applying the functor $* \hat{\otimes}_A \mathcal{O}(U)$ to (12), from the long exact sequence of $\hat{Tor}$'s we get the isomorphisms:

$$\hat{Tor}_j^A(E, \mathcal{O}(U)) \cong \hat{Tor}_{j+n}^A(C, \mathcal{O}(U)), \ j \geqslant 1.$$

Since the Fréchet algebra A has global $\hat{Tor}$-dimension equal to n (due to the existence of Koszul's resolution), we infer $\hat{Tor}_j^A(E, \mathcal{O}(U)) = 0$ for $j \geqslant 1$. Moreover, denoting $Z^1 = S^0/E$, the same argument shows that

$$\hat{Tor}_1^A(Z^1, \mathcal{O}(U)) \cong \hat{Tor}_n^A(C, \mathcal{O}(U)).$$

But the last space is the kernel of the first boundary in the Koszul complex $K_{\cdot}(z-w, \mathcal{O}(U) \hat{\otimes} C)$, thus it is a separated locally convex space. Accordingly, from the exact sequence

$$0 \longrightarrow Tor_1^A(Z^1, \mathcal{O}(U)) \longrightarrow E \hat{\otimes}_A \mathcal{O}(U) \longrightarrow S^0 \hat{\otimes}_A \mathcal{O}(U)$$

we find that $E \hat{\otimes}_A \mathcal{O}(U)$ is a Fréchet space, q.e.d.

Corollary 2.3. A Fréchet $\mathcal{O}(\mathbb{C})$-module is quasi-coherent if and only if it is a closed submodule of a Fréchet soft $\mathcal{O}(\mathbb{C})$-module.

Corollary 2.4. A Fréchet $\mathcal{O}_M$-module $\mathcal{F}$ on a Stein manifold M of dimension n is quasi-coherent if and only if it is acyclic on Stein open sets and there exist, locally on M, exact complexes of Fréchet $\mathcal{O}$-modules, of the form

(13) $\quad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{S}^0 \longrightarrow \mathcal{S}^1 \longrightarrow \dots \longrightarrow \mathcal{S}^{n-1} \longrightarrow \mathcal{C} \longrightarrow 0,$

with $\mathcal{S}^j$ *soft sheaves*, $j=0,1,\dots,n-1$.

Proof. Suppose $\mathcal{F}$ fulfills the last two conditions. Take a Stein open set $U \subset M$, on which a complex like (13) exists, and such that U is isomorphic to a domain of $\mathbb{C}^n$. By Proposition 2.2 we find that $\mathcal{F}|_U$ is a quasi-coherent $\mathcal{O}_U$-module. The equivalence a)$\Leftrightarrow$c) in Proposition 1.1 ends the proof.

The next result is relevant for applications to multiparameter spectral theory.

Proposition 2.5. *A Hilbert quasi-coherent* $\mathcal{O}(\mathbb{C}^n)$*-module has a finite resolution to the right with Hilbert soft* $\mathcal{O}(\mathbb{C}^n)$*-modules*.

Proof. Let E denote a Hilbert quasi-coherent $\mathcal{O}(\mathbb{C}^n)$-module. The continuity of the multiplication with scalars map implies $\mathrm{supp}(E) \subset\subset \mathbb{C}^n$, see [17]. We consider a bounded polydisc D which contains $\mathrm{supp}(E)$, and the Bergman space $A^2(D)$ of analytic functions which are square summable on D.

Our first aim is to prove by induction on n that $A^2(D) \perp_{\widehat{\mathcal{O}(\mathbb{C}^n)}} E$ and $A^2(D) \hat{\otimes}_{\widehat{\mathcal{O}(\mathbb{C}^n)}} E \cong E$.

Let us assume $n=1$. Fix a second polydisc D', with the properties $\mathrm{supp}(E) \subset D' \subset\subset D$. We must compute the homology of the complex $K_\bullet(z-w, A^2(D)\tilde{\otimes} E)$, where "$\tilde{\otimes}$" may be chosen to be the hilbertian tensor product.

Let us consider the morphisms of complexes induced by the restriction maps:

$$\begin{array}{ccc} K_\bullet(z-w,\mathcal{O}(\mathbb{C})\tilde{\otimes} E) & \xrightarrow{\quad\alpha\quad} & K_\bullet(z-w,A^2(D)\tilde{\otimes} E) \\ & {}_{\gamma}\searrow \qquad \swarrow_{\beta} & \\ & K_\bullet(z-w,\mathcal{O}(D')\tilde{\otimes} E) & \end{array}$$

Notice that $\gamma = \beta\alpha$ and that the three morphisms are one to one and with dense range on components. Since $\mathcal{O}(D') \perp_{\mathcal{O}(\mathbb{C})} E$ and $\mathcal{O}(D') \hat{\otimes}_{\mathcal{O}(C)} E \cong E$, one finds that γ is a quasi-isomorphism. Then one easily derives that α and β are quasi-isomorphisms, too. This proves that the complex

$$(14) \qquad K_\bullet(z-w,A^2(D)\tilde{\otimes} E) \longrightarrow E \longrightarrow 0$$

is exact for $n=1$.

If $n>1$, we decompose $D=D'\times D''$, where D' and D'' are polydiscs in $\mathbb{C}^{n-1}$, respectively in $\mathbb{C}$. Correspondingly, the coordinates are denoted by $z=(z',z'')$. A fundamental property of Koszul's complexes enables us to identify the simple complex associated to the double complex

$$K_{\cdot}(z''-w'',A^2(D'')\tilde{\otimes}K_{\cdot}(z'-w',A^2(D')\tilde{\otimes}E))$$

with

$$K_{\cdot}(z-w,A^2(D)\tilde{\otimes}E).$$

Recall that $A^2(D)\cong A^2(D')\tilde{\otimes}A^2(D'')$ as Hilbert spaces.

By the induction hypothesis the complex (14) is exact for D',z' instead of D,z. Since all components of (14) are Hilbert spaces, the complex splits over $\mathbb{C}$. Therefore it remains exact after tensorization with $*\tilde{\otimes}K_p(z''-w'',A^2(D''))$, $p=0,1$. Again the induction hypothesis yields:

$$H_pK_{\cdot}(z''-w'',A^2(D'')\tilde{\otimes}H_qK_{\cdot}(z'-w',A^2(D')\tilde{\otimes}E)) = \begin{cases} E & \text{if } p=q=0, \\ 0 & \text{otherwise.} \end{cases}$$

Then a familiar argument on double complexes proves the exactness of the sequence (14).

The L^2-estimates for the $\bar{\partial}$-operator on pseudoconvex domains [11] give a resolution

$$0 \longrightarrow A^2(D) \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \dots \longrightarrow W^n \longrightarrow 0,$$

where W^j are Hilbert $\mathcal{E}(\mathbb{C}^n)$-modules endowed with Sobolev type norms. See also [17]§4 for the needed details.

By replacing the module $A^2(D)$ by the above complex, denoted in short $W^{\cdot}$, one obtains a double complex

$$K_{\cdot}(z-w,W^{\cdot}\tilde{\otimes}E)$$

whose associated simple complex has nontrivial homology only in degree zero, and there it is E.

Then by repeating the proof of Theorem 1 (namely implication (i)$\Rightarrow$(iii)), one constructs the desired resolution of E with Hilbert soft $\mathcal{O}(\mathbb{C}^n)$-modules.

The proof of Proposition 2.5 is complete.

Remark 2.6. The proof of Proposition 2.5 applies with minor modifications to the case of Banach modules. Namely, by replacing the hilbertian tensor product with an appropriate Banach spaces tensor product, for instance the projective one, one proves by induction on n=dimD that the exact complex (14) splits over $\mathbb{C}$, whenever E is a quasi-coherent Banach $\mathcal{O}(\mathbb{C}^n)$-module with supp(E)$\subset$D. Then the proof runs analogously.

It would be interesting to know whether the conclusion of Proposition 2.5 remains valid on Stein spaces with singularities.

3. Resolutions with $\mathcal{E}$-Modules

Theorem 1 raises the natural question whether a quasi-coherent module on a Stein manifold possesses a resolution to the right with $\mathcal{E}$-modules rather than only with soft analytic modules. As Corollary 1.4 shows, the coherent analytic modules do have this property. The present section gives a partial answer to this question, which covers all the important cases needed in applications. The main tools are Mittag-Leffler's exactness criterion given by Proposition 1.5 and the next technical fact proved in the paper [7] Corollary 4.5.

Proposition 3.1. Let E be a Banach $\mathcal{E}(\mathbb{C})$-module. Then $E \perp_{\mathcal{O}(\mathbb{C})} \mathcal{E}(\mathbb{C})$.

In the same paper [7] it was proved that E is a Banach $\mathcal{O}(\mathbb{C})$-submodule of a Banach $\mathcal{E}(\mathbb{C})$-module if and only if $E \perp_{\mathcal{O}(\mathbb{C})} \mathcal{E}(\mathbb{C})$.

The multidimensional analogue of Proposition 3.1 is easily obtained by enlarging the class of Banach modules to that of Fréchet modules having the (ML) property. See the last part of Section 1 for definition.

Lemma 3.2. Let $E = \varprojlim E_p$ be a Fréchet $\mathcal{O}(\mathbb{C}^n)$-module with the (ML) property

(<u>on the inverse system of Banach modules</u> $(E_p)_{p\in\mathbb{N}}$).

<u>If</u> $E_p \perp_{\widehat{\Theta(\mathbb{C}^n)}} \mathcal{E}(\mathbb{C}^n)$ <u>for any</u> $p\geq 0$, <u>then</u> $E \perp_{\widehat{\Theta(\mathbb{C}^n)}} \mathcal{E}(\mathbb{C}^n)$.

Proof. Condition $E_p \perp_{\widehat{\Theta(\mathbb{C}^n)}} \mathcal{E}(\mathbb{C}^n)$ is equivalent the the exactness of the following augmented Koszul complex of Fréchet spaces:

$$K.(z-w, E_p \hat{\otimes} \mathcal{E}(\mathbb{C}^n)) \longrightarrow E_p \hat{\otimes}_{\widehat{\Theta(\mathbb{C}^n)}} \mathcal{E}(\mathbb{C}^n) \longrightarrow 0.$$

Since tensorization with a Fréchet nuclear space preserves the dense range property of a linar continuous map of Banach spaces, the inverse system $(E_p \hat{\otimes} \mathcal{E}(\mathbb{C}^n))_{p\in\mathbb{N}}$ still possesses the (ML) property. Accordingly, Lemma 1.7 implies the exactness of the complex of Fréchet spaces

$$K.(z-w, E \hat{\otimes} \mathcal{E}(\mathbb{C}^n)) \longrightarrow E \hat{\otimes}_{\widehat{\Theta(\mathbb{C}^n)}} \mathcal{E}(\mathbb{C}^n) \longrightarrow 0.$$

This in turn is equivalent to condition $E \perp_{\widehat{\Theta(\mathbb{C}^n)}} \mathcal{E}(\mathbb{C}^n)$, q.e.d.

Proposition 3.3. <u>Let</u> E <u>be a Banach</u> $\mathcal{E}(\mathbb{C}^n)$-<u>module</u>. <u>Then</u> $E \perp_{\widehat{\Theta(\mathbb{C}^n)}} \mathcal{E}(\mathbb{C}^n)$.

Proof. Induction on n. The case n=1 coincides with Proposition 3.1.

For the proof of the induction step, we decompose $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$ and $z=(z', z'')$, correspondingly. We have already remarked that the simple complex attached to the double complex

$$K.(z''-w'', \mathcal{E}(\mathbb{C}) \hat{\otimes} K.(z'-w', \mathcal{E}(\mathbb{C}^{n-1}) \hat{\otimes} E))$$

coincides with

$$K.(z-w, \mathcal{E}(\mathbb{C}^n) \hat{\otimes} E).$$

We have to prove that the last complex is exact in positive degree and has separated homology in degree zero.

The induction hypothesis implies the exactness of the following augmented Koszul complex:

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$$K_\cdot(z'-w', \mathcal{E}(\mathbb{C}^{n-1})\hat{\otimes} E) \longrightarrow C \longrightarrow 0,$$

where $C = \mathrm{Coker}(\delta : K_1 \longrightarrow K_0)$ is a Fréchet $\mathcal{E}(\mathbb{C}^{n-1})$-module.

As in the proof of Proposition 2.5 one relates by a standard argument the homology of the simple complex to that of its generating double complex. Therefore it suffices to prove that

$$0 \longrightarrow \mathcal{E}(\mathbb{C})\hat{\otimes} C \xrightarrow{z''-w''} \mathcal{E}(\mathbb{C})\hat{\otimes} C \longrightarrow \mathrm{Coker}(z''-w'') \longrightarrow 0$$

is an exact sequence of Fréchet spaces.

Notice that C is also an $\mathcal{E}(\mathbb{C}^n)$-module. Moreover, its explicit form yields

$$(15) \qquad C = \varprojlim \left[W^p(D)\hat{\otimes}_\pi E/(\mathrm{Im}\,\delta)^- \right],$$

where $D \subset \mathbb{C}^{n-1}$ is a polydisc containing the support of the Banach $\mathcal{E}(\mathbb{C}^{n-1})$-module E, $W^p(D)$ is the hilbertian Sobolev space of order p and the closure of the range of the boundary operator δ is taken in the norm of the projective tensor product $W^p(D)\hat{\otimes}_\pi E$.

The definition of the projective norm on tensor products shows that the canonical maps $W^p(D)\hat{\otimes}_\pi E \longrightarrow W^q(D)\hat{\otimes}_\pi E$ have dense range for any $p \geq q$. Moreover, the quotients appearing in (15) are Banach $\mathcal{E}(\mathbb{C})$-modules in a natural way, with respect to the variable z''. Hence we have proved that C is a Fréchet $\mathcal{E}(\mathbb{C})$-module wiht the property (ML).

In view of Lemma 3.2 one obtains $C \perp_{\mathcal{O}(\mathbb{C})} \mathcal{E}(\mathbb{C})$, and consequently $E \perp_{\widehat{\mathcal{O}(\mathbb{C}^n)}} \mathcal{E}(\mathbb{C}^n)$.

This concludes the proof of Proposition 3.3.

Corollary 3.4. <u>A Fréchet</u> $\mathcal{O}(\mathbb{C}^n)$-<u>module</u> E <u>with property</u> (ML) <u>satisfies</u> $E \perp_{\widehat{\mathcal{O}(\mathbb{C}^n)}} \mathcal{E}(\mathbb{C}^n)$.

The proof of this corollary consists in a direct application of Lemma 1.7 and Proposition 3.3.

At this point we are able to prove a first criterion for an analytic module to admit right resolutions with $\mathcal{E}$-modules.

Proposition 3.5. <u>A Fréchet</u> $\mathcal{O}(\mathbb{C}^n)$-<u>module</u> E <u>with property</u> (ML) <u>admits a finite resolution to the right with Fréchet</u> $\mathcal{E}(\mathbb{C}^n)$-<u>modules having property</u> (ML) <u>if and only if</u> $E \perp_{\mathcal{O}(\mathbb{C}^n)} \mathcal{E}(\mathbb{C}^n)$.

Proof. Throughout this proof one denotes for simplicity $A = \mathcal{O}(\mathbb{C}^n)$.

By assumption, E is isomorphic in the category of Fréchet A-modules with $\varprojlim E_p$, where $(E_p)_{p \in \mathbb{N}}$ is an inverse system of Banach A-modules which has property (ML). Suppose $E \perp_A \mathcal{E}(\mathbb{C}^n)$.

Let us consider Dolbeault's complex:

$$0 \longrightarrow A \longrightarrow \mathcal{E}^{(0,0)}(\mathbb{C}^n) \xrightarrow{\bar\partial} \cdots \longrightarrow \mathcal{E}^{(0,n)}(\mathbb{C}^n) \longrightarrow 0.$$

Since every space of (0,q) forms is isomorphic to a finite direct sum of $\mathcal{E}(\mathbb{C}^n)$, Corollary 3.4 yields $E \perp_A \mathcal{E}^{(0,q)}(\mathbb{C}^n)$, $q \geq 0$. Therefore, by applying the functor $E \hat{\otimes}_A *$ to the above exact complex, one gets a resolution of the A-module E with Fréchet $\mathcal{E}(\mathbb{C}^n)$-modules:

$$0 \longrightarrow E \longrightarrow \mathcal{E}^{(0,0)}(\mathbb{C}^n)\hat{\otimes}_A E \longrightarrow \cdots \longrightarrow \mathcal{E}^{(0,n)}(\mathbb{C}^n)\hat{\otimes}_A E \longrightarrow 0.$$

It remains to prove that the A-module $\mathcal{E}(\mathbb{C}^n)\hat{\otimes}_A E$ has property (ML). Let D_p denote the polydisc centered at 0, of multiradius (p,p,...,p). Then

$$\mathcal{E}(\mathbb{C}^n)\hat{\otimes} E \cong \varprojlim W^p(D_p)\hat{\otimes}_A E_p$$

as left and right A-modules. By passing to quotients one proves that $\mathcal{E}(\mathbb{C}^n)\hat{\otimes}_A E$ is a left A-module with the (ML) property, as desired.

Conversely, if the Fréchet A-module E admits a finite resolution to the right, with $\mathcal{E}(\mathbb{C}^n)$-modules with property (ML), then Corollary 3.4 and a repeated use of the long exact sequence of $\hat{\text{Tor}}$'s gives $E \perp_A \mathcal{E}(\mathbb{C}^n)$.

The proof of Proposition 3.5 is complete.

The main result of this section can be stated as follows.

Theorem 3.6. <u>Let X be a Stein manifold and let</u> $\mathcal{F}$ <u>be a Fréchet</u> $\mathcal{O}$-<u>module with a topologically free resolution</u>:

$$\cdots \xrightarrow{d_2} \mathcal{O}\hat{\otimes}E_1 \xrightarrow{d_1} \mathcal{O}\hat{\otimes}E_0 \longrightarrow \mathcal{F} \longrightarrow 0.$$

<u>Assume that the</u> $\mathcal{O}(X)$-<u>module</u> $\mathcal{F}(X)$ <u>has property</u> (ML).

<u>Then the following conditions are equivalent</u>:

(i) <u>The complex</u>

$$\cdots \longrightarrow \mathcal{E}\hat{\otimes}E_1 \xrightarrow{d_1} \mathcal{E}\hat{\otimes}E_0$$

<u>is exact and</u> Coker(d_1) <u>is a sheaf of Fréchet spaces</u>;

(ii) <u>There exists a finite resolution to the right of the</u> $\mathcal{O}$-<u>module</u> $\mathcal{F}$, <u>with Fréchet</u> $\mathcal{E}$-<u>modules with property</u> (ML).

Proof. Condition (i) is obviously equivalent to $F \perp_{\mathcal{O}(X)} \mathcal{E}(X)$, where $F = \mathcal{F}(X)$. Moreover, this condition may be localized with respect to the second term. Namely, if $\mathcal{U}$ denotes an open covering of X with finite dimensional nerve, then the following Cech complex

$$0 \longrightarrow \mathcal{E}(X) \longrightarrow \mathcal{C}^0(\mathcal{U},\mathcal{E}) \longrightarrow \cdots \longrightarrow \mathcal{C}^n(\mathcal{U},\mathcal{E}) \longrightarrow 0$$

is exact. By applying it the functor $F\hat{\otimes}_{\mathcal{O}(X)}*$ one easily gets the implication (i)' ⇒ (i), where we have marked for convenience by (i)' the condition:

(i)' $F \perp_{\mathcal{O}(X)} \mathcal{E}(U)$ for every Stein open coordinate patch U of X.

Conversely, if (i) holds, then with the notations above

$$\cdots \longrightarrow \mathcal{E}(X)\hat{\otimes}E_1 \longrightarrow \mathcal{E}(X)\hat{\otimes}E_0 \longrightarrow \mathcal{E}(X)\hat{\otimes}_{\mathcal{O}(X)}F \longrightarrow 0$$

is an exact sequence of Fréchet soft $\mathcal{O}(X)$-modules. By Theorem 2.1 in [17] these modules are quasi-coherent, whence the exactness of the complex is preserved after tensorizing it with $\mathcal{O}(U)\hat{\otimes}_{\mathcal{O}(X)}*$, where U is a Stein open subset of X. Since $\mathcal{O}(U)\hat{\otimes}_{\mathcal{O}(X)}\mathcal{E}(X) \cong \mathcal{E}(U)$, one gets assertion (i)'.

In conclusion (i)$\Leftrightarrow$(i)'.

(i)'$\Rightarrow$(ii). One starts with Dolbeault's complex on the manifold X, $(\mathcal{E}^{(0,\cdot)}(X),\bar{\partial})$.Since every $\mathcal{E}^{(0,q)}$ is a locally free $\mathcal{E}$-module,condition (i)' implies $F\perp_{\mathcal{O}(X)}\mathcal{E}^{(0,q)}(U)$ for every U as before and $q\geq 0$.The same argument resorting to a Čech complex yields $F\perp_{\mathcal{O}(X)}\mathcal{E}^{(0,q)}(X),q\geq 0$.An application of the long exact sequence of $\hat{\mathrm{Tor}}$'s gives assertion (ii),by remarking as in the proof of Proposition 3.5 that $F\hat{\otimes}_{\mathcal{O}(X)}\mathcal{E}^{(0,p)}(X)$ are Fréchet modules with the (ML) property.

(ii)$\Rightarrow$(i). Fix a point $x\in X$.Cartan's theorem A provides a Stein open neighbourhood $U\ni x$ and an n-tuple $f=(f_1,\dots,f_n)$ of global analytic functions on X,with the property that $f|_U:U\longrightarrow f(U)\subset\mathbb{C}^n$ is an isomorphism.

Because $f(x)-f(y)$ is a regular sequence on $X\times U\ni(x,y)$,the next Koszul complex

$$K.(f(x)-f(y),\mathcal{O}(X\times U))\longrightarrow\mathcal{O}(U)\longrightarrow 0$$

is exact.Consequently,the quasi-coherence of $\mathcal{E}(X)$ implies that the sequence

$$K.(f(x)-f(y),\mathcal{O}(X)\hat{\otimes}\mathcal{E}(U))\longrightarrow\mathcal{E}(U)\longrightarrow 0$$

is exact.Thus,in order to prove assertion (i)' it suffices to verify that the complex

$$K.(f-f(y),F\hat{\otimes}\mathcal{E}(U))$$

is exact in positive degree and has separated homology in degree zero.

Let us assume that condition (ii) holds.Regarding F as an $\mathcal{O}(\mathbb{C}^n)$-module via the map $f^*:\mathcal{O}(\mathbb{C}^n)\longrightarrow\mathcal{O}(X)$,we have to prove that the complex

$$K.(z-w,F\hat{\otimes}\mathcal{E}(f(U)))$$

is exact in positive degree and has separated homology.This assertion follows from Proposition 3.5.

The proof of Theorem 3.7 is complete.

Remarks 3.7.a) Assumption (ML) is superfluous for the implication (i)$\Rightarrow$(ii).

b) It is worth mentioning that condition (i) is local. Moreover, it can be replaced by familiar glueing methods by a similar request imposed only on local topologically free resolutions.

c) A sheaf as in the theorem which satisfies one of the conditions (i) or (ii) has a resolution with Fréchet $\mathcal{E}$-modules of length at most than the complex dimension of X.

d) Conditions for a complex of Banach free analytic modules on a manifold to be exact at the level of analytic, smooth or Sobolev sections have been recently obtained by Jantz [11] and Slodkowski [20].

We close this section by a complement to Proposition 2.2. Below all modules are supposed to have property (ML).

Proposition 3.8. *Let* E *be a Fréchet* $\mathcal{O}(\mathbb{C}^n)$*-module. Then* $E \perp_{\mathcal{O}(\mathbb{C}^n)} \mathcal{E}(\mathbb{C}^n)$ *if and only if there exists an exact sequence of Fréchet* $\mathcal{O}(\mathbb{C}^n)$*-modules:*

$$(16) \qquad 0 \longrightarrow E \longrightarrow S^0 \longrightarrow S^1 \longrightarrow \dots \longrightarrow S^{n-1} \longrightarrow \mathbb{C} \longrightarrow 0,$$

where the S^j *are Fréchet* $\mathcal{E}(\mathbb{C}^n)$*-modules*, $j=0,1,\dots,n-1$.

The proof is a simple adaptation of the proof of Proposition 2.2.

Corollary 3.9. *A Fréchet* $\mathcal{O}(\mathbb{C})$*-module* E *satisfies* $E \perp_{\mathcal{O}(\mathbb{C})} \mathcal{E}(\mathbb{C})$ *if and only if* E *is a closed submodule of a Fréchet* $\mathcal{E}(\mathbb{C})$*-module*.

By the open map principle and Corollary 3.8 we infer that a Hilbert (Banach) $\mathcal{O}(\mathbb{C})$-module E which satisfies $E \perp_{\mathcal{O}(\mathbb{C})} \mathcal{E}(\mathbb{C})$ is a submodule of a Hilbert (respectively Banach) $\mathcal{E}(\mathbb{C})$-module. However, we ignore whether in higer dimensions a resolution like (16) with Hilbert (Banach) $\mathcal{E}(\mathbb{C})$-modules exists.

4. Applications to Multidimensional Spectral Theory

Let a be a commutative n-tuple of linear continuous operators acting on the Fréchet space E. Throughout this section we assume as in [17] that all n-tuples admit a continuous functional calculus with entire functions. Recall that a is said to possess Bishop's condition (β) if, for instance, the following Koszul complex

$$K.(z-a, \mathcal{O}(D)\hat{\otimes}E)$$

is exact in positive degree and has separated homology in degree zero for every open polydisc $D\subset\mathbb{C}^n$, see [17]. A useful condition which ensures that the system a fulfills (β) is the existence of a so called Fréchet soft sheaf model for a. In other terms this means that the $\mathcal{O}(\mathbb{C}^n)$-module E attached to a coincides with the global sections space of a Fréchet soft $\mathcal{O}$-module.

The restriction of the n-tuple a to an invariant subspace F is denoted below in short by $a|F$. Similarly, the quotient system is denoted by a/F.

We are now able to restate Proposition 2.2.

Proposition 4.1. Let a be a commutative n-tuple of linear bounded operators on a Fréchet (Banach or Hilbert) space E. Then a has property (β) if and only if there exist a sequence of commutative n-tuples $a_0,\dots,a_{n-1}$, possesing Fréchet soft sheaf models and acting on the Fréchet (Banach, respectively Hilbert) spaces $S_0,\dots,S_{n-1}$, and closed invariant subspaces $E_0,\dots,E_{n-1}$, respectively, such that

$$a\sim a_0|E_0,\ a_0/E_0\sim a_1|E_1,\ \dots\ ,a_{n-2}/E_{n-2}\sim a_{n-1}|E_{n-1}. \tag{17}$$

Above "$\sim$" stands for the joint similarity equivalence relation.

In the case n=1 one obtains the main result of [1].

Condition $(\beta)_{\mathcal{E}}$ for the n-tuple a can be defined as in [7], by requiring that the complex

$$K.(z-a, \mathcal{E}(\mathbb{C}^n)\hat{\otimes}E)$$

be exact in positive degree and have separated homology in degree zero. Then Proposition 3.8 may be restated as follows.

Proposition 4.2. Let a be a commutative n-tuple of linear bounded operators on a Banach space E. The system a has property $(\beta)_{\mathcal{E}}$ if and only if there exists a sequence of inverse limits of Banach space n-tuples $a_0,\ldots,a_{n-1}$ with smooth joint functional calculus, defined on the Fréchet spaces $S_0,\ldots,S_{n-1}$, respectively, and closed invariant subspaces $E_j\subset S_j$, $j=0,\ldots,n-1$, such that relations (17) hold.

The analogue statement for Fréchet spaces and inverse limits of systems of operators on them is left to the reader.

For instance, the examples of subnormal n-tuples considered in [17] have property $(\beta)_{\mathcal{E}}$ by virtue of the above criterion.

Proposition 4.2 contains as a particular case (n=1) the main result of [7].

To draw a conclusion from the last two technical propositions, the right multidimensional analogues of subdecomposable or subscalar operators seem to be the n-tuples a which fit into a chain like (17), with a_j decomposable or scalar n-tuples, rather than simple restrictions to an invariant subspace of a single decomposable or scalar n-tuple. It is worth mentioning that all the significant examples of subnormal n-tuples satisfy these conditions.

We close this section by a stability property of condition $(\beta)_{\mathcal{E}}$ which is less transparent from its very definition.

Proposition 4.3. Let a be a commutative n-tuple of linear continuous operators on a Banach space E and let $f:U\longrightarrow \mathbb{C}^m$ be an analytic map defined on an open neighbourhood of the joint spectrum of a.

If a has property $(\beta)_{\mathcal{E}}$, then the m-tuple f(a) has also this property.

Proof. Assume a fulfills condition $(\beta)_{\mathcal{E}}$. By Propositions 4.1 and 4.2 one finds that a possesses (β). In other terms E is a quasi-coherent $\mathcal{O}(\mathbb{C}^n)$-module supported by the joint spectrum σ of a, see [17]. Consequently $E=\mathcal{F}(\mathbb{C}^n)$ and $\operatorname{supp}\mathcal{F}\subset U$, where $\mathcal{F}$ is a quasi-coherent module on $\mathbb{C}^n$. Since E is a Banach space it has a fortiori the (ML) property. Moreover, by our assumption $\mathcal{F}(D)\perp_{\mathcal{O}(D)}\mathcal{E}(D)$ for every polydisc $D\subset\subset\mathbb{C}^n$. Hence $\mathcal{F}\hat{\otimes}_{\mathcal{O}}\mathcal{E}$ is a Fréchet sheaf, also supported by σ.

The $\bar{\partial}$-resolution (Proposition 3.5) provides an exact complex

$$0 \to E \to S^0 \to S^1 \to \dots \to S^n \to 0$$

of $\mathcal{O}(\mathbb{C}^n)$-modules supported on the set σ. Moreover, S^j are $\mathcal{E}(U)$-modules for $j=0,\dots,n-1$ and E is a $\mathcal{O}(U)$-module, all with the (ML) property.

By restricting the scalars via the map $f^*: \mathcal{O}(\mathbb{C}^m) \longrightarrow \mathcal{O}(U)$, the above complex becomes a resolution of E with $\mathcal{O}(\mathbb{C}^m)$-modules and the S^j are $\mathcal{E}(\mathbb{C}^m)$-modules. Then Proposition 4.2 finishes the proof of the fact that f(a) is an m-tuple with property $(\beta)_{\mathcal{E}}$, q.e.d.

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