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ON A PROBLEM RAISED BY ALPERIN AND BASS

by

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Serban A. Basarab

Abstract

In their paper [1], R.Alperin and H.Bass raised the following:

Fundamental problem. Find the group theoretic information carried by a Λ-tree action, analoguous to that presented in (Serre's book) Trees for the case $\Lambda = \mathbb{Z}$.

The aim of this paper is to give a possible answer to the question above by extending to arbitrary ordered abelian groups Λ the main constructions and results contained in the first chapter of Serre's book.

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The aim of this paper is to give a possible answer to the question above by extending to arbitrary ordered abelian groups Λ the main constructions and results contained in the first chapter of Serre's book [3].

The principal objects investigated in [3] Ch.I are ordinary graphs, in particular ordinary trees, and group actions on ordinary graphs. The group theoretic information carried by a group action on an ordinary tree is contained in the graph of groups, assigned to such an action, and its fundamental group.

To get the required generalization for an arbitrary ordered abelian group Λ, we first need a suitable extension of the concept of ordinary graph. In Section 1 we introduce such a general concept, called Λ-graph, and we show that the category of ordinary graphs is equivalent with a category of $\underline{Z}$-graphs. On the other hand, the Λ- trees investigated in [2], [1] are identified with particular Λ -graphs.

Next it is necessary to generalize the basic concept of graph of groups. This task is achieved in Section 2, where the more general and quite technical cnocept of Λ -graph of groups is defined and investigated.

Section 3 is devoted to the fundamental group of a strongly connected Λ -graph of groups, which turns out to be a natural, but more technical, generalization of the fundamental group of an ordinary graph of groups.

The transition from Λ-graphs of groups to group actions on Λ-trees is considered in Section 4. The main result of the paper, an extension of structure Theorem 13 from [3] Ch.I, is proved in Section 5, where the basic concept of the universal cover of a group action on a connected Λ -graph is defined and investigated.

Finally, Section 6 provides two examples in order to illustrate the general theory.

1. Λ - graphs.

By a groupoid we understand a small category $\underline{\underline{X}}$ whose morphisms are isomorphisms. Denote by X the set of objects of $\underline{\underline{X}}$, called also the vertices of $\underline{\underline{X}}$. For $x,y \in X$, denote by $\underline{\underline{X}}(x,y)$ the set of morphisms, called also arrows, of $\underline{\underline{X}}$ from x into y. For $f \in \underline{\underline{X}}(x,y)$ call the vertex $\underline{\underline{o}}(f)=x$ the origin of f, and the vertex $\underline{\underline{t}}(f)=y$ the terminus of f. Denote by 1_x the identity of $x \in X$, by $f^{-1} \in \underline{\underline{X}}(y,x)$ the inverse of $f \in \underline{\underline{X}}(x,y)$, and by $fg \in \underline{\underline{X}}(x,z)$ the composite of $f \in \underline{\underline{X}}(x,y)$ and $g \in \underline{\underline{X}}(y,z)$.

The groupoid $\underline{\underline{X}}$ is connected if $\underline{\underline{X}}(x,y)$ is non-empty for all pairs $(x,y) \in X^2$. A grupoid $\underline{\underline{X}}$ is the direct sum in the category of groupoids of its connected components.

Now let Λ be an ordered abelian group. By a Λ- metric groupoid we understand a groupoid $\underline{\underline{X}}$ with a "length" function $| \; | :$ arrow $\underline{\underline{X}} \to \Lambda : f \mapsto |f|$, satisfying:

i) $|f| \geqslant 0$ for every arrow f of $\underline{\underline{X}}$;

ii) $|f|=0$ iff $f=1_x$ for some $x \in X$;

iii) $|f^{-1}|=|f|$ for every arrow f of $\underline{\underline{X}}$;

iv) $|fg| \leqslant |f|+|g|$ for every pair $(f,g) \in \underline{\underline{X}}(x,y) \times \underline{\underline{X}}(y,z)$, $x,y,z \in X$.

The Λ-metric spaces as defined in [2], [1], are identified with the Λ- metric connected groupoids $\underline{\underline{X}}$ for which the set $\underline{\underline{X}}(x,y)$ has only one element for all $x,y \in X$.

The Λ-metric groupoids forms a category $\Lambda\underline{\underline{MG}}$ having as morphisms $F:(\underline{\underline{X}}, | \; |) \to (\underline{\underline{Y}}, | \; |)$ the morphisms $F:\underline{\underline{X}} \to \underline{\underline{Y}}$ of groupoids, i.e. functors. F is an isometry if $F:\underline{\underline{X}} \to \underline{\underline{Y}}$ is an isomorphism of groupoids and $|F(f)|=|f|$ for every arrow f of $\underline{\underline{X}}$.

Now we put in evidence a subcategory of $\Lambda\underline{\underline{MG}}$ which is a natural generalization of the category of the ordinary graphs as well as of the category of Λ- trees as defined [2], [1].

Definition 1.1. A Λ- graph is a groupoid $\underline{\underline{X}}$ with a function $|\ |$: arrow $\underline{\underline{X}}\to\Lambda$ satisfying:

a) the axioms i), ii), iii) from the definition of a Λ- metric groupoid;

b) for every pair $(f,g)\in\underline{\underline{X}}(x,y)\times\underline{\underline{X}}(y,z)$, $x,y,z\in X$, $\delta(f,g):=|f|+|fg|-|g|\in 2\Lambda$;

c) $\delta(f,g)\geqslant\min(\delta(fg,h),\delta(f,gh))$ for every triple $(f,g,h)\in\underline{\underline{X}}(x,y)\times\underline{\underline{X}}(y,z)\times\underline{\underline{X}}(z,u)$, $x,y,z,u\in X$;

d) the map $[g]=\{f:\underline{\underline{t}}(f)=\underline{\underline{o}}(g),\ \delta(f,g)=0\}\to[0,|g|]=$ $=\{\alpha\in\Lambda:0\leqslant\alpha\leqslant|g|\}:f\mapsto|f|$ is onto.

Lemma 1.2. Any Λ-graph $\underline{\underline{X}}$ is a Λ-metric groupoid.

Proof. We have to verify the triangle inequality $|fg|\leqslant|f|+|g|$ for $(f,g)\in\underline{\underline{X}}(x,y)\times\underline{\underline{X}}(y,z)$. Consider the triple $(f^{-1}:y\to x,\ fg:x\to z,\ g^{-1}:z\to y)$. We get $\delta(f^{-1},fg)=|f|+$ $+|g|-|fg|$, $\delta(f^{-1}fg,g^{-1})=\delta(g,g^{-1})=0$, $\delta(f^{-1},fgg^{-1})=\delta(f^{-1},f)=0$. It remains to apply the axiom c) of Definition 1.1. ∎

Denote by $\Lambda\underline{\underline{GRAPHS}}$ the full subcategory of $\Lambda\underline{\underline{MG}}$ whose objects are the Λ-graphs.

Given a Λ- metric groupoid $\underline{\underline{X}}$ and a pair $(f,g)\in\underline{\underline{X}}(x,y)\times$ $\times\underline{\underline{X}}(y,z)$, let $Y(f,g)=\{h\in[f]\cap[fg]:\ hf\in[g]\}$. The following lemma provides an equivalent definition of Λ-graphs.

Lemma 1.3. The necessary and sufficient condition for a Λ-metric groupoid to be a Λ-graph is that

i) the map $[g]\to[0,|g|]:f\mapsto|f|$ is bijective for every arrow g of $\underline{\underline{X}}$, and

ii) the set $Y(f,g)$ is non-empty (in fact a singleton) for every pair $(f,g)\in\underline{\underline{X}}(x,y)\times\underline{\underline{X}}(y,z)$, $x,y,z\in X$.

Proof. Assume $\underline{\underline{X}}$ is a Λ-graph, and let $g\in\underline{\underline{X}}(y,z)$. First we have to show that the canonic map $[g]\to[0,|g|]$ is injective. In fact we prove a stronger fact: the map above is an isometry, i.e., $|fh^{-1}|=\big||f|-|h|\big|$ for $f,h\in[g]$. As $\underline{\underline{X}}$ is a Λ-metric groupoid, we get $|f|\leq|fh^{-1}|+|h|$, $|h|\leq|hf^{-1}|+|f|=|f^{-1}h|+|f|$, and hence $|fh^{-1}|\geq\big||f|-|h|\big|$. On the other hand, applying the axiom c) of Definition 1.1 to the triple $(f,g,g^{-1}h^{-1})$, we get $0=\delta(f,g)\geq\min(\delta(fg,g^{-1}h^{-1}),\ \delta(f,h^{-1}))$. As $f,h\in[g]$, it follows $\delta(fg,g^{-1}h^{-1})=|fg|+|fh^{-1}|-|hg|=|fh^{-1}|-|f|+|h|$. Since $\delta(f,h^{-1})=|f|+|fh^{-1}|-|h|$, we conclude that $|fh^{-1}|\leq\big||f|-|h|\big|$.

Next we have to show that the set $Y(f,g)$ is a singleton for every pair (f,g) of arrows of $\underline{\underline{X}}$ such that $\underline{\underline{t}}(f)=\underline{\underline{s}}(g)$. By the axiom b) of Definition 1.1, there exists a unique $\gamma\in\Lambda$ such that $\delta(f,g)=2\gamma$. Obviously, $0\leq\gamma\leq\min(|f|,\ |fg|)$. As the canonic map $[f]\to[0,|f|]$ is bijective, there is a unique $h\in[f]$ such that $|h|=\gamma$. It remains to show that $h\in[fg]$ and $hf\in[g]$ to conclude that $Y(f,g)=\{h\}$. Considering the triple (h,f,g) we get $0=\delta(h,f)\geq\min(\delta(hf,g),\ \delta(h,fg))$. As $\delta(hf,g)=$

$=|hf|+|hfg|-|g|-|f|-|h|+|hfg|-|g|=\gamma+|hfg|-|fg|=\delta(h,fg)$,

it follows $\delta(hf,g)=\delta(h,fg)=0$, i.e., $h\in[fg]$ and $hf\in[g]$.

Conversely, assume the Λ-metric groupoid $\underline{\underline{X}}$ satisfies the conditions i) and ii) of the statement above. We have to show that $\underline{\underline{X}}$ is a Λ-graph. The conditions a), b) and d) of Definition 1.1 are trivially satisfied, so it remains to verify the condition c). Consider a triple (f,g,h) of arrows of $\underline{\underline{X}}$ such that $\underline{\underline{t}}(f)=\underline{\underline{o}}(g)$ and $\underline{\underline{t}}(g)=\underline{\underline{o}}(h)$. Denote by $\underline{\underline{Y}}(f,g)$ the unique element of the set $Y(f,g)$. Note that $\delta(f,g)=2|\underline{\underline{Y}}(f,g)|$. Assuming $\delta(f,gh)\leq\delta(fg,h)$, it follows by the conditions i) that $\underline{\underline{Y}}(f,gh)\in[f]\cap[fg]$, and hence $|\underline{\underline{Y}}(f,g)|\geq|\underline{\underline{Y}}(f,gh)|$, i.e., $\delta(f,g)\geq\delta(f,gh)$. The case $\delta(fg,h)\leq\delta(f,gh)$ is similar. ∎

Lemma 1.4. Let $\underline{\underline{X}}$ be a Λ-graph, $x\in X$ and $f\in\underline{\underline{X}}(x,x)$. The following statements are equivalent:

i) $f^2=1_x$ and $|f|\in 2\Lambda$;

ii) $f=1_x$.

Proof. We have only to show that i)$\Rightarrow$ii).
Let $|f|=2\gamma$. There is a unique arrow g such that $\underline{\underline{t}}(g)=x$ and $|g|=|gf|=\gamma$. As $f^2=1_x$ it follows $g=gf$ and hence $f=1_x$. ∎

Lemma 1.5. Assume Λ has a smallest positive element, so the ordered group of integers $\underline{\underline{Z}}$ is identified with a convex subgroup of Λ. Assume also that the factor group $\Lambda/\underline{\underline{Z}}$ is 2-divisible, i.e., $\Lambda=\underline{\underline{Z}}+2\Lambda$. The next statements are equivalent for a Λ- graph $\underline{\underline{X}}$:

i) There exist $x\in X$ and $f\in\underline{\underline{X}}(x,x)$ such that $f^2=1_x$ and

$f \neq 1_x$;

ii) There exist $y \in X$ and $g \in \underline{\underline{X}}(y,y)$ such that $g^2 = 1_y$ and $|g| = 1$.

Proof. We have only to show that i) $\Rightarrow$ ii). Let $f \in \underline{\underline{X}}(x,x)$ be such that $f^2 = 1_x$ and $f \neq 1_x$. By Lemma 1.4, we must have $|f| = 2\gamma + 1$ with $\gamma \in \Lambda$, $\gamma \geq 0$. As $\underline{\underline{X}}$ is a Λ- graph there is a unique arrow f_α such that $\underline{\underline{t}}(f_\alpha) = x$, $|f_\alpha| = \alpha$ and $|f_\alpha f| = 2\gamma + 1 - \alpha$ for each $\alpha \in \Lambda$ such that $0 \leq \alpha \leq 2\gamma + 1$. Moreover, $|f_\alpha f_\beta^{-1}| = |\alpha - \beta|$ for $0 \leq \alpha, \beta \leq 2\gamma + 1$. Since $f^2 = 1_x$ we get $f_\alpha f = f_{2\gamma + 1 - \alpha}$ for $0 \leq \alpha \leq 2\gamma + 1$. Let $g = f_\gamma f_{\gamma+1}^{-1} = f_\gamma f f_\gamma^{-1} \in \underline{\underline{X}}(y,y)$, where $y = \underline{\underline{o}}(f_\gamma)$. Thus $g^2 = 1_y$ and $|g| = 1$. $\square$

The Λ-graphs which are also Λ- metric spaces, i.e., $\underline{\underline{X}}(x,y)$ is a singleton for every pair $(x,y) \in X^2$, are exatly the Λ - trees defined in [2], [1].

In the following we show that the concept of Λ-graph is also a natural generalization of the concept of ordinary graph. An ordinary graph $\underline{\underline{X}}$ consists of a set X = vert $\underline{\underline{X}}$, a set edge $\underline{\underline{X}}$ and two maps edge $\underline{\underline{X}} \to X^2 : f \mapsto (\underline{\underline{o}}(f), \underline{\underline{t}}(f))$ and edge $\underline{\underline{X}} \to$ edge $\underline{\underline{X}} : f \mapsto \bar{f}$ which satisfy the following condition: for each $f \in$ edge $\underline{\underline{X}}$ we have $\bar{\bar{f}} = f$, $\bar{f} \neq f$ and $\underline{\underline{o}}(f) = \underline{\underline{t}}(\bar{f})$.

Let $\underline{\underline{C}}$ be the subcategory of $\underline{\underline{Z}}$ GRAPHS having as objects the Z-graphs $\underline{\underline{X}}$ satisfying the condition: $f = 1_x$ if $f \in \underline{\underline{X}}(x,x)$ and $f^2 = 1_x$, and as morphisms the functors $F : \underline{\underline{X}} \to \underline{\underline{Y}}$ subject to $|F(f)| \leq |f|$ and $|F(f)| \equiv |f| \bmod 2$ for each arrow f of $\underline{\underline{X}}$. Denote by $\underline{\underline{D}}$ the category of ordinary graphs, and consider the functor $G : \underline{\underline{C}} \to \underline{\underline{D}}$ which assigns to an object $\underline{\underline{X}}$ of $\underline{\underline{C}}$ the ordinary graph

$G(\underline{\underline{X}})$ with vert $G(\underline{\underline{X}})=X$, edge $G(\underline{\underline{X}})=\{f\in \text{arrow } \underline{\underline{X}} : |f|=1\}$, and $\underline{\underline{o}}(f)=x$, $\underline{\underline{t}}(f)=y$, $\bar{f}=f^{-1}$ for $f\in\underline{\underline{X}}(x,y)$, $|f|=1$. If $F:\underline{\underline{X}}\to\underline{\underline{Y}}$ is a morphism in $\underline{\underline{C}}$ then $G(F):G(\underline{\underline{X}})\to G(\underline{\underline{Y}})$ is canonically induced by F.

Proposition 1.6. The functor $G:\underline{\underline{C}}\to\underline{\underline{D}}$ is an equivalence of categories.

Proof. First let us show that G is totally faithfull, i.e., the map $\underline{\underline{C}}(\underline{\underline{X}},\underline{\underline{Y}})\to\underline{\underline{D}}(G(\underline{\underline{X}}),\ G(\underline{\underline{Y}}))$: $F\mapsto G(F)$ is bijective for each pair $(\underline{\underline{X}},\ \underline{\underline{Y}})$ of objects of $\underline{\underline{C}}$. Let us check the injectivity.

Let $F_1,F_2\in\underline{\underline{C}}(\underline{\underline{X}},\underline{\underline{Y}})$ be such that $G(F_1)=G(F_2)$. By definition of G, we have $F_1(x)=F_2(x)$ for each $x\in X$ and $F_1(f)=F_2(f)$ for each arrow f of $\underline{\underline{X}}$ such that $|f|=1$. In order to show that $F_1(f)=F_2(f)$ for each arrow f of $\underline{\underline{X}}$ we proceed by induction on $n=|f|$. The cases $n=0,1$ are verified by assumption. Assume $|f|=n+1$, $n\geq 1$. As $\underline{\underline{X}}$ is a Z-graph, there exists an arrow g of $\underline{\underline{X}}$ such that $\underline{\underline{t}}(g)=\underline{\underline{o}}(f)$, $|g|=n$ and $|gf|=1$. Thus $F_1(gf)=F_2(gf)$ and $F_1(g)=F_2(g)$ by induction hypothesis, and hence $F_1(f)=$ $=F_2(f)$.

Next let us check the surjectivity. Let $H:G(\underline{\underline{X}})\to G(\underline{\underline{Y}})$ be a morphism of ordinary graphs. We look for $F\in\underline{\underline{C}}(\underline{\underline{X}},\ \underline{\underline{Y}})$ such that $G(F)=H$. We are forced to put $F(x)=H(x)$ for $x\in X$, $F(1_x)=$ $=1_{H(x)}$ for $x\in X$ and $F(f)=H(f)$ for every arrow f of $\underline{\underline{X}}$ with $|f|=1$. In order to extend H to $F\in\underline{\underline{C}}(\underline{\underline{X}},\ \underline{\underline{Y}})$ we proceed as follows. Let f be an arrow of $\underline{X}$ and let $|f| = n$. As $\underline{X}$ is a $\underline{Z}$-graph there exists a unique sequence $\underline{f}=(f_i)_{1\leq i\leq n}$ of arrows such that $\underline{o}(f_1)=\underline{o}(f)$, $\underline{t}(f_n)=\underline{t}(f)$, $\underline{t}(f_i)=\underline{o}(f_{i+1})$ for $1\leq i<n$, $|f_i| = 1$ for $1\leq i\leq n$ and $f=f_1f_2\ldots f_n$.

Let us put $F(f)=H(f_1)H(f_2)\ldots H(f_n)$. First let us show that $F(fg)=F(f)F(g)$ for each pair (f,g) of arrows of $\underline{X}$ with $\underline{t}(f)=\underline{o}(g)$. Let $n=|f|$, $m=|g|$ and $\underline{f}$, $\underline{g}$ be the sequences assigned to f, g as above. Let $k = \dfrac{\delta(f,g)}{2} \in \underline{N}$. Then $f_{k+i}=$ $=g^{-1}_{n-k-i+1}$ for $1\leq i\leq n-k$ and $(f_1,\ldots,f_k,g_{n-k+1},\ldots,g_m)$ is the sequence assigned to fg. The required equality is immediate. It remains to verify the conditions: $|F(f)|\leq|f|$ and $|F(f)|\equiv|f|$ mod 2. The first one is immediate by the triangle inequality. To check the second one we proceed by induction on $n=|f|$. The cases $n=0,1$ are trivial, so we may assume $|f| = n+1$, $n\geq 1$. Write $f=gh$ with $|g|=n$, $|h|=1$. Then $F(f)=F(g)F(h)$, $|F(h)| = 1$ and $|F(g)| \equiv n$ mod 2 by induction hypothesis. Since $\delta(F(g), F(h)) = |F(g)| + |F(f)| - 1 \equiv 0$ mod 2 it follows $|F(f)|\equiv n+1$ mod 2, as required.

Finally we have to show that for every ordinary graph $\underline{Y}$ there exists an object $\underline{X}$ of $\underline{C}$ such that $G(\underline{X})$ and $\underline{Y}$ are isomorphic. Given $\underline{Y}$, let us define the $\underline{Z}$-graph $\underline{X}$ as follows. Let $X=$ $=\mathrm{vert}\,\underline{Y}$. If $x,y\in X$, let $\underline{X}(x,y)$ be the set of paths without backtracking of $\underline{Y}$ with origin in x and terminus in y, i.e., sequences $f=(f_1,\ldots,f_n)$ of edges of $\underline{Y}$ such that $\underline{o}(f_1)=x$, $\underline{t}(f_n)=y$, $\underline{t}(f_i)=\underline{o}(f_{i+1})$ and $f_{i+1}\neq\bar{f_i}$ for $1\leq i<n$. If $f\in\underline{X}(x,y)$ and $g\in\underline{X}(y,z)$, let $fg\in\underline{X}(x,z)$ be the path without backtracking assigned to the

path obtained by the usual composition of the paths f and g. For $f\in\underline{\underline{X}}(x,y)$ let $|f|$ be the length of the path f. One checks easily that $\underline{\underline{X}}$ is an object of $\underline{\underline{C}}$ and $G(\underline{\underline{X}})\cong\underline{\underline{Y}}$. $\square$

Let us mention the special case of the Λ-graphs $\underline{\underline{X}}$ for which the set X of vertices of $\underline{\underline{X}}$ is a singleton. Then arrow $\underline{\underline{X}}$ becomes a group. Assuming in addition that $f^2=1\Rightarrow f=1$ for each arrow f, we get a natural generalization of the concept of free group. Indeed, if $\Lambda=\mathbb{Z}$ and A is an orientation of $\underline{\underline{X}}$, i.e., $A\subset\{f\in \text{arrow } \underline{\underline{X}} ; |f|=1\}$ and for each arrow f of $\underline{\underline{X}}$ with $|f|=1$, $f\in A$ iff $f^{-1}\notin A$, then the group arrow $\underline{\underline{X}}$ is freely generated by A. Conversely, if $F=F(A)$ is the free group generated by a set A, let us put arrow $\underline{\underline{X}}=F$ and let $|f|$ be the length of $f\in F$ seen as a reduced word over A.

2. Λ - graphs of groups

In this section we introduce a more general concept than the concept of Λ-graph defined in Section 1, which turns out to be a technical extension of the notion of *graph of groups* as defined in [3].

Let Λ be an ordered abelian group. By a Λ-*graph of groups* we understand a structure $\mathbf{X}$ consisting of the following data:

1) a non-empty set X = vert $\mathbf{X}$, a set $E=E_{\mathbf{X}}$ and two maps $E \to X^2 : f \mapsto (\mathbf{o}(f), \mathbf{t}(f))$ and $E \to E : f \mapsto \bar{f}$ such that $\mathbf{o}(f)=\mathbf{t}(\bar{f})$ and $\bar{\bar{f}}=f$; set $E(x,y)=\{f\in E : \mathbf{o}(f)=x, \mathbf{t}(f)=y\}$; thus (X,E) is an "ordinary graph" with the possibility that $\bar{f}=f$ for some $f\in E$ with $\mathbf{o}(f)=\mathbf{t}(f)$;

2) a family $(1_x)_{x\in X}$, where $1_x\in E(x,x)$ and $\bar{1}_x=1_x$;

3) a family of groups $(G_f)_{f\in E}$ such that G_f is a subgroup of $G_{\mathbf{o}(f)}$, where $G_x:G_{1_x}$ for $x\in X$;

4) a family of isomorphisms $(\omega_f : G_f \to G_{\bar{f}})_{f\in E}$ such that $\omega_x := \omega_{1_x}$ is the identity of G_x for $x\in X$;

5) a family $(\theta_f)_{f\in E}$, where $\theta_f\in G_f$ such that $\theta_f=1$ if $f\neq\bar{f}$ or $f=1_x$ for $x\in X$, $\omega_{\bar{f}}(\omega_f(s))=\theta_f^{-1}s\theta_f$ for $s\in G_f$, and $\omega_f(\theta_f)=$ $=\theta_{\bar{f}}$;

6) a map $\varepsilon : E(x,y)\times G_y\times E(y,z)\to E(x,z)$ for $x,y,z\in X$; a coset $\rho(A)\in G_x/G_{\varepsilon(A)}$ and a bijection $\lambda_A : \rho(A)\to\rho(\bar{A})$ for $A=$ $=(f,s,g)$, $\bar{A}=(\bar{g},\theta_g^{-1}s^{-1}\theta_{\bar{f}}^{-1},\bar{f})$, satisfying:

i) $\overline{\mathcal{E}(A)}=\mathcal{E}(\bar{A})$, $\lambda_{\bar{A}}(\lambda_A(p))=p\theta_{\mathcal{E}(A)}$ for $p\in\rho(A)$ and λ_A is compatible with $\omega_{\mathcal{E}(A)}$, i.e., $\lambda_A(pa)=\lambda_A(p)\omega_{\mathcal{E}(A)}(a)$ for $p\in\rho(A)$ and $a\in G_{\mathcal{E}(A)}$;

ii) $\mathcal{E}(A)=f$, $\rho(A)=G_f$ and $\lambda_A(1)=s^{-1}$ for $A=(f,s,1_y)$, $f\in E(x,y)$, $s\in G_y$;

iii) $\mathcal{E}(A)=1_x$ and $\lambda_A(1)=\theta_f^{-1}s^{-1}$ for $A=(f,\omega_f(s),\bar{f})$, $f\in E(x,y)$, $s\in G_f$;

iv) $\mathcal{E}(A)=\mathcal{E}(B)$, $a\rho(A)=\rho(B)$ and $\lambda_B(ap)=\lambda_A(p)$ for $A=(f,s,g)$, $B=(f,\omega_f(a)s,g)$, $a\in G_f$, $p\in\rho(A)$;

v) $\mathcal{E}(C)=\mathcal{E}(D)$, $u\rho(C)=\rho(D)$ and $\lambda_D(up)=\lambda_B(v)^{-1}\lambda_C(p)$ for $A=(f,s,g)$, $B=(g,t,h)$, $u\in\rho(A)$, $v\in\rho(B)$, $C=(\mathcal{E}(A),\lambda_A(u)^{-1}t,h)$, $D=(f,sv,\mathcal{E}(B))$, $p\in\rho(C)$;

7) a "length" function $E\to\Lambda : f\mapsto|f|$ satisfying:

i) $|f|\geq 0$, with equality iff $f=1_x$ for some $x\in X$;

ii) $|f|=|\bar{f}|$ for each $f\in E$;

iii) $\delta(f,s,g):=|f|+|\mathcal{E}(f,s,g)|-|g|\in 2\Lambda$ for each suitable triple (f,s,g);

iv) $\delta(A)\geq\min(\delta(C),\delta(D))$ for A,C,D as in 6.v);

v) for each $g\in E$ and for each $\gamma\in[0,|g|]$ there exist $f\in E$ and $s\in G_{\underline{o}(g)}$ such that $\underline{t}(f)=\underline{o}(g)$, $\delta(f,s,g)=0$ and $|f|=\gamma$.

The Λ-graphs defined in Section 1 are obviously identified with the Λ-graphs of groups $\underline{X}$ for which the groups G_x are trivial for all $x\in X$.

<u>Lemma 2.1.</u> Let $\underline{X}$ be a structure satisfying the conditions 1)-6) above. Then $\underline{X}$ has the following properties:

6.vi) $\mathcal{E}(A)=f$, $\rho(A)=sG_f$ and $\lambda_A(s)=1$ for $A=(1_x,s,f)$, $f\in E(x,y)$, $s\in G_x$;

$B=(f,\omega_f(a)sb,g)$, $a\in G_f$, $b\in G_g$, $p\in\rho(A)\}$ 3

6.viii) Let $f\in E(x,y)$, $g\in E(y,x)$ and $s\in G_y$. Then $\varepsilon(f,s,g)=1_x$ iff $f=\bar g$ and $s\in G_g$.

Proof. 6.vi) is immediate by 6.i) and 6.ii).

To prove 6.vii), let $C=(f,\omega_f(a)s,g)$ and apply 6.iv) to the triples A and C. We get $\varepsilon(A)=\varepsilon(C)$, $a\rho(A)=\rho(C)$, $\rho(\bar A)=\rho(\bar C)$ and $\lambda_A(p)=\lambda_C(ap)$. Applying 6.iv) to the triples $\bar C$ and $\bar B$, we get $\varepsilon(\bar C)=\varepsilon(\bar B)$, $\omega_g(b)^{-1}\rho(\bar A)=\rho(\bar B)$, $a\rho(A)=\rho(B)$ and $\lambda_{\bar B}(\omega_g(b)^{-1}\lambda_A(p))=\lambda_{\bar C}(\lambda_A(p))$. By 6.i) it follows $\varepsilon(A)=\varepsilon(C)=\varepsilon(B)$ and $\omega_g(b)^{-1}\lambda_A(p)\theta_{\varepsilon(\bar A)}=\lambda_B(\lambda_{\bar B}(\omega_g(b)^{-1}\lambda_A(p)))=\lambda_B(\lambda_{\bar C}(\lambda_C(ap)))=$ $=\lambda_B(ap\,\theta_{\varepsilon(A)})=\lambda_B(ap)\,\omega_{\varepsilon(A)}(\theta_{\varepsilon(A)})=\lambda_B(ap)\theta_{\varepsilon(\bar A)}$, and hence $\lambda_B(ap)=\omega_g(b)^{-1}\lambda_A(p)$, as required.

To prove 6.viii), let $A=(f,s,g)$, $B=(g,1,\bar g)$, $C=(\varepsilon(A),u^{-1},\bar g)=(1_x,u^{-1},\bar g)$ and $D=(f,sv,\varepsilon(B))$ with $u\in\rho(\bar A)$, $v\in\rho(B)$. By 6.iii), $\varepsilon(B)=1_y$ and hence we may take $v=1$ and $D=(f,s,1_y)$. By 6.ii), 6.v) and 6.vi), we get $\bar g=\varepsilon(C)=\varepsilon(D)=f$ and $s\in G_g$, as required. ◻

Remark. By 6.vii), $\varepsilon(f,s,g)$ depends only on f,g and the double coset $G_f sG_g$, so we may write $\varepsilon(f,\alpha,g)$ and $\delta(f,\alpha,g)$ for $f\in E(x,y)$, $g\in E(y,z)$, $\alpha\in G_f\backslash G_y/G_g$, or $\alpha\in G_f\backslash G_y$, or $\alpha\in G_y/G_g$. Similarly, $\rho(f,s,g)$ depends only on f,g and the coset $s\,G_g$, so we may write $\rho(f,\alpha,g)$ for $\alpha\in G_y/G_g$.

Lemma 2.2. In the presence of the conditions 1)-6), the following versions of the triangle inequality are equivalent for a map $E\to\Lambda: f\mapsto|f|$ satisfying 7.ii):

7.vi) $\delta(f,s,g)\geqslant 0$ for each triple $(f,s,g)\in E(x,y)\times G_y\times E(y,z)$ $x,y,z\in X$;

7.vi)' $|\varepsilon(f,s,g)|\leqslant|f|+|g|$ for each triple (f,s,g) as above.

<u>Proof.</u> 7.vi) $\Rightarrow$ 7.vi)': Let $A=(f,s,g)$, $B=(\bar{f},\rho(A),\ \varepsilon(A))$, $C=(\bar{f},1,f)$. By 6.v), 6.iii), 6.vi) and 6.vii) we get $\varepsilon(B)= =\varepsilon(\varepsilon(C),\ \rho(\bar{C})^{-1}s,g)=\varepsilon(1_y,1,g)=g$, and hence $0\leq\delta(B)=|f|+|g|- -|\varepsilon(A)|$ by 7.ii) and 7.vi).

7.vi)' $\Rightarrow$ 7.vi): Applying 7.ii) and 7.vi)' to the triple B above we get $\delta(A)\geq 0$. $\square$

The following statement is an extension of Lemmata 1.2 and 1.3.

<u>Proposition 2.3.</u> Let $\underline{\underline{X}}$ be a structure satisfying the conditions 1)–6) and 7.i), 7.ii). For $g\in E(y,z)$, let $[g]= =\{(f,p): f\in E,\ \underline{\underline{t}}(f)=y,\ p\in G_{\bar{f}}\setminus G_y \text{ and } \delta(f,p,g)=0\}$, and $\alpha_g\colon [g]\to[0,|g|]$ be the map given by $\alpha_g(f,p)=|f|$. For $(f,s,g)\in E(x,y)\times G_y\times E(y,z)$, let $Y(f,s,g)=\{(h,t)\in[f]: \delta(h,t\rho(f,s,g),\ \varepsilon(f,s,g))=\delta(\varepsilon(h,t,f),\ \rho(\bar{f},\ \theta_f^{-1}t^{-1},\bar{h})^{-1}s,g)=0\}$. $[g]$, α_g and $Y(f,s,g)$ are well-defined according to the remark above. The following statements are equivalent:

a) $\underline{\underline{X}}$ is a Λ-graph of groups.

b) $\underline{\underline{X}}$ has the next properties:

7.vii) α_g is bijective for each $g\in E$;

7.viii) $Y(f,s,g)$ is non-empty for each suitable triple (f,s,g).

c) $\underline{\underline{X}}$ satisfies the triangle inequality 7.vi) and the following conditions:

7.vii)' α_g is an isometry for each $g\in E$, i.e., α_g is bijective and $d((f,s),(h,t)):=|\varepsilon(f,st^{-1},\bar{h})|=||f|-|h||$ for

(f,s), $(h,t)\in[g]$;

7.viii)' $Y(f,s,g)$ is a singleton for each suitable triple (f,s,g), depending only on f,g and the double coset $G_f^{*}sG_g$.

Proof. a)$\Rightarrow$ c): Assume $\underline{X}$ is a Λ-gruph of groups. First let us check 7.vi). Let $A=(f,s,g)\in E(x,y)\times G_y\times E(y,z)$, $p\in\rho(A)$ and $C=(\varepsilon(A), 1, \varepsilon(\bar{A}))$. We get $\varepsilon(\bar{A})=\overline{\varepsilon(A)}$ and $\varepsilon(C)=1_x$ by 6.i) and 6.iii), and hence $\delta(C)=0$ by 7.i) and 7.ii). Let $F=(g,\lambda_A(p), \varepsilon(\bar{A}))$ and $D=(f,sq, \varepsilon(F))$, with $q\in\rho(F)$. Applying 6.v) to the quintuple $(f,s,g,\lambda_A(p), \varepsilon(\bar{A})$, we get $\varepsilon(D)=\varepsilon(C)=1_x$, and hence $\varepsilon(F)=\bar{f}$, by 6.viii). It follows $\delta(D)=0$, by 7.i) and 7.ii). As $\delta(A)\geq\min(\delta(C), \delta(D)$, by 7.iv), it follows $\delta(A)\geq 0$, as required.

Next let us check 7.viii)'. Let $g\in E(y,z)$. As α_g is onto by 7.v), it remains to show that $d:=|\varepsilon(f,st^{-1},\bar{h})|=||f|-|h||$ if $\delta(f,s,g)=\delta(h,t,g)=0$. Indeed, if so, then the injectivity of α_g follows by 7.i) and 6.viii), $([g], d)$ becomes a Λ-metric space and α_g is an isometry. Let $A=(f,s,g)$, $B=(h,t,g)$, $C=(\varepsilon(A),p^{-1}q, \varepsilon(\bar{B}))$ with $p\in\rho(\bar{A})$, $q\in\rho(\bar{B})$, $F=(g,q, \varepsilon(\bar{B}))$, $D=(f,st^{-1}, \varepsilon(F))$ and $H=(1_y,t^{-1}G_h^{-1},\bar{h})$. By 6.V, 6.iii) and 6.vi), we get $\varepsilon(F)=\varepsilon(\varepsilon(g,1,\bar{g}), t^{-1}G_h^{-1},\bar{h})=\varepsilon(H)=\bar{h}$ and $\rho(F)=\rho(H)=t^{-1}G_{\bar{h}}$, and hence $D=(f,st^{-1},\bar{h})$. On the other hand, $\varepsilon(C)=\varepsilon(D)$ by 6.v), and $\overline{\varepsilon(B)}=\varepsilon(\bar{B})$ by 6.i). Consequently, $\delta(C)=|\varepsilon(A)|+d-|\varepsilon(B)|=d-|f|+|h|$ since $\delta(A)=\delta(B)=0$ by assumption, and $\delta(D)=|f|+d-|h|$. As $0=\delta(A)\geq\min(\delta(C), \delta(D))$ by 7.iv), it follows $d=||f|-|h||$, as required.

Finally let us check 7.viii)'. Let $A=(f,s,g)\in E(x,y)\times G_y\times E(y,z)$. By 7.iii), 7.vi) and 7.vi)', $\delta(f,s,g)=2\gamma$ with $\gamma\in\Lambda$ such that $0\leq\gamma\leq\min(|f|,|\varepsilon(A)|)$. As α_f is onto there exists a pair (h,t) such that $\underline{\underline{t}}(h)=x$, $t\in G_x$, $\delta(h,t,f)=0$ and $|h|=\gamma$. Let us show that $(h,G_{\bar h}t)\in Y(f,s,g)$. Let $B=(h,t,f)$, $C=(h,tp,\varepsilon(A))$, $D=(\varepsilon(B),q^{-1}s,g)$ with $p\in\rho(A)$, $q\in\rho(\bar B)$. As $0=\delta(B)\geq\min(\delta(C),\delta(D))$ by 7.iv), it remains to show that $\delta(C)=\delta(D)$. We get $\delta(C)=|h|+|\varepsilon(C)|-|\varepsilon(A)|=\gamma+|\varepsilon(C)|-(2\gamma-|f|+|g|)=|\varepsilon(C)|+|f|-|g|-\gamma$ and $\delta(D)=|\varepsilon(B)|+|\varepsilon(D)|-|g|=(|f|-|h|)+|\varepsilon(D)|-|g|=|\varepsilon(D)|+|f|-|g|-\gamma$, and hence $\delta(C)=\delta(D)$ since $\varepsilon(C)=\varepsilon(D)$ by 6.v). Thus $(h,G_{\bar h}t)\in Y(f,s,g)$. The unicity of $(h,G_{\bar h}t)$ is immediate thanks to the injectivity of α_f. Thus $Y(f,s,g)$ is a singleton depending obviously only on f,g and the double coset $G_{\bar f}\,s\,G_y$.

$b)\Leftrightarrow c)$: We have only to show that $b)\Rightarrow c)$ 7.vi) is immediate. Indeed, let $(h,t)\in Y(f,s,g)$. Then $2|h|=\delta(f,s,g)$ and hence $\delta(f,s,g)\geq 0$ as $|h|\geq 0$. Using the same argument and the injectivity of α_f we get also 7.viii)'. So it remains to show that α_g is an isometry for each $g\in E$. In this order we verify the following useful property:

7.ix). If $\delta(f,s,g)=\delta(h,t,\bar f)=0$ then $\delta(h,ts,g)=0$.

Let $A=(f,s,g)$, $B=(h,t,\bar f)$, $C=(\varepsilon(B),q^{-1}\theta_{\bar f}^{-1}p,\varepsilon(A))$, $F=(\varepsilon(B),q^{-1}\theta_{\bar f}^{-1},f)$, $D=(\varepsilon(F),ts,g)$, with $p\in\rho(A)$, $q\in\rho(\bar B)$. By 6.v), 6.iii) and 6.ii), we get $\varepsilon(F)=\varepsilon(h,t,1_{\underline{\underline{t}}(f)})=h$ and $\rho(\bar F)=t^{-1}G_{\bar h}$. By 6.v) applied to the quintuple $(\varepsilon(B),q^{-1}\theta_{\bar f}^{-1},f,s,g)$, it follows $\varepsilon(C)=\varepsilon(D)=\varepsilon(h,ts,g)$. As $\delta(A)=\delta(B)=0$ by assumption, it follows by 7.vi) and 7.vi)' that $0\leq\delta(D)=|h|+|\varepsilon(C)|-$

$-|g| \leqslant (|f| - |\varepsilon(B)|) + (|\varepsilon(B)| + |\varepsilon(A)|) \vee |g| = \delta(A) = 0$, and hence $\delta(D) = 0$, as required.

Now we are ready to finish the proof of the implication b) $\Rightarrow$ c). Let (f,s,g), (h,t,g) be such that $\delta(f,s,g) = \delta(h,t,g) = 0$ and $|f| \leqslant |h|$. We have to show that $\delta(f, st^{-1}, \bar{h}) = 0$. Since $\alpha_{\bar{h}}$ is onto there exists a pair (u,p) such that $\delta(u,p,\bar{h}) = 0$ and $|u| = |f|$. By 7.ix) it follows $\delta(u,pt,g) = 0$. As α_g is injective, we get $u = f$ and $pts^{-1} \in G_{\bar{f}}$, and hence $\delta(f, st^{-1}, \bar{h}) = \delta(u, p, \bar{h}) = 0$.

c) $\Rightarrow$ a): 7.iii) and 7.v) are immediate, so it remains to check 7.iv). Let (f,s,g,t,h) be a suitable quintuple, $A = (f,s,g)$, $B = (g,t,h)$, $C = (\varepsilon(A), \lambda_h(p)^{-1}t, h)$, $D = (f, sq, \varepsilon(B))$ with $p \in \rho(A)$, $q \in \rho(B)$, and assume that $\delta(C) \geqslant \delta(D)$. Let (l,v) be such that $(l, G_{\bar{l}}v) \in Y(D)$. As $\delta(D) = 2|l|$, $\delta(l,v,f) = 0$, $\varepsilon(l, vp, \varepsilon(A)) = \varepsilon(\varepsilon(l,v,f), \rho(\bar{f}, G_{\bar{f}}^{-1}v^{-1}, \bar{l})^{-1}s, g)$ and $\delta(\varepsilon(l,v,f), \rho(\bar{f}, G_{\bar{f}}^{-1}v^{-1}, \bar{l})^{-1}s, g) \geqslant 0$ it suffices to show that $\delta(l, vp, \varepsilon(A)) = 0$ to conclude that $\delta(A) \geqslant \delta(D)$. Since $\varepsilon(C) = \varepsilon(D)$ and $p\rho(C) = \rho(D)$ by 6.v), and $\delta(l, v\rho(D), \varepsilon(D)) = 0$ by assumption, it follows $\delta(l, vp\rho(C), \varepsilon(C)) = 0$ by 6.vii). Let (k,w) be such that $(k, G_{\bar{k}}w) \in Y(C)$. Then $|l| \leqslant |k|$, as $\delta(C) \geqslant \delta(D)$, by assumption, and $\delta(k, w, \varepsilon(A)) = \delta(k, w\rho(C), \varepsilon(C)) = 0$. Since $\alpha_{\varepsilon(C)}$ is an isometry, we get $\delta(l, vpw^{-1}, \bar{k}) = 0$, and hence $\delta(l, vp, \varepsilon(A)) = 0$ by 7.ix). The case $\delta(C) \leqslant \delta(D)$ is quite similar. $\square$

<u>Corollary 2.4.</u> Let $\underline{\underline{X}}$ be a Λ-graph of groups. Then $\underline{\underline{X}}$ has the following property:

7.x) If $(f,s,g) \in E(x,y) \times G_y \times E(y,z)$, $x,y,z \in X$, and $\delta(f,s,g) = 0$ then $sG_g s^{-1} \subset G_{\bar{f}}$.

<u>Proof</u>. As $\delta(f, G_{\bar f}sG_g, g)=\delta(f,s,g)=0$ and α_g is injective, it follows $G_{\bar f}\, s=G_{\bar f}sG_g$, i.e., $sG_gs^{-1}\subset G_{\bar f}$.

<u>Definition 2.5</u>. A <u>subpretree</u> $\underline{\underline{T}}$ of a Λ-graph of groups $\underline{\underline{X}}$ consists of a non empty subset T of X and a family $(1_{xy})_{(x,y)\in T^2}$, where $1_{xy}\in E(x,y)$, such that $1_{xx}=1_x$, $\overline{1_{xy}}=1_{yx}$ and $\varepsilon(A)=1_{xz}$, $\rho(A)=G_{xz}:=G_{1_{xz}}$ and $\lambda_A=\omega_{xz}:=\omega_{1_{xz}}$ for $A=(1_{xy}, 1, 1_{yz})$, $x,y,z\in T$.

<u>Definition 2.6</u>. The Λ-graph of groups $\underline{\underline{X}}$ is <u>strongly connected</u> if there exists a subpretree $\underline{\underline{T}}$ of $\underline{\underline{X}}$ such that T=X.

If $\underline{\underline{X}}$ is strongly connected then $\underline{\underline{X}}$ is <u>connected</u>, i.e., $E(x,y)$ is non-empty for arbitrary $x,y\in X$. The converse is also true if G_x is trivial for each $x\in X$, i.e., $\underline{\underline{X}}$ is a Λ-graph as defined in Section 1.

In the following we extend to an arbitrary ordered abelian group Λ the construction given in [3] assigning a graph of groups to a group action on a connected graph.

Given a connected Λ- graph $\underline{\underline{X}}$ and a group G acting from the left on $\underline{\underline{X}}$, let $Y=G\backslash X$, $E=G\backslash \text{arrow}\,\underline{\underline{X}}$, $E(Ga,Gb)=\{Gf:\underline{\underline{o}}(f)\in Ga,$ $\underline{\underline{t}}(f)\in Gb\}$, $\overline{Gf}=Gf^{-1}$, $1_{Ga}=G1_a$. Choose a section $j:Y\to X$ of the projection map $X\to Y: a\mapsto Ga$ and a family $(1_{xy})_{(x,y)\in Y^2}$ of arrows $1_{xy}: jx\to jy$ in $\underline{\underline{X}}$ in such a way that $1_{xy}1_{yz}=1_{xz}$ for $x,y,z\in Y$; in particular $1_{xx}=\mathbf{1}_{jx}$ and $1_{xy}^{-1}=1_{yx}$. This choice is possible since $\underline{\underline{X}}$ is connected. Set $\mathbf{1}_{xy}=G1_{xy}\in E(x,y)$ for $x,y\in X$. Thus $\mathbf{1}_{xx}=\mathbf{1}_x$ and $\overline{\mathbf{1}_{xy}}=\mathbf{1}_{yx}$. Extend j to a section $j:E\to \text{arrow}\,\underline{\underline{X}}$ of the projection map arrow $\underline{\underline{X}}\to E$ such that $\underline{\underline{o}}(jf)=jo(f)$ for each

$f \in E$ and $j\mathbb{1}_{xy} = 1_{xy}$ for $x, y \in Y$. Choose a map $\sigma: E \to G$ in such a way that $(jf)^{-1} = \sigma(f)j\bar{f}$ for each $f \in E$; in particular, $\underline{\underline{t}}(jf) = \sigma(f)j\underline{\underline{t}}(f)$. We may assume that $\sigma(\bar{f}) = \sigma(f)^{-1}$ if $f \neq \bar{f}$ and $\sigma(\mathbb{1}_{xy}) = 1$ for $x, y \in Y$. Set $G_f := G_{jf} = \{s \in G : sjf = jf\}$ for $f \in E$, and $G_x := G_{1_x}$ for $x \in Y$. Thus G_f is a subgroup of $G_{\underline{\underline{o}}(f)}$ and $G_{\bar{f}} = \sigma(f)^{-1}G_f\sigma(f)$. Define the isomorphism $\omega_f: G_f \to G_{\bar{f}}$ by $\omega_f(s) = \sigma(f)^{-1}s\sigma(f)$, and let $\theta_f = \sigma(f)\sigma(\bar{f}) \in G_f$.

For $(f, s, g) \in E(x, y) \times G_y \times E(y, z)$, let $\varepsilon(f, s, g) \in E(x, z)$ be the G-orbit of the composite arrow $(jf)(\sigma(f)sjg) \in \underline{\underline{X}}(jx, \sigma(f)s\sigma(g)jz)$. Consequently, there exists $p \in G_x$ such that $(jf)(\sigma(f)sjg) = pj\varepsilon(f, s, g)$. Obviously, the element p is uniquely determined modulo $G_{\varepsilon(f,s,g)}$, so let $\rho(f, s, g) \subset G_x$ be the coset $pG_{\varepsilon(f,s,g)}$. Let $A = (f, s, g)$, $\bar{A} = (\bar{g}, \theta_g^{-1}s^{-1}\theta_f^{-1}, \bar{f})$ and $p \in \rho(A)$. We get $(j\bar{g})(\sigma(\bar{g})\theta_g^{-1}s^{-1}\theta_f^{-1}j\bar{f}) = (j\bar{g})(\sigma(g)^{-1}s^{-1}j\bar{f}) = \sigma(g)^{-1}s^{-1}\sigma(f)^{-1}\left[(jf)(\sigma(f)sjg)\right]^{-1} = \sigma(g)^{-1}s^{-1}\sigma(f)^{-1}p\sigma(\varepsilon(A))j\overline{\varepsilon(A)}$, and hence $\overline{\varepsilon(A)} = \varepsilon(\bar{A})$, $\sigma(g)^{-1}s^{-1}\sigma(f)^{-1}p\sigma(\varepsilon(A)) \in G_z$ and $\rho(\bar{A}) = \sigma(g)^{-1}s^{-1}\sigma(f)^{-1}p\sigma(\varepsilon(A))G_{\varepsilon(\bar{A})}$. Define the bijective map $\lambda_A: \rho(A) \to \rho(\bar{A})$ by $\lambda_A(p) = \sigma(g)^{-1}s^{-1}\sigma(f)^{-1}p\sigma(\varepsilon(A))$. Finally, let us note that the length function $|\ |: \text{arrow}\ \underline{\underline{X}} \to \Lambda$ induces a map $|\ |: E \to \Lambda$ given by $|f| = |jf|$. A routine verification shows that the structure $\underline{\underline{Y}} = \underline{\underline{Y}}(G, \underline{\underline{X}}; j, \sigma) = (Y, E, \theta, \varepsilon, \rho, \lambda, |\ |)$ constructed above is a strongly connected Λ-graph of groups and $(Y, (\mathbb{1}_{xy})_{(x,y) \in Y^2})$ is a subpretree of $\underline{\underline{Y}}$.

3. The fundamental group of a strongly connected Λ- graph of groups

In this section we extend the construction of the fundamental group of a graph of groups given in [3] to the general case of arbitrary ordered abelian groups.

Let $\mathbf{X}$ be a strongly connected Λ-graph of groups and $\mathbf{T}=(X, (l_{xy})_{(x,y)\in X^2})$ be a subpretree of $\mathbf{X}$. Denote by $\mathbf{F}$ the free product of the groups G_x for $x\in X$ and the free group generated by $E=E_{\mathbf{X}}$. Let $\pi_1(\mathbf{X}, \mathbf{T})$ be the quotient of $\mathbf{F}$ by the normal subgroup $R_{\mathbf{T}}$ generated by the elements l_{xy} for $x,y\in X$, and the elements $p\,\varepsilon(A)\lambda_A(p)^{-1}g^{-1}s^{-1}f^{-1}$ for $A=(f,s,g)\in E(x,y)\times G_y\times E(y,z)$, $x,y,z\in X$, $p\in\rho(A)$. We shall see (Proposition 3.5) that the group $\pi_1(\mathbf{X}, \mathbf{T})$ does not depend, up to an isomorphism, on the choice of the subpretree $\mathbf{T}$, and so we way speak on the <u>fundamental group</u> $\pi_1(\mathbf{X})$ of $\mathbf{X}$.

Choose $\rho_A\in\rho(A)$ for each suitable triple $A=(f,s,g)$, and let $\lambda'_A=\lambda_A(\rho_A)^{-1}\in\rho(\bar{A})$. Denote by $R'_{\mathbf{T}}$ the normal subgroup of $\mathbf{F}$ generated by the elements l_{xy} for $x,y\in X$, and the elements $\rho_A\,\varepsilon(A)\lambda'_A g^{-1}s^{-1}f^{-1}$ for each suitable triple $A=(f,s,g)$.

<u>Lemma 3.1.</u> $R_{\mathbf{T}} = R'_{\mathbf{T}}$.

<u>Proof.</u> We have to show that the generators of $R_{\mathbf{T}}$ belong to $R'_{\mathbf{T}}$. First note that $f\bar{f}\theta_f^{-1}\in R'_{\mathbf{T}}$ and $f\omega_f(a)f^{-1}a^{-1}\in R'_{\mathbf{T}}$ for $a\in G_f$, by 6.iii) applied to the triples $(f,1,\bar{f})$ and $(f,\omega_f(a),\bar{f})$. Now let $A=(f,s,g)$ and $p\in\rho(A)$. As λ_A is compatible with ω

we get $p\varepsilon(A)\lambda_A(p)^{-1} \equiv p\varepsilon(A)\omega_{\varepsilon(A)}(p^{-1}\rho_A)\lambda'_A \equiv \rho_A\varepsilon(A)\lambda'_A \equiv fsg$ mod $R_{\underline{\underline{T}}}$, as required. $\square$

Let us give an alternative description of the fundamental group of $\underline{\underline{X}}$. Fix a vertex $x \in X$ and let $S = S_x = G_x \times E(x,x) \times G_x$. Define an equivalence relation $\equiv$ on S as follows: $(s,f,t) \equiv (p,g,q)$ iff $f=g$, $s^{-1}p \in G_f$, $tq^{-1} \in G_f$ and $\omega_f(s^{-1}p) = tq^{-1}$. If $\alpha=(s,f,t)$, $\beta=(p,g,q) \in S$, let us put $\alpha\circ\beta = (s\rho_A, \varepsilon(A), \lambda'_A q)$, where $A=(f,tp,g)$.

Lemma 3.2. Let $\alpha,\beta,\alpha',\beta' \in S$ and assume $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$. Then $\alpha\circ\beta \equiv \alpha'\circ\beta'$.

Proof. Let $\alpha=(s,f,t)$, $\alpha'=(s',f,t') \equiv \alpha$ and $\beta=(p,g,q)$. Let us show that $\alpha\circ\beta \equiv \alpha'\circ\beta$. Let $A=(f,tp,g)$, $A'=(f,t'p,g)$. As $A'=(f,\omega_f(s'^{-1}s)tp,g)$, it follows by 6.iv) that $\varepsilon(A)=\varepsilon(A')$, $\rho_{A'}^{-1}s'^{-1}s\rho_A \in G_{\varepsilon(A)}$ and $\omega_{\varepsilon(A)}(\rho_{A'}^{-1}s'^{-1}s\rho_A) = \lambda'_{A'}\lambda_A^{-1}$, and hence $\alpha\circ\beta \equiv \alpha'\circ\beta$. Similarly, we get $\alpha\circ\beta \equiv \alpha\circ\beta'$ if $\beta \equiv \beta'$. $\square$

Let $\pi_1(\underline{\underline{X}},x)$ be the quotient set $S/_{\equiv}$. By Lemma 3.2, $\pi_1(\underline{\underline{X}},x)$ is equipped with a binary composition law induced by the map $S^2 \to S : (\alpha,\beta) \mapsto \alpha\circ\beta$, which does not depend on the choice of the representatives ρ_A of the cosets $\rho(A)$.

Lemma 3.3. $\pi_1(\underline{\underline{X}},x)$ is a group with respect to the composition law above.

Proof. Let $\alpha=(s,f,t)$, $\beta=(p,g,q)$, $\gamma=(r,l,u) \in S$. We have to show that $(\alpha\circ\beta)\circ\gamma \equiv \alpha\circ(\beta\circ\gamma)$. Let $A=(f,tp,g)$, $B=(g,qr,l)$,

$C=(\varepsilon(A),\lambda'_A\, qr,\ 1)$, $D=(f, tp\rho_B, \varepsilon(B))$. By definition we get $(\alpha\circ\beta)\circ\gamma=(s\rho_A\rho_C, \varepsilon(C), \lambda'_C u)$ and $\alpha\circ(\beta\circ\gamma)=(s\rho_D, \varepsilon(D), \lambda'_D\lambda'_B u)$. The required congruence is immediate by 6.v).

By 6.ii), it follows that the class of $(1, 1_x, 1)$ is the neutral element of $\pi_1(\underline{\underline{X}}, x)$, while $(t^{-1}, \bar{f}, \theta_f^{-1} s^{-1})$ is a representative of the inverse of the class of (s,f,t), by 6.iii).

The next lemma is immediate.

Lemma 3.4. The maps $G_x \to S$: $s\mapsto(s,1_x,1)$ and $E(x,x)\to S$: $f\mapsto(1,f,1)$ identify G_x with a subgroup of $\pi_1(\underline{\underline{X}},x)$ and $E(x,x)$ with a subset of $\pi_1(\underline{\underline{X}},x)$. Let $\underline{\underline{F}}_x$ be the free product of G_x and the free group generated by $E(x,x)$, and R_x be its normal subgroup generated by 1_x and the elements $\rho_A\ \varepsilon(A)\lambda'_A g^{-1}s^{-1}f^{-1}$ for $A=(f,s,g)\in E(x,x)\times G_x\times E(x,x)$. Then R_x does not depend on the representatives ρ_A of the cosets $\rho(A)\in G_x/G_{\varepsilon(A)}$ and F_x/R_x, $\pi_1(\underline{\underline{X}}, x)$ are canonically isomorphic.

Let $A_s=(1_{xy},s,1_{yx})$, $A_f=(1_{xy},1,f)$, $B_f=(\varepsilon(A_f), \lambda'_{A_f}, 1_{zx})$ for $s\in G_y$, $f\in E(y,z)$, $y,z\in X$. Note that $\rho_{A_s}, \lambda'_{A_s}, \rho_{A_f}, \rho_{B_f}, \lambda'_{B_f}\in G_x$ and $\varepsilon(A_s)$, $\varepsilon(B_f)\in E(x,x)$, so we may identify them with elements of the group $\pi_1(\underline{\underline{X}}, x)$. Define a map $\eta: (\bigcup_{y\in X} G_y)\cup E\to\pi_1(\underline{\underline{X}}, x)$ by $\eta(s)=\rho_{A_s}\varepsilon(A_s)\lambda'_{A_s}$, $\eta(f)=\rho_{A_f}\rho_{B_f}\varepsilon(B_f)\lambda'_{B_f}$ for $s\in\bigcup_{y\in X} G_y$, $f\in E$. Note that the map η depends on the choice of the pretree $\underline{\underline{T}}$, but not on the choice of the representatives $\rho_{A_s}, \rho_{A_f}, \rho_{B_f}$.

Proposition 3.5. The map η above induces an isomorphism

$\eta_{\underline{T},x}: \pi_1(\underline{X}, \underline{T}) \to \pi_1(\underline{X}, x)$.

Proof. a) First we have to show that η induces a homomorphism $\underline{F} \to \pi_1(\underline{X}, x)$, i.e., $\eta(st)=\eta(s)\eta(t)$ for $s,t \in G_y$, $y \in X$. Since, by Lemma 3.4, the identity $fqg = p\,\varepsilon(A)\lambda_A(p)^{-1}$ holds in $\pi_1(\underline{X}, x)$ for $A=(f,q,g)$, $f,g\in E(x,x)$, $q\in G_x$, $p\in\rho(A)$, we get

$\eta(s)\,\eta(t)=\rho_{A_s}\rho_C\;\varepsilon(C)\lambda'_C\,\lambda'_{A_t}$, where $C=(\varepsilon(A_s), \lambda'_{A_s}\rho_{A_t}, \varepsilon(A_t))$.

Thus we have to show that $\varepsilon(C)=\varepsilon(A_{st})$, $\rho^{-1}_{A_{st}}\rho_{A_s}\rho_C \in G_{\varepsilon(C)}$ and $\omega_{\varepsilon(C)}(\rho^{-1}_{A_{st}}\rho_{A_s}\rho_C) = \lambda'_{A_{st}}\,\lambda'^{-1}_{A_t}\,\lambda'^{-1}_C$. Let $H=(1_{yx}, \rho_{A_t}, \varepsilon(A_t))$, $I=(1_{yx},1,1_{xy})$, $J=(\varepsilon(I),t,1_{yx})=(1_y,t,1_{yx})$. By 6.vi), we get $\varepsilon(J)=1_{yx}$, $\rho(J)=tG_{yx}$ and $\lambda_J(t)=1$. Since $\rho(I)=G_{yx}$ and $\lambda_I=\omega_{yx}$ by assumption, it follows by 6.v) that $\varepsilon(H)=\varepsilon(J)=1_{yx}$ and $\lambda_H(t)=\lambda'_{A_t}$. Applying the rule 6.v) to the triples C and $A_{st}=(1_{xy}, st, \varepsilon(H))$ we get $\varepsilon(C)=\varepsilon(A_{st})$, $\rho_{A_s}\,\rho(C)=\rho(A_{st})$ and $\lambda'^{-1}_{A_{st}}=\lambda_{A_{st}}(\rho_{A_s}(\rho^{-1}_{A_s}\rho_{A_{st}}))=\lambda_H(t)^{-1}\lambda_C(\rho^{-1}_{A_s}\rho_{A_{st}}) =$
$=\lambda'^{-1}_{A_t}\lambda'^{-1}_C\,\omega_{\varepsilon(C)}(\rho^{-1}_C\,\rho^{-1}_{A_s}\,\rho_{A_{st}})$, as required.

b) Next we have to show that the normal subgroup $R_{\underline{T}}$ of $\underline{F}$ is contained in the kernel of $\eta:\underline{F}\to\pi_1(\underline{X}, x)$, i.e., $\eta(1_{yz})=1$ for $y,z\in X$, and $\eta(f)\eta(s)\eta(g)=\eta(\rho_O)\eta(\varepsilon(O))\,\eta(\lambda'_O)$ for $O=(f,s,g)\in E(y,z)\times G_z\times E(z,u)$. The first condition is immediate, while the second one will follows after a long chain of computations.

By Lemma 3.4, we get $\eta(f)\,\eta(s)\,\eta(g)=\rho_{A_f}\rho_{B_f}\rho_C\rho_D\;\varepsilon(D)\lambda'_D\,\lambda'_{B_g}$, where $C=(\varepsilon(B_f), \lambda'_{B_f}\rho_{A_s}, \varepsilon(A_s))$ and $D=(\varepsilon(C), \lambda'_C\lambda'_{A_s}\rho_{A_g}\rho_{B_g}$,

$\mathcal{E}(B_g))$.

Consider the triples $F=(\mathcal{E}(A_s),\lambda'_{A_s}\rho_{A_g}\rho_{B_g}, \mathcal{E}(B_g))$.

$D_1=(\mathcal{E}(B_f),\lambda'_{B_f}\rho_{A_s}\rho_F, \mathcal{E}(F))$, $H=(1_{zx},\rho_{A_s}\rho_F, \mathcal{E}(F))$,

$D_2=(\mathcal{E}(A_f),\lambda'_{A_f}\rho_H, \mathcal{E}(H))$, $I=(f,\rho_H,\mathcal{E}(H))$, $D_3=(1_{xy},\rho_I,\mathcal{E}(I))$,

$J=(f,1,1_{zx})$, $I_1=(\mathcal{E}(J),\lambda'_J\rho_{A_s}\rho_F,\mathcal{E}(F))$, $K=(\mathcal{E}(J),\lambda'_J\rho_{A_s},\mathcal{E}(A_s))$,

$I_2=(\mathcal{E}(K),\lambda'_K\lambda'_{A_s}\rho_{A_g}\rho_{B_g},\mathcal{E}(B_g))$, $L=(\mathcal{E}(K),\lambda'_K\lambda'_{A_s}\rho_{A_g},\mathcal{E}(A_g))$,

$I_3=\mathcal{E}(L),\lambda'_L\lambda'_{A_g}, 1_{ux})$, $M=(\mathcal{E}(K),\lambda'_K\lambda'_{A_s}, 1_{xz})$, $L_1=(\mathcal{E}(M)$, $\lambda'_M, g)$, $N=(\mathcal{E}(A_s), \lambda'_{A_s}, 1_{xz})$, $M_1=(\mathcal{E}(J),\lambda'_J\rho_{A_s}\rho_N, \mathcal{E}(N))$,

$P=(1_{zx}\rho_{A_s}\rho_N, \mathcal{E}(N))$, $M_2=(f,\rho_P, \mathcal{E}(P))$, $L_2=(f,s,g)$,

$I_4=(\mathcal{E}(L_2),\lambda'_{L_2}, 1_{ux})$ and $D_4=(1_{xy},\rho_{L_2}\rho_{I_4},\mathcal{E}(I_4))$.

We get, step by step:

I) $\mathcal{E}(D)=\mathcal{E}(D_3)$, $\rho^{-1}_{D_3}\rho_{A_f}\rho_{B_f}\rho_C\rho_D\in G_{\mathcal{E}(D)}$ and its image through $\omega_{\mathcal{E}(D)}$ is $\lambda'_{D_3}\lambda'_I\lambda'_M\lambda'_F\lambda'^{-1}_D$, by 6.v) applied to D,D_1,D_2,D_3;

II) $\mathcal{E}(I)=\mathcal{E}(I_3)$, $\rho^{-1}_I\rho_J\rho_K\rho_L\rho_{I_3}\in G_{\mathcal{E}(I)}$ and its image through $\omega_{\mathcal{E}(I)}$ is $\lambda'_I\lambda'_H\lambda'_F\lambda'_{B_g}\lambda'^{-1}_{I_3}$, by 6.v) applied to I,I_1,I_2,I_3;

III) $\mathcal{E}(L)=\mathcal{E}(L_1)$, $\rho^{-1}_L\rho_M\rho_{L_1}\in G_{\mathcal{E}(L)}$ and its image through $\omega_{\mathcal{E}(L)}$ is $\lambda'_L\lambda'_{A_g}\lambda'^{-1}_{L_1}$, by 6.v);

IV) $\mathcal{E}(M)=\mathcal{E}(M_2)$, $\rho^{-1}_{M_2}\rho_J\rho_K\rho_M\in G_{\mathcal{E}(M)}$ and its image through $\omega_{\mathcal{E}(M)}$ is $\lambda'_{M_2}\lambda'_P\lambda'_N\lambda'^{-1}_M$, by 6.v) applied to M,M_1,M_2;

V) $\mathcal{E}(N)=1_{xz}$, $\rho_{A_s}\rho_N\in G_{xz}$ and $\omega_{xz}(\rho_{A_s}\rho_N)=s\lambda'^{-1}_N$, by 6.v), 6.ii) and 6.iii);

VI) $P=(1_{zx},\rho_{A_s}\rho_N, 1_{xz})=(1_{zx},\omega_{zx}(s\lambda'^{-1}_N),1_{xz})$, $\mathcal{E}(P)=1_z$ and $\lambda'_P=\rho^{-1}_P s\lambda'^{-1}_N$, by 6.iii);

VII) $M_2=(f,1,1_z)$, $\mathcal{E}(M_2)=f$, $\rho(M_2)=G_f$ and $\lambda'_{M_2}=\omega_f(\rho_{M_2})^{-1}\rho_P$, by 6.ii); thus $\rho_J\rho_K\rho_M\in G_f$ and $\omega_f(\rho_J\rho_K\rho_M)=s\,\lambda'^{-1}_M$ by IV), VI);

VIII) $L_1=(f,\lambda'_M,g)$, $\mathcal{E}(L_1)=\mathcal{E}(L_2)$, $\rho^{-1}_{L_2}\rho_J\rho_K\rho_M\rho_{L_1}\in G_{\mathcal{E}(L_2)}$ and its image through $\omega_{\mathcal{E}(L_2)}$ is $\lambda'_{L_2}\lambda'^{-1}_{L_1}$, by 6.iv); consequently $\rho^{-1}_{L_2}\rho_J\rho_K\rho_L\in G_{\mathcal{E}(L_2)}$ and its image through $\omega_{\mathcal{E}(L_2)}$ is $\lambda'_{L_2}\lambda'_{A_g}\lambda'_L$.

IX) $I_3=(\varepsilon(L_2), \lambda'_L \lambda'_{A_g}, 1_{ux})$, $\varepsilon(I_3)=\varepsilon(I_4)$, $\rho^{-1}_{I_4}\rho^{-1}_{L_2}\rho_y\rho_K\rho_L\rho_{I_3} \in G_{\varepsilon(I_4)}$ and its image through $\omega_{\varepsilon(I_4)}$ is $\lambda'_{I_4}\lambda'^{-1}_{I_3}$, by 6.iv); thus $\varepsilon(I)=\varepsilon(I_4)$, $\rho^{-1}_I\rho_{L_2}\rho_{I_4} \in G_{\varepsilon(I)}$ and $\omega_{\varepsilon(I)}(\rho^{-1}_I\rho_{L_2}\rho_{I_4})= \lambda'_I\lambda'_H\lambda'_F\lambda'_{B_g}\lambda'_{I_4}$, by II);

X) $D_3=(1_{xy}, \rho_I, \varepsilon(I_4))$, $\varepsilon(D_3)=\varepsilon(D_4)$, $\rho^{-1}_{D_3}\rho_{D_4} \in G_{\varepsilon(D_4)}$ and $\omega_{\varepsilon(D_4)}(\rho^{-1}_{D_3}\rho_{D_4})=\lambda'_{D_3}\lambda'_I\lambda'_H\lambda'_F\lambda'_{B_g}\lambda'^{-1}_{I_4}\lambda'^{-1}_{D_4}$, by 6.iv); consequently, $\varepsilon(D)=\varepsilon(D_4)$, $\rho^{-1}_{D_4}\rho_{A_f}\rho_{B_f}\rho_C\rho_D \in G_{\varepsilon(D)}$ and its image through $\omega_{\varepsilon(D)}$ is $\lambda'_{D_4}\lambda'_{I_4}\lambda'^{-1}_{B_g}\lambda'^{-1}_D$, by I).

It follows $\eta(f)\eta(s)\eta(g)=\rho_{D_4}\varepsilon(D_4)\lambda'_{D_4}\lambda'_{I_4}$. One may show similarly that $\eta(\rho_{L_2})\eta(\varepsilon(L_2))\eta(\lambda'_{L_2}) = \rho_{D_4}\varepsilon(D_4)\lambda'_{D_4}\lambda'_{I_4}$, geting the required equality. Thus we obtain a homomorphism $\eta : \pi_1(\underline{\underline{X}},\underline{\underline{T}}) \to \pi_1(\underline{\underline{X}}, x)$.

c) In order to show that the η above is an isomorphism, let us define the map $\zeta : G_x \cup E(x,x) \to \pi_1(\underline{\underline{X}},\underline{\underline{T}})$ by $\zeta(s)=sR_{\underline{\underline{T}}}$, $\zeta(f)=fR_{\underline{\underline{T}}}$ for $s\in G_x$, $f\in E(x,x)$. By Lemma 3.4, the map ζ induces a homomorphism $\zeta : \pi_1(\underline{\underline{X}}, x) \to \pi_1(\underline{\underline{X}}, \underline{\underline{T}})$ and it is obvious that η and ζ are inverse each to other. $\square$

Proposition 3.5 shows that the fundamental group $\pi_1(\underline{\underline{X}}) \simeq \pi_1(\underline{\underline{X}},\underline{\underline{T}}) \simeq \pi_1(\underline{\underline{X}}, x)$ does not depend, up to an isomorphism on the choice of the pretree $\underline{\underline{T}}$ or the vertex x. It follows also that for each $x\in X$, the canonic homomorphism $G_x \to \pi_1(\underline{\underline{X}},\underline{\underline{T}})$ and the canonic map $E(x,x) \to \pi_1(\underline{\underline{X}}, \underline{\underline{T}})$ are injective. Moreover, we get:

Corollary 3.6. Let $x, y \in X$. Then

i) the canonic map $i: E(x,y) \to \pi_1(\underline{\underline{X}}, \underline{\underline{T}}): f \mapsto fR_{\underline{\underline{T}}}$ is injective;

ii) the map $E(x,y) \to G_x \backslash \pi_1(\underline{\underline{X}}, \underline{\underline{T}})/G_y: f \mapsto G_x i(f) G_y$ is bijective;

iii) the canonic map $G_x \times E(x,y) \times G_y \to \pi_1(\underline{\underline{X}}, \underline{\underline{T}}): (s,f,t) \mapsto sftR_{\underline{\underline{T}}}$ is onto, and $sft \equiv pgq \bmod R_{\underline{\underline{T}}}$ iff $f=g$, $s^{-1}p \in G_f$ and $\omega_f(s^{-1}p) = tq^{-1}$.

Proof. As i) and ii) are immediate consequences of iii), it remains to prove iii). First let us prove the surjectivity of the map above. Let $a \in \pi := \pi_1(\underline{\underline{X}}, \underline{\underline{T}})$. We have to show that there exist $f \in E(x,y)$, $s \in G_x$ and $t \in G_y$ such that the identity $a = sft$ holds in π. By Proposition 3.5, there exist $h \in E(x,x)$ and $p, q \in G_x$ such that the identity $a = phq$ is true in π. We get the sequence of identities in π:
$a = phq = phq1_{xy} = sft$, where $s = p\rho_C$, $f = \varepsilon(C)$, $t = \lambda_C^{\ell}$, $C = (h, q, 1_{xy})$.

Next let $f, g \in E(x,y)$, $s \in G_x$ and $t \in G_y$. We have to show that the identity $sft = g$ holds in π iff $f = g$, $s \in G_f$ and $\omega_f(s) = t^{-1}$. We get the sequence of identities in π: $sft = sft1_{yx} = s\rho_A \varepsilon(A)\lambda_A^{\ell}$, where $A = (f, t, 1_{yx})$. Similarly we get $g = \rho_B \varepsilon(B)\lambda_B^{\ell}$, with $B = (g, 1, 1_{yx})$. By Proposition 3.5 and the definition of $\pi_1(\underline{\underline{X}}, x)$ it follows that $sft = g$ is true in π iff the following condition is satisfied:

$$(*)\quad \varepsilon(A) = \varepsilon(B),\ \rho_B^{-1} s \rho_A \in G_{\varepsilon(A)} \text{ and } \omega_{\varepsilon(A)}(\rho_B^{-1} s \rho_A) = \lambda_B^{\ell} \lambda_A^{\ell -1}.$$

If $f = g$, $s \in G_f$ and $\omega_f(s) = t^{-1}$ then $(*)$ follows by 6.iv).

Conversely, assuming (*), let $C=(\varepsilon(A), \lambda^{\ell}_{A}, 1_{xy})$ and $D=(\varepsilon(B), \lambda^{\ell}_{B}, 1_{xy})$. By 6.iv) it follows $\varepsilon(C)=\varepsilon(D)$, $\rho_D^{-1}\rho_B^{-1}s\rho_A\rho_C \in G_{\varepsilon(C)}$ and $\omega_{\varepsilon(C)}(\rho_D^{-1}\rho_B^{-1}s\,\rho_A\rho_C)=\lambda^{\ell}_{D}\lambda^{\ell -1}_{C}$. On the other hand, by 6.v) and 6.ii), we get $\varepsilon(E)=\varepsilon(f,t\rho_F, \varepsilon(F))=\varepsilon(f,t\rho_F,1_y)=f$, $\rho_A\rho_C \in G_f$ and $\omega_f(\rho_A\rho_C)=t\lambda^{\ell -1}_{C}$, where $F=(1_{yx},1,1_{xy})$. Similarly, we get $\varepsilon(D)=g$, $\rho_B\rho_D \in G_g$ and $\omega_g(\rho_B\rho_D)=\lambda^{\ell -1}_{D}$. Consequently, $f=g$, $s\in G_f$ and $\omega_f(s)=t^{-1}$, as required. ∎

Remark. If G_x is trivial for each $x\in X$, i.e, $\underline{\underline{X}}$ is a connected Λ-graph as defined in Section 1, then $\pi_1(\underline{\underline{X}}) \simeq \mathrm{Aut}(x)$ for $x\in X$.

4. From strongly connected Λ-graphs of groups to group actions on Λ-trees.

In this section we extend to arbitrary ordered abelian groups the construction of the universal covering relative to a graph of groups given in [3].

Let $\underline{\underline{X}}$ be a strongly connected Λ-graph of groups and $\underline{\underline{T}}=(T,(1_{xy})_{x,y\in X^2})$ be a subpretree of $\underline{\underline{X}}$. We construct a Λ-tree $\widetilde{\underline{\underline{X}}}=\widetilde{\underline{\underline{X}}}_{\underline{\underline{T}}}$ on which the fundamental group $\pi_1(\underline{\underline{X}},\underline{\underline{T}})$ acts from the left, in such a way that the pair $(\pi_1(\underline{\underline{X}},\underline{\underline{T}}),\widetilde{\underline{\underline{X}}}_{\underline{\underline{T}}})$ does not depend, up to an isomorphism, on the choice of $\underline{\underline{T}}$.

Let $\check{X}$ be the disjoint union $\bigcup_{x\in X} \pi/G_x$, where $\pi=\pi_1(\underline{\underline{X}},\underline{\underline{T}})$ and G_x is identified with a subgroup of π by Proposition 3.5. By Corollary 3.6, we may also identify the sets $E(x,y)$ for $x,y\in X$ with subsets of π. The group π acts canonically from the left on the set $\tilde{X}$. Given $\underline{\underline{a}}=aG_x$, $\underline{\underline{b}}=bG_y\in\tilde{X}$, the double coset $G_xa^{-1}bG_y$ does not depend on the choice of the representatives a and b of the cosets $\underline{\underline{a}}$, $\underline{\underline{b}}$. By Corollary 3.6., there is a unique $f_{\underline{\underline{a}},\underline{\underline{b}}}\in E(x,y)$ such that $G_xa^{-1}bG_y=G_xf_{\underline{\underline{a}},\underline{\underline{b}}}G_y$. Note that $f_{\underline{\underline{a}},\underline{\underline{b}}}=1_{xy}$ if $a^{-1}b\in G_xG_y$ (in particular, $f_{\underline{\underline{a}},\underline{\underline{a}}}=1_x$) and $f_{\underline{\underline{b}},\underline{\underline{a}}}=\tilde{f}_{\underline{\underline{a}},\underline{\underline{b}}}$. Define a map $d:\tilde{X}^2\to\Lambda$ by $d(\underline{\underline{a}},\underline{\underline{b}})=|f_{\underline{\underline{a}},\underline{\underline{b}}}|$ for $\underline{\underline{a}},\underline{\underline{b}}\in\tilde{X}$.

Proposition 4.1. $\underline{\underline{\tilde{X}}}=(\tilde{X}, d)$ is a Λ-tree on which the group $\pi=\pi_1(\underline{\underline{X}}, \underline{\underline{T}})$ acts.

Proof. Obviously, $d(\underline{\underline{a}},\underline{\underline{b}})\geqslant 0$ for $\underline{\underline{a}},\underline{\underline{b}}\in\tilde{X}$, with equality iff $\underline{\underline{a}}=\underline{\underline{b}}$. As $f_{\underline{\underline{b}},\underline{\underline{a}}}=\tilde{f}_{\underline{\underline{a}},\underline{\underline{b}}}$, we get $d(\underline{\underline{a}},\underline{\underline{b}})=d(\underline{\underline{b}},\underline{\underline{a}})$. For $\underline{\underline{a}},\underline{\underline{b}}\in\tilde{X}$, let $[\underline{\underline{a}},\underline{\underline{b}}]=[\underline{\underline{b}},\underline{\underline{a}}]=\{\underline{\underline{c}}\in\tilde{X}: d(\underline{\underline{a}},\underline{\underline{c}})+d(\underline{\underline{c}},\underline{\underline{b}})=d(\underline{\underline{a}},\underline{\underline{b}})\}$ and $i_{\underline{\underline{a}},\underline{\underline{b}}}:[\underline{\underline{a}},\underline{\underline{b}}]\to[0, d(\underline{\underline{a}}, \underline{\underline{b}})]$ be the map $\underline{\underline{c}}\mapsto d(\underline{\underline{a}}, \underline{\underline{c}})$. For $\underline{\underline{a}},\underline{\underline{b}},\underline{\underline{c}}\in\tilde{X}$, let $Y(\underline{\underline{a}},\underline{\underline{b}},\underline{\underline{c}})=[\underline{\underline{a}}, \underline{\underline{b}}]\cap[\underline{\underline{b}}, \underline{\underline{c}}]\cap[\underline{\underline{c}}, \underline{\underline{a}}]$. To conclude that $\underline{\underline{\tilde{X}}}=(\tilde{X},d)$ is a Λ- tree, it suffices to show that the map $i_{\underline{\underline{a}},\underline{\underline{b}}}$ is bijective and the set $Y(\underline{\underline{a}},\underline{\underline{b}},\underline{\underline{c}})$ is non-empty for arbitrary $\underline{\underline{a}},\underline{\underline{b}},\underline{\underline{c}}\in\tilde{X}$.

First let us show that $i_{\underline{\underline{a}},\underline{\underline{b}}}$ is bijective. Let $f=f_{\underline{\underline{a}},\underline{\underline{b}}}$ and consider the map $\alpha_f: [f]\to[0,|f|]$ introduced in Section 2, which is bijective by Proposition 2.3. We define a map $\beta: [\underline{\underline{a}},\underline{\underline{b}}]\to[f]$ in such a way that $\alpha_f\circ\beta=i_{\underline{\underline{a}},\underline{\underline{b}}}$. Let $\underline{\underline{a}}=aG_x$, $\underline{\underline{b}}=bG_y$ and $\underline{\underline{c}}=cG_z\in[\underline{\underline{a}},\underline{\underline{b}}]$, $g=f_{\underline{\underline{c}},\underline{\underline{a}}}$, $h=f_{\underline{\underline{c}},\underline{\underline{b}}}$. By assumption $|f|=|g|+|h|$ amd there exist $p,t\in G_x$, $q\in G_y$ and $s\in G_z$ such that the following identities hold in π: $a^{-1}b=pfq$ and $c^{-1}a=sgt$. Consequently, $c^{-1}b=sgtpfq=s\rho_A\varepsilon(A)\lambda'_Aq$, where $A=(g,tp,f)$, and hence $\varepsilon(A)=h$

and $(g, G_{\underline{g}}tp)\in[f]$. As α_f is injective and $|g|=d(\underline{a},\underline{c})$, it follows that $\beta(\underline{c}):=(g,G_{\underline{g}}tp)$ is uniquely determined by $\underline{a},\underline{b},\underline{c}$, and $\alpha_f\circ\beta=\iota_{\underline{a},\underline{b}}$. It remains to check that β is bijective. Let $\underline{c}'=c'G_u\in[\underline{a},\underline{b}]$ be such that $\beta(\underline{c})=\beta(\underline{c}')$. Let $g'=f_{\underline{c}',\underline{a}}$, $c'^{-1}a=s'g't'$ with $s'\in G_u$, $t'\in G_x$. By assumption, $g=g'$, $z=u$ and $tt'^{-1}\in G_{\underline{g}}$. It follows $c^{-1}c'=sgtt'^{-1}g^{-1}s'^{-1}=s\,\omega_g^{-1}(tt'^{-1})s'^{-1}\in G_z$, i.e., $\underline{c}=\underline{c}'$, and hence β is injective. Let now $g\in E(z,x)$, $v\in G_x$ be such that $(g,G_{\underline{g}}v)\in[f]$. Let $\underline{c}=cG_z$, where $c=apv^{-1}g^{-1}$; remind that $a^{-1}b=pfq$ with $p\in G_x$, $q\in G_y$. Then $c^{-1}a=gvp^{-1}$ and $c^{-1}b=gvfq=\rho_B\,\varepsilon(B)\lambda_B'q$, where $B=(g,v,f)$. Thus $f_{\underline{c},\underline{a}}=g$, $f_{\underline{c},\underline{b}}=\varepsilon(B)$, $\underline{c}\in[\underline{a},\underline{b}]$ and $\beta(\underline{c})=(g,G_{\underline{g}}v)$, and hence β is onto.

Let now $\underline{a}=aG_x$, $\underline{b}=bG_y$, $\underline{c}=cG_z\in\tilde{X}$. We have to show that $Y(\underline{a},\underline{b},\underline{c})$ is non-empty. Let $f=f_{\underline{a},\underline{b}}$ $g=f_{\underline{b},\underline{c}}$, $a^{-1}b=pfq$, $b^{-1}c=sgt$ with $p\in G_x$, $q,s\in G_y$, $t\in G_z$. Then $a^{-1}c=pfqsgt=p\rho_A\,\varepsilon(A)\lambda_A't$, where $A=(f,qs,g)$, and hence $f_{\underline{a},\underline{c}}=\varepsilon(A)$. By Proposition 3.2, there exist $u\in X$ and $(h,v)\in E(u,x,)\times G_x$ such that $|h|+|\varepsilon(B)|=|f|$, $|h|+|\varepsilon(C)|=|\varepsilon(A)|$ and $|\varepsilon(B)|+|\varepsilon(C)|=|g|$, where $B=(h,v,f)$ and $C=(h,v\rho_A,\varepsilon(A))$. Let $\underline{e}=eG_u$, where $e=apv^{-1}h^{-1}$. It follows $e^{-1}a=hvp^{-1}$, $e^{-1}b=hvfq=\rho_B\,\varepsilon(B)\lambda_B'q$, $e^{-1}c=hv\rho_A\,\varepsilon(A)\lambda_A't=\rho_C\,\varepsilon(C)\lambda_C'\lambda_A't$, and hence $d(\underline{e},\underline{a})=|h|$, $d(\underline{e},\underline{b})=|\varepsilon(B)|$ and $d(\underline{e},\underline{c})=|\varepsilon(C)|$. Consequently, $\underline{e}\in Y(\underline{a},\underline{b},\underline{c})$, as required.

It remains to observe that $d(s\underline{a}, s\underline{b})=d(\underline{a},\underline{b})$ for $\underline{a},\underline{b}\in\tilde{X}$, $s\in\pi$, i.e., the group π acts on the Λ- tree $\tilde{X}$. $\square$

Proposition 4.2. The pair $(\pi_1(\underline{X},\underline{T}), \tilde{X}_{\underline{T}})$ does not depend, up to an isomorphism, on the choice of $\underline{T}$.

Proof. Let $\underline{T}'=(X,(\Phi'_{xy})_{(x,y)\in X^2})$ be another subpretree of $\underline{X}$ and x_o be a vertex of $\underline{X}$. Consider the isomorphisms $\eta_{\underline{T},x_o}:\pi_1(\underline{X},\underline{T})\to\pi_1(\underline{X},x_o)$ and $\eta_{\underline{T}',x_o}:\pi_1(\underline{X},\underline{T}')\to\pi_1(\underline{X},x_o)$ given by Proposition 3.5, and let $F_1:\pi_1(\underline{X},\underline{T})\to\pi_1(\underline{X},\underline{T}')$ be the composite isomorphism $\eta^{-1}_{\underline{T}',x_o}\circ\eta_{\underline{T},x_o}$. Identifying G_x, $x\in X$, with a common subgroup and $E(x, y), x, y\in X$, with a common subset of $\pi_1(\underline{X},\underline{T})$ and $\pi_1(\underline{X},\underline{T}')$, we get $F_1(s)=1_{x_ox}s1_{xx_o}$ and $F_1(f)=$

$=1_{x_0x}f\,1_{yx_0}$ for $s\in G_x$, $f\in E(x,y)$. Define the map $F_2:\tilde{X}_{\underline{\underline{T}}}\to\tilde{X}_{\underline{\underline{T}}'}$ by $F_2(aG_x)=F_1(a)1_{x_0x}\,G_x$ for $a\in\pi_1(\underline{\underline{X}},\underline{\underline{T}})$; the independence on the representative a is immediate. The map F_2 is bijective and $F_2^{-1}(bG_x)=F_1^{-1}(b)1'_{x_0x}G_x$ for $b\in\pi_1(\underline{\underline{X}},\underline{\underline{T}}')$. Clearly, the pair (F_1,F_2) is compatible with the group actions. It remains to show that $d'(F_2(\underline{\underline{a}}),F_2(\underline{\underline{b}}))=d(\underline{\underline{a}},\underline{\underline{b}})$ for $\underline{\underline{a}}=aG_x$, $\underline{\underline{b}}=bG_y\in\tilde{X}_{\underline{\underline{T}}}$. Let $f=f_{\underline{\underline{a}},\underline{\underline{b}}}$ and $a^{-1}b=sft$ with $s\in G_x$, $t\in G_y$. It follows $(F_1(a)1_{x_0x})^{-1}(F_1(b)1_{x_0y})=1_{x_0y})=1_{xx_0}\ F_1(a^{-1}b)1_{x_0y}=1_{xx_0}F_1(s)F_1(f)F_1(t)1_{x_0y}=$ $=sft$, i.e., $f'_{F_2(\underline{\underline{a}}),F_2(\underline{\underline{b}})}=f$, and hence $d'(F_2(\underline{\underline{a}}),\ F_2(\underline{\underline{b}}))=|f|=$ $=d(\underline{\underline{a}},\underline{\underline{b}})$, as required. ☐

5. Structure of a group acting on a Λ- tree

The aim of this section is to extend to arbitrary ordered abelian groups the structure theorem for a group acting on a $\underline{Z}$-tree [3] Ch.I, Theorem 13.

Consider two pairs $(G,\underline{\underline{X}})$, $(G',\underline{\underline{X}}')$, where $\underline{\underline{X}}$, $\underline{\underline{X}}'$ are connected Λ-graphs and G,G' are groups acting from the left respectively on $\underline{\underline{X}}$, $\underline{\underline{X}}'$. By a morphism from $(G,\underline{\underline{X}})$ to $(G',\underline{\underline{X}}')$ we understand a pair $F=(F_1,F_2)$ where $F_1:G\to G'$ is a homomorphism and $F_2:\underline{\underline{X}}\to\underline{\underline{X}}'$ is a morphism of Λ-graphs which are compatible with the actions of the groups G and G', i.e., $F_2(sh)=F_1(s)F_2(h)$ for $s\in G$, $h\in$arrow $\underline{\underline{X}}$. We say that the morphism F is an <u>epi</u> if F_1,F_2 are both onto.

<u>Definition 5.1.</u> The epi $F=(F_1,F_2):(G,\underline{\underline{X}})\to(G',\underline{\underline{X}}')$ is a <u>cover</u> (we may also say that $(G,\underline{\underline{X}})$ is a cover of

$(G', \underline{\underline{X}}'))$ if the following conditions are satisfied:

i) if h_1, h_2 are arrows of $\underline{\underline{X}}$ and $F_2(h_1)=F_2(h_2)$ then $h_2=sh_1$ for some $s\in G$;

ii) F_1 is <u>locally isomorphism</u>, i.e., for each vertex x of $\underline{\underline{X}}$ the homomorphism $G_x \to G'_{F_2(x)}$, between the stabilizers of x, $F_2(x)$ in G, G', induced by F_1, is an isomorphism;

iii) F_2 preserves the length, i.e., $|F_2(h)|=|h|$ for each arrow h of $\underline{\underline{X}}$.

<u>Lemma 5.2.</u> If $F=(F_1,F_2):(G,\underline{\underline{X}}) \to (G,\underline{\underline{X}}')$ is a cover then the following conditions are satisfied:

a) The maps $G\backslash X \to G'\backslash X'$ and $G\backslash \text{arrow } \underline{\underline{X}} \to G'\backslash \text{arrow } \underline{\underline{X}}'$ induced by F_2 are bijective.

b) For each arrow h of $\underline{\underline{X}}$ the homomorphism $G_h \to G'_{F_2(h)}$ between the stabilizers of the arrows h and $F_2(h)$ in G, G', induced by F_1, is an isomorphism.

c) F_2 is <u>locally bijective</u>, i.e., for each $x\in X$, the map F_2 induces a bijection between $\underline{\underline{X}}(x,-)=\bigcup_{y\in X}\underline{\underline{X}}(x,y)$ and $\underline{\underline{X}}'(F_2(x), -)=\bigcup_{y'\in X'}\underline{\underline{X}}'(F_2(x), y')$.

d) If $\underline{\underline{X}}'$ is an Λ- tree then $\underline{\underline{X}}$ is a Λ- tree and F is an isomorphism.

<u>Proof.</u> a) is immediate by 5.1.i) since F_1 and F_2 are onto.

b) Let $h\in \underline{\underline{X}}(x,y)$, $x,y\in X$. The injectivity of the homomorphism $G_h \to G'_{F_2(h)}$ is immediate by 5.1.ii). To check its surjectivity, let $t\in G'_{F_2(h)} \subset G'_{F_2(x)}$. By 5.1. ii), there is $s\in G_x$

such that $F_1(s)=t$. Consequently, $\underline{\underline{o}}(sh)=\underline{\underline{o}}(h)=x$ and $F_2((sh)h^{-1})=$ $=1_{F_2(x)}$. As F_2 preserves the length by 5.1 iii), we get $sh=h$, i.e., $s\in G_h$, as required.

c) First let us show that the map $F_{2,x}:\underline{\underline{X}}(x,-)\to\underline{\underline{X}}'(F_2(x),-)$ is injective. Let $h_1,h_2\in\underline{\underline{X}}(x,-)$ be such that $F_2(h_1)=F_2(h_2)$. By 5.1 i), $h_2=sh_1$ for some $s\in G_x$. It follows $F_1(s)\in G'_{F_2(h_1)}$, and hence $s\in G_{h_1}$ by b), i.e., $h_1=h_2$, as required. Next let us show that the map $F_{2,x}$ is onto. Let $h'\in\underline{\underline{X}}'(F_2(x),-)$. As F_2: arrow $\underline{\underline{X}}\to$ arrow $\underline{\underline{X}}'$ is onto, there is an arrow h of $\underline{\underline{X}}$ such that $F_2(h)=h'$. Let $y:=\underline{\underline{o}}(h)$. As $F_2(y)=\underline{\underline{o}}(h')=F_2(x)$, it follows by 5.1 i) that $x=sy$ for some $s\in G$. Thus $F_1(s)\in G'_{F_2(x)}$, and hence, by 5.1 ii), $F_1(s)=F_1(t)$ for some $t\in G_x$. Consequently, $t^{-1}sh\in\underline{\underline{X}}(x,-)$ and $F_2(t^{-1}sh)=F_1(t^{-1}s)F_2(h)=h'$, as required.

d) As F is an epi, which preserves the length, it remains to show that F_1 and F_2 are injective. First let us show that $F_2:X\to X'$ is injective. Let $x,y\in X$ be such that $F_2(x)=$ $=F_2(y)$. Since $\underline{\underline{X}}$ is connected, there exists an arrow $h\in\underline{\underline{X}}(x,y)$. As $\underline{\underline{X}}'$ is assumed to be a Λ- tree, we get $F_2(h)=1_{F_2(x)}$ and hence $h=1_x$ by c). Thus $x=y$, as required.

Next let us show that F_1 is injective. Let $s\in\mathrm{Ker}\,F_1$ and $x\in X$. As $F_2(sx)=F_2(x)$ it follows $sx=x$ by the first part of the proof. Since $s\in G_x$ and $F_1(s)=1$, we get $s=1$ by 5.1 ii).

Finally let us show that F_2: arrow $\underline{\underline{X}}\to$ arrow $\underline{\underline{X}}'$ is injective. Let h_1, h_2 be arrows of $\underline{\underline{X}}$ such that $F_2(h_1)=F_2(h_2)$. In particular, $F_2(\underline{\underline{o}}(h_1))=F_2(\underline{\underline{o}}(h_2))$, and hence $\underline{\underline{o}}(h_1)=\underline{\underline{o}}(h_2)$

thanks to the injectivity of $F_2: X \to X'$. According to c), we get $h_1 = h_2$, as required. ■

The next lemma is immediate.

Lemma 5.3. Let $F:(G, \underline{\underline{X}}) \to (G', \underline{\underline{X}}')$ and $F':(G, \underline{\underline{X}}') \to (G'', \underline{\underline{X}}'')$ be epis and assume that F' is a cover. Then $F' \circ F$ is a cover iff F is a cover.

Definition 5.4. The cover $\hat{F}:(\hat{G}, \hat{\underline{\underline{X}}}) \to (G, \underline{\underline{X}})$ is universal if for each cover $F':(G', \underline{\underline{X}}') \to (G, \underline{\underline{X}})$ there is an epi (not necessarily unique) $F:(\hat{G}, \hat{\underline{\underline{X}}}) \to (G', \underline{\underline{X}}')$ such that $F' \circ F = \hat{F}$.

The next lemma plays a key role in the following.

Lemma 5.5. Let $\hat{F}:(\hat{G}, \hat{\underline{\underline{X}}}) \to (G, \underline{\underline{X}})$ be a cover and assume that $\hat{\underline{\underline{X}}}$ is a Λ- tree. Then the cover $\hat{F}$ is universal.

Proof. Consider a cover $F':(G, \underline{\underline{X}}') \to (G, \underline{\underline{X}})$. To construct an epi (in fact a cover by Lemma 5.3) $F:(\hat{G}, \hat{\underline{\underline{X}}}) \to (G', \underline{\underline{X}}')$ such that $F' \circ F = \hat{F}$, we proceed step by step as follows.

1) Fix a point x_0 of the Λ-tree $\hat{\underline{\underline{X}}}$. As the map $F_2': X' \to X$ is onto we may choose a vertex x_0' of $\underline{\underline{X}}'$ such that $F_2'(x_0') = \hat{F}_2(x_0)$. Define a map $F_2: \hat{X} \to X'$ with $F_2(x_0) = x_0'$, as follows. Given $x \in \hat{X}$, it follows by Lemma 5.2, c) that there exists a unique arrow l_x of the Λ-graph $\underline{\underline{X}}'$ such that $\underline{\underline{o}}(l_x) = x_0'$ and $F_2'(l_x)$ is the image $\hat{F}_2(x_0, x)$ of the unique arrow in the Λ-tree $\hat{\underline{\underline{X}}}$ with origin in x_0 and terminus in x. Let us put $F_2(x) = \underline{\underline{t}}(l_x)$.

2) Extend $F_2:\hat{X}\to X'$ to a map $F_2:\hat{X}^2=\text{arrow }\underline{\underline{\hat{X}}}\to\text{arrow }\underline{\underline{X}}'$ by $F_2(x,y)=1_x^{-1}1_y$. Obviously, $\underline{\underline{o}}(F_2(x,y))=F_2(x)$, $\underline{\underline{t}}(F_2(x,y))=F_2(y)$, $F_2(x,x)=1_{F_2(x)}$ and $F_2(x,y)F_2(y,z)=F_2(x,z)$ for $x,y,z\in\hat{X}$. Thus $F_2:\underline{\underline{\hat{X}}}\to\underline{\underline{X}}'$ is a morphism of Λ-graphs and $F_2'\circ F_2=\hat{F}_2$. In fact, according to Lemma 5.2, c), F_2 is the unique morphiom of Λ-graphs subject to $F_2'\circ F_2=\hat{F}_2$ and $F_2(x_o)=x_o'$.

3) Given $s\in\hat{G}$, let us show that there exists a unique $t\in G$ satisfying: $F_1'(t)=\hat{F}_1(s)$ and $F_2(sx)=tF_2(x)$ for each $x\in\hat{X}$. Fix for a moment some point $x\in\hat{X}$. Since $F_1':G'\to G$ is onto, there is $p\in G'$ such that $F_1'(p)=\hat{F}_1(s)$. Consequently, $F_2'(pF_2(x))=\hat{F}_2(sx)=F_2'(F_2(sx))$. As F' is a cover, there is $q\in G'$ such that $qpF_2(x)=F_2(sx)$, and hence $F_1'(q)\in G_{\hat{F}_2(sx)}$. By 5.1. ii), there exists $r\in G'_{F_2(sx)}$ with $F_1'(r)=F_1'(q)$. Let us put $t=r^{-1}qp$. Obviously, $F_1'(t)=\hat{F}_1(s)$ and $F_2(sx)=tF_2(x)$. Now let $y\in\hat{X}$, may be $y=x$, and $t'\in G'$ be such that $F_1'(t')=\hat{F}_1(s)$ and $F_2(sy)=t'F_2(y)$. We have to show that $t=t'$. Consider the arrows $F_2(x,y)\in\underline{\underline{X}}'(F_2(x),F_2(y))$ and $t^{-1}F_2(sx,sy)\in\underline{\underline{X}}'(t^{-1}F_2(sx),t^{-1}F_2(sy))=\underline{\underline{X}}'(F_2(x),t^{-1}t'F_2(y))$. Since the arrows above have the common origin $F_2(x)$ and $F_2'(F_2(x,y))=\hat{F}_2(x,y)=F_2'(t^{-1}F_2(sx,sy))$, it follows by Lemma 5.2, c) that $F_2(x,y)=t^{-1}F_2(sx,sy)$ and hence $t^{-1}t'\in G'_{F_2(y)}$. As $F_1'(t^{-1}t')=1$, we get $t=t'$ by 5.1. ii). Thus there exists a unique map $F_1:\hat{G}\to G'$ subject to $F_1'\circ F_1=\hat{F}_1$ and $F_2(s(x,y))=F_1(s)F_2(x,y)$ for each $s\in\hat{G}$ and each arrow (x,y) in the Λ-tree $\underline{\underline{\hat{X}}}$. The unicity of the map F_1 satisfying the conditions above implies

also that F_1 is a homomorphism. Thus the pair $F=(F_1,F_2)$ is the unique morphism from $(\hat{G},\underline{\underline{\hat{X}}})$ to $(G',\underline{\underline{X}}')$ with the properties $F'\circ F=\hat{F}$ and $F_2(x_0)=x_0'$. To finish the proof we have to show that the morphism F is an epi. First let us chech that the homomorphism $F_1:\hat{G}\to G'$ is onto. Fix some $x\in\hat{X}$ and let $t\in G'$. As $\underline{\underline{X}}'$ is connected, there is an arrow $h\in\underline{\underline{X}}'(F_2(x),\ tF_2(x))$. Since $\hat{F}$ is a cover, there exists, by Lemma 5.2, c), an unique point $y\in\hat{X}$ such that $\hat{F}_2(x,y)=F_2'(h)$. As $F_2'(h)\in\underline{\underline{X}}(\hat{F}_2(x),\ F_2'(t)\hat{F}_2(x))$, it follows by 5.1. i), that $y=sx$ for some $s\in\hat{G}$. Applying Lemma 5.2, c) to the arrows h and $F_2(x,\ sx)$ in $\underline{\underline{X}}'$ having the common origin $F_2(x)$ and the same image through F_2', we get $h=F_2(x,sx)$, and hence $tF_2(x)=\underline{t}(h)=\underline{t}(F_2(x,sx))=F_1(s)F_2(x)$. Thus $t^{-1}F_1(s)\in G'_{F_2(x)}$, and hence $F_1'(t)^{-1}\hat{F}_1(s)\in G_{\hat{F}_2(x)}$. By 5.1. ii), there is $p\in\hat{G}_x$ such that $\hat{F}_1(p)=F_1'(t)^{-1}\hat{F}_1(s)$. Since $t^{-1}F_1(sp^{-1})$ $\in G'_{F_2(x)}$ and $F_1'(t^{-1}F_1(sp^{-1}))=1$, we get $t=F_1(sp^{-1})$, by 5.1.ii), proving that F_1 is onto. It remains to show that $F_2:\hat{X}^2\to$ arrow $\underline{\underline{X}}'$ is onto. Let h be an arrow of $\underline{\underline{X}}'$. As $\hat{F}_2:\hat{X}^2\to$ arrow $\underline{\underline{X}}$ is onto there is an arrow (x,y) of $\underline{\underline{\hat{X}}}$ such that $\hat{F}_2(x,y)=F_2'(h)$. Since the arrows h and $F_2(x,y)$ in $\underline{\underline{X}}'$ have the same image through F_2', it follows by 5.1.i) that $h=tF_2(x,y)$ for some $t\in G'$. As F_1 is onto, there is $s\in\hat{G}$ such that $F_1(s)=t$. Consequently, $h=F_1(s)$ $F_2(x,y)=F_2(sx,\ sy)$, as required. ∎

We are now prepared to prove the main result of the paper.

Theorem 5.6. Given a group G which acts on the connected Λ- graph $\underline{\underline{X}}$, there exists a universal cover $\hat{F}:(\hat{G},\hat{\underline{\underline{X}}})\to(G,\underline{\underline{X}})$. $\hat{\underline{\underline{X}}}$ is a Λ-tree and $\hat{F}$ is unique up to an isomorphism (not necessarily unique).

Proof. Existence: To the pair $(G,\underline{\underline{X}})$ we may assign as in Section 2 a strongly connected Λ-graph of groups $\underline{\underline{Y}}=\underline{\underline{Y}}(G,X;j,\sigma)$ and a maximal subpretree $\underline{\underline{T}}=\underline{\underline{T}}(G,X;j,\sigma)=(Y,(1_{xy})_{(x,y)\in Y^2})$ depending on the choice of the maps $E=G\setminus$ arrow $\underline{\underline{X}}\to$ arrow $\underline{\underline{X}}$ and $\sigma:E\to G$. Further, to the pair $(\underline{\underline{Y}},\underline{\underline{T}})$ onemay assign as in Section 4 a Λ-tree $\tilde{\underline{\underline{Y}}}=\tilde{\underline{\underline{Y}}}_{\underline{\underline{T}}}$ together with an action of the fundamental group $\pi=\pi_1(\underline{\underline{Y}},\underline{\underline{T}})$ on $\tilde{\underline{\underline{Y}}}$. By construction, the maps $G_x\to G:s\mapsto s\in G_{jx}\subset G$ and $\sigma:E\to G$ induce a homomorphism $\hat{F}_1:\pi\to G$. Note that $G_x=G_{jx}$ is identified with a common subgroup of π and G and $\hat{F}_1$ is the identity on G_x for each $x\in Y$. Moreover, $\hat{F}_1$ is onto. Indeed, let $a\in G$ and $x\in Y$. Since $\underline{\underline{X}}$ is connected, the set $\underline{\underline{X}}(jx, ajx)$ is non-empty, and hence there exist $f\in E(x,x)$ and $s\in G_x$ such that $\underline{\underline{t}}(sjf)=ajx$. As $\underline{\underline{t}}(jf)=$ $=\sigma(f)jx$, we get $a^{-1}s\sigma(f)\in G_x$, i.e., $a=s\sigma(f)t$ for some $t\in G_x$. Therefore $a=\hat{F}_1(sft \bmod R_{\underline{\underline{T}}})$, as required.

On the other hand, we may define a morphism of Λ-graphs $\hat{F}_2:\tilde{\underline{\underline{Y}}}\to\underline{\underline{X}}$ as follows. Let us put $\hat{F}_2(\underline{\underline{a}})=\hat{F}_1(a)jx$ for $\underline{\underline{a}}=aG_x\in\tilde{Y}$; obviously, the definition does not depend on the representative a of the coset $\underline{\underline{a}}\in\pi/G_x$. For an arrow of the Λ-tree $\tilde{\underline{\underline{Y}}}$, i.e., an ordered pair $(\underline{\underline{a}},\underline{\underline{b}})$ with $\underline{\underline{a}}=aG_x$, $\underline{\underline{b}}=bG_y\in\tilde{Y}$, we have to define an arrow $\hat{F}_2(\underline{\underline{a}},\underline{\underline{b}})\in\underline{\underline{X}}(\hat{F}_2(\underline{\underline{a}}),\hat{F}_2(\underline{\underline{b}}))$. By Corollary 3.6, there exist $f_{\underline{\underline{a}},\underline{\underline{b}}}\in E(x,y)$, uniquely determined by the cosets $\underline{\underline{a}},\underline{\underline{b}}$, and some $s\in G_x$, $t\in G_y$ such that the identity $a^{-1}b=sf_{\underline{\underline{a}},\underline{\underline{b}}}t$ holds in $\mathcal{T}$. Let us put $\widehat{F}_2(\underline{\underline{a}},\underline{\underline{b}})=\widehat{F}_1(a)sjf_{\underline{\underline{a}},\underline{\underline{b}}}$. By Corollary 3.6, the definition does not depend on the choice of a,b,s,t, and $\widehat{F}_2(\underline{\underline{a}},\underline{\underline{b}})\in \underline{\underline{X}}(\widehat{F}_2(\underline{\underline{a}}),\widehat{F}_2(\underline{\underline{b}}))$. Obviously, $\widehat{F}_2(\underline{\underline{a}},\underline{\underline{a}})=1_{\widehat{F}_2(\underline{\underline{a}})}$ and $|\widehat{F}_2(\underline{\underline{a}},\underline{\underline{b}})|=|f_{\underline{\underline{a}},\underline{\underline{b}}}|=d(\underline{\underline{a}},\underline{\underline{b}})$, so it remains to show that $\widehat{F}_2(\underline{\underline{a}},\underline{\underline{b}})\widehat{F}_2(\underline{\underline{b}},\underline{\underline{c}})=\widehat{F}_2(\underline{\underline{a}},\underline{\underline{c}})$ for $\underline{\underline{a}},\underline{\underline{b}},\underline{\underline{c}}\in\widetilde{Y}$ to conclude that $\widehat{F}_2:\underline{\underline{\widetilde{Y}}}\to\underline{\underline{X}}$ is a morphism of Λ-graphs preserving the length. Let $\underline{\underline{a}}=aG_x$, $\underline{\underline{b}}=bG_y$, $\underline{\underline{c}}=cG_z$, $f:=f_{\underline{\underline{a}},\underline{\underline{b}}}$, $g:=f_{\underline{\underline{b}},\underline{\underline{c}}}$, $a^{-1}b=sft$, $b^{-1}c=$ $=pgq$ with $s\in G_x$, $t,p\in G_y$, $q\in G_z$. We get $a^{-1}c=sftpgq=s\,\rho_A\,\varepsilon(A)\lambda_A'q$, where $A=(f,tp,g)$, $\rho_A\in\rho(A)$ and $\lambda_A'=\lambda_A(\rho_A)^{-1}$. Thus $f_{\underline{\underline{a}},\underline{\underline{c}}}=\varepsilon(A)$ and $\widehat{F}_2(\underline{\underline{a}},\underline{\underline{c}})=\widehat{F}_1(a)s\rho_A\,j\varepsilon(A)=\widehat{F}_1(a)s\left[(jf)(\sigma(f)tpjg\right]=$ $=(\widehat{F}_2(\underline{\underline{a}},\underline{\underline{b}}))(\widehat{F}_1(asftp)jg)=(\widehat{F}_2(\underline{\underline{a}},\underline{\underline{b}}))(\widehat{F}_1(b)pjg)=\widehat{F}_2(\underline{\underline{a}},\underline{\underline{b}})\widehat{F}_2(\underline{\underline{b}},\underline{\underline{c}})$, asrequired. Note also that the map $\widehat{F}_2:\widetilde{Y}^2\to$ arrow $\underline{\underline{X}}$ is onto. Indeed, let h be an arrow of $\underline{\underline{X}}$. Then there exist $f\in E(x,y)$, $x,y\in Y$, and $v\in G$ such that $h=vjf$. Since the homomorphism $\widehat{F}_1:\mathcal{T}\to G$ is onto, there is $a\in\mathcal{T}$ such that $\widehat{F}_1(a)=v$. Let $\underline{\underline{a}}=aG_x$, $\underline{\underline{b}}=bG_y\in\widetilde{Y}$ with $b=af$. As $a^{-1}b=f$ it follows $\widehat{F}_2(\underline{\underline{a}},\underline{\underline{b}})=$ $=\widehat{F}_1(a)jf=vjf=h$, as required.

Obviously, the pair $(\widehat{F}_1,\widehat{F}_2)$ is compatible with the actions of $\mathcal{T}$ and G. Thus $\widehat{F}=(\widehat{F}_1,\widehat{F}_2):(\mathcal{T},\underline{\underline{\widetilde{Y}}})\to(G,\underline{\underline{X}})$ is an epi which preserves the length. To conclude that $\widehat{F}$ is an universal cover it suffices, by Lemma 5.5) to show that $\widehat{F}$ satisfies the conditions 5.1.i), ii).

First let us check 5.1.i). Let $\underline{\underline{a}}=aG_x$, $\underline{\underline{b}}=bG_y$.

$\underline{\underline{a}}'=a'G_{x'}, \underline{\underline{b}}'=b'G_{y'}\in\widetilde{Y}$, $f:=f_{\underline{\underline{a}},\underline{\underline{b}}}$, $f':=f_{\underline{\underline{a}}',\underline{\underline{b}}'}$, $a^{-1}b=sft$, $a'^{-1}b'=s'f't'$ with $s\in G_x$, $t\in G_y$, $s'\in G_{x'}$, $t'\in G_{y'}$, and assume that $\widehat{F}_2(\underline{\underline{a}},\underline{\underline{b}})=\widehat{F}_2(\underline{\underline{a}}',\underline{\underline{b}}')$, i.e., $\widehat{F}_1(a)sjf=\widehat{F}_1(a')s'jf'$. It follows $f=f'$, $x=x'$, $y=y'$ and $p:=s'^{-1}\widehat{F}_1(a'^{-1}a)s\in G_f$. Consequently, $\widehat{F}_1(a'^{-1}a)=s'ps^{-1}\in G_x$, so we may see it as an element of $\mathcal{T}$ and consider the element $c:=a'\widehat{F}_1(a'^{-1}a)a^{-1}$ of $\mathcal{T}$. We get the identities in $\mathcal{T}$: $c=a's'ps^{-1}a^{-1}=b't'^{-1}f^{-1}pftb^{-1}=$ $=b't'^{-1}\omega_f(p)tb^{-1}$. It follows $c\underline{\underline{a}}=caG_x=a'\widehat{F}_1(a'^{-1}a)G_x=a'G_x=$ $=\underline{\underline{a}}'$ and $c\underline{\underline{b}}=cbG_y=b't'^{-1}\omega_f(p)tG_y=b'G_y=\underline{\underline{b}}'$. Thus $c(\underline{\underline{a}},\underline{\underline{b}})=$ $=(\underline{\underline{a}}',\underline{\underline{b}}')$ and 5.1.i) is verified.

The condition 5.1 ii) is immediate since given $\underline{\underline{a}}=aG_x\in\widetilde{Y}$, the stabilizers of $\underline{\underline{a}}$ and $\widehat{F}_2(\underline{\underline{a}})=\widehat{F}_1(a)jx$ in $\mathcal{T}$ and G are respectively $\mathcal{T}_{\underline{\underline{a}}}=aG_xa^{-1}$, $G_{F_2(\underline{\underline{a}})}=\widehat{F}_1(a)G_x\widehat{F}_1(a)^{-1}$, and $\widehat{F}_1$ is the identity on G_x.

<u>Unicity</u>: Immediate by Lemmata 5.2, d) and 5.3. $\square$

<u>Corollary 5.7.</u> (<u>Structure theorem for groups acting on Λ- trees</u>). With the notations above the following statements are equivalent:

i) $\underline{\underline{X}}$ is a Λ- tree.

ii) $\widehat{F}_1:\mathcal{T}\to G$ is an isomorphism.

iii) $\widehat{F}_2:\underline{\underline{\widetilde{Y}}}\to\underline{\underline{X}}$ is an isomorphism of Λ-graphs.

<u>Proof.</u> i)$\Rightarrow$ii) and i)$\Rightarrow$iii) are immediate by Lemma 5.2, d) since $\widehat{F}$ is a cover. iii)$\Rightarrow$i) is trivial since $\underline{\underline{\widetilde{Y}}}$ is a

Λ- tree.

ii)$\Rightarrow$iii) : As $\widehat{F}_2$ is onto and preserves the length, we have only to show that $\widehat{F}_2$ is injective. Let h_1,h_2 be arrows of $\widetilde{\underline{\underline{Y}}}$ such that $\widehat{F}_2(h_1)=\widehat{F}_2(h_2)$. Since $\widehat{F}$ is a cover, it follows by 5.1.i) that $h_2=ah_1$ for some $a\in\widetilde{\pi}$. Thus $\widehat{F}_1(a)$ $\in G_{\widehat{F}_2(h_1)}$, and hence $a\in\widetilde{\pi}_{h_1}$ since by assumption $\widehat{F}_1$ is an isomorphism. Consequently, $h_1=ah_1=h_2$, i.e., $\widehat{F}_2$ is injective.

iii)$\Rightarrow$ ii): As $\widehat{F}_1:\widetilde{\pi}\to G$ is onto, we have only to show that $\widehat{F}_1$ is injective. Let $\underline{\underline{a}}\in\widetilde{Y}$, $s\in\widetilde{\pi}$ and assume $\widehat{F}_1(s)=1$. Since $\widehat{F}_2$ is an isomorphism by assumption and $\widehat{F}_2(s\underline{\underline{a}}) = \widehat{F}_2(\underline{\underline{a}})$, it follows $s\in\widetilde{\pi}_{\underline{\underline{a}}}$, and hence $s=1$, by 5.1.ii). ▫

The most interesting assertion of the corollary above is the implication i)$\Rightarrow$ ii) which gives a description of a group acting on a Λ- tree in terms of generators and relations as in Section 3.

We end this section with the following version of Theorem 5.6 which is an immediate consequence of Theorem 5.6 and of the proof of Lemma 5.5.

<u>Corollary 5.8.</u> Given a group action on a rooted connected Λ- graph, i.e., a triple $(G,\underline{\underline{X}},x_o)$ with $x_o\in X$, there exists a group action on a rooted Λ- tree $(\widehat{G}, \widehat{\underline{\underline{X}}}, \widehat{x}_o)$ and a cover $\widehat{F}=(\widehat{F}_1,\widehat{F}_2) = (\widehat{G},\widehat{\underline{\underline{X}}}) \to (G,\underline{\underline{X}})$ preserving the root, i.e., $\widehat{F}_2(\widehat{x}_o)=$ $=x_o$, such that the following universality property holds:

Given a group action on a rooted connected Λ- graph $(G',\underline{\underline{X}}',x_0')$ and a cover $F'=(F_1',F_2'):(G',\underline{\underline{X}}')\to(G,\underline{\underline{X}})$ with $F_2'(x_0')=$ $=x_0$, there exists a unique cover $F=(F_1,F_2):(\hat{G},\hat{\underline{\underline{X}}})\to(G',\underline{\underline{X}}')$ with $F_2(\hat{x}_0)=x_0'$ such that $F'\circ F=\hat{F}$.

6. Examples

To illustrate the general theory above we give in this section two examples. The first one is quite simple, while the second one concerning the SL_2 of a valued field is more elaborated and susceptible of further developments.

6.1. The isometry group of an ordered abelian group

Given an ordered abelian group Λ, we may assign to Λ two significant structures of connected Λ- graphs. The first one is the Λ- tree $\hat{\underline{\underline{X}}}$ considered in [2], [1], whose universe $\hat{X}$ is the underlying set of the group Λ and whose metric is given by $d(\alpha,\beta)=|\alpha-\beta|$ for $\alpha,\beta\in\Lambda$, with $|\alpha|=\max(\alpha,-\alpha)$. The second one is the Λ- graph $\underline{\underline{X}}$ having a unique vertex, with arrow $\underline{X}=\Lambda$. The groupoid structure on $\underline{\underline{X}}$ is identified with the additive group structure of Λ and the length function $\Lambda\to\Lambda$ is nothing else than the module function $\alpha\mapsto|\alpha|$.

Denote by $\hat{G}$ the isometry group of the Λ- tree $\hat{\underline{\underline{X}}}$ consisting of the translations $t_\alpha: x\mapsto x+\alpha$ and the reflections $r_\alpha: x\mapsto\alpha-x$, for $\alpha\in\Lambda$. On the other hand, denote by G the cyclic group of order two with generator r_0. The group G acts on the Λ- graph $\underline{\underline{X}}$ according to the rule $r_0x=-x$ for $x\in\Lambda$. Let $\hat{F}_1:\hat{G}\to G$ be the homomorphism given by $\hat{F}_1(t_\alpha)=1$, $\hat{F}_1(r_\alpha)=r_0$ for $\alpha\in\Lambda$. Denote by $\hat{F}_2:\hat{\underline{\underline{X}}}\to\underline{\underline{X}}$ the morphism of Λ - graphs which sends the points of $\hat{\underline{\underline{X}}}$ in the unique vertex of $\underline{\underline{X}}$ and each arrow of $\hat{\underline{\underline{X}}}$, i.e., an ordered pair $(\alpha,\beta)\in\Lambda^2$, in the arrow $\beta-\alpha$ of $\underline{\underline{X}}$. It is easy to see that the pair $\hat{F}=(\hat{F}_1,\hat{F}_2):(\hat{G},\hat{\underline{\underline{X}}})\to(G;\underline{\underline{X}})$ is a cover, and hence a universal cover, by Lemma 5.5. Thus we may obtain a description of the group $\hat{G}$ in terms of generators and relations applying to the pair $(G, \underline{\underline{X}})$ the procedure described in the existence part of Theorem 5.6.

First we have to assign a strongly connected Λ-graph of groups $\underline{\underline{Y}}=\underline{\underline{Y}}(G,\underline{\underline{X}};j,\sigma)$ to the pair $(G,\underline{\underline{X}})$. Obviously, $\underline{\underline{Y}}$ has a unique vertex and the space of G-orbits $E=G\backslash\Lambda$ may be identified with the subset of non-negative elements of Λ. Then the maps $\alpha\mapsto\bar{\alpha}$ and $\alpha\mapsto|\alpha|$ are both the identity of E. Let $j:E\to\Lambda$ be the inclusion and $\sigma:E\to G$ be defined by

$$\sigma(\alpha)=\begin{cases}1 & \text{if } \alpha=0\\ r_0 & \text{if } \alpha\neq0\end{cases}$$

. Thus $\theta_\alpha=1$ for each $\alpha\in E$, the stabilizer G_α is trivial if $\alpha\neq0$, respectively G if $\alpha=0$, and $\omega_\alpha=1_{G_\alpha}$. The map $\varepsilon:E\times G\times E\to E$ given by $\varepsilon(\alpha,s,\beta)=|\alpha+\sigma(\alpha)s\beta|=$

$$=\begin{cases}|\alpha-\beta| & \text{if } s=1\\ \alpha+\beta & \text{if } s=r_0\end{cases}$$

. For $A=(\alpha,s,\beta)\in E\times G\times E$, we get:

$$\rho(\Lambda)=\begin{cases} \{1\} & \text{if } \underline{\text{either}}\ s=1 \text{ and } (\alpha>\beta \text{ or } 0=\alpha<\beta) \\ & \quad \underline{\text{or}} \quad s=r_0 \text{ and } \alpha\neq 0. \\ \{r_0\} & \text{if } \underline{\text{either}}\ s=1 \text{ and } 0<\alpha<\beta \\ & \quad \underline{\text{or}} \quad s=r_0 \text{ and } 0=\alpha<\beta \\ G & \text{if } \underline{\text{either}}\ s=1 \text{ and } \alpha=\beta \\ & \quad \underline{\text{or}} \quad s=r_0 \text{ and } \alpha=\beta=0. \end{cases}$$

Choose $\rho_\Lambda \in \rho(\Lambda)$ as follows:

$$\rho_\Lambda=\begin{cases} 1 & \text{if } \underline{\text{either}}\ s=1 \text{ and } (\alpha=0 \text{ or } \alpha\geqslant\beta) \\ & \quad \underline{\text{or}} \quad s=r_0 \text{ and } \alpha\neq 0 \\ r_0 & \text{if } \underline{\text{either}}\ s=1 \text{ and } 0<\alpha<\beta \\ & \quad \underline{\text{or}} \quad s=r_0 \text{ and } \alpha=0 \end{cases}$$

Let $\lambda'_\Lambda=\lambda_\Lambda(\rho_\Lambda)^{-1}=\lambda_\Lambda(\rho_\Lambda)=\sigma(\beta)s\sigma(\alpha)\rho_\Lambda\,\sigma(\varepsilon(\Lambda))$. We get:

$$\lambda'_\Lambda=\begin{cases} 1 & \text{if } \underline{\text{either}}\ s=1 \text{ and } (\alpha\leqslant\beta \text{ or } \beta=0) \\ & \quad \underline{\text{or}} \quad s=r_0 \text{ and } (\alpha=0 \text{ or } \beta\neq 0) \\ r_0 & \text{if } \underline{\text{either}}\ s=1 \text{ and } 0<\beta<\alpha \\ & \quad \underline{\text{or}} \quad s=r_0 \text{ and } 0=\beta<\alpha. \end{cases}$$

The fundamental group $\pi_1(\underline{\underline{Y}})\simeq\hat{G}$ is generated by the elements r_α for $\alpha\in E=\Lambda_{\geqslant 0}$ subject to the relations: $r_0^2=1$ and $r_\alpha s r_\beta = \rho_\Lambda r_{\varepsilon(\Lambda)}\lambda'_\Lambda$ for $\Lambda=(\alpha,s,\beta)\in E\times G\times E$. By explicit computations we get finally the well-known relations: $r_0^2=1$ and $r_\alpha r_0 = r_{\alpha+\beta}r_\beta$ for $\alpha,\beta\in\Lambda_{\geqslant 0}$. We get also easily the well-known presentation of $\hat{G}$ as a semi-direct product of the normal subgroup of translations, which is isomorphic to Λ, and the group $G=\{1, r_o\}$.

6.2. SL_2 of a valued field

Let K be a valued field with valuation v, valuation ring $\mathcal{O}$ and value group $\Lambda=vK$. For $s=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(K)$, let $v(s)=\min(va, vb, vc, vd)$ and $|s|= v(\det s)-2v(s)$. Note that $|s|\geqslant 0$ for each $s\in GL_2(K)$. Consider the equivalence relation $\sim$ on $GL_2(K)$ induced by the map $|\;|: GL_2(K)\to\Lambda$ as follows : $s\sim t \Leftrightarrow |s^{-1}t|=0$, and let X be the quotient set $GL_2(K)/_\sim$. The group $GL_2(K)$ acts obviously from the left on X. The map $GL_2(K)^2\to\Lambda:(s,t)\mapsto |s^{-1}t|$ induces a distance function $d:X^2\to\Lambda$ on X. It is shown in [2], [1] that $\underline{\underline{X}}=(X,d)$ is a Λ- tree and $GL_2(K)$ acts by isometries on $\underline{\underline{X}}$. Consider the pair $(G, \underline{\underline{X}})$, where $G=SL_2(K)$. Our goal is to illustrate Corollary 5.7 by giving an explicit presentation of G by generators and relations.

First we have to construct the strongly connected Λ-graph of groups $\underline{\underline{Y}}=\underline{\underline{Y}}(G,X;j,\sigma)$ associated to the pair $(G,\underline{\underline{X}})$. The set $Y=G\backslash X$ is identified with the underlying set of the factor group $\Lambda/_{2\Lambda}$. Indeed, the canonic epimorphism $v(\det)$: $GL_2(K)\to\Lambda$ $:s\mapsto v(\det(s))$ induces a map $X\to\Lambda/_{2\Lambda}:\hat{s}\mapsto v(\det(s))$ mod 2Λ, which does not depend on the choice of the representative s of $\hat{s}$. Obviously, the map above is onto. One gets a bijection $Y\to\Lambda/_{2\Lambda}$ thanks to the following exact sequence

$$1 \to K^{\times}GL_2(O)SL_2(K) \to GL_2(K) \xrightarrow{v(\det)\bmod 2} \Lambda/2\Lambda \to 0.$$

Fix a section $t:\Lambda \to K^{\times} : \alpha \mapsto t_{\alpha}$ of the valuation v. We may assume $t_0=1$ and $t_{-\alpha}=t_{\alpha}^{-1}$. Fix a section $u:\Lambda/2\Lambda \to \Lambda$ of the canonic map $\Lambda \to \Lambda/2\Lambda$ and let us put $\xi_x=\begin{pmatrix} t_{u(x)} & 0 \\ 0 & 1 \end{pmatrix}$ for $x\in Y$. We may assume $u(0)=0$ and $\xi_0=1$. Thus we get a section $j:X\to X: x\mapsto \hat{\xi}_x$ of the projection map $X \to Y=G\backslash X$.

For $x\in Y$, let G_x : be the stabilizer of jx in G. Thus $G_0=SL_2(O)$ and $G_x=\xi_x G_0 \xi_x^{-1}=$

$$=\left\{\begin{pmatrix} a & t_{u(x)}b \\ t_{u(x)}^{-1}c & t_{u(x)}^{-1}d \end{pmatrix} ; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0\right\}.$$

<u>Lemma 6.2.1.</u> Let $x,y\in Y$. The correspondence $G_x s G_y \mapsto -v(\xi_x^{-1} s\, \xi_y)$ induces a bijection of $G_x\backslash G/G_y$ onto the subset $\{\alpha\in\Lambda : 2\alpha \geqslant u(x)-u(y)\}$ of Λ. Its inverse sends α to $G_x T_\alpha G_y$, where $T_\alpha=\begin{pmatrix} t_\alpha & 0 \\ 0 & t_\alpha^{-1} \end{pmatrix}$.

<u>Proof.</u> Let $p\in G_x\, s\, G_y$. Then $d(jx, pjy)=d(jx,sjy)=$

$=\left|\xi_x^{-1}s\,\xi_y\right| = u(y)-u(x)-2v(\xi_x^{-1}s\,\xi_y)\geqslant 0$. Thus the map is well-defined and onto. It remains to show that $G_x s G_y=G_x T_\alpha G_y$, where $\alpha=-v(\xi_x^{-1}s\,\xi_y)$. Let us put $M=\xi_x^{-1}s\,\xi_y=$

$$= \begin{pmatrix} t_{u(x)}^{-1} t_{u(y)} a & t_{u(x)}^{-1} b \\ t_{u(y)} c & d \end{pmatrix} \quad \text{for} \quad s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

We distinguish 4 cases:

<u>Case 1</u>: $v(M) = v\left(t_{u(x)}^{-1} t_{u(y)} a\right)$. We get

$$s = \begin{pmatrix} 0 & t_{u(x)} \\ -t_{u(x)}^{-1} & t_{u(x)} c a^{-1} \end{pmatrix} \begin{pmatrix} t_{u(x)} t_{u(y)}^{-1} a^{-1} & 0 \\ 0 & t_{u(x)}^{-1} t_{u(y)} a \end{pmatrix} \begin{pmatrix} 0 & -t_{u(y)} \\ t_{u(y)}^{-1} & b a^{-1} t_{u(y)}^{-1} \end{pmatrix}$$

and hence $s \in G_x T_\alpha G_y$.

<u>Case 2:</u> $v(M)=v(t_{u(x)}^{-1} b)$. Then

$$s = \begin{pmatrix} 0 & t_{u(x)} \\ -t_{u(x)}^{-1} & t_{u(x)} d b^{-1} \end{pmatrix} \begin{pmatrix} t_{u(x)} b^{-1} & 0 \\ 0 & t_{u(x)}^{-1} b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a b^{-1} & 1 \end{pmatrix} \in G_x T_\alpha G_y.$$

<u>Case 3:</u> $v(M)=v(t_{u(y)} c)$. It follows

$$s = \begin{pmatrix} 1 & a c^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_{u(y)}^{-1} c^{-1} & 0 \\ 0 & t_{u(y)} c \end{pmatrix} \begin{pmatrix} 0 & -t_{u(y)} \\ t_{u(y)}^{-1} & d c^{-1} t_{u(y)}^{-1} \end{pmatrix} \in G_x T_\alpha G_y.$$

<u>Case 4:</u> $v(M)=v(d)$. We get

$$s = \begin{pmatrix} 1 & b d^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c d^{-1} & 1 \end{pmatrix} \in G_x T_\alpha G_y. \quad \square$$

For $x,y\in Y$, let us put $E(x,y)=\{(x,\alpha,y):\alpha\in\Lambda, 2\alpha\geqslant u(x)-u(y)\}$, and let $E=\bigcup_{(x,y)\in Y^2} E(x,y)$. The map arrow $\underline{\underline{X}}=X^2\to E$: $:(sjx,\ pjy)\mapsto(x,-v(\xi_x^{-1}s^{-1}p\xi_y),y)$ identifies E with the space of orbits $G\backslash$ arrow $\underline{\underline{X}}$.

We get $\overline{(x,\alpha,y)}=(y,\alpha+u(y)-u(x),x)$; in particular, $\overline{(x,\alpha,x)}=(x,\alpha,x)$. Let us put $1_{xy}=\begin{cases}(x,0,y) & \text{if } u(x)\leqslant u(y)\\ (x,u(x)-u(y),y), & \text{otherwise.}\end{cases}$

In other words, $1_{xy}=(x,\max(0,\ u(x)-u(y)),y)$.

Thus $1_x:=1_{xx}=(x,0,x)$, $\overline{1_{xy}}=1_{yx}$. Define the length function $|\ |:E\to\Lambda$ by $|(x,\alpha,y)|=u(y)-u(x)+2\alpha$. For $\alpha\in\Lambda$,

let $S_\alpha=\begin{pmatrix}0 & t_\alpha\\ -t_\alpha^{-1} & 0\end{pmatrix}$; S_α has order 4 and $S_\alpha^2=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. Define a map $\sigma:E\to G$ by

$$\sigma(x,\alpha,y)=\begin{cases} T_\alpha & \text{if } u(x)<u(y)\\ T_{u(x)-u(y)-\alpha} & \text{if } u(x)>u(y)\\ T_0=1 & \text{if } x=y,\ \alpha=0\\ S_{u(x)+\alpha} & \text{if } x=y,\ \alpha>0\end{cases}$$

Thus $\sigma(\overline{(x,\alpha,y)})=\sigma(x,\alpha,y)^{-1}$ *if $x\neq y$. Extend $j:Y\to X$,* to a section $j:E\to$ arrow $\underline{\underline{X}}$ of the projection map arrow $\underline{\underline{X}}\to E$ by $j(x,\alpha,y)=(jx,\ \sigma(x,\alpha,y)jy)$. In particular, we get $j1_{xy}=(jx,\ jy)$. Denote by $G_{(x,\alpha,y)}$ the stabilizer in G of $j(x,\alpha,y)$. Thus $G_{(x,\alpha,y)}=G_x\cap\sigma(x,\alpha,y)G_y\sigma(x,\alpha,y)^{-1}$. Let $\Theta_{(x,\alpha,y)}=\sigma(x,\alpha,y)\ \sigma(\overline{(x,\alpha,y)})\in G_{(x,\alpha,y)}$. We get

$$\Theta(x,\alpha,y)=\begin{cases} 1 & \text{if } x\neq y \text{ or } x=y \text{ and } \alpha=0 \\ \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} & \text{if } x=y \text{ and } \alpha>0.\end{cases}$$ Denote

by $\omega_{(x,\alpha,y)}: G_{(x,\alpha,y)} \to G_{\overline{(x,\alpha,y)}}$ the isomorphism

$s \mapsto \sigma(x,\alpha,y)^{-1}\, s\, \sigma(x,\alpha,y)$.

Now let us define the map $\varepsilon: E(x,y)\times G_y\times E(y,z) \to E(x,z)$ for $x,y,z\in Y$, by $\varepsilon((x,\alpha,y),s\ ,(y,\beta,z))=(x,\gamma,z)$, where $\gamma=-v(\zeta_x^{-1}\,\sigma(x,\alpha,y)s\,\sigma(y,\beta,z)\,\zeta_z)$. For $A=((x,\alpha,y),\ s,(y,\beta,z))$ $\in E(x,y)\times G_y\times E(y,z)$, let

$\bar{A}=(\overline{(y,\beta,z)},\ \Theta_{(y,\beta,z)}s^{-1}\,\Theta_{\overline{(x,\alpha,y)}}\ ,\ \overline{(x,\alpha,y)})$,

$\rho(A)=G_x\cap\sigma(x,\alpha,y)s\sigma(y,\beta,z)G_z\,\sigma(\varepsilon(A))^{-1}\in G_x/G_{\varepsilon(A)}$,

$\lambda_A:\rho(A)\to\rho(\bar{A})$ be the bijection given by $\lambda_A(p)=$ $=\sigma(y,\beta,z)^{-1}s^{-1}\sigma(x,\alpha,y)^{-1}p\,\sigma(\varepsilon(A))$. Thus we constructed a strongly connected Λ-graph of groups $\underline{\underline{Y}}$ and a maximal subprotree $(1_{xy})_{(x,y)\in Y^2}$. To get the desired presentation of G it suffices to compute $\pi_1(\underline{\underline{Y}},\ x_0)$ for some vertex x_0 of $Y=\Lambda/_{2\Lambda}$. Take for simplicity $x_0=0 \bmod 2\Lambda$. Then $E(x_0,x_0)$ is identified with $\Lambda_{\geqslant 0}$, $\sigma(\alpha)=\begin{cases} 1 & \text{if } \alpha=0\\ S_\alpha & \text{if } \alpha>0\end{cases}$, $G_\alpha=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in G_0 : v(b)\geqslant 2\alpha\right\}$, the automorphism ω_α of G_α is the identity if $\alpha=0$ and $\omega_\alpha(s)=\begin{pmatrix} d & -t_\alpha^2 c\\ -bt_\alpha^{-2} & a\end{pmatrix}$ if $s=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in G_\alpha$, $\alpha>0$. Furhter, we get $\varepsilon(A)=\max(\alpha+\beta-v(b),|\alpha-\beta|)$ for $A=(\alpha,s,\beta)\in\Lambda_{\geqslant 0}\times G_0\times\Lambda_{\geqslant 0}$,

$s=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Doing the explicit computations it follows that $G \simeq \pi_1(\underline{\underline{Y}}, 0)$ is generated by $G_0=SL_2(\mathcal{O})$ and the elements

$S_\alpha = \begin{pmatrix} 0 & t_\alpha \\ -t_\alpha^{-1} & 0 \end{pmatrix}$ for $0<\alpha \in \Lambda$, subject to the relations:

(1) $\quad S_\alpha^2 = -I$

(2) $\quad S_\alpha\, \omega_\alpha(s) = s\, S_\alpha$ for $s \in G_\alpha$

(3) $$S_\alpha \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} S_\beta = \begin{pmatrix} 1 & -t_\alpha^2 b^{-1} \\ 0 & 1 \end{pmatrix} S_\gamma \begin{pmatrix} -t_\alpha^{-1} t_\beta^{-1} t_\gamma b & t_\alpha^{-1} t_\beta t_\gamma \\ 0 & -t_\alpha t_\beta t_\gamma^{-1} b^{-1} \end{pmatrix}$$

for $0 \leq v(b) < 2 \min(\alpha, \beta)$, $\gamma = \alpha + \beta - v(b)$.

In particular, if $\Lambda = \underline{\mathbb{Z}}$, i.e., the valuation v is discrete with $v(\pi)=1$ for some $\pi \in K$, the group $G=SL_2(K)$ is generated by $G_0=SL_2(\mathcal{O})$ and the element $S=\begin{pmatrix} 0 & \pi \\ -\frac{1}{\pi} & 0 \end{pmatrix}$ subject to the relations:

(1)' $\quad S^2 = -I$

(2)' $$S \begin{pmatrix} d & -\pi^2 c \\ -\frac{b}{\pi^2} & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} S \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0,\ v(b) \geq 2.$$

(3)' $$S \begin{pmatrix} 1 & \pi b \\ 0 & 1 \end{pmatrix} S = \begin{pmatrix} 1 & -\pi b^{-1} \\ 0 & 1 \end{pmatrix} S \begin{pmatrix} -b & \pi \\ 0 & -b^{-1} \end{pmatrix}$$ for

$v(b)=0$.

Let $G_1 = \left\{ \begin{pmatrix} a & \pi^{-1}b \\ \pi c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0 \right\}$ and

$B = G_0 \cap G_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0 : v(c) \geq 1 \right\}$. Let $\underline{\underline{F}}$ be the free product of the group G_0 and the infinite cyclic group with generator $\underline{\underline{S}}$. Consider the maps $G_0 \to \underline{\underline{F}} : s \mapsto s$ and $G_1 \to \underline{\underline{F}}$:

$$s \mapsto s \text{ if } s \in B,\quad s \mapsto \begin{pmatrix} 0 & 1 \\ -1 & \pi d b^{-1} \end{pmatrix} \underline{\underline{S}} \begin{pmatrix} -\pi a & -b \\ b^{-1} & 0 \end{pmatrix} \text{ if } s = \begin{pmatrix} a & \pi^{-1}b \\ \pi c & d \end{pmatrix},$$

$v(b)=0$. One checks easily that the maps above induce an isomorphism from the sum $G_0 *_B G_1$ of the groups G_0 and G_1 amalgamated along their intersection B onto the quotient of $\underline{\underline{F}}$ by its normal subgroup generated by the relations (1)', (2)', (3)'. Thus we recover Theorem 7 from [3] Ch. II.

Remark. A quite similar result may be obtained by applying the procedure above to the tree considered by Morgan in [4]. Let (K,v) be a valued field with valuation ring $\mathcal{O}$ and value group Λ and assume that the residue field is formally real. Consider the non-degenerate symmetric bilinear form $xy = x_0y_1 + x_1y_0 + \sum_{i=2}^{n} x_iy_i$ on the vector space K^{n+1}, $n \geq 1$. Let $G = SO(n,1)_K$ be the subgroup of $SL_{n+1}(K)$ consisting of the matrices which preserve the bilinear form above. To an $\mathcal{O}$-lattice L in K^{n+1} ~~, n≥1. Let G=SO(n,1)~~ $~~_K~~$ ~~be the subgroup of SL~~ $~~_{n+1}~~$ ~~(K) consisting of the matrices which preserve the bilinear form above. To an —— lattice L in K~~ $~~^{n+1}~~$ one assigns its dual lattice $L^* = \{ x \in K^{n+1} : x.L \subset \mathcal{O} \}$. L is unimodular if $L = L^*$. It is shown in [4] that the set X of unimodular lattices in K^{n+1}

has a canonic structure of Λ- tree on which the group G acts transitively. Denote by L_o the standard lattice $\mathcal{O}^{n+1}$ and $G_o=G\cap SL_{n+1}(\mathcal{O})$ be its stabilizer in G. The distance function $d:X^2\to\Lambda$ is given by $d(sL_o, tL_o)=-v(s^{-1}t)$ for $s,t\in G$, where $v(s)=\min_{i,j} v(s_{ij})$ for $s=(s_{ij})_{0\leq i,j\leq n}$. For n=1, the Λ-tree $\underline{\underline{X}}=(X,d)$ is identified with Λ, $d(\alpha,\beta)=|\alpha-\beta|$, and $G\simeq K^\times$ acts on Λ through translations : $s\alpha=v(s)+\alpha$ for $s\in K^\times$, $\alpha\in\Lambda$. Thus we may assume in the following that $n\geq 2$.

For $\alpha\in\Lambda$, $\alpha>0$, choose $t_\alpha\in K^\times$ such that $v(t_\alpha)=\alpha$, and denote by S_α the matrix in G

$$\begin{array}{c c}
 & \begin{array}{cccc} 1 & 1 & n-2 & 1 \end{array} \\
\begin{array}{c} 1 \\ 1 \\ n-2 \\ 1 \end{array} & \begin{pmatrix} 0 & t_\alpha & 0 & 0 \\ t_\alpha^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1_{n-2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{array}$$

Put $S_o=1_{n+1}$. The map $j:\Lambda_{\geq 0}\to X^2:\alpha\mapsto(L_o,S_\alpha L_o)$ identifies $\Lambda_{\geq 0}$ with the space of G-orbits $G\backslash X^2$. Note that $(j\alpha)^{-1}=$ $=(S_\alpha L_o,L_o)=S_\alpha j\alpha$ for each $\alpha\in\Lambda_{\geq 0}$. Thus the Λ-graph of groups $\underline{\underline{Y}}$ assigned to the pair $(G,\underline{\underline{X}})$ has a unique vertex and the set $E=E_{\underline{\underline{Y}}}$ of edges is identified with $\Lambda_{\geq 0}$. It follows that $\overline{\alpha}=\alpha$, $G_\alpha:=G_{j\alpha}=G_o\cap S_\alpha G_o S_\alpha=\{s\in G_o : v(s_{01})\geq 2\alpha\}$, $\theta_\alpha=S_\alpha^2=1_{n+1}$ and the automorphism ω_α of G_α is given by $\omega_\alpha(s)=S_\alpha sS_\alpha$ for $s\in G_\alpha$. Now we have to describe explicitely the map $\varepsilon:E\times G_o\times E\to E$.

Claim. $\mathcal{E}(\alpha,s,\beta)=\max(|\alpha-\beta|,\alpha+\beta-v(s_{01}))$ for $\alpha,\beta\in E=\Lambda_{\geq 0}$, $s=(s_{ij})_{0\leq i,j\leq n}\in G_0$.

As the case $\alpha=0$ or $\beta=0$ is trivial we may assume $\alpha,\beta>0$. We have to verify that $v(S_\alpha sS_\beta)=\min(-|\alpha-\beta|,v(s_{01})-\alpha-\beta)$. Taking into account the form of the matrix $S_\alpha s S_\beta$ it suffices to check that $v(s_{00})=v(s_{11})=0$, $v(s_{01})=2\min_{2\leq i\leq n} v(s_{i1})$ and $\min_{2\leq i\leq n} v(s_{0i})\geq \min_{2\leq i\leq n} v(s_{i1})$ if $v(s_{01})>0$.

Assuming $v(s_{01})>0$ it follows $2\min_{2\leq i\leq n} v(s_{i1})=v(s_{01}s_{11})$ since $2s_{01}s_{11}+\sum_{i=2}^{n}s_{i1}^2=0$ and the residue field of v is formally real. On the other hand, the identities $s_{00}s_{11}+s_{10}s_{01}+\sum_{i=2}^{n}s_{i0}s_{i1}=1$ and $s_{01}s_{1j}+s_{11}s_{0j}+\sum_{i=2}^{n}s_{i1}s_{ij}=0$ for $2\leq j\leq n$ imply $v(s_{00})=v(s_{11})=0$, $v(s_{01})=2\min_{2\leq i\leq n} v(s_{i1})$ and $\min_{2\leq i\leq n} v(s_{0i})\geq \min_{2\leq i\leq n} v(s_{i1})$.

Claim. The group G is described in terms of generators and relations as follows:

Generators: G_0 and the matrices S_α for $\alpha>0$.

Relations: (1) $S_\alpha^2=1_{n+1}$ for $\alpha>0$.

(2) $S_\alpha s=\omega_\alpha(s)S_\alpha$ for $\alpha>0$, $s\in G_\alpha$.

(3) $S_\alpha s S_\beta=s'S_\gamma s''$ for $\alpha,\beta>0$,

$s\in G_0$ with $s_{00}=s_{11}=1$, $s_{ij}=\delta_{ij}$ for $i\neq 0$, $j\neq 1$ (thus $s_{i1}=-s_{0i}$ for $2\leq i\leq n$ and $2s_{01}+\sum_{i=2}^{n}s_{i1}^2=0$), $v(s_{01})<2\min(\alpha,\beta)$, $\gamma=\alpha+\beta-v(s_{01})$, $s'\in G_0$ with $s'_{00}=s'_{11}=1$, $s'_{ij}=\delta_{ij}$ for $i\neq 0$,

$j\neq 1$, $s'_{01}=t_\alpha^2 s_{01}^{-1}$, $s'_{i1}=t_\alpha s_{i1}s_{01}^{-1}$ for $2\leq i\leq n-1$, $s'_{n1}=$ $=-t_\alpha s_{n1}s_{01}^{-1}$, $s'_{0i}=-s'_{i1}$ for $2\leq i\leq n$, and $s''\in G_0$ with

$$s''_{i0}=t_\alpha^{-1}t_\beta^{-1}t_\gamma s_{01}\delta_{i0}\quad,\quad s''_{1i}=t_\alpha t_\beta t_\gamma^{-1}s_{01}^{-1}\delta_{1i}$$

for $0\leq i\leq n$, $s''_{01}=t_\alpha^{-1}t_\beta t_\gamma$, $s''_{0i}=-t_\alpha^{-1}t_\gamma s_{i1}$ for $2\leq i\leq n-1$, $s''_{0n}=t_\alpha^{-1}t_\gamma s_{n1}$, $s''_{i1}=-t_\beta s_{i1}s_{01}^{-1}$ for $2\leq i\leq n$,

$s''_{ij}=\delta_{ij}+s_{i1}s_{j1}s_{01}^{-1}$ for $2\leq i\leq n$, $2\leq j\leq n-1$,

$s''_{in}=-\delta_{in}-s_{i1}s_{n1}s_{01}^{-1}$ for $2\leq i\leq n$.

The claim above is a consequence of the general theory and of the following fact: Let $s=(s_{ij})_{0\leq i,j\leq n}\in G_0$. We distinguish two cases.

<u>Case 1:</u> $v(s_{00})>0$. As $2s_{00}s_{10}+\sum_{i=2}^{n}s_{i0}^2=0$, $s_{00}s_{11}+$ $+s_{10}s_{01}+\sum_{i=2}^{n}s_{i0}s_{i1}=1$ and the residue field of v is formally real it follows $\min_{2\leq i\leq n} v(s_{i0})>0$ and $v(s_{01})=v(s_{10})=0$. Let $p\in\bigcap_{\alpha>0}G_\alpha$ and $q\in G_0$ be the matrices given by $p_{0i}=s_{01}\delta_{0i}$, $p_{i1}=s_{11}^{-1}\delta_{i1}$ for $0\leq i\leq n$, $p_{10}=-\frac{1}{2}s_{01}^{-1}\sum_{i=2}^{n}(1+s_{i1})^2$, $p_{i0}=1+s_{i1}$ for $2\leq i\leq n$, $p_{i1}=-s_{01}^{-1}(1+s_{i1})$ for $2\leq i\leq n$, $p_{ij}=\delta_{ij}$ for $2\leq i,j\leq n$; $q_{00}=q_{11}=1$, $q_{ij}=\delta_{ij}$ for $i\neq 0$, $j\neq 1$, $q_{0i}=-\frac{2}{n-1}$ for $1\leq i\leq n$ and $q_{i1}=\frac{2}{n-1}$ for $2\leq i\leq n$. One gets $(pq)^{-1}s\in\bigcap_{\alpha>0}G_\alpha$.

<u>Case 2:</u> $v(s_{00})=0$. Then $s=pqr$, where $p\in\bigcap_{\alpha>0}G_\alpha$, $q\in G_0$ and $r\in\bigcap_{\alpha>0}G_\alpha$ are given by $p_{i0}=s_{i0}$ for $0\leq i\leq n$, $p_{11}=s_{00}^{-1}$, $p_{1i}=-s_{00}^{-1}s_{i0}$ for $2\leq i\leq n$, $p_{ij}=\delta_{ij}$ for $i\neq 1$, $j\neq 0$;

$q_{00}=q_{11}=1$, $q_{01}=s_{00}^{-1}s_{01}$, $q_{0i}=-q_{i1}=s_{00}^{-1}s_{01}s_{i0}-s_{i1}$ for $2\leq i\leq n$, $q_{ij}=\delta_{ij}$ for $i\neq 0$, $j\neq 1$; $r_{ij}=\delta_{ij}$ for $i=0,1$ or $j=0,1$ and $r_{ij}=s_{ij}-s_{00}^{-1}s_{i0}s_{0j}$ for $2\leq i,j\leq n$.

In particular, if the valuation v is discrete, the group G is generated by G_0 and the matrix $S=S_1$ subject to the relations $S^2=1_{n+1}$ and $Ss=\omega_1(s)S$ for $s\in G_1=\{p\in G_0 : v(p_{01})\geqslant 2\}$. Thus G is the group derived from (G_1, G_0, ω_1) by a HNN-like construction. It follows that G is the semi-direct product of the cyclic group of order two generated by S and the normal subgroup U generated by G_0, consisting of the matrices s for which $v(s)\equiv 0 \bmod 2$. Moreover U is the amalgamated sum of G_0 and its conjugate SG_0S along the intersection G_1.

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